Chapter 1

Preliminaries

1.1 Numbers and Sets

The numbers we will use in this book are the real numbers. These are all the numbers that can be written in decimal notation. We often think of them as corresponding to points on a number line (see Figure 1.1). The simplest real numbers are the integers: the numbers 0, 1, −1, 2, −2, 3, −3, and so on. A real number is said to be rational if it can be written as an integer divided by an integer; for example, 2/3 and −13/5 are rational numbers. Notice that every integer is also a rational number, since, for example, we can write 3 as 3/1. Real numbers that are not rational are called irrational. For example, it can be shown that \( \sqrt{2} \approx 1.41421 \ldots \) and \( \pi \approx 3.14159 \ldots \) are irrational. Both rational and irrational numbers are spread throughout the number line; in fact, between any two real numbers there are infinitely many rational numbers and also infinitely many irrational numbers.

We will also often work with collections of numbers. In mathematics, a collection of objects is called a set, and the objects in the collection are called elements of the set. The simplest way to specify a set is to list the elements of the set between braces. For example, \( \{−1, 0, 3/2\} \) is the set whose elements are the three numbers −1, 0, and 3/2. If we let the letter \( A \) stand for this set, then we write \( 3/2 \in A \) to say that 3/2 is an element of \( A \), while \( 4 \not\in A \) means that 4 is not an element of \( A \).

Another way to specify a set is to give a rule for determining which objects belong to the set and which do not. For example, if we write

\[
B = \{x : 2x^3 - x^2 - 3x = 0\},
\]  

(1.1)
then this means that $B$ is the set whose elements are all values of $x$ that satisfy the equation $2x^3 - x^2 - 3x = 0$. Equation (1.1) is read “$B$ is equal to the set of all $x$ such that $2x^3 - x^2 - 3x = 0$.” The equation $2x^3 - x^2 - 3x = 0$ that appears in the definition of $B$ is a statement that is true for some values of $x$ and false for others. You should think of it as an elementhood test for the set $B$; those values of $x$ that make the equation true pass the test and are elements of $B$, while those that make the equation false are not. To determine which numbers belong to $B$ we simply have to solve the equation, which we can do by factoring the left-hand side. We have

$$2x^3 - x^2 - 3x = x(x + 1)(2x - 3),$$

so the equation can be rewritten $x(x + 1)(2x - 3) = 0$, and the solutions are $0$, $-1$, and $3/2$. These are the elements of the set $B$. Notice that these are exactly the same as the elements of the set $A$ defined earlier. Thus $B = A$; they are both the same collection of numbers, described in two different ways.

Although we will usually use the letter $x$ when writing an elementhood test to define a set, as we did in the definition of $B$, in fact any letter can be used. For example, the set $C = \{y : 2y^3 - y^2 - 3y = 0\}$ is the set of all values of $y$ that satisfy the equation $2y^3 - y^2 - 3y = 0$. Of course, this is the same equation that we used in the definition of $B$, but with $x$ replaced by $y$. The values of $y$ that satisfy the equation are therefore once again the numbers $0$, $-1$, and $3/2$. Therefore $C = B = A$; we have the same set of numbers described in yet another way.

Here is another example of a set defined by an elementhood test: $I = \{x : 2 < x < 5\}$. This time the elementhood test is $2 < x < 5$, which is a shorthand way of saying that $2 < x$ and $x < 5$. In this case $3 \in I$, since the statement $2 < 3 < 5$ is true, but $5 \notin I$, since the statement $2 < 5 < 5$ is false. The elements of $I$ are all the numbers strictly between 2 and 5. There are infinitely many numbers in this range, so we cannot list all the elements of $I$, as we did for $A$. But we can mark them on a number line, as in Figure 1.2.

The set $I$ is an example of a kind of set called an open interval. For any numbers $a$ and $b$ with $a < b$, the set of all numbers strictly between $a$ and $b$ is an open interval, and it is denoted $(a, b)$. In other words, 

$$(a, b) = \{x : a < x < b\}.$$
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The numbers $a$ and $b$ are called the *endpoints* of the interval. Thus $I = (2, 5)$; it is the open interval with endpoints 2 and 5.

The endpoints of an open interval are not elements of the interval. But sometimes we will want to include the endpoints, so we define

$$[a, b] = \{ x : a \leq x \leq b \}.$$ 

This set is called a *closed interval*. For example, $[2, 5] = \{ x : 2 \leq x \leq 5 \}$. This set is exactly the same as the open interval $(2, 5)$ considered earlier, except that it includes the endpoints 2 and 5 as elements. If we include only one endpoint, we get a *half-open interval*. As you might guess, we write half-open intervals like this:

$$(a, b] = \{ x : a < x \leq b \},$$

$$[a, b) = \{ x : a \leq x < b \}.$$ 

In general, we use a square bracket to indicate that an endpoint is included in an interval, and a parenthesis to indicate that it is not. Figure 1.3 shows examples of closed and half-open intervals.

The *interior* of an interval is the set containing all numbers in the interval except the endpoints. Thus, the interior of the closed interval $[2, 5]$ is the open interval $(2, 5)$. The interiors of $(2, 5)$ and $[2, 5)$ are also $(2, 5)$, and we will even say that the interior of $(2, 5)$ is $(2, 5)$.

Finally, we sometimes want to consider intervals that extend infinitely far in some direction, so we introduce the notation:

$$(a, \infty) = \{ x : x > a \},$$

$$[a, \infty) = \{ x : x \geq a \},$$

$$(-\infty, b) = \{ x : x < b \},$$

$$(-\infty, b] = \{ x : x \leq b \}.$$ 

Some examples of infinite intervals are shown in Figure 1.4. The interior of $[a, \infty)$ is $(a, \infty)$, and the interior of $(-\infty, b]$ is $(-\infty, b)$. The set of all real numbers is often denoted $\mathbb{R}$, but we could also think of it as the interval $(-\infty, \infty)$. We consider any
interval that does not include its endpoints to be an open interval. Thus, the intervals 
\((a, \infty), (-\infty, b),\) and \((-\infty, \infty)\) are open intervals, and the interior of any interval is an open interval.

Since this is our first use of the infinity symbol \(\infty\), it might be worthwhile to pause at this point to explain what this symbol means. The most important thing to understand about the infinity symbol is that there is no such number as infinity. You might wonder, then, how it can be correct to use this symbol in mathematical notation like \((a, \infty)\). The answer is that, according to the definition we have given, this notation is simply a shorthand for \(\{x : x > a\}\), and this last expression makes no mention of infinity. Every time we make a statement using the symbol \(\infty\), it will be a similar shorthand for a statement that makes no mention of infinity. We will never use the symbol \(\infty\) as if it stood for a number. Thus, for example, we would never set \(x\) equal to \(\infty\) in some formula, and we would never talk about the “closed interval” \([2, \infty]\).

There are two ways of combining sets that we will sometimes make use of. If \(A\) and \(B\) are sets, then the intersection of \(A\) and \(B\), denoted \(A \cap B\), is the set whose elements are those objects that belong to both \(A\) and \(B\). Thus

\[
A \cap B = \{x : x \in A \text{ and } x \in B\}.
\]

For example,

\[
[2, \infty) \cap (-\infty, 5) = \{x : x \in [2, \infty) \text{ and } x \in (-\infty, 5)\} = \{x : 2 \leq x \text{ and } x < 5\} = [2, 5).
\]

Looking at Figure 1.4, you can see that the elements of \([2, \infty) \cap (-\infty, 5)\) are those numbers that are in the overlap of the sets \([2, \infty)\) and \((-\infty, 5)\). In general, you can think of \(A \cap B\) as the overlap of \(A\) and \(B\).

