Chapter 6

Harmonic Measure

Prologue: It is unfortunate that most graduate complex analysis courses do not treat harmonic measure. Indeed, most complex analysis texts do not cover the topic.

The term “harmonic measure” was introduced by R. Nevanlinna in the 1920s. But the ideas were anticipated in work of the Riesz brothers and others. It is still studied intensively today.

Harmonic measure is a natural outgrowth of the study of the Dirichlet problem. It is a decisive tool for measuring the growth of functions, and to seeing how they bend. It has been used in such diverse applications as the corona problem and in mapping problems. Harmonic measure has particularly interesting applications in probability theory—especially in the consideration of diffusions and Brownian motion.

It requires a little sophistication to appreciate harmonic measure. And calculating the measure—working through the examples—requires some elbow grease. But the effort is worth it and the results are substantial.

This chapter gives a basic introduction to the idea of harmonic measure.
CHAPTER 6. HARMONIC MEASURE

6.1 The Idea of Harmonic Measure

Capsule: Harmonic measure is a way of manipulating the data that comes from the Dirichlet problem. It is a way of measuring the geometry of a domain, and can be used to control the growth of harmonic functions.

Let $\Omega \subseteq \mathbb{C}$ be a domain with boundary consisting of finitely many Jordan curves (we call such a domain a Jordan domain). Let $E$ be a finite union of arcs in $\partial \Omega$ (we allow the possibility of an entire connected boundary component—a simple, closed curve—to be one of these arcs). Then the harmonic measure of $E$ at the point $z \in \Omega$ with respect to $\Omega$ is the value at $z$ of the bounded harmonic function $\omega$ on $\Omega$ with boundary limit 1 at points of $E$ and boundary limit 0 at points of $\partial \Omega \setminus E$ (except possibly at the endpoints of the arcs that make up $E$). We denote the harmonic measure by $\omega(z, \Omega, E)$.

The first question to ask about harmonic measure is that of existence and uniqueness: does harmonic measure always exist? If it does exist, is it unique? The answer to both these queries is “yes.” Let us see why.

Fix a Jordan domain $\Omega$. It is a classical result (see [AHL2], as well as our Section 1.5) that there is a conformal mapping $\Phi : \Omega \to U$ of $\Omega$ to a domain $U$ bounded by finitely many circles. If $F \subseteq \partial \Omega$ consists of finitely many circular arcs, then it is obvious that the harmonic measure of $F$ exists. It would just be the Poisson integral of the characteristic function of $F$, and that Poisson integral can be shown to exist by elementary conformal mapping arguments (i.e., mapping to the disc). Now a classic result of Carathéodory guarantees that $\Phi$ and its inverse extend continuously to the respective boundaries, so that the extended function is a homeomorphism of the closures. Thus we may pull the harmonic measure back from $U$ to $\Omega$ via $\Phi$.

As for uniqueness: if $\Omega$ is bounded, then uniqueness follows from the standard maximum principle. For the general case, we need an extended maximum principle that is due to Lindelöf:

**Proposition 6.1** Let $\Omega$ be a domain whose boundary is not a finite set. Let $u$ be a real-valued harmonic function on $\Omega$, and assume that there is a real constant $M > 0$ such that

$$u(z) \leq M, \quad \text{for } z \in \Omega.$$
6.1. THE IDEA OF HARMONIC MEASURE

Suppose that $m$ is a real constant and that
\[
\limsup_{z \to \zeta} u(z) \leq m \tag{6.1.1}
\]
for all except possibly finitely many points $\zeta \in \partial \Omega$. Then $u(z) \leq m$ for all $z \in \Omega$.

**Remark 6.2** Observe that any Jordan domain certainly satisfies the hypotheses of the proposition. Furthermore, by classical results coming from potential theory (see [AHL2]), any domain on which the Dirichlet problem is solvable will satisfy the hypotheses of the theorem. Note, in particular, that $\Omega$ certainly need not be bounded.

We conclude by noting that this result is a variant of the famous Phragmen-Lindelöf theorem (see [RUD2]).

**Proof of Proposition 6.1:** First assume that $\Omega$ is bounded. We will remove this extra hypothesis later. Let the diameter of $\Omega$ be $d$. Let the exceptional boundary points (at which (6.1.1) does not hold) be called $\zeta_1, \ldots, \zeta_k$. Let $\epsilon > 0$. We then may apply the ordinary textbook maximum principle (see [GRK1]) to the auxiliary function
\[
h(z) = u(z) + \epsilon \sum_{j=1}^{k} \log \frac{|z - \zeta_j|}{d}.
\]
Notice that $h$ (instead of $u$) satisfies the hypotheses of the proposition at every boundary point. So $h \leq m$ on all of $\Omega$. Then we let $\epsilon \to 0$ and the result follows.

Now consider the case of $\Omega$ unbounded. If $\Omega$ has an exterior point (i.e., a point in the interior of the complement of the closure of $\Omega$), then we may apply an inversion and reduce to the case in the preceding paragraph.

