A Teacher's Manual

showing how this book can be used with:

- liberal arts students
- education majors or practicing teachers
- mathematics or science majors

Third Edition

MATHEMATICS
THE MAN-MADE UNIVERSE

An Introduction to the Spirit of Mathematics

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CONTENTS

GUIDE WITH ANSWERS AND DISCUSSIONS OF SELECTED EXERCISES

The text can serve as an aid in courses varying from the gentle to the demanding. The emphasis is at the discretion of the instructor. The manual indicates some of the options the instructor has in treating each chapter. These options are expressed in terms of three types of students whom the text may serve: the general student, the education student, and the math student.

THREE MAIN TYPES OF STUDENTS

The general student: a student who wishes to or must broaden his or her math background; typically a liberal arts student with at least a year of algebra and a year of geometry in high school.

The education student: a prospective elementary or secondary teacher, possibly with a negative attitude toward mathematics. He or she should already have had a course that covers the elementary curriculum and basic arithmetical skills. Also teachers returning for enrichment.

For such a student a course based on this text should focus on open-ended questions rather than proofs and answers. The student should be encouraged to develop (and test) classroom projects derived from or related to the material. Proofs should be omitted almost entirely. Cuisenaire rods, geoboards, beans, etc. should be introduced to help convey the techniques that will help bring excitement to the teacher's own classroom.

The math student: a math or science-oriented freshman. Such a student would obtain a view of mathematics, both pure and applied, that would broaden and balance that obtained from calculus. Moreover, the course would introduce the style of thinking more often found in upper division math courses.

Chapter 1: Weighing

This chapter is easily covered in one or two meetings. I usually just pose a specific problem or two and let the class develop the conjecture. Almost everyone eventually comes up with "the greatest common divisor" though it is not mentioned in the chapter. To avoid one student's short circuiting the discussions, you may break the class into small groups of 2 to 4, to work on some cases.

GENERAL STUDENT. Already, by using a variety of the later exercises, one can make the point that many different problems may readily be just versions of one mathematical problem. Some will feel the conjecture is "obviously true" on the basis of experimental evidence. However, Polya's problem in Chapter 19 provides an equally "obvious" assertion that holds up through 900,000,000 but is false. It could be presented here.

EDUCATION STUDENT. Bring Cuisenaire rods to class and approach the chapter as in Exercise 42. Also emphasize the number line as in Exercise 41. Exercises 26, 28, 29, and 30 also provide resources for enriching and teaching arithmetic. (Omit the proof given later in Chapter 3.)

MATH STUDENT. This chapter introduces principal ideal rings. Ask them to try to "prove" their conjectures. Also ask them to consider the following problem, which generalizes Exercise 41: Let a, b, c, d be four positive integers. A rabbit jumps about on the points (x, y) where x and y are integers. When he is at one point he may either go a to the right and b upward or c to the right and d upward. Or he may go a to the left and b downward or c to the left and d downward Starting at (0,0), what points can he reach? What must be true about a, b, c, d if he can reach all (x, y) by repeated jumps? Answer: $ad-bc = \pm 1$.

The case of three measures A, B and C may also be considered.

Chapter 2: Primes

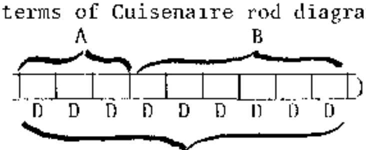
The emphasis here depends on what will be done with Chapter 3. If the Fundamental Theorem of Arithmetic will be assumed, then all that is necessary from Chapter 2 are the notions, "prime", "divides", and "factor". If the Fundamental Theorem will be proved, then the Lagado system at the end of the chapter should be covered. It persuades some, but not all, students that unique factorization into primes is not obvious. Some criticize it "because lots of integers are omitted." A more sophisticated example is the ring of numbers of the form $a + b\sqrt{10}$, where a and b are integers. In this ring, 2, 3, $-2 + \sqrt{10}$ and $2 + \sqrt{10}$ are primes, yet

$$2 \cdot 3 = 6 = (2 + \sqrt{10}) \quad (-2 + \sqrt{10})$$

GENERAL STUDENT. The proof that the set of primes is infinite is a jewel of mathematical thinking. It provides an opportunity to distinguish between "experimental evidence" and "mathematical truth". The debates over the Lagado system and proposed "proofs" of unique factorization in the usual ring of integers may take two class meetings before the students are sufficiently frustrated to appreciate Chapter 3 They will naturally come upon the importance of the fact that if a prime divides a product, it divides at least one of the factors.