The union of \(A\) and \(B\), denoted \(A \cup B\), is the set whose elements are all those objects that are elements of either \(A\) or \(B\) (or both). That is,

\[
A \cup B = \{x : x \in A \text{ or } x \in B\}.
\]

You could think of \(A \cup B\) as the set you get if you take all the elements of \(A\), and all the elements of \(B\), and throw them together into one set. For example, if we take all the numbers in the intervals \((2, 4]\) and \([4, 5]\) and put them together into one set, we get the
interval \((2, 5]\). That is,
\[
(2, 4] \cup [4, 5] = \{x : x \in (2, 4) \text{ or } x \in [4, 5]\} = \{x : 2 < x \leq 5\} = (2, 5].
\]

If \(A\) and \(B\) are sets, then \(A\) is called a subset of \(B\) if every element of \(A\) is also an element of \(B\). We write \(A \subseteq B\) to indicate that \(A\) is a subset of \(B\). For example, \((2, 4) \subseteq [2, 4]\), and \([2, 4] \subseteq (1, 5)\).

One reason that intervals are important in calculus is that they often come up as solution sets of inequalities. In particular, we will often be concerned with inequalities involving absolute values. Recall that the absolute value of a number \(x\) is defined as follows:
\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0, \\
  -x, & \text{if } x < 0.
\end{cases}
\]

This notation means that if \(x \geq 0\) then \(|x| = x\), and if \(x < 0\) then \(|x| = -x\). For example, \(|3| = 3\) and \(|-4| = -(-4) = 4\).

The fact that \(|x|\) is defined by cases, with one formula when \(x \geq 0\) and a different formula when \(x < 0\), suggests a method that can be used when solving any problem involving absolute values: reasoning by cases. As a simple example of this kind of reasoning, notice that if \(x \geq 0\) then \(|x| = x\), and if \(x < 0\) then \(|x| = -x > 0\). In both cases the statement \(|x| \geq 0\) is true, so we conclude that for every number \(x\), \(|x| \geq 0\). You can also use reasoning by cases to show that for every number \(x\), \(\sqrt{x^2} = |x|\) (see Exercise 15).

Here’s an example of how reasoning by cases can be used to solve an inequality involving absolute values:

**Example 1.1.1.** Solve:
\[
|x| < 3.
\]

**Solution.** Motivated by the definition of \(|x|\), we will consider \(x \geq 0\) and \(x < 0\) separately.

**Case 1.** \(x \geq 0\). In this case, according to the definition of absolute value we have \(|x| = x\), so the inequality \(|x| < 3\) means \(x < 3\).

**Case 2.** \(x < 0\). Now the definition of absolute value says that \(|x| = -x\), and substituting this into our inequality \(|x| < 3\) gives us \(-x < 3\). Multiplying by \(-1\) (and remembering that when multiplying an inequality by a negative number, we must reverse the direction of the inequality) we get \(x > -3\).

So what’s the solution to our inequality? Is it \(x < 3\), as we found in case 1, or \(x > -3\), as in case 2? To answer this question, we must think about what it means to solve an inequality. To solve the inequality \(|x| < 3\) means to determine which values of \(x\) make the inequality true. Our reasoning in case 1 shows that, for \(x \geq 0\), the inequality means \(x < 3\). Thus the inequality is true if \(0 \leq x < 3\) and false if \(x \geq 3\). We can’t tell from this reasoning whether the inequality is true or false when \(x < 0\); that’s the purpose of case 2. Case 2 tells us that, for \(x < 0\), the inequality will be true precisely when \(x > -3\). Thus, the inequality is true if \(-3 < x < 0\) and false if \(x \leq -3\). Putting all this
Figure 1.5: The solution to Example 1.1.1. The parts of the number line marked with solid lines are in the solution set, and outlined areas are not. The blue lines were determined in case 1 of the solution, and the red lines in case 2.

information together, we conclude that the inequality is true if \(-3 < x < 3\) and false if either \(x \geq 3\) or \(x \leq -3\), as shown in Figure 1.5. This means that the solution set of the inequality is an open interval:

\[
\{ x : |x| < 3 \} = \{ x : -3 < x < 3 \} = (-3, 3).
\]

Notice that in case 1 we determined that all numbers in the interval \([0, 3)\) are in the solution set, and in case 2 we determined that the numbers in \((-3, 0)\) are also in the solution set. The solution set is therefore the union of these two intervals: \([0, 3) \cup (-3, 0) = (-3, 3)\).

Here’s another way of describing the answer to Example 1.1.1. Our reasoning shows that the statements \(|x| < 3\) and \(-3 < x < 3\) are true for exactly the same values of \(x\); the two statements are equivalent. In other words, for any number \(x\), if \(|x| < 3\), then \(-3 < x < 3\), and if \(-3 < x < 3\), then \(|x| < 3\). Mathematicians usually describe this situation by saying that \(|x| < 3\) is true if and only if \(-3 < x < 3\). The phrase “if and only if” comes up often in mathematics, and we will see it many times later in this book.

Of course, there is nothing special about the number 3 in this example. Similar reasoning, using the variable \(y\) in place of the number 3, can be used to establish the following theorem. Parts 3 and 4 of the theorem follow directly from parts 1 and 2, by negating the statements involved.

**Theorem 1.1.2.** For any numbers \(x\) and \(y\), the following statements are true:

1. \(|x| < y\) if and only if \(-y < x < y\).
2. \(|x| \leq y\) if and only if \(-y \leq x \leq y\).
3. \(|x| \geq y\) if and only if either \(x \leq -y\) or \(x \geq y\).
4. \(|x| > y\) if and only if either \(x < -y\) or \(x > y\).

The most important use of absolute values in calculus is to compute distances between numbers on the number line. To find the distance between two numbers, we subtract the smaller number from the larger. For example, the distance from \(-4\) to 3 on the number line is \(3 - (-4) = 7\). In general, if \(a \leq b\) then the distance from \(a\) to \(b\) is \(b - a\). But if \(a > b\) then the distance is \(a - b\). Is there a single formula that gives the distance from \(a\) to \(b\) no matter which of the numbers is larger?

It turns out that the formula \(|b - a|\) does the trick. We can see this by once again using reasoning by cases.
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Case 1. $a \leq b$. Then $b - a \geq 0$, so $|b - a| = b - a$, which, in this case, is the distance from $a$ to $b$ on the number line.

Case 2. $a > b$. Now $b - a < 0$, so $|b - a| = -(b - a) = a - b$, which is, once again, the distance from $a$ to $b$ in this case.

Thus, no matter which of the numbers $a$ and $b$ is larger, we have

$$|b - a| = \text{the distance from } a \text{ to } b \text{ on the number line}.$$ 

This fact provides a nice way to see why our solution to Example 1.1.1 makes sense. We have

$$|x| = |x - 0| = \text{distance from } 0 \text{ to } x \text{ on the number line}.$$ 

With this interpretation for $|x|$, the inequality $|x| < 3$ can be thought of as saying

$$(\text{distance from } 0 \text{ to } x \text{ on the number line}) < 3.$$ 

It is clear geometrically that the values of $x$ for which this statement is true are the numbers in the open interval $(-3, 3)$, exactly as we found in our solution to Example 1.1.1.

Example 1.1.3. Solve:

$$|3t + 2| \leq 4.$$ 

Solution. By part 2 of Theorem 1.1.2, the inequality to be solved is equivalent to

$$-4 \leq 3t + 2 \leq 4.$$ 

Subtracting 2 all the way through the inequality gives us

$$-6 \leq 3t \leq 2,$$ 

and then dividing by 3 we get

$$-2 \leq t \leq 2/3.$$ 

Thus the solution set for this inequality is a closed interval:

$$\{t : |3t + 2| \leq 4\} = \{t : -2 \leq t \leq 2/3\} = [-2, 2/3].$$

Sometimes it is useful to be able to simplify absolute values of complicated expressions. In such situations, the following theorem can be helpful.

Theorem 1.1.4. For any numbers $x$ and $y$, the following statements are true:

1. $|xy| = |x||y|$.

2. If $y \neq 0$ then $\frac{|x|}{|y|} = \frac{|x|}{|y|}$.

Proof. We will just prove part 1; the proof of part 2 is similar. First note that if either $x$ or $y$ is 0 then both sides of the equation $|xy| = |x||y|$ are 0, so the equation is true.
If neither of them is 0, then each is either positive or negative. This leaves us with four cases to consider:

**Case 1.** \( x > 0, \ y > 0 \). Then by the definition of absolute value, \(|x| = x\) and \(|y| = y\). Also, since the product of two positive numbers is positive, \(xy > 0\). Therefore \(|xy| = xy = |x||y|\).