Finally, if $\Omega$ has no exterior point, then let $R > |\zeta_j|$ for all $j = 1, \ldots, k$. Let
\[
\Omega_1 = \{z \in \Omega : |z| < R\}
\]
and
\[
\Omega_2 = \{z \in \Omega : |z| > R\}.
\]
These may not be domains (i.e., they could be disconnected), but they are certainly open sets. Let
\[
S = \{z \in \Omega : |z| = R\}.
\]
If $u \leq m$ on $S$, then we apply the result of the first paragraph on $\Omega_1$ and the result of the second paragraph on $\Omega_2$. 
If it is not the case that $u \leq m$ on $S$, then $u$ will have a maximum $N$ on $S$ with $N > m$. But this will be a maximum for $u$ on all of $\Omega$. Since $u \leq m$ at the ends of the arcs of $S$, it follows that $u$ actually achieves the maximum $N$ on $S$. Therefore, by the usual maximum principle, $u$ is identically equal to the constant $N$. But then the boundary condition (6.1.1) cannot obtain. That is a contradiction.

It is important to realize that, if $\Omega$ has reasonably nice boundary, then $\omega(z, \Omega, E)$ is nothing other than the Poisson integral of the characteristic function of $E$. (We used this fact in our discussion of the existence of harmonic measure.) In any event, by the maximum principle the function $\omega$ takes real values between 0 and 1. Therefore $0 < \omega(z, \Omega, E) < 1$.

One of the reasons that harmonic measure is important is that it is a conformal invariant. Essential to this fact is Carathéodory’s theorem, that a conformal map of Jordan domains will extend continuously to the closures. We now formulate the invariance idea precisely.

**Proposition 6.3** Let $\Omega_1, \Omega_2$ be domains with boundaries consisting of Jordan curves in $\mathbb{C}$ and $\varphi : \Omega_1 \rightarrow \Omega_2$ a conformal map. If $E_1 \subseteq \partial \Omega_1$ is an arc and $z \in \Omega_1$ then

$$\omega(z, \Omega_1, E_1) = \omega(\varphi(z), \Omega_2, \varphi(E_1)).$$

Here $\varphi(E_1)$ is well defined by Carathéodory’s theorem.

**Proof:** Let $h$ denote the harmonic function $\omega(\varphi(z), \Omega_2, \varphi(E_1))$. Then $h \circ \varphi$ equals harmonic measure for $\Omega_1$ (by Carathéodory’s theorem).

**Remark 6.4** It is worth recording here an important result of F. and M. Riesz (see [KOO, page 72]). Let $\Omega \subseteq \mathbb{C}$ be a Jordan domain bounded by a rectifiable boundary curve. Let $\varphi : D \rightarrow \Omega$ be a conformal mapping. If $E \subseteq \partial D$ has Lebesgue linear measure zero, then $\varphi(E)$ also has Lebesgue linear measure zero.

## 6.2 Some Examples

**Capsule:** An idea like harmonic measure is a bit hollow without some concrete examples. In this section, we produce several very concrete instances.
6.2. SOME EXAMPLES

Harmonic measure helps us to understand the growth and value distribution of harmonic and holomorphic functions on Ω. It has become a powerful analytic tool. We begin to understand harmonic measure by first calculating some examples.

Example 6.5 Let \( U \) be the upper-half-plane. Let \( E \) be an interval \([-T, T]\) on the real axis, centered at 0. Let us calculate \( \omega(z, U, E) \) for \( z = x + iy \cong (x, y) \in \Omega \).

It is a standard fact (see [GRK1]) that the Poisson kernel for \( U \) is

\[
P(x, y) = \frac{1}{\pi} \cdot \frac{y}{x^2 + y^2}.
\]

Here we use the traditional real notation for the kernel. In other words, the harmonic function \( \Omega \) that we seek is given by

\[
\omega(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \chi_{[-T,T]}(t) \cdot \frac{y}{(x-t)^2 + y^2} dt
\]

\[
= \frac{1}{\pi} \int_{-T}^{T} \frac{y}{(x-t)^2 + y^2} dt.
\]

Now it is a simple matter to rewrite the integral as

\[
\omega(x + iy, \Omega, E) = \frac{1}{\pi y} \int_{-T}^{T} \frac{1}{(x-t)^2 + y^2} dt
\]

\[
= -\frac{1}{\pi} \tan^{-1} \left( \frac{x-t}{y} \right) \bigg|_{-T}^{T}
\]

\[
= \frac{1}{\pi} \tan^{-1} \left( \frac{x+T}{y} \right) - \frac{1}{\pi} \tan^{-1} \left( \frac{x-T}{y} \right).
\]

The reader may check that this function is harmonic, tends to 1 as \((x, y)\) approaches \( E \subseteq \partial \Omega \), and tends to 0 as \((x, y)\) approaches \( \partial \Omega \setminus E \).

Glancing at Figure 6.1, we see that the value of \( \omega(z, \Omega, E) \) is simply \( \alpha/\pi \), where \( \alpha \) is the angle subtended at the point \((x, y)\) by the interval \( E \).

More generally, if \( E \) is any bounded, closed interval in the real line, then the harmonic measure \( \omega(z, U, E) \) will be \( 1/\pi \) times the angle subtended at the point \((x, y)\) by the interval \( E \). But we can
say more. If now $E$ is the finite disjoint union of closed, bounded intervals, 

$$E = I_1 \cup \cdots \cup I_k,$$

then the harmonic measure of $E$ is just the sum of the harmonic measures $\omega(z, \Omega, I_j)$ for each of the individual intervals. So the harmonic measure at $z$ is just $1/\pi$ times the sum of the angles subtended at $(x, y)$ by each of the intervals $I_j$.

**Example 6.6** Let $D$ be the unit disc. Let $E$ be an arc of the circle with central angle $\alpha$. Then, for $z \in D$,

$$\omega(z, D, E) = \frac{2\theta - \alpha}{2\pi},$$

where $\theta$ is the angle subtended by $E$ at $z$. See Figure 6.2.