EDUCATION STUDENT. Introduce the primes by "rectangular flags". Also, relate them to the natural numbers that do not appear in the multiplication table except in the 1- row and 1- column. Cuisenaire rods may also be used: determine whether a given number can be "measured" by copies of any smaller number other than 1. The Lemma and Theorem 1 are easily discussed in terms of Cuisenaire rod diagrams.

(e.g. D divides A + B:

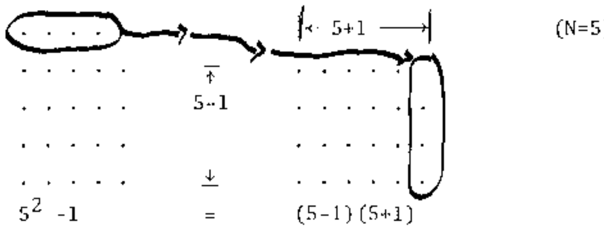


Proofs may be omitted. All that is needed is the definition of "prime", "divides", and "factor".

Students may bring beans to class to experiment "concretely" with such open ended questions as Goldbach's conjecture. (Take an even number of beans. Can you break them into two prime piles?) The use of such models in open-ended questions may help attain three goals:

(a) diminish fear and increase confidence
(b) clarify the questions
(c) provide resources for the prospective teacher.

Other examples: Exercise 4; Remove 1 bean from a square array of beans. Is the number of beans left ever prime? The explanation, $N^2-1 = (N + 1)(N - 1)$ can be shown with beans:



The assertion that every positive integer is the sum of at most four squares can also be translated into beans; Can a pile of beans always be broken into at most four square piles? (Note. 1 is a square.)

MATH STUDENT. The later exercises and references offer additional topics for reading, discussion, or reports.

Chapter 3: Fundamental theorem

There are various levels at which this chapter can be treated.

1. Simply assume the Fundamental Theorem of Arithmetic and assign a few exercises or common divisors.
2. Assume the weighing lemma and show that it implies the Fundamental Theorem of Arithmetic.
3. Prove the weighing lemma, but do not get involved with computations of M and N. This can be done as follows. The Euclidean algorithm for (945,219) yields, after a slight change.

$$69 = \underline{945} - 4 \cdot \underline{219}$$
$$12 = \underline{219} - 3 \cdot \underline{69}$$
$$9 = \underline{69} - 5 \cdot \underline{12}$$
$$3 = \underline{12} - \underline{9}$$

The last equation tells us "3 can be weighed with 9's and 12's." The next-to-the-last equation tells us "9 can be weighed with 69's and 12's." Together, they show "3 can be weighed with 12's and 69's." By following this example equation by equation (without doing any arithmetic) the student eventually finds that "3 can be weighed with 945's and 219's."
4. Cover everything except "the concealed theorem" which can be read by the interested student.

In any case, it is not necessary to refer to "special numbers".

GENERAL STUDENT. Chapter 3 covered almost completely provides a companion to Chapter 1. Essential to the success of this chapter is the conviction in the student that there really is something to prove.

Other more topical and traditional beliefs might also be questioned. The contrast of mathematical proof with persuasion by experience may be discussed. Incidentally, Chapter 3 is probably the most difficult in the text. It also illustrates the notion of "algorithm".

EDUCATION STUDENT. Omit proofs, and concentrate on the concepts of common divisor, greatest common divisor, and least common multiple. The arithmetic in the unwinding of the Euclidean algorithm may be appropriate. The weighing lemma "explains" Chapter 1 and provides an algorithm for finding M and N.