**Case 2.** \( x > 0, \ y < 0 \). Then \(|x| = x\) and \(|y| = -y\). Since a positive number times a negative number is negative, \(xy < 0\). Thus \(|xy| = -xy = x(-y) = |x||y|\).

**Case 3.** \( x < 0, \ y > 0 \). Then \(|x| = -x\), \(|y| = y\), and \(xy < 0\), so \(|xy| = -xy = |x||y|\).

**Case 4.** \( x < 0, \ y < 0 \). Then \(|x| = -x\) and \(|y| = -y\). Since a negative times a negative is positive, we have \(xy > 0\), so \(|xy| = xy = (-x)(-y) = |x||y|\).

To illustrate the use of Theorem 1.1.4, we briefly revisit Example 1.1.3. Notice that \(|3t + 2| = |3(t + 2/3)| = |3||t + 2/3| = 3|t - (-2/3)|\), where we have used part 1 of Theorem 1.1.4 in the second step. Thus, the inequality in Example 1.1.3 can be rewritten

\[ 3|t - (-2/3)| \leq 4. \]

Dividing through by 3, this is equivalent to

\[ |t - (-2/3)| \leq 4/3, \]

which can be interpreted as meaning

\( (\text{distance from } t \text{ to } -2/3 \text{ on the number line}) \leq 4/3. \)

Thus the solution set of the inequality is

\[ [-2/3 - 4/3, -2/3 + 4/3] = [-2, 2/3], \]

exactly as we found in Example 1.1.3.

Theorem 1.1.4 tells us how to simplify an absolute value of a product or a quotient. What about the absolute value of a sum? Is it always true that \(|x + y| = |x| + |y|\)? A little experimentation shows that, unfortunately, this equation is not always true. For example, \(|5 + (-3)| = 2\), but \(|5| + |-3| = 8\). However, we do have the following important fact, which is known as the **triangle inequality**.

**Theorem 1.1.5 (Triangle Inequality).** For all numbers \( x \) and \( y \), \(|x + y| \leq |x| + |y|\).

**Proof.** According to part 2 of Theorem 1.1.2, the triangle inequality is equivalent to the statement

\[ -(|x| + |y|) \leq x + y \leq |x| + |y|, \]

so it will suffice to prove this inequality. We will leave it to you to verify (using reasoning by cases) that

\[ -|x| \leq x \leq |x| \]

and

\[ -|y| \leq y \leq |y| \]

(see Exercise 18). Adding these two inequalities gives inequality (1.2).
The theorems we have proven about absolute values could be regarded as shortcuts that allow us to solve problems more easily. For example, using Theorem 1.1.2 we were able to solve Example 1.1.3 without having to resort to reasoning by cases. This is a pattern that we will see repeated many times in this book. When a new concept is introduced, we will initially rely on the definition to tell us how to solve problems involving that concept. But often solutions based on the definition will be long and complicated, so we will develop theorems that provide shortcuts that allow us to solve problems more easily.

You may wonder, why bother learning the definitions? Why not just learn the shortcuts, if they provide easier ways of solving problems? One answer is that shortcuts, while helpful, are also often somewhat limited. They usually allow us to solve only a restricted range of problems. Sometimes you come across a problem for which the shortcuts are not helpful, and then you have no choice but to return to the definitions of the relevant concepts to solve the problem.

As an illustration of this, we close this section by solving an inequality involving absolute values for which our various shortcuts don’t seem to be very helpful.

**Example 1.1.6.** Solve:

\[ 2|x| - 3 \geq |x - 1|. \]

**Solution.** Although it is possible to use Theorem 1.1.2 to solve this inequality (see Exercise 16), the solution is not easy. Here we use the more straightforward approach of working from the definition of absolute value, which suggests reasoning by cases. Since we need to work with both \(|x|\) and \(|x - 1|\), we are led to consider three cases: \(x < 0\), \(0 \leq x < 1\), and \(x \geq 1\).

**Case 1.** \(x < 0\). In this case we also have \(x - 1 < 0\), so according to the definition of absolute value, \(|x| = -x\) and \(|x - 1| = -(x - 1) = 1 - x\). Substituting into our inequality, we find that we must solve

\[ -2x - 3 \geq 1 - x. \]

Adding \(2x\) to both sides and subtracting \(1\), we get \(-4 \geq x\). Thus, for negative values of \(x\), the inequality is true if and only if \(x \leq -4\). In other words, it is true if \(x \leq -4\) and false if \(-4 < x < 0\).

**Case 2.** \(0 \leq x < 1\). We have \(x \geq 0\), so \(|x| = x\), and \(x < 1\), so \(x - 1 < 0\) and therefore \(|x - 1| = 1 - x\). Thus in this case the inequality means

\[ 2x - 3 \geq 1 - x, \]

which can be simplified to \(x \geq 4/3\). But this inequality is false for all values of \(x\) in the range \(0 \leq x < 1\). Thus, none of the numbers in the interval \([0, 1)\) will be included in the solution set of our inequality.

**Case 3.** \(x \geq 1\). Then \(x \geq 0\) and \(x - 1 \geq 0\), and therefore \(|x| = x\) and \(|x - 1| = x - 1\). Filling in these formulas in our inequality we get

\[ 2x - 3 \geq x - 1, \]

and simplifying leads to \(x \geq 2\). Thus, the inequality is true for \(x \geq 2\) and false for \(1 \leq x < 2\).
Figure 1.6: The solution set for Example 1.1.6 is \((-\infty, -4] \cup [2, \infty)\).

Combining the information from all three cases, we find that the solution set is the set containing all numbers in the interval \((-\infty, -4]\) and also all numbers in the interval \([2, \infty)\). In other words,

\[
\{x : 2|x| - 3 \geq |x - 1|\} = \{x : \text{either } x \leq -4 \text{ or } x \geq 2\} = (-\infty, -4] \cup [2, \infty).
\]

\[\square\]

**Exercises 1.1**

1–13: Solve the inequality. If possible, write the solution set as an interval or a union of intervals.

1. \(|x - 5| < 7\).
2. \(|4x + 2| \leq 6\).
3. \(|5 - 2t| \geq 4\).
4. \(|x - 4| < x\).
5. \(2|x - 4| < x\).
6. \(|x + 4| < x\).
7. \(|x + 4| < 2x\).
8. \(|6 - 2u| - 3 < u\).
9. \(|3y| \leq |y| + 10\).
10. \(|x - 3| \geq |x| - 1\).
11. \(|2x - 3| \geq |x| - 1\).
12. \(|2x - 2| > |x| - 1\).
13. \(|3x - 6| \leq |3 - 6x|\).

14. Write the following sets as intervals or unions of intervals. Use interval notation.

   (a) \(\{x : x^2 < 64\}\).
   (b) \(\{x : x^3 < 64\}\).
   (c) \(\{x : 4 \leq x^2 < 9\}\).
   (d) \((-\infty, 5] \cap (3, \infty)\).
   (e) \((-3, 5] \cup (3, 7]\).
15. Use reasoning by cases to prove that for every number \( x \), \( \sqrt{x^2} = |x| \).

16. Solve Example 1.1.6 by using Theorem 1.1.2.

17. Prove part 1 of Theorem 1.1.2. (Hint: Treat the cases \( y > 0 \) and \( y \leq 0 \) separately.)

18. Prove that for every number \( x \), \(-|x| \leq x \leq |x|\). (This fact is used in the proof of Theorem 1.1.5.)

19. Prove that for all numbers \( x \) and \( y \), \(|x - y| \geq |x| - |y|\). (Hint: Start by applying the triangle inequality to \(|(x - y) + y|.|\).

20. Use the triangle inequality to show that for every number \( x \), if \(|x| \leq 2\) then \(|x^3 - 7x + 3| \leq 25\).