It turns out to be convenient to first treat the case where $E$ is the arc from $-i$ to $i$. See Figure 6.3.

We may simply calculate the Poisson integral of $\chi(-\pi/2, \pi/2)$, where the argument of this characteristic function is the angle in radian measure on the circle. Thus, for $z = re^{i\lambda}$,

$$\omega(z, D, E) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{1-r^2}{1 - 2r \cos(\lambda - t) + r^2} dt$$

$$= \frac{1-r^2}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dt}{(1 + r^2) - 2r \cos(\lambda - t)}.$$
6.2. SOME EXAMPLES

Figure 6.2: The angle subtended by $E$ at $z$.

Figure 6.3: The case $E$ is the arc from $-i$ to $i$. 
At this point, it is useful to recall from calculus (and this may be calculated using the Weierstrass \( w = \tan x/2 \) substitution) that

\[
\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left( \sqrt{\frac{a - b}{a + b}} \tan \frac{x}{2} \right).
\]

Therefore

\[
\omega(z, D, E) = \frac{1}{\pi} \left[ \tan^{-1} \left( \frac{1 + r}{1 - r} \tan \frac{\lambda + \pi}{2} \right) 
- \tan^{-1} \left( \frac{1 + r}{1 - r} \tan \frac{\lambda - \pi}{2} \right) \right].
\]

Some easy but tedious manipulations now allow us to rewrite this last expression as

\[
\omega(z, D, E) = \frac{1}{\pi} \left[ \tan^{-1} \left( \left( \frac{1 + r}{1 - r} \right) \cdot \left( \frac{\sin \lambda + 1}{\cos \lambda} \right) \right) 
- \tan^{-1} \left( \left( \frac{1 + r}{1 - r} \right) \cdot \left( \frac{\sin \lambda - 1}{\cos \lambda} \right) \right) \right].
\] (6.6.1)

Recall, however, that

\[
\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta};
\]

hence

\[
\tan^{-1} a - \tan^{-1} b = \tan^{-1} \left( \frac{a - b}{1 + ab} \right). \tag{6.6.2}
\]

Applying this simple idea to (6.6.1) yields

\[
\omega(z, D, E) = \frac{1}{\pi} \tan^{-1} \left( \frac{\left( \frac{1 + r}{1 - r} \right) \left( \frac{\sin \lambda + 1}{\cos \lambda} \right) - \left( \frac{1 + r}{1 - r} \right) \left( \frac{\sin \lambda - 1}{\cos \lambda} \right)}{1 + \left( \frac{1 + r}{1 - r} \right) \left( \frac{\sin \lambda + 1}{\cos \lambda} \right) \cdot \left( \frac{1 + r}{1 - r} \right) \left( \frac{\sin \lambda - 1}{\cos \lambda} \right)} \right).
\]

Elementary simplifications finally lead to

\[
\omega(z, D, E) = \frac{1}{\pi} \tan^{-1} \left( \frac{1 - r^2}{2r \cos \lambda} \right).
\]
6.3. HADAMARD’S THREE-CIRCLES THEOREM

Now it is helpful to further rewrite the right-hand side of this last expression as

\[
\frac{2 \left[ \tan^{-1} \left( \frac{1-r \sin \lambda}{r \cos \lambda} \right) + \tan^{-1} \left( \frac{1+r \sin \lambda}{r \cos \lambda} \right) \right] - \pi}{2\pi}
\]

(remembering, of course, that \( \pi/2 - \tan^{-1}(\gamma) = \tan^{-1}(1/\gamma) \)). But the preceding expression is just the formula

\[
\frac{2\theta - \alpha}{2\pi},
\]

for the special angle \( \alpha = \pi \), which was enunciated at the start of the example.

For the case of general \( E \), we may first suppose that \( E \) is an arc centered at the point \( 1 \in \partial D \). Second, we may reduce the general case to the one just calculated by using a Möbius transformation.

**Example 6.7** Suppose that \( U \) is an annulus with radii \( 0 < r_1 < r_2 < \infty \). Then one may verify by inspection that, if \( E \) is the outer boundary circle of the annulus, then

\[
\omega(z, U, E) = \frac{\log(|z|/r_1)}{\log(r_2/r_1)}.
\]

In general, it is a tricky business to calculate exactly the harmonic measure for a given domain \( U \) and a given \( E \). But one may often obtain useful estimates. The ensuing discussion will bear out this point.

### 6.3 Hadamard’s Three-Circles Theorem

**Capsule:** One of the big ideas in modern harmonic analysis is interpolation of operators. If a linear operator \( T \) maps a Banach space \( X \) to itself and another Banach space \( Y \) to itself, then is there automatically a range of “intermediate” spaces on which \( T \) is bounded? (This question goes back to M. Riesz’s proof of the \( L^p \) boundedness of the Hilbert transform in 1926.)

The first real interpolation theorem (which treats the concept of intermediate space) is the Riesz–Thorin theorem. Its proof uses Hadamard’s three-circles theorem.

Hadamard’s three-circles theorem (sometimes called the “three-lines theorem”) is an important sharpening of the classical maximum
principle. It has proved useful in various parts of analysis, notably in proving the Riesz–Thorin interpolation theorem for linear operators. Here we shall give a thorough treatment of the three circles theorem from the point of view of harmonic measure. Afterwards we shall discuss the result of Riesz and Thorin.