MATH STUDENT. Point out the slight changes needed to prove the unique factorization of polynomials with real coefficients. Lemma 4 may also be proved by showing that the smallest positive number of the form MA+NB is (A,B). Contrast this existence proof with the constructive proof in the chapter. Exercises 55-63 would be appropriate.

Chapter 4: Rationals and Irrationals

An effective way to start the chapter is simply to draw the two large squares (I and II) that prove the Pythagorean Theorum, then allow two minutes or so of silence as students ponder the question, "What does this show about the areas of the three smaller squares in the figures?"

Another approach is to have each student cut eight (8) congruent right triangles out of paper, tagboard or wood and the three squares determined by the three sides. Ask them to assemble four of the triangles and the two smallest squares to form a square. Then ask them to use the remaining four triangles and the largest square to form another square. Comparison proves the Pythagorean Theorem. (Exercise 66 provides a more complicated puzzle.)

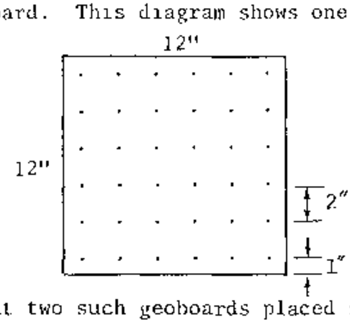
If there are students whose arithmetic with rational numbers is rusty, refer them to Appendix A and emphasize the high points, namely the addition and multiplication of rationals.

Discuss cube roots (E 26) as soon as possible. Otherwise, the student may simply memorize, "even exponents, rational, odd exponents, irrational."

Take time to outline the logical flow and unity of the first four chapters.

GENERAL STUDENT. Point out the gradual development of the number system. You might jump ahead and discuss algebraic numbers and complex numbers, which could be treated now.

EDUCATION STUDENT. The chapter can be approached with the aid of geoboards A geoboard consists of a square array of nails set in a square board. This diagram shows one convenient size·



3/8" plywood,
linoleum nails or
1" brads

Observe that two such geoboards placed side by side continue the 2" grid.

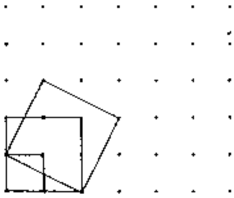
Rubber bands are placed around the nails to form triangles or polygons. The square formed by four "adjacent" nails is usually assigned area "1". Sketched below is a series of projects, which each student may do at his own geoboard.

1. Make a triangle on the geoboard. Find its area. (Do not use formula 1/2 · base · height)
2. What areas of triangles on the geoboard are possible?
3. Show that the area of triangle or geoboard is n/2 for some integer n.

   Solution: Surround the triangle by a rectangle and delete right triangles or rectangles.

   (This reduces the problem to the easier ones of finding the areas of rectangles or halves of rectangles.)
4. Make a quadrilateral on the geoboard. Find its area. (This can always be done by cutting it into triangles.)
5. What area squares can be made on the geoboard? Experiment. (After the Pythagorean Theorem, the answer can be obtained areas of the form $a^2 + b^2$ where a and b are non-negative integers. Thus 1, 2, 4, 5, 8, 9, 10, for instance, are possible.)
6. Make a (small) right triangle on the geoboard. Find the areas of the squares on its three sides. Tabulate the results for several triangles. They should pick a "corner" right triangle and draw the squares overlapping, as in this figure:

   Include the 3-4 triangle (yielding hypotenuse 5). Students have been told so often that "the 3 - 4 - 5 triangle suggests the Pythagorean Theorem", though it was probably never shown that the hypotenuse is in fact exactly 5. With the geoboard this can be done.

The above exercises 1 - 6 will prepare the student for the Pythagorean Theorem, familiarize him with the geoboard, and indicate a rich source of geometry for the elementary grades.

7 A student may lay out a "messy" triangle in the geoboard and use the Pythagorean Theorem to find the lengths of the three sides and thus its perimeter. (Hand-held calculator would be an appropriate aid here.)