21. Show that for every number \( x \), if \(|x| \leq 2\) then \(\left| \frac{x^3 - 7x + 3}{9 - x^2} \right| \leq 5\). (Hint: Use Exercise 19 to show that \(|9 - x^2| \geq 5\).

1.2 Graphs in the Plane

In Section 1.1 we considered statements involving a single variable, usually \( x \), and we illustrated these statements by marking on a number line the values of the variable for which the statement is true. Often in calculus we will work with statements involving two variables, usually \( x \) and \( y \). To illustrate such statements we use the coordinate plane. Each point in the plane represents an ordered pair of numbers \((x, y)\), as illustrated in Figure 1.7.

For example, consider the statement

\[ 1 < x \leq 3 \text{ and } -2 \leq y < 1. \]  

The points \((x, y)\) whose coordinates make this statement true are shown in Figure 1.8. This set of points is called the graph of the statement. For example, the statement is true if \( x = 3 \) and \( y = -2 \), and therefore the point \((3, -2)\) is included in the graph. On the other hand, it is false if \( x = 3 \) and \( y = 1 \), so the point \((3, 1)\) is not included.

The statements we will be most concerned with in this book are equations involving two variables, most often \( x \) and \( y \). The graph of such an equation is usually a curve in the plane. Among the most important examples are equations whose graphs are straight lines. We assume you are familiar with equations of straight lines, but since they will be so important in calculus we briefly review the most important facts.

For any numbers \( m \) and \( b \), the graph of the equation \( y = mx + b \) is a straight line with slope \( m \) and \( y\)-intercept \( b \). This equation is called the slope-intercept equation of the line. For example, the line \( y = 2x - 3 \), which has slope 2 and \( y\)-intercept \(-3\), is shown in Figure 1.9a. The fact that the \( y\)-intercept is \(-3\) means that the line crosses the \( y\)-axis

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\[ \text{\footnote{It is an unfortunate fact that the notation for a point in the plane is the same as the notation for an open interval, namely a pair of numbers in parentheses. You will have to tell from context whether, for example, the notation (3, 5) stands for the open interval from 3 to 5 or the point in the plane whose coordinates are 3 and 5.}} \]
at \(-3\). In other words, the line passes through the point \((0, -3)\), as you can verify by checking that the equation \(y = 2x - 3\) is true when \(x = 0\) and \(y = -3\). To understand the meaning of the slope, it might be helpful to imagine a point \((x, y)\) moving along the line from left to right. The fact that the slope is 2 means that every time the \(x\)-coordinate of our moving point increases by 1, the \(y\)-coordinate increases by 2. More generally, if we add some number \(h\) to the \(x\)-coordinate of a point on the line, then we must add \(2h\) to the \(y\)-coordinate to reach another point on the line. To see why this is true, suppose that some point \((x_1, y_1)\) is on the line \(y = 2x - 3\). This means that the equation
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\[ y = 2x - 3 \] is true when \( x = x_1 \) and \( y = y_1 \), or in other words, \( y_1 = 2x_1 - 3 \). It follows that \( y_1 + 2h = 2x_1 - 3 + 2h = 2(x_1 + h) - 3 \), so the point \((x_1 + h, y_1 + 2h)\) is also on the line. For example, Figure 1.9a shows that if we move along the line from \((0, -3)\) to \((3, 3)\), \(x\) increases by 3 and \(y\) increases by \(2 \cdot 3 = 6\). The change in \(y\) is always twice as big as the change in \(x\), and we therefore say that the slope gives the rate of change of \(y\) with respect to \(x\).

Lines with positive slope slant upwards as we move from left to right, and lines with negative slope slant downwards. For example, Figure 1.9b shows the line \( y = -\frac{1}{3}x + 4 \). This time, when we add \( h \) to \( x \), we must add \(-\frac{1}{3}h\) to \( y\)—in other words, if \( x \) increases by \( h \), then \( y \) decreases by \( \frac{1}{3}h \). Figure 1.9b shows that when we move from \((0, 4)\) to \((3, 3)\), \(x\) increases by 3 and \(y\) decreases by \( \frac{1}{3} \cdot 3 = 1 \).

A line with slope 0 has an equation of the form \( y = 0 \cdot x + b \), or in other words \( y = b \). The graph of this equation is a horizontal line passing through the point \( b \) on the \( y\)-axis. Similarly, the equation of a vertical line passing through the point \( a \) on the \( x\)-axis is \( x = a \). Vertical lines are the only lines whose equations cannot be written in the slope-intercept form \( y = mx + b \). The slope of a vertical line is undefined.

**Example 1.2.1.** Find an equation of the line with slope 3 that passes through the point \((-1, 2)\).

**Solution.** The slope-intercept equation of the line must have the form \( y = 3x + b \), where \( b \) is the \( y\)-intercept. We must find \( b \).

Since the line passes through the point \((-1, 2)\), the equation for the line must be true when \( x = -1 \) and \( y = 2 \). In other words, \( 2 = 3(-1) + b \), and therefore \( b = 5 \). So the equation of the line is \( y = 3x + 5 \).
More generally, suppose we are looking for the line with slope $m$ passing through the point $(x_1, y_1)$. As in Example 1.2.1, we can substitute $x_1$ and $y_1$ for $x$ and $y$ in the equation $y = mx + b$ to conclude that $y_1 = mx_1 + b$, and therefore $b = y_1 - mx_1$. Thus, the slope-intercept equation for the line is $y = mx + (y_1 - mx_1)$. It is sometimes convenient to rearrange this equation to put it in the form

$$y - y_1 = m(x - x_1).$$

Equation (1.4) is called the point-slope form of the equation for the line.

If a line with equation $y = mx + b$ passes through two different points $(x_1, y_1)$ and $(x_2, y_2)$, then we can substitute the coordinates of both points into the equation of the line to conclude that $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$. Subtracting the first equation from the second we conclude that $y_2 - y_1 = m(x_2 - x_1)$, and therefore

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$  \hspace{1cm} (1.5)

This gives a convenient formula for the slope of a nonvertical line, given two points on the line. The formula confirms our earlier interpretation of the slope as the rate of change of $y$ with respect to $x$, since it says that if you move along the line from $(x_1, y_1)$ to $(x_2, y_2)$, then the slope is equal to the number of units of change in $y$ per unit of change in $x$.

**Example 1.2.2.** Find an equation of the line through the points $(-2, 3)$ and $(4, -1)$.

**Solution.** Applying equation (1.5), we find that the slope of the line is

$$m = \frac{-1 - 3}{4 - (-2)} = -\frac{2}{3}.$$  

Plugging this slope and the given point $(-2, 3)$ into the point-slope formula (1.4), we see that one equation for the line is

$$y - 3 = -\frac{2}{3}(x + 2).$$

Of course, we could have used the other given point $(4, -1)$ in the point-slope formula, so another answer is

$$y + 1 = -\frac{2}{3}(x - 4).$$

These two answers are equivalent, as you can see by checking that both can be rearranged to give the same slope-intercept equation

$$y = -\frac{2}{3}x + \frac{5}{3}.$$  

**Example 1.2.3.** Find the intersection point of the two lines

$$y = 4x + 1,$$

$$y = -2x + 3.$$
Solution. We are looking for the unique point \((x, y)\) whose coordinates satisfy both equations. If both equations are to be true, then we must have \(4x + 1 = -2x + 3\), and therefore \(6x = 2\) and \(x = 1/3\). Substituting this value into the first equation, we find that \(y = 4(1/3) + 1 = 7/3\). (Of course, substituting into the second equation would give the same answer for \(y\).) Thus, the intersection point is \((1/3, 7/3)\).

Since the slope of a line measures how steeply it is inclined, parallel lines have the same slope. Figure 1.10 illustrates that if a line has slope \(m \neq 0\), then a perpendicular line will have slope \(-1/m\). (Of course, if a line has slope 0 then it is horizontal, so a perpendicular line will be vertical.)