We begin by treating some general comparison principles regarding harmonic measure. It will facilitate our discussion to first introduce a slightly more general concept of harmonic measure.

Let \( U \subseteq \mathbb{C} \) be, as usual, a domain. Let \( A \) be a closed set in the extended plane \( \hat{\mathbb{C}} \). Let \( E \) denote that part of \( \partial(U \setminus A) \) that lies in \( A \). [In what follows, we will speak of pairs \( (U, A) \).] Then \( \omega(z, U \setminus A, E) \) will be called the harmonic measure of \( A \) with respect to \( U \), assuming that the geometry is simple enough that we can compute this number component-by-component (of \( U \setminus A \)). Generally speaking, in the present chapter, we assume that all boundaries that arise consist of finitely many Jordan curves. This standing hypothesis is made so that we can readily apply Lindelöf’s Proposition 6.1. Often this standing hypothesis will go unspoken.

The first new comparison tool is called the majorization principle.

**Prelude:** We consider here a transformation property for harmonic measure.

**Theorem 6.8** Consider two pairs \((U, A)\) and \((\tilde{U}, \tilde{A})\). Let

\[
f : U \setminus A \to \tilde{U}
\]

be holomorphic. Assume furthermore that, if \( U \ni z \to A \), then \( \tilde{U} \ni f(z) \to \tilde{A} \). Then

\[
\omega(z, U, A) \leq \omega(f(z), \tilde{U}, \tilde{A})
\]

for \( z \in f^{-1}(\tilde{U} \setminus \tilde{A}) \).

**Remark 6.9** Of course we consider only holomorphic \( f \) because we want a mapping that preserves harmonic functions under composition.

**Proof of Theorem 6.8:** Let us abbreviate

\[
\omega = \omega(z, U, A) \quad \text{and} \quad \tilde{\omega} = (f(z), \tilde{U}, \tilde{A}).
\]

Now we apply the maximum principle to \( \omega - \tilde{\omega} \) on a connected component \( V \) of \( f^{-1}(\tilde{U} \setminus \tilde{A}) \). As \( z \) approaches the boundary of \( V \), either
6.3. Hadamard’s Three-Circles Theorem 179

$z$ tends to a boundary point of $U$ that is not on $A$, or else $f(z)$ tends to $\tilde{A}$. In either of these circumstances, $\limsup_{z \to \partial V}(\omega - \tilde{\omega}) \leq 0$ except when $f(z)$ tends to an endpoint of the boundary arcs of $\tilde{U} \setminus \tilde{A}$ that lie on $\tilde{A}$. Since there are only finitely many such points, Lindelöf’s Proposition 6.1 tells us that the maximum principle still remains valid. We therefore conclude that $\omega \leq \tilde{\omega}$ on all of $f^{-1}(\tilde{U} \setminus \tilde{A})$. □

**Corollary 6.10** The function $\omega(z, U, A)$ increases if either $U$ increases or $A$ increases. That is to say, if $U \subseteq U^*$ and $A \subseteq A^*$, then

$$\omega(z, U, A) \leq \omega(z, U^*, A) \quad \text{and} \quad \omega(z, U, A) \leq \omega(z, U, A^*).$$

**Proof:** We prove the first statement and leave the second for the reader.

We apply the theorem with $f : U \setminus A \to U^*$ being the identity mapping. The result is now immediate. □

Now let $U^*$ be an open disc of radius $R > 0$, and let $A^*$ be a smaller closed disc of radius $0 < r < R$. One may see by inspection that

$$\omega(f(z), U^*, A^*) = \frac{\log(R/|f(z)|)}{\log R/r}.$$  

We do not yet say what $f$ is, and there is considerable latitude in practice. Nonetheless, we see immediately that the function $\omega$ is identically equal to 1 when $f(z) \in \partial A^*$ and identically equal to 0 when $f(z) \in \partial U^*$. Certainly it is harmonic.

Now we have our first result where the quantitative properties of the harmonic measure play a decisive role.

**Prelude:** This next result is reminiscent of our arguments (below) using the three-circles theorem to establish the Riesz–Thorin theorem.

**Theorem 6.11** Let $f$ be a holomorphic function on a domain $U$. Let $A$ be a closed set. If $|f(z)| \leq M$ in $\tilde{U}$ and $|f(z)| \leq m < M$ on $A$, then, for $0 \leq \theta \leq 1$,

$$|f(z)| \leq m^\theta M^{1-\theta}$$

at points $z$ where $\omega(z, U, A) \geq \theta$.

**Remark 6.12** This result is of fundamental philosophical importance. It shows how the harmonic measure is a device for interpolating information about the function $f$. 
CHAPTER 6. HARMONIC MEASURE

Proof: Of course \( f : U \rightarrow D(0, M) \). We take \( \tilde{A} = D(0, m) \), \( A = f^{-1}(\tilde{A}) \), and \( \tilde{U} = f(U) \). Then we apply the previous theorem and the remark following. So

\[
\omega(z, U, A) \leq \omega(f(z), \tilde{U}, \tilde{A}) = \frac{\log(M/|f(z)|)}{\log M/m}.
\]

At a point \( z \) for which \( \omega(z, U, A) \geq \theta \), we have

\[
\theta \leq \frac{\log(M/|f(z)|)}{\log M/m}.
\]

This inequality is equivalent to the desired conclusion. \( \square \)

If we take \( U \) and \( A \) in this last theorem to be annuli, then we can draw an important and precise conclusion. Namely, we have the following version of the three-circles theorem.