8. Students can use the geoboard to develop the general formulas for area of rectangle, parallelogram, and triangle in terms of height and base.
9. Then with the aid of the Pythagorean Theorem, it can be shown that the area of an equilateral triangle is $s^2 \sqrt{3}/4$, where s is the length of the side. (This is the basis of Exercise 59: Since s^2 is an integer, if the triangle is on the geoboard, such a triangle would have rational area, making $\sqrt{3}$ rational, hence cannot be equilateral.)
10. Students can experiment with area of geoboard triangles that have no nail in the interior. (This area is always 1/2.)
11. Students can experiment with the relation between the area (A) of a polygon on the geoboard, the number of interior nails (I), and number of border nails (B). It turns out that

$$A = I + (1/2)\ B - 1.$$

This provides a good opportunity for studying patterns and also a neat shortcut for finding area. (See R 8 for further information on this result, known as Pick's theorem.)

It is important that education students master the arithmetic of rational numbers and decimals. Take time for Appendix A. You might go to Chapter 9 and 10 next.

Some students may object, "why do I have to know this stuff if I don't have to teach it. Who needs to know the distinction between rational and irrational?" After all, about a third of education majors tend to have a very fearful and antagonistic attitude toward mathematics. Indeed, they may go further, "With hand-held computers and the metric system, who needs fractions? Indeed, who needs arithmetic?"

There are a few possible responses; each instructor will certainly add his own.

1. This is not a methods course. Such a course should focus directly on the curriculum and be tied to extensive concurrent work with pupils in the classroom.
2. This course provides the resources from which you may draw so that when you teach you will be able to invent open-ended projects for your class. Otherwise your students will be restricted to doing long lists of routine computations, as you probably were.
3. The number line is important in grades K - 12. The placing of certain rationals and irrationals on it is important. It would be quite natural to ask where $\sqrt{2}$ is, and whether it is a fraction.
4. A teacher who knows only as much as his pupils will lack the confidence to encourage an open and questioning classroom. Such a teacher will focus on routine computation, the kind that "turned him off" when he was a child.
5. The distinctions between integers and non-integers and between rationals and irrationals are fundamental in understanding the number line.
6. Even with the metric system (voted down by Congress in 1974) and hand-held computers, fractions will still be needed. They provide the foundation for understanding the decimal

representation of numbers such as 0.307. (What does the "3" mean? The "0"? The "7"?) Why is (0.2) x (0.3) = 0.06? They are essential in even the most rudimentary study of probability or statistics. They are at the heart of elementary algebra. (If they say, "How many really need algebra," this excerpt from a study made by Lucy W. Sells, University of California at Berkeley, in 1973 may help suggest a reply:

HIGH SCHOOL MATHEMATICS AS THE
CRITICAL FILTER IN THE JOB MARKET

"We know that the job market is dismal for untrained people. We know that it is somewhat better for people with high school and college degrees. We know that the fields which are opening up in the next ten or fifteen years are based on mathematics training. We know that certain groups of students are less likely to take any more mathematics in high school than is required for admission to the university. These include girls and non-Asian minority students.

A study of admission applications of Berkeley Freshmen shows that while 57% of the boys had taken four years of high school math (first year algebra, geometry, second year algebra, trigonometry and solid geometry), only 8% of the girls had done so.

The four year mathematics sequence is required for admission to mathematics 1A at Berkeley, which in turn is required for majoring in every field at the University of California <u>except</u> the traditionally female, and hence lower paying, fields of humanities, social sciences, education and social welfare."

Of course, the need for the arithmetic of fractions may be with us longer than we think, even with the metric system and hand-held computers. After all, there have been slide rules since the 17th century and France has been on the metric system since the 18th century; but fractions are still an essential part of the curriculum there.

Incidentally, I omit almost all proofs for education students, preferring to focus on patterns, material aids, and trying to change their attitude.

MATH STUDENT. You might start off with the question: Are there three points on the plane, with integer coordinates, that determine an angle of 60°?" The answer, "no", depends on trigonometry and the irrationality of $\sqrt{3}$, as follows:

First of all, the area, A, of any triangle whose vertices have integer coordinates is rational.