In Section 1.1, we determined that the distance between two points \(x_1\) and \(x_2\) on the number line is \(|x_2 - x_1|\). How do we compute the distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) in the plane? Applying the Pythagorean theorem to the triangle in Figure 1.11, we see that the distance \(d\) satisfies the equation \(d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2\), and therefore

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\] (1.6)

Equation (1.6) is known as the distance formula.

Example 1.2.4. Find the area of the triangle whose vertices are the points \((-1, 0), (5, 3),\) and \((1, 6)\).

Solution. See Figure 1.12. We take the side from \((-1, 0)\) to \((5, 3)\) as the base of the triangle. By the distance formula, it has length

\[
b = \sqrt{(5 - (-1))^2 + (3 - 0)^2} = \sqrt{45} = 3\sqrt{5}.
\]

Let \(L_1\) be the line containing the base of the triangle. Then \(L_1\) has slope \(m = (3 - 0)/(5 - (-1)) = 1/2\), and by the point-slope formula its equation is \(y - 0 = (1/2)(x - (-1))\),

\[
y = \frac{1}{2}(x + 1).
\]
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Figure 1.11: By the Pythagorean theorem, 
\[ d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2. \]

Figure 1.12: The solution to Example 1.2.4.

or equivalently \( y = (1/2)x + 1/2 \). To find the altitude of the triangle, we need to find the line \( L_2 \) that is perpendicular to \( L_1 \) and passes through the point \((1, 6)\). This line will have slope \(-1/m = -2\), so the point-slope formula now tells us that \( L_2 \) has equation 
\[ y - 6 = -2(x - 1), \]
or equivalently \( y = -2x + 8 \). As in Example 1.2.3, by solving the equation \((1/2)x + 1/2 = -2x + 8\) we find that \( L_1 \) and \( L_2 \) intersect at the point \((3, 2)\). Thus, the altitude of the triangle is the distance from \((3, 2)\) to \((1, 6)\), which is

\[ h = \sqrt{(1 - 3)^2 + (6 - 2)^2} = \sqrt{20} = 2\sqrt{5}. \]

Finally, we can conclude that the area of the triangle is

\[ \frac{1}{2}bh = \frac{1}{2} \cdot 3\sqrt{5} \cdot 2\sqrt{5} = 15. \]
Note that in our solution to Example 1.2.4 we left the base and altitude of the triangle in the form $b = 3\sqrt{5}$ and $h = 2\sqrt{5}$. We could have used a calculator to determine that $b \approx 6.708$ and $h \approx 4.472$, leading to the area calculation $(1/2)bh \approx 14.999$. But notice that this approach involves more work and leads to a less accurate answer. We usually find it easier and more useful to leave all of our numbers in exact form, as we did in Example 1.2.4, even when this means using an expression like $\sqrt{5}$.

So far we have focused on equations whose graphs are straight lines. Perhaps the most fundamental geometric shape other than a straight line is a circle. The circle of radius $r$ centered at the point $(a, b)$ consists of all points $(x, y)$ whose distance from $(a, b)$ is equal to $r$. Thus, to find an equation for this circle, we simply have to write an equation that expresses the statement

\[(\text{distance from } (x, y) \text{ to } (a, b)) = r.\]

Using the distance formula for the left-hand side of this equation, we get

$$\sqrt{(x-a)^2 + (y-b)^2} = r.$$  

Squaring both sides gives the simpler, equivalent formula

$$(x-a)^2 + (y-b)^2 = r^2.$$  

For example, the graph of the equation

$$x^2 + y^2 = 1$$

is the circle of radius 1 centered at the origin, $(0, 0)$.

**Exercises 1.2**

1–14: Graph the given statement in the plane.

1. $x = 3$ and $1 \leq y \leq 4$.
2. $y = x - 2$.
3. $y = -3x + 5$.
4. $y = (5 - x)/2$.
5. $2x + 3y = 5$.
6. $y = 3$.
7. $x = -2$.
8. $(y - 3)(x + 2) = 0$.
9. $x^2 - y^2 = 0$.
10. $y < x - 2$. 
11. \(0 \leq x \leq y \leq 1\).
12. \(x^2 + (y - 2)^2 = 4\).
13. \(x^2 + (y - 2)^2 \leq 4\).
14. \(x^2 + 2x + y^2 - 6y = 6\). (Hint: Add something to both sides of the equation to get it into the form \((x - a)^2 + (y - b)^2 = r^2\).)
15. Show that the points that are equidistant from the points \((0, 3)\) and \((6, 0)\) form a straight line. What are the slope and y-intercept of this line?
16. Show that the points that are twice as far from \((6, 0)\) as they are from \((0, 3)\) form a circle. What are the center and radius of this circle?

17–23: Find an equation for the line.
17. The line passing through the points \((-1, 1)\) and \((2, 7)\).
18. The line passing through the points \((-1, 7)\) and \((2, 1)\).
19. The line passing through the points \((-1, 7)\) and \((2, 7)\).
20. The line passing through the points \((2, 1)\) and \((2, 7)\)
21. The line through the point \((2, 3)\) that is parallel to the line \(x - 3y = 2\).
22. The line through the point \((2, 3)\) that is perpendicular to the line \(x - 3y = 2\).
23. The line tangent to the circle \((x - 3)^2 + y^2 = 13\) at the point \((1, 3)\). (Hint: The line tangent to a circle at a point is perpendicular to the radius from the center of the circle to that point.)
24. Find the point of intersection of the lines \(y = 4x - 5\) and \(y = -2x + 4\).
25. Find all points of intersection of the line \(y = -2x + 1\) and the circle \((x + 1)^2 + (y - 1)^2 = 4\).
26. Show that the points \((0, 0)\), \((3, 1)\), \((1, 7)\), and \((-2, 6)\) are the vertices of a rectangle. What is the area of this rectangle?
27. Give an alternative solution to Example 1.2.4 by computing the area of the gray region in Figure 1.13 and then subtracting the area of the part of that region that is striped.

1.3 Functions

High school mathematics is mostly concerned with numbers and operations on numbers, such as addition, subtraction, multiplication, and division. One of the most distinctive features of calculus is that it is primarily concerned with operations on functions rather than operations on numbers.

A function is a rule that associates, with every number, exactly one corresponding number. For example, the rule might associate with any number \(x\) the square of that number, \(x^2\). Or it might associate with each number \(x\) the number \(3x^5 - 7x\).
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We will often use letters to stand for functions, most often the letters $f$ and $g$. If $f$ is a function, then we write $f(x)$ to denote the number associated with $x$ by the function $f$. For example, if $f$ and $g$ are the two functions defined in the last paragraph, then we could say that for every number $x$, $f(x) = x^2$ and $g(x) = 3x^5 - 7x$. The fact that these equations hold for every number $x$ means that we can substitute any number we please for $x$. For example, we have $f(5) = 5^2 = 25$ and $g(2) = 3(2^5) - 7(2) = 82$. In other words, the number associated with 5 by the function $f$ is 25, and the number associated with 2 by the function $g$ is 82. We can also replace $x$ in these formulas by a more complicated formula. For example, for any numbers $a$ and $b$ we have $g(a - 3b) = 3(a - 3b)^5 - 7(a - 3b)$.

In a formula like $f(x) = x^2$, it may be helpful to think of the number $x$ as the input to the function $f$. The function can be thought of as an operation or calculation that is performed on this input to find the number associated with $x$ by the function—in this case, $x^2$. We therefore sometimes speak of the function $f$ as being applied to the number $x$, and we say that $f(x)$ is the result of applying the function $f$ to $x$. It is also sometimes called the value of $f$ at $x$, or simply $f$ of $x$. Another image that may be helpful is to think of a function $f$ as a machine, as shown in Figure 1.14. We feed a number $x$ into the machine, and the machine spits out the number $f(x)$, which in this example is $x^2$. For example, if we feed the number 5 into this machine, the number 25 comes out.