Prelude: We have the celebrated Hadamard three-circles theorem, whose uses in analysis are many and varied.

**Theorem 6.13 (Hadamard)** Let \( f \) be a holomorphic function on an annulus \( A = \{ z \in \mathbb{C} : c < |z| < C \} \). For \( c < \tau < C \), we let

\[
M(\tau) \equiv \max_{|z|=\tau} |f(z)|.
\]

If \( c < r < \rho < R < C \), then we have

\[
\log M(\rho) \leq \frac{\log R - \log \rho}{\log R - \log r} \cdot \log M(r) + \frac{\log \rho - \log r}{\log R - \log r} \cdot \log M(R).
\]

(This inequality says quite plainly that \( \log M(s) \) is a convex function of \( \log s \).)

Proof: Let

\[
U = \{ z \in \mathbb{C} : r < |z| < R \},
\]

and let

\[
A = \{ z \in \mathbb{C} : r \leq |z| \leq r + \epsilon \}, \text{ some } \epsilon > 0.
\]

We assume that \( f \) is holomorphic on \( U \), \( |f| \leq M \) on \( U \), and \( |f| \leq m \) on \( A \). The preceding theorem tells us that

\[
|f(z)| \leq m^\theta M^{1-\theta}
\] (6.13.1)
6.3. HADAMARD’S THREE-CIRCLES THEOREM

on the set where \( \omega(z, U, A) \geq \theta \), that is, on the set where

\[
\frac{\log(|z|/R)}{\log((r+\epsilon)/R)} \geq \theta.
\]

(Note that the roles of \( r+\epsilon \) and \( R \) are reversed from their occurrence in Example 6.7, just because now we are creating harmonic measure for the inside circle of the annulus.)

In particular, inequality (6.13.1) holds when \( |z| = (r+\epsilon)^\theta R^{1-\theta} \).

We take

\[
\theta = \frac{\log R - \log \rho}{\log R - \log(r+\epsilon)}
\]

and

\[
|z| = (r+\epsilon)^\theta R^{1-\theta} \equiv \rho_\epsilon.
\]

Then (6.13.1) translates to

\[
M(\rho_\epsilon) \leq M(r+\epsilon)^\theta \cdot M(R)^{1-\theta}.
\]

Letting \( \epsilon \to 0^+ \) and taking the logarithm of both sides yields the desired result.

We note that Ahlfors [AHL2], from which our exposition derives, likes to express the conclusion of this last result as

\[
\det \begin{pmatrix} 1 & 1 & 1 \\ \log r & \log \rho & \log R \\ \log M(r) & \log M(\rho) & \log M(R) \end{pmatrix} \geq 0.
\]

We close this section by formulating the three-lines version of Hadamard’s theorem. It is proved from Theorem 6.13 simply with conformal mapping, and we leave the details to the interested reader.

**Prelude:** We have the “three-lines” version of the Hadamard theorem.

**Theorem 6.14** Let \( f \) be a holomorphic function on the strip \( S = \{ z \in \mathbb{C} : 0 < \Re z < 1 \} \). For \( 0 < x < 1 \) we let

\[
M(x) \equiv \max_{\Re z=x} |f(z)|.
\]

If \( 0 < a < \rho < b < 1 \), then we have

\[
\log M(\rho) \leq \frac{\log b - \log \rho}{\log b - \log a} \cdot \log M(a) + \frac{\log \rho - \log a}{\log b - \log a} \cdot \log M(b).
\]

(This inequality says quite plainly that \( \log M(s) \) is a convex function of \( \log s \).)
6.4 A Discussion of Interpolation of Linear Operators

Capsule: As predicted in the last Capsule, we now apply the three-lines theorem to prove a result about interpolation of operators. This will be a new idea for you, and a profound one.

Marcel Riesz discovered the idea of interpolating linear operators in the following context. Perhaps the most important linear operator in all of analysis is the Hilbert transform

\[ H : f \mapsto \int_{\mathbb{R}} \frac{f(t)}{x-t} \, dt. \]

This operator arises naturally in the study of the boundary behavior of conjugates of harmonic functions on the disc, in the existence and regularity theory for the Laplacian, in the summability theory of Fourier series, and, more generally, in the theory of singular integral operators. It had been an open problem for some time to show that \( H \) is a bounded operator on \( L^p, 1 < p < \infty \).

It turns out that the boundedness on \( L^2 \) is easy to prove and follows from Plancherel’s theorem in Fourier analysis. Riesz cooked up some extremely clever tricks to derive boundedness on \( L^p \) when \( p \) is an even integer. (These ideas are explained in detail in [KRA3].) He needed to find some way to derive therefrom the boundedness on the “intermediate” \( L^p \) spaces. Now we introduce enough language to explain precisely what the concept of “intermediate space” means.