Second, let θ be the alleged 60° angle, c the side opposite, and a, b the sides adjacent. Then

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

and

$$A = (1/2)\, ab \sin \theta.$$

Since A, a^2, b^2 are rational

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

is rational. Since $\tan 60° = \sqrt{3}/2$ is not rational, the answer is "no".

The last few exercises in the chapter may serve as topics for students to report on.

Chapter 5: Tiling

Theorems 1-4 can easily be developed by the "discovery" discussion method. Observe that implicit in them is the Greek notion of commensurability, which was the basis of their approach to rational numbers. This part of the chapter, together with Exercise 33, would constitute a sound reinforcement of the notion of rational versus irrational.

The remainder of the chapter is not referred to elsewhere in the text, even in the following chapter.

Note that the argument after Theorem 6 provides a nice application of algebra.

GENERAL STUDENT. If there is sufficient interest, cover everything. However, it may be better to save time here and devote it to the application of mathematics to electricity in the next chapter.

EDUCATION STUDENT. Exercises 1-7 reinforce earlier chapters and provide a resource of open-ended problems for the classroom. Exercise 26 also has immediate classroom appeal. Exercise 36 should also be called to his attention.

MATH STUDENT. A student could report on some of the later exercises, such as E 28, 29, 33, 34.

Chapter 6: Tiling and electricity

Later chapters do not depend on this one. There are at least three options.

1. Just show how linear equations are used in a "practical" problem. (Many students may not have seen an application of them.)

2. Cover the whole chapter, but minimize the tedious solving of the equations. Emphasize the underlying ideas: an axiomatic system and a model for the system.

3 The complete treatment.

GENERAL STUDENT Probably the second option is most appropriate.

EDUCATION STUDENT The secondary teacher would benefit. Perhaps one of them could bring in electrical circuits, voltmeters, resistances etc. and see how close experiment and theory are.

MATH STUDENT. This chapter is an introduction to the axiomatic method and algebraic topology. The voltages are a 0-cochain and the currents are a 1-cochain. The coboundary operator relates them. A student might try to prove Theorem 1 or find a suitable reference for it.

Chapter 7: Inspector and salesman

Chapters 7 and 8, together, provide a simple, interesting illustration of how the mathematics developed for one reason, perhaps "pure", may later be applied to a variety of practical problems.

Chapter 7 can be presented easily by the discovery-discussion approach, the notion of degree being introduced only when required

The lecture might open with Exercise 9 or Exercises 25 and 26. If Chapter 8 will be covered, then the salesman's problem should be discussed, as well as Theorem 6 and the highway system with loops (the one with 8 towns and 16 one-way sections).

If you wish to emphasize Chapter 8, avoid getting tangled up with exercises later than Exercise 11 in Chapter 7. Observe that some of the later exercises come in groups.

GENERAL STUDENT Do enough of Chapter 7 to set the stage for Chapter 8. Chapter 8 (and Chapter 6: electricity) both illustrate the interplay of pure and applied mathematics - indeed the impossibility of distinguishing between them in some cases.

EDUCATION STUDENT. This chapter provides a resource directly applicable to an elementary or secondary classroom.

MATH STUDENT The later exercises might serve as the basis of oral reports.

Chapter 8: Memory wheels

This chapter illustrates two main ideas.

1. the interplay between fields of mathematics;
2. the wide range of applications that one mathematical idea may have.

When discussing Chapter 10, Congruences, you may wish to show how algebra is used to produce memory wheels. The technique is described in the teacher's manual for that chapter. Note R 16.

GENERAL STUDENT. Avoid getting tangled with the messy details of producing memory wheels. Emphasize ideas such as those pointed out above. (Remember: mathematicians are now on the defensive.)

EDUCATION STUDENT. It can be omitted or touched lightly, especially if there is little interest.