![Figure 1.14: The function $f$ viewed as a machine.](image)
We usually define a function \( f \) by giving a formula that tells us, for any number \( x \), the value of \( f(x) \). For example, here are two more definitions of functions:

\[
\text{For every number } x, \quad f(x) = \frac{x^3 - 3}{x^2 + 2}.
\]

\[
\text{For every number } x, \quad g(x) = \sqrt{x}.
\]

However, a function need not be defined by a formula. Any rule that specifies unambiguously, for each number \( x \), the corresponding value \( f(x) \) counts as a function. For example, here are two more examples of functions:

\[
\text{For every number } x, \quad f(x) = \text{the greatest integer that is less than or equal to } x.
\]

\[
\text{For every number } x, \quad g(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}
\]

Of course, you recognize \( g(x) \) as \(|x|\), the absolute value of \( x \). Thus, we may speak of the absolute value function. The function \( f \) is called the greatest integer function, and \( f(x) \) is often denoted \([x]\). For example, \([3.7] = 3\), \([-3.7] = -4\), and \([6] = 6\).

Although we usually define a function \( f \) by specifying the value of \( f(x) \) for every \( x \), it is important to understand the distinction between the expressions \( f \) and \( f(x) \). The letter \( f \) is the name of a function, and the expression \( f(x) \) denotes the number associated with \( x \) by \( f \). In particular, \( f \) is a function, while \( f(x) \) is a number.

Can we define a function \( f \) by saying that for every number \( x \), \( f(x) = \sqrt{x} \)? There is a problem with this definition: the formula for \( f(x) \) can’t be used for every number \( x \), since \( \sqrt{x} \) is undefined if \( x < 0 \). But we will certainly want to be able to work with formulas like \( \sqrt{x} \) in calculus. We are thus led to extend our definition by allowing a function to be a rule that applies to only some numbers. The set of numbers to which the function applies is called the domain of the function, and it could be any subset of \( \mathbb{R} \), the set of all real numbers. Thus, our new definition is that if \( D \subseteq \mathbb{R} \), then a function \( f \) with domain \( D \) is a rule that associates, with each number \( x \in D \), exactly one corresponding number \( f(x) \). We write \( f : D \to \mathbb{R} \) to indicate that \( f \) is a function with domain \( D \); in other words, it is a rule that associates, with each element of \( D \), a corresponding element of \( \mathbb{R} \). In all of our previous examples we had \( D = \mathbb{R} \), but sometimes our functions will have smaller domains. For example, the equation \( f(x) = \sqrt{x} \) can be used to define a function with domain \( \{x : x \geq 0\} = [0, \infty) \). For this example we would say \( f : [0, \infty) \to \mathbb{R} \).

Here are two more examples of functions whose domains are not all of \( \mathbb{R} \). In each case, we define the function by first specifying the domain and then giving the result of applying the function to any element of the domain.

The domain of \( f \) is \([-1, 1]\), and for all \( x \in [-1, 1] \), \( f(x) = \sqrt{1 - x^2} \).

The domain of \( g \) is \((0, \infty)\), and for all \( x \in (0, \infty) \), \( g(x) = 1/x \).

For example, \( f(1/2) = \sqrt{3}/2 \) and \( g(7) = 1/7 \). However, \( g(-7) \) is undefined, because \(-7\) is not in the domain of \( g \); the equation \( g(x) = 1/x \) was declared to hold only for \( x \in (0, \infty) \), so it does not apply to \( x = -7 \).
There are certain shortcuts that mathematicians often take when defining a function. First, mathematicians often don’t specify the domain of the function they are defining. If no domain is specified, it is understood that the domain is the set of all values for which the definition of the function makes sense. Second, when defining a function \( f \) by giving a formula for \( f(x) \), mathematicians usually don’t say explicitly that this formula applies to all \( x \) in the domain of the function. Again, this is to be understood. Finally, mathematicians sometimes write things like “the function \( f(x) = x^2 \)” when what they really mean is “the function \( f \) defined by the equation \( f(x) = x^2 \).”

**Example 1.3.1.** Find the domains of the functions \( f \), \( g \), and \( h \) defined by the following formulas:

\[
\begin{align*}
f(x) &= \frac{x^2 - 4}{x - 2}, \\
g(x) &= x + 2, \\
h(x) &= \sqrt{x} + \frac{1}{\sqrt{3 - x}}.
\end{align*}
\]

**Solution.** The formula for \( f(x) \) makes sense for all values of \( x \) except \( x = 2 \), for which we have a 0 in the denominator. Thus, the domain of \( f \) is \( \{x : x \neq 2\} \); we could also write this set as the union of two intervals, \( (-\infty, 2) \cup (2, \infty) \). Of course, the formula for \( g(x) \) makes sense for all values of \( x \), so the domain of \( g \) is \( \mathbb{R} \).

The formula for \( h(x) \) involves the expressions \( \sqrt{x} \) and \( \sqrt{3-x} \), and these are undefined for some values of \( x \). In order for \( \sqrt{x} \) to make sense we must have \( x \geq 0 \), and in order for \( \sqrt{3-x} \) to make sense we must have \( 3-x \geq 0 \), and therefore \( x \leq 3 \). But there is one more restriction; if \( x = 3 \) then the formula for \( h(x) \) will involve division by 0, so we must exclude 3 from the domain. Thus, the domain of \( h \) is the half-open interval \( [0, 3) \). \( \square \)

The formula for \( f(x) \) in Example 1.3.1 can be simplified by factoring the numerator and then canceling:

\[
f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2.
\]

You might think, therefore, that the functions \( f \) and \( g \) in Example 1.3.1 are really the same. However, the algebraic simplification we have done is correct only for \( x \neq 2 \); when \( x = 2 \), the fraction \( (x-2)(x+2)/(x-2) \) is undefined, whereas \( x+2 \) is equal to 4. In other words, for all \( x \neq 2 \) we have \( f(x) = g(x) \), but \( f(2) \) is undefined, whereas \( g(2) = 4 \). Thus, we will insist that \( f \) and \( g \) are different functions. It may seem that we are being unnecessarily fussy about this, but as we will see in the next chapter, maintaining this distinction will be crucial to understanding some of the central concepts of calculus.

The **graph** of a function \( f \) is the graph of the equation \( y = f(x) \). For example, if we define a function \( f \) by the equation \( f(x) = 2x - 3 \), then the graph of \( f \) is the graph of the equation \( y = 2x - 3 \), which is the line in Figure 1.9a. In the equation \( y = f(x) \), the variable \( x \) is sometimes called the *independent variable*, and \( y \) is the *dependent variable*. The reason for this terminology is that we are free to choose any value for \( x \) (as long as
the value is in the domain of \( f \), but once a value for \( x \) has been chosen, the equation \( y = f(x) \) determines a corresponding value for \( y \). Thus, the value of \( y \) depends on the value of \( x \), but the value of \( x \) doesn’t depend on anything. We may sometimes use other letters for the independent and dependent variables, but the letters \( x \) and \( y \) are used most often.

You can learn a lot about a function \( f \) very easily by looking at its graph, and we will therefore use graphs of functions extensively throughout this book. Perhaps most important is the fact that the value of \( f(a) \) for any particular number \( a \) can be read off from the graph. To see how, simply note that a point \((a, b)\) will be on the graph of \( f \) if and only if the assignment of values \( x = a, y = b \) makes the equation \( y = f(x) \) true, that is, if and only if \( b = f(a) \). Thinking of the function \( f \) as a machine, this means that a point \((a, b)\) will be on the graph of \( f \) if and only if when you feed the number \( a \) into the machine, the number \( b \) comes out. Thus, if \( a \) is in the domain of \( f \) then there will be exactly one point on the graph of \( f \) whose \( x \)-coordinate is \( a \); the \( y \)-coordinate of that point will be \( f(a) \). If \( a \) is not in the domain, then there will be no point on the graph with \( x \)-coordinate \( a \).