Let \( X_0, X_1, Y_0, Y_1 \) be Banach spaces. Intuitively, we will think of a linear operator \( T \) such that

\[ T : X_0 \to Y_0 \]

continuously and

\[ T : X_1 \to Y_1 \]

continuously. We wish to posit the existence of spaces \( X_\theta \) and \( Y_\theta \), \( 0 \leq \theta \leq 1 \) such that these new spaces are natural “intermediaries” of \( X_0, X_1 \) and \( Y_0, Y_1 \) respectively. Furthermore, we want that

\[ T : X_\theta \to Y_\theta \]

continuously. There is interest in knowing how the norm of \( T \) acting on \( X_\theta \) depends on the norms of \( T \) acting on \( X_0 \) and \( X_1 \).
6.4. INTERPOLATION OF OPERATORS

One rigorous method for approaching this situation is as follows (see [BEL], [STW], and [KAT] for our inspiration). Let \( X_0, Y_0, X_1, Y_1 \) be given as in the last paragraph. Suppose that

\[
T : X_0 \cap X_1 \rightarrow Y_0 \cup Y_1
\]

is a linear operator with the properties that

(i) \( \|Tx\|_{Y_0} \leq C_0\|x\|_{X_0} \) for all \( x \in X_0 \cap X_1 \);

(ii) \( \|Tx\|_{Y_1} \leq C_0\|x\|_{X_1} \) for all \( x \in X_0 \cap X_1 \).

Then we want to show that there is a collection of norms \( \| \cdot \|_{X_{\theta}} \) and \( \| \cdot \|_{Y_{\theta}} \), \( 0 < \theta < 1 \), such that

\[
\|Tx\|_{Y_{\theta}} \leq C_{\theta}\|x\|_{X_{\theta}}.
\]

So the problem comes down to how to construct the “intermediate norms” \( \| \cdot \|_{X_{\theta}} \) and \( \| \cdot \|_{Y_{\theta}} \) from the given norms \( \| \cdot \|_{X_0}, \| \cdot \|_{Y_0}, \| \cdot \|_{X_1}, \) and \( \| \cdot \|_{Y_1} \).

There are in fact a number of paradigms for effecting the indicated construction. Most prominent among these are the “real method” (usually attributed to Lyons and Peetre—see [LYP]) and the “complex method” (usually attributed to Calderón—see [CAL]). In this text we shall concentrate on the complex method, which was inspired by the ideas of Riesz and Hadamard that were described at the beginning of this section.

Instead of considering the general paradigm for complex interpolation, we shall concentrate on the special case that was of interest to Riesz and Thorin. We now formulate our main theorem.

Prelude: The Riesz–Thorin theorem was the very first theorem in the study of interpolation of operators. This is now a very well-developed part of analysis, and there are entire monographs devoted to the subject.

**Theorem 6.15** Let \( 1 \leq p_0 < p_1 \leq \infty \) and \( 1 \leq q_0 < q_1 \leq \infty \). Suppose that

\[
T : L^{p_0}(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2) \rightarrow L^{q_0}(\mathbb{R}^2) \cup L^{q_1}(\mathbb{R}^2)
\]

is a linear operator on Lebesgue spaces satisfying

\[
\|Tf\|_{L^{q_0}} \leq C_0\|f\|_{L^{p_0}}
\]

and

\[
\|Tf\|_{L^{q_1}} \leq C_0\|f\|_{L^{p_1}}.
\]
Define $p_\theta$ and $q_\theta$ by

$$\frac{1}{p_\theta} = (1 - \theta) \cdot \frac{1}{p_0} + \theta \cdot \frac{1}{p_1}$$

and

$$\frac{1}{q_\theta} = (1 - \theta) \cdot \frac{1}{q_0} + \theta \cdot \frac{1}{q_1},$$

with $0 \leq \theta \leq 1$. Then $T$ is a bounded operator from the $L^{p_\theta}$ norm to the $L^{q_\theta}$ norm, and

$$\|Tf\|_{L^{q_\theta}} \leq C_0^{1-\theta} C_1^\theta \|f\|_{L^{p_\theta}}$$

for all $f \in L^{p_0} \cap L^{p_1}$.

**Proof:** Fix a nonzero function $f$ that is continuous and with compact support in $\mathbb{R}^2$. We will prove an a priori inequality for this $f$, and then extend to general $f$ at the end. Now certainly $f \in L^{p_0} \cap L^{p_1}$. We consider the holomorphic function

$$H : \zeta \mapsto \frac{\left(\int_{\mathbb{R}^2} |Tf(x)|^{(1-\zeta)q_0+\zeta q_1} \, dx\right)^{(1-\zeta)/q_0+\zeta/q_1}}{\left(\int_{\mathbb{R}^2} |f(x)|^{(1-\zeta)p_0+\zeta p_1} \, dx\right)^{(1-\zeta)/p_0+\zeta/p_1}}$$
on the strip $S = \{\zeta \in \mathbb{C} : 0 < \text{Re} \, \zeta < 1\}$. We define

$$M(s) = \sup_{\text{Re} \, \zeta = s} |H(\zeta)|, \; 0 < s < 1.$$

Notice that

$$M(0) = \sup_{t \in \mathbb{R}} \left| \frac{\int_{\mathbb{R}^2} |Tf(x)|^{(1-it)q_0+it q_1} \, dx^{(1-it)/q_0+it/q_1}}{\int_{\mathbb{R}^2} |f(x)|^{(1-it)p_0+it p_1} \, dx^{(1-it)/p_0+it/p_1}} \right|$$

$$\leq \sup_{t \in \mathbb{R}} \frac{\int_{\mathbb{R}^2} |Tf(x)|^{q_0} \, dx^{1/q_0}}{\int_{\mathbb{R}^2} |f(x)|^{p_0} \, dx^{1/p_0}}$$

$$\leq C_0.$$

A similar calculation shows that

$$M(1) \leq C_1.$$

Obviously this is grist for the three-lines theorem. We may conclude that $M(s) \leq C_0^{1-s} \cdot C_1^s$, $0 < s < 1$. 