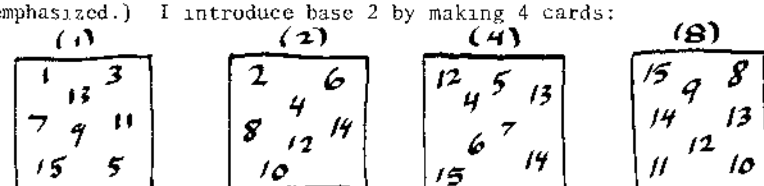
MATH STUDENT. They might report on some of the ways of producing memory wheels described in the later exercises.

Chapter 9: Representation

The decimal representation of the positive integers is used in the next chapter in developing divisibility tests. The present chapter can be covered without referring to bases other than ten The discussion of the metric system is independent of the rest of the chapter. It reinforces the idea of decimal notation.

GENERAL STUDENT. Since the various parts of the chapter are presented independently, there are many options. If the metric system is discussed, one student may report on the Congressional debates which led to the latest rejection of the metric system, May 7, 1974. Another student might bring in a meter stick, etc., and illustrate measurements in the metric system.

EDUCATION STUDENT. This is a key chapter and deserves a great deal of time. It surveys decimal arithmetic. (Other bases can be deemphasized.) I introduce base 2 by making 4 cards:



On the back of a card appears the number in parentheses. A student picks a number from 1 to 15 and points to the cards on which it appears. By adding the digits on the back, I can tell him what the number is.

Base 2 could be covered just enough so the student knows what it is and perhaps gets a fresh view of base ten.

The metric system should be covered in detail, with the aid of actual metric sticks, liters, and gram weights. Perhaps the students could buy or make some. Note E 55-72.

MATH STUDENT. The chapter can be covered quickly. Exercises 73-82 will be of special interest. Note that E 81 contains an unsolved problem.

Chapter 10: Congruence

There are several possible emphases.

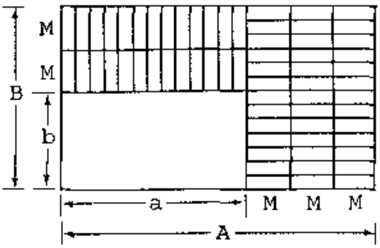
1. Emphasize the miniature arithmetics (mod M) to offer perspective on ours. Call attention to $\sqrt{-1}$, especially if you go on to cover complex numbers, discussed in Chapter 16. That treatment of the complex numbers, being very geometric, could be taken up at this point.

2. Emphasize the various divisibility tests, including casting out 9's, based on congruence. Of course, as Exercises 16 and 17 show, congruences can, in some cases, be by-passed.

3. Emphasize the number theory.

GENERAL STUDENT. Omit or downplay Theorems 6 and 7. Perhaps a spectacular exercise like E 33 might appeal, or the more philosophical E 70.

EDUCATION STUDENT. Obtain the divisibility tests, casting out 9's, and cover E 16, 17, 25-32, 35, 45-47. Use Cuisenaire rods to illustrate congruence. Can a train made of M-rods measure the difference between a train of length A and a train of length B? If so, $A \equiv B \pmod{M}$. The proof that congruences can be multiplied can be shown geometrically.



AB - ab is a sum of M's (that is, the difference of the two rectangles is tiled by 1 by M blocks.)

MATH STUDENT. Exercises 69 and 70 will be provocative. The answer to E 72 (a) for arbitrary squares is not known. The student may report on the following topic, which ties many ideas together.

In a mathematically oriented class it is possible to relate congruences, the fundamental theorem of arithmetic, polynomials, and memory wheels. This outlines application of polynomials with coefficients mod 2.