Another way to say this is that if \( a \) is in the domain of \( f \) then the vertical line \( x = a \) intersects the graph of \( f \) exactly once, at the point \((a, f(a))\), and if \( a \) is not in the domain of \( f \) then the vertical line \( x = a \) does not intersect the graph of \( f \) at all. Thus, no vertical line intersects the graph of \( f \) more than once. Conversely, if some curve does not intersect any vertical line more than once, then the curve is the graph of some function \( f \). We can define \( f \) by saying that the domain of \( f \) is the set of numbers \( a \) such that the line \( x = a \) intersects the curve, and for each such number \( a \), \( f(a) \) is the \( y \)-coordinate of the unique point where the line \( x = a \) intersects the curve. Thus, we can say that a curve is the graph of a function if and only if no vertical line intersects the curve more than once. This is sometimes called the vertical line test.

We have seen that the domain of a function \( f \) is equal to the set of all numbers that appear as the \( x \)-coordinate of a point on the graph of \( f \). There is also a name for the set of numbers appearing as the \( y \)-coordinate of a point on the graph of \( f \); it is called the range of \( f \). Thinking of \( f \) as a machine, we could say that the domain of \( f \) is the set of all numbers that can be fed into the machine, and the range is the set of all numbers that come out.

Let’s try out these ideas in some examples. The curve in Figure 1.15a passes the vertical line test, so it is the graph of a function \( f \). Although we don’t have a formula for \( f(x) \), we can see that \( f(-1) = 3 \), \( f(1) = 2 \), \( f(3) = 4 \), and \( f(4) = 5 \), because the points \((-1, 3), (1, 2), (3, 4), \) and \((4, 5)\) are on the graph. We can also see that the domain of \( f \) is the interval \([-1, 5]\), and the range is the interval \([2, 5]\). On the other hand, the black curve in Figure 1.15b is not the graph of a function, because it fails the vertical line test. The line \( x = 2 \), shown in red, crosses the curve three times.

Figure 1.16 shows the graphs of the functions \( g(x) = |x| \) and \( h(x) = [x] \). The graph of \( g \) is the graph of the equation \( y = |x| \), and filling in the definition of \( |x| \) this means

\[
y = \begin{cases} x, & \text{if } x \geq 0, \\
-x, & \text{if } x < 0. \end{cases}
\]
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Figure 1.15: The curve in (a) is the graph of a function, but the curve in (b) is not.

Figure 1.16: Graphs of the functions $g(x) = |x|$ and $h(x) = \lfloor x \rfloor$.

For $x \geq 0$ this is the same as the graph of $y = x$, which is a line with slope 1 and passing through the origin. For $x < 0$, it is the same as the graph of $y = -x$, a line through the origin with slope $-1$. Putting these two pieces together gives us the graph shown in Figure 1.16a. Similarly, the equation $y = h(x) = \lfloor x \rfloor$ means that $y = 0$ when $0 \leq x < 1$, $y = 1$ when $1 \leq x < 2$, and so on. This leads to the infinitely many horizontal line segments that make up the graph of $h$ in Figure 1.16b. The domains of $g$ and $h$ are both $\mathbb{R}$. The range of $g$ is the interval $[0, \infty)$, and the range of $h$ is the set of all integers, which is usually denoted $\mathbb{Z}$.

To illustrate the usefulness of graphs of functions, let us return to Example 1.1.6, in which we solved the inequality $2|x| - 3 \geq |x - 1|$. Define functions $f$ and $g$ by the formulas

$$f(x) = 2|x| - 3, \quad g(x) = |x - 1|.$$
Figure 1.17: The red curve is the graph of \( y = f(x) = 2|x| - 3 \), and the blue curve is the graph of \( y = g(x) = |x - 1| \).

To graph \( f \), we graph the equation \( y = 2|x| - 3 \). As in Figure 1.16a, this consists of the line \( y = 2x - 3 \) for \( x \geq 0 \) together with the line \( y = -2x - 3 \) for \( x < 0 \). Thus, the graph of \( f \) is the red curve in Figure 1.17, and similar reasoning shows that the graph of \( g \) is the blue curve. We will leave it to you to compute the intersection points that are shown in Figure 1.17. The inequality from Example 1.1.6 is \( f(x) \geq g(x) \), and since \( f(x) \) and \( g(x) \) give the \( y \)-coordinates of points on the red and blue curves, the solution set of the inequality is just the set of \( x \) values for which the red curve is at least as high as the blue curve. Looking at Figure 1.17, it is clear that this solution set is \((-\infty, -4] \cup [2, \infty)\), exactly as we found in Example 1.1.6.

Since graphs of functions are so useful, given a function it will be important to be able to draw its graph. We will see later how calculus can be used to draw accurate graphs of functions, but for now we will just mention a method that can be used to get started on drawing the graph of a function \( f \). Pick a collection of values of \( x \) in the domain of the function and compute, for each value of \( x \), the corresponding value of \( y = f(x) \). This will give you the coordinates of a collection of points on the graph of \( f \). For example, in Figure 1.18 we have computed a table of values for the function \( f \) defined by the equation \( f(x) = x^3 - 4x^2 + 2x + 2 \) and plotted the corresponding points. The points seem to lie along a smooth curve, so we have filled in such a curve passing through the plotted points as the graph of \( f \). Filling in this curve is, at this point, guesswork. The only points we can be sure are on the graph are the ones whose coordinates we computed. The graph of \( f \) might have extra wiggles between the plotted points, and it might have further bends beyond the part of the graph we have drawn. Of course, we could plot more points to try to detect further features of the graph, but no matter how many points we plot we will always be guessing if we fill in a curve connecting these points. One important application of calculus is that it will allow us to get many important features of the graph of a function right without guesswork.
1.3. FUNCTIONS

<table>
<thead>
<tr>
<th>x</th>
<th>y = f(x)</th>
</tr>
</thead>
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<tr>
<td>−1</td>
<td>−5</td>
</tr>
<tr>
<td>−1/2</td>
<td>−1/8</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1/2</td>
<td>17/8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
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<tr>
<td>3/2</td>
<td>−5/8</td>
</tr>
<tr>
<td>2</td>
<td>−2</td>
</tr>
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<td>−19/8</td>
</tr>
<tr>
<td>3</td>
<td>−1</td>
</tr>
<tr>
<td>7/2</td>
<td>23/8</td>
</tr>
</tbody>
</table>

Figure 1.18: Graphing the function $f(x) = x^3 - 4x^2 + 2x + 2$.

You might think that another way to get an accurate graph of a function without guesswork is to use a computer or graphing calculator. But what does a computer do when you ask it to graph a function? The answer is that it simply plots some points and then draws a smooth curve through those points! Thus, the computer-produced graph is also based on guesswork. In practice, computer-drawn graphs are usually fairly accurate, because the computer is able to plot many more points than you could plot by hand. For example, in Figure 1.18 we plotted 10 points, but a computer might plot hundreds of points when graphing a function. But computers do sometimes miss important features of graphs of functions. In contrast, the calculus-based methods we will learn later never miss important features of graphs of functions, because they involve no guesswork.

Exercises 1.3

1–9: Let $f$, $g$, and $h$ be the functions defined by the following equations:

$$f(x) = x^2 - 4, \quad g(x) = \sqrt{x^2 + 5}, \quad h(x) = \lfloor 5x - 2 \rfloor.$$  

Evaluate each expression, simplifying as much as possible.

1. $f(3)$.
2. $g(2)$.
3. $h(2.7)$.
4. $f(2 - a)$.
5. $g(2x + 1)$.
6. $h([x] + 1)$.  

7. $f(g(x))$.
8. $g(f(x))$.
9. $h(h(x))$.

10–18: Find the domain of each function. Write your answer as a union of intervals, if possible.

10. $f(x) = \frac{2x - 5}{5x - 2}$.
11. $g(x) = \frac{x^2 - 9}{x^2 + x - 6}$.
12. $f(x) = \sqrt{2 - x}$.
13. $g(x) = \sqrt{16 - x^2}$.
14. $h(x) = \frac{\sqrt{16 - x^2}}{\sqrt{2 - x}}$.
15. $h(x) = \sqrt{\frac{16 - x^2}{2 - x}}$.
16. $f(x) = \sqrt{2 - |x|}$.
17. $g(x) = \sqrt{2 - \lfloor x \rfloor}$.
18. $f(x) = \sqrt{x^4 - x^2}$.