Now let $p_\zeta$, $q_\zeta$ be given by
\[
\frac{1}{p_\zeta} = (1 - \zeta) \cdot \frac{1}{p_0} + \zeta \cdot \frac{1}{p_1}
\]
and
\[
\frac{1}{q_\zeta} = (1 - \zeta) \cdot \frac{1}{q_0} + \zeta \cdot \frac{1}{q_1}.
\]
Let $\zeta = s + it$. Then
\[
\|Tf\|_{L^{q_\zeta}} \leq M(s) \leq C_0^{1-s}C_1^s.
\]
In other words,
\[
\|Tf\|_{L^{q_s}} \leq C_0^{1-s}C_1^s\|f\|_{L^{p_s}}.
\]
This is the desired conclusion for a function $f$ that is continuous with compact support.

The result for general $f \in L^{p_\theta}$ can be achieved with a simple approximation argument.

6.5 The F. and M. Riesz Theorem

Capsule: One of the big ideas of twentieth century function theory is that due to the brothers F. and M. Riesz. The result has many interpretations—in terms of the absolute continuity of harmonic measure, in terms of the vanishing of negative Fourier coefficients, and many other ideas as well.

One of the classic results of function theory was proved by the brothers F. and M. Riesz. It makes an important statement about the absolute continuity of harmonic measure. We begin with a preliminary result that captures the essence of the theorem. Our exposition here follows the lead of [GARM, Ch. 6].

For $0 < p < \infty$ we define the Hardy space
\[
H^p(D) = \left\{ f \text{ holomorphic on } D : \sup_{0<r<1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} \right\}
\]
\[
\equiv \|f\|_{H^p} < \infty
\].

Hardy spaces are generalizations of the more classical space $H^\infty(D)$ of bounded, holomorphic functions.
CHAPTER 6. HARMONIC MEASURE

The most important fact about a function in a Hardy space is that it has a boundary function. We enunciate that result here, and refer the reader to [KRA2, Chapter 8] for the details.

Prelude: The theory of boundary limits for functions in Hardy spaces goes back to work of Fatou in 1906. Later on, Riesz and Hardy played a decisive role in fleshing out the theory and pointing in the directions that we study today.

Theorem 6.16 Let $0 < p \leq \infty$ and let $f \in H^p(D)$. Then there is a function $f^* \in L^p(\partial D)$ such that:

(6.16.1) \[ \lim_{r \to 1^-} f(re^{i\theta}) = f^*(e^{i\theta}) \text{ for almost every } \theta \in [0, 2\pi). \]

(6.16.2) Let $f_r(e^{i\theta}) = f(re^{i\theta})$ for $0 < r < 1$. Assume that $0 < p < \infty$. Then $\lim_{r \to 1^-} \|f - f_r\|_{L^p(\partial D)} = 0$.

In the following discussions we shall consider rectifiable curves. Basically, a rectifiable curve is a curve with finite length. The technical definition is that the curve is locally the Lipschitz image of the unit interval (see [FED]). But context will make it clear that the intuitive notion of rectifiability will suffice for our purposes.

Prelude: Here it is important that we relate rectifiability for a curve with the theory of Hardy spaces. Hausdorff measure also comes into play.

Theorem 6.17 Let $U$ be a domain such that $\gamma = \partial U$ is a Jordan curve. Let

$\varphi : D \to U$

be a conformal map. Then the curve $\gamma$ is rectifiable if and only if $\varphi' \in H^1$. In case $\varphi' \in H^1$, then we have

$\|\varphi'\|_{H^1} = \text{length}(\gamma) = \mathcal{H}^1(\gamma). \quad (6.17.1)$

Here $\mathcal{H}^1$ is one-dimensional Hausdorff measure.

Remark 6.18 It is known (see [KRA2, Chapter 8]) that a function in $H^1$ has a boundary limit function that is in $L^1(\partial D)$. Thus the hypothesis of the theorem says, essentially, that $\varphi$ is absolutely continuous on the boundary of $D$. It makes sense, then, that $\varphi$ would preserve length on the boundary. Theorem 6.17 is essentially equivalent to the result of F. and M. Riesz that we described in Remark 6.4.
6.5. THE F. AND M. RIESZ THEOREM

Proof of Theorem 6.17: Once again, we invoke Carathéodory’s theorem; thus we know that \( \varphi \) and its inverse extend continuously and univalently to their respective boundaries. Let us assume that \( \varphi' \in H^1 \). Let

\[
0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k = 2\pi
\]

be a partition of \([0, 2\pi]\). Then of course

\[
\sum_{j=1}^{k} |\varphi(e^{i\theta_j}) - \varphi(e^{i\theta_{j-1}})| = \lim_{r \to 1^{-}} \sum_{j=1}^{k} |\varphi(re^{i\theta_j}) - \varphi(re^{i\theta_{j-1}})|
\]

\[
= \lim_{r \to 1^{-}} \sum_{j=1}^{k} \left| \int_{\theta_{j-1}}^{\theta_j} \varphi'(re^{i\theta})ire^{i\theta} d\theta \right|
\]

\[
\leq \|\varphi'\|_{H^1}. \tag{6.17.2}
\]

But plainly the length of \( \gamma \) is the supremum, over all partitions of \([0, 2\pi]\), of the left-hand side of (6.17.2). We conclude that \( \gamma \) is rectifiable and

\[
\operatorname{length}(\gamma) \leq \|\varphi'\|_{H^1}.
\]