1. Introduce polynomials P in x, coefficients 0 or 1 mod 2. Show how to add, subtract, and multiply them.
2. Show how to carry out a division such as:

$$1+X^2+X^3 \overline{\big)\;1\;\;\;\;\;}\,,$$

which is

$$\begin{array}{r l}
 & 1+0X+X^2+X^3+X^4+0X^5+0X^6+X^7+0X^8+X^9+\cdots \\
1+0X+X^2+X^3 & \overline{\big)\,1+0X+0X^2+0X^3+\ldots} \\
 & \underline{1+0X+X^2+X^3} \\
 & \quad X^2+X^3 \\
 & \quad \underline{X^2+0X^3+X^4+X^5} \\
 & \qquad X^3+X^4+X^5 \\
 & \qquad \underline{X^3+0X^4+X^5+X^6} \\
 & \qquad\quad X^4+0X^5+X^6+\cdots \\
 & \qquad\quad \underline{X^4+0X^5+X^6+X^7} \\
 & \qquad\qquad\qquad X^7 \\
 & \qquad\qquad\qquad \underline{X^7+0X^8+X^9+X^{10}} \\
 & \qquad\qquad\qquad\qquad X^9+X^{10} \\
 & \qquad\qquad\qquad\qquad\qquad \cdots
\end{array}$$

The quotient is a "power series" whose coefficients repeat in blocks of seven:

$$\overbrace{1011100}\overbrace{1011100}\cdots$$

Insert an extra 0 with the two adjacent 0's and the memory word

1011100010

results.

3. When you carry out the division of P into 1 for a polynomial P, why must the coefficients, which are 0 or 1, eventually repeat in blocks? The length of the block is called "the period", denoted "q". The degree of P is denoted "n".

4. What would be meant by a "prime" polynomial? Note: $1+x^2$ $(=(1+x)\ (1+x))$ is not prime.

5. Why is there an infinitude of prime polynomials? (Proof as in Chapter 2.) Advanced algebra shows that there are prime polynomials of all degrees.

6. Why is factorization into prime polynomials unique? (Proof similar to that in Chapter 3, depending on a Euclidean algorithm.)

The following theorems can be proved.

<u>Theorem.</u> If P has degree n, then the period q of 1/P is at most 2^n-1. (Proof not hard.)

<u>Theorem.</u> If $q = 2^n-1$, then P is a prime polynomial.

<u>Theorem.</u> If P is a prime polynomial then q divides 2^n-1.

<u>Corollary.</u> If P is a prime polynomial and if 2^n-1 is a prime integer, then the period of 1/P is 2^n-1.

(Examples of such n are 3, 5, and 7.)

The following data may be checked by straightforward long division. In each case the polynomial is prime.

Polynomial P	degree n	period $q = 2^n - 1$
$x^3 + x + 1$	3	7
$x^4 + x + 1$	4	15
$x^5 + x^2 + 1$	5	31
$x^5 + x^3 + 1$	5	31
$x^6 + x + 1$	6	63

It is with such polynomials, but of much higher degree, that the radar cited in Chapter 8 operates. See S. Golomb, Shift register sequences, a book that discusses this topic fully.

Chapter 11: Strange algebras

Almost all of this chapter can be covered without congruences (Chapter 10). Observe that it does relate to the real world (design of experiments) though the emphasis is algebraic. The identities of commutivity and associativity are contrasted with other identities. Note the discussion of the distributive law, whose careless use is responsible for so many errors in introductory algebra.

Tables (also called "Latin squares") are used in Chapter 12.

GENERAL STUDENT. This chapter has the appeal of crossword puzzles, but call attention both to its pure and applied interest.

EDUCATION STUDENT. Emphasize the three basic identities of an arithmetic. (The chapter itself could serve as enrichment material.)

MATH STUDENT. The identity Xo (XoY) = YoX of Theorems 5 and 6 is not thoroughly understood. (Late in 1974 I was informed in a letter that a mathematician may have proved that there are such tables for all but a finite number of orders.) Call attention to E 48 - 50 on groups. A student could read up on groups and show why any group of prime order is commutative.

Chapter 12: Orthogonal tables

The chapter could be introduced several ways: either by the problem of the 36 officers, as the text starts, the design of experiments, magic squares, or round-robin tournaments, all mentioned later in the chapter. Incidentally, not all magic squares are obtainable from orthogonal tables, e.g. one that is six by six. A pupil could see how many "boxes" he or she could find, in a given table, no two on the same row or column or having the same color.