19. Which of the graphs in Figure 1.19 are graphs of functions?

20–30: Graph each function.

20. $f(x) = x - 2$.
21. $g(x) = 2 - x$.
22. $h(x) = \frac{x + 4}{3}$.

23. $f(x) = \begin{cases} 3 - x, & \text{if } x > 2, \\ 1, & \text{if } -2 \leq x \leq 2, \\ x + 4, & \text{if } x < -2. \end{cases}$
24. $f(x) = 2|x| - 1$.
25. $g(x) = |2x - 1|$.
26. $h(x) = |2|x| - 1|$.
27. $f(x) = 2[x] - 1$.
28. $g(x) = |2x - 1|$.
1.3. FUNCTIONS

Figure 1.19: Which are graphs of functions?

29. \( f(x) = |x| \).

30. \( g(x) = |x| \).

31. The graph of a function \( f \) is shown in Figure 1.20.

   (a) What is the domain of \( f \)?
   (b) What is the range of \( f \)?
   (c) What is \( f(2) \)?
   (d) What is \( f(5) \)?
   (e) Solve the inequality: \( f(x) > 1 \). Write the solution set as a union of intervals.
   (f) For what value of \( x \), if any, is \( f(x) \) smallest?
   (g) For what value of \( x \), if any, is \( f(x) \) largest?

32. Let \( f(x) = x^5 - 30x^{5/3} + 36x - 8\sqrt[3]{x} \).

   (a) Use a calculator to evaluate \( f(x) \) at \( x = -2, -1, 0, 1, 2 \). Plot the corresponding points and draw a smooth curve through these points to get a guess at the shape of the graph of \( f \).
(b) Evaluate \( f(x) \) at \( x = -3/2, -1/2, 1/2, 3/2 \) and plot the corresponding points. Do these points lie on the curve you drew in part (a)? Adjust your graph to accommodate these new points.

(c) Evaluate \( f(-1/40) \) and \( f(1/40) \). Do you need to change your graph again?

1.4 Combining Functions

Sometimes we will want to combine functions to create new functions. For example, suppose \( f \) and \( g \) are functions. We can define a new function \( h \) by the equation

\[
h(x) = f(x) + g(x).
\]

(1.7)

The function \( h \) is called the sum of \( f \) and \( g \), and it is denoted \( f + g \). Thus, we can write \( h = f + g \), and substituting \( f + g \) for \( h \) in equation (1.7) we have

\[
(f + g)(x) = f(x) + g(x).
\]

(1.8)

It is important to understand that the two plus signs in equation (1.8) have different meanings. On the right-hand side of the equation, the plus sign appears between \( f(x) \) and \( g(x) \), which are numbers, and the plus sign stands for ordinary addition of numbers. But on the left-hand side, it appears between \( f \) and \( g \), which are functions, and it stands for a new kind of addition, addition of functions. The expression \( f + g \) is the name of a function—the function that we originally called \( h \) in equation (1.7). Equation (1.8) tells us how to apply the function \( f + g \) to a number \( x \). Figure 1.21 shows the function \( f + g \) as a machine, constructed from machines for the functions \( f \) and \( g \).
1.4. COMBINING FUNCTIONS

\[ f + g \]

For example, suppose \( f \) and \( g \) are the functions defined by the equations

\[ f(x) = \sqrt{x}, \quad g(x) = \frac{1}{\sqrt{3-x}}. \]

Then \( f + g \) is the function defined by

\[ (f + g)(x) = f(x) + g(x) = \sqrt{x} + \frac{1}{\sqrt{3-x}}. \]

This is just the function that we called \( h \) in Example 1.3.1. Notice that in order for the formula for \( (f + g)(x) \) to make sense for some particular value of \( x \), both \( f(x) \) and \( g(x) \) must make sense; in other words, \( x \) must be in the domains of both \( f \) and \( g \). It follows that the domain of \( f + g \) is the intersection of the domains of \( f \) and \( g \). We will leave it for you to verify that in this example, the domain of \( f \) is \([0, \infty)\) and the domain of \( g \) is \((-\infty, 3)\), so the domain of \( f + g \) is \([0, \infty) \cap (-\infty, 3) = [0, 3)\). Of course, this is in agreement with what we got for the domain of \( h \) in Example 1.3.1.

The ideas we have just discussed for addition of functions can also be applied to subtraction, multiplication, and division. We summarize these ideas with the following definition.

**Definition 1.4.1.** Suppose \( f \) and \( g \) are functions. Then the functions \( f + g \), \( f - g \), \( f \cdot g \), and \( f/g \) are defined as follows:

\[
(f + g)(x) = f(x) + g(x), \\
(f - g)(x) = f(x) - g(x), \\
(f \cdot g)(x) = f(x) \cdot g(x), \\
(f/g)(x) = f(x)/g(x).
\]

Notice that if the domains of \( f \) and \( g \) are \( D_f \) and \( D_g \), then the domains of \( f + g \), \( f - g \), and \( f \cdot g \) are all \( D_f \cap D_g \). For \( f/g \), we must add the further restriction that \( g(x) \neq 0 \) for all \( x \) in the domain of \( f/g \).
must not be 0. Thus, the domain of $f/g$ is

$$D_{f/g} = \{ x : x \in D_f \cap D_g \text{ and } g(x) \neq 0 \}.$$  

There is one more important way of combining functions that we will make extensive use of:

**Definition 1.4.2.** Suppose $f$ and $g$ are functions. Then the composition of $f$ and $g$ is the function $f \circ g$ defined by the formula

$$(f \circ g)(x) = f(g(x)).$$

Thus, to apply the function $f \circ g$ to a number $x$, we first apply the function $g$ to $x$ to get $g(x)$, and then we apply $f$ to $g(x)$ to get $f(g(x))$. Figure 1.22 shows how machines for the functions $f$ and $g$ can be combined to construct a machine for $f \circ g$.

![Figure 1.22: The function $f \circ g$, viewed as a machine.](image)

For example, let $f(x) = 3x + 1$ and $g(x) = x^2 - 2$. To compute $(f \circ g)(3) = f(g(3))$, we first compute $g(3) = 3^2 - 2 = 7$, and then we apply $f$ to get $f(g(3)) = f(7) = 3(7) + 1 = 22$. More generally, for every number $x$ we have

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 2) = 3(x^2 - 2) + 1 = 3x^2 - 5.$$  

Notice that

$$(g \circ f)(x) = g(f(x)) = g(3x + 1) = (3x + 1)^2 - 2 = 9x^2 + 6x - 1,$$

which is an entirely different function. For example, we have seen that $(f \circ g)(3) = 22$, but $(g \circ f)(3) = 9(3^2) + 6(3) - 1 = 98$. Thus, $f \circ g$ is not the same as $g \circ f$.

For any functions $f$ and $g$, in order for the formula $(f \circ g)(x) = f(g(x))$ to make sense, $x$ must be in the domain of $g$, so that $g(x)$ is defined, and then $g(x)$ must be in the domain of $f$, so that $f(g(x))$ is defined. Thus, if the domains of $f$ and $g$ are $D_f$ and $D_g$, as before, then the domain of $f \circ g$ is

$$D_{f \circ g} = \{ x : x \in D_g \text{ and } g(x) \in D_f \}.$$
-designed for undergraduate mathematics majors, this rigorous and rewarding treatment covers the usual topics of first-year calculus: limits, derivatives, integrals, and infinite series. Author Daniel J. Velleman focuses on calculus as a tool for problem solving rather than the subject’s theoretical foundations. Stressing a fundamental understanding of the concepts of calculus instead of memorized procedures, this volume teaches problem solving by reasoning, not just calculation. The goal of the text is an understanding of calculus that is deep enough to allow the student to not only find answers to problems, but also achieve certainty of the answers’ correctness.

No background in calculus is necessary. Prerequisites include proficiency in basic algebra and trigonometry, and a concise review of both areas provides sufficient background. Extensive problem material appears throughout the text and includes selected answers. Complete solutions are available to instructors.