For the converse, assume that \( \gamma \) is rectifiable. If \( 0 < r < 1 \) is fixed, then let \( \gamma_r = \varphi(\{ z \in \mathbb{C} : |z| = r \}) \). Let \( \epsilon > 0 \). Now choose a partition \( \{\theta_0, \theta_1, \ldots, \theta_k\} \) of the interval \([0, 2\pi]\) as before so that

\[
\sum_{j=1}^{k} |\varphi(re^{i\theta_j}) - \varphi(re^{i\theta_{j-1}})| \geq \operatorname{length}(\gamma_r) - \epsilon.
\]

We write

\[
\eta(z) = \sum_{j=1}^{k} |\varphi(z e^{i\theta_j}) - \varphi(z e^{i\theta_{j-1}})|.
\]

Then \( \eta \), being the sum of absolute values of holomorphic functions, is subharmonic. By Carathéodory’s theorem, \( \eta \) is continuous on \( \overline{D} \). Hence

\[
\sup_{D} \eta(z) = \sup_{\theta} \eta(e^{i\theta}) \leq \operatorname{length}(\gamma).
\]

We conclude that

\[
\int_{0}^{2\pi} |\varphi'(re^{i\theta})| d\theta = \operatorname{length}(\gamma_r) \leq \eta(r) + \epsilon \leq \operatorname{length}(\gamma) + \epsilon.
\]

It follows that \( \phi' \in H^1 \) and equality (6.17.1) is valid. We conclude by noting that, for a rectifiable curve, the ordinary notion of length and the one-dimensional Hausdorff measure are the same. \( \square \)
The reason that we include Theorem 6.17 in the present chapter—apart from its general aesthetic interest—is that it can be interpreted in the language of harmonic measure. Let notation be as in Theorem 6.17. To wit, let $\gamma = \partial U$ be a rectifiable Jordan curve as above and let $F \subseteq \gamma$ be a subcurve. Let $\varphi$ be a conformal mapping of $U$ to the unit disc $D$. Let $F = \varphi(E)$ and $\alpha = \varphi(0)$. Then Carathéodory’s theorem, Example 6.6, and Hadamard’s theorem tell us that

$$\omega(\alpha, U, F) = \omega(0, D, E) = \frac{1}{2\pi} |E|,$$

where the $| \cdot |$ denotes arc length. Since $F$ is an arc, we see that Theorem 6.17 and the proof of (6.17.1) tell us that

$$\mathcal{H}^1(F) = \mathcal{H}^1(\varphi(E)) = \lim_{r \to 1^-} \int_E |\varphi'(re^{i\theta})| \, d\theta = \int_E |\varphi'(e^{i\theta})| \, d\theta. \quad (6.18)$$

Of course if (6.18) holds for arcs, then, by passing to unions and intersections, we see that it holds for Borel sets $F$. Thus we derive the important conclusion

$$\omega(\alpha, U, F) = 0 \Rightarrow \mathcal{H}^1(F) = 0.$$

Conversely, since (by a standard uniqueness theorem—see [KOO])

$$|\{\theta : |\varphi'(e^{i\theta})| = 0\}| = 0,$$

we see that

$$\mathcal{H}^1(F) = 0 \Rightarrow \omega(\alpha, U, F) = 0.$$

What we have proved, then, is that when $\gamma = \partial U$ is rectifiable then harmonic measure for $U$ and linear measure on $\gamma$ are mutually absolutely continuous. We now summarize this result in a formally enunciated theorem.

**Prelude:** The F. and M. Riesz theorem is one of the celebrated results of classical function theory. There are many different renditions of the result, and many different proofs. We give here a geometric rendition of the theorem.

**Theorem 6.19 (F. and M. Riesz, 1916)** Let $U$ be a simply connected planar domain such that $\gamma = \partial U$ is a rectifiable Jordan curve.
Assume that 
\[ \varphi : D \to U \]
is conformal. Then \( \varphi' \in L^1(\partial D) \). For any Borel set \( E \subseteq \partial D \) it holds that 
\[ \mathcal{H}^1(\varphi(E)) = \int_E |\varphi'(e^{i\theta})| \, d\theta. \]
Also, for any Borel set \( F \subseteq \partial U \) and any point \( \alpha \in U \),
\[ \omega(\alpha, U, F) = 0 \iff \mathcal{H}^1(F) = 0. \]

We conclude by noting that Theorems 6.17 and 6.19 are equivalent, just because formula (6.17.1) is valid precisely when \( \varphi' \in H^1 \).
The Theory and Practice of Conformal Geometry

STEVEN G. KRANTZ

In this original text, prolific mathematics author Steven G. Krantz addresses conformal geometry, a subject that has occupied him for four decades and for which he helped to develop some of the modern theory. This book takes readers with a basic grounding in complex variable theory to the forefront of some of the current approaches to the topic. "Along the way," the author notes in his Preface, "the reader will be exposed to some beautiful function theory and also some of the rudiments of geometry and analysis that make this subject so vibrant and lively."

More up-to-date and accessible to advanced undergraduates than most of the other books available in this specific field, the treatment discusses the history of this active and popular branch of mathematics as well as recent developments. Topics include the Riemann mapping theorem, invariant metrics, normal families, automorphism groups, the Schwarz lemma, harmonic measure, extremal length, analytic capacity, and invariant geometry. A helpful Bibliography and Index complete the text.

Dover (2016) Aurora Original.