GENERAL STUDENT. Point out the interplay of algebra and combinatorics, also the unpredictable diverse applications of a single idea. Call attention to the unsolved problem of "partial" sampling systems.

EDUCATION STUDENT. Omit proofs. Note that the construction of partial sampling systems can be explored in elementary grades using letters, or colored pieces of paper. (A wide-open elementary problem for exploration.) Emphasize the relation to magic squares.

MATH STUDENT. Note the "simple" unsolved problem concerning partial sampling systems (technical name: partial transversals). It is easy to prove that there is one with at least n/2 elements.

PROOF. Proceeding box by box, build a partial transversal to which no boxes can be added. Say it has t boxes. Permuting rows and columns, and relabeling, we may assume that these boxes lie along the top part of the main diagonal and contain the entries 1, 2, 3, . . ., t. The bottom right square, n - t by n - t, contains entries only in the set {1, 2, . . ., t}. Since no row has a duplication, $n-t \leq t$; hence $t \geq n/2$. Note that this proof does not use the fact that a column has no duplications.

Chapter 13: Chance

This chapter does not depend on earlier chapters. It focuses on the basic addition and multiplication rules together with the notion of expectation. The emphasis can be as "applied" or as "pure" as the instructor wishes.

The chapter utilizes dice and roulette as an introduction. However, the instructor could introduce the concept of probability with such applied exercises as E 59-61 or such a pure exercise as E 8, 48, 49, or 50.

Observe that dependent events are not considered, nor is the binomial distribution, though E 13, 14, 16, and 17 came close and the instructor could easily extrapolate.

It is hoped that this introduction to probability will convince the student that probability is important in daily life and that he or she should consider taking at least a semester of statistics.

GENERAL STUDENT. Consider the interests of the student when assigning or suggesting later exercises. Note that they concern decision in the face of risk, Gallup polls, the Buffon needle problem, nuclear reactors, SST, baseball, astrologers, and psychics. The use of actual dice and roulette, etc. in the classroom might be effective.

EDUCATION STUDENT. Emphasize the importance of rational numbers in even the simplest calculation. Allow lots of class time for experiments, which can be used to illustrate data collection, bar graphs, per cents, decimals, etc.

MATH STUDENT. The hat check problem, E 46, and the Buffon needle, E 48, would appeal to the "pure". However, some of the applied exercises, as well as R 11, would show that there are serious practical problems left to be solved.

Incidentally, the limitation of expectation as a measure of utility is illustrated in M. H. De Groot, Optimal Statistical decisions, McGraw Hill, 1970, pp. 93-94.

Chapter 14: Fifteen puzzle

Students could bring Fifteen-puzzles to class. It is clumsy to write down all the steps in a solution, though it is easy enough, using chalk and eraser, to show each move at the board. Exercise 38 presents an algorithm for solving the solvable puzzles. Start the chapter with the Fifteen-puzzle, then do the Weaver's problem. See E 46 for a very short proof of the key theorem. Exercise 35 might be covered in a lecture right after the case of the two-thread belt.

GENERAL STUDENT. The chapter concludes by indicating some of the deeper applications of the central idea. Such a student should not think of mathematicians as just playing games. Rather, a mathematician derives his inspiration from problems, whether they are recreational, applied, or pure. In the case of orthogonal tables (Euler) and the highway inspecter (Euler) the origin happened to be recreational, but, as the pertinent chapters show, the consequences are significant in diverse areas.

EDUCATION STUDENT. He should at least develop skill in working the Fifteen-puzzle, and deciding whether it can be solved. The proof could be omitted.

MATH STUDENT. He might show how determinants of orders 2, 3, and 4 are defined and used to solve equations.

Chapter 15: Map Coloring

If time is a constraint, treat the 2- color and 3-color theorems superficially, even omitting them, and focus on the five-color theorem.

GENERAL STUDENT. While map coloring is generally of interest, it can be overdone. Allow the class to form its own conjectures about the number of colors required, before giving any answers or proofs.

EDUCATION STUDENT. Omit proofs. Many topics in the chapter provide enrichment material at elementary or secondary levels. For instance, the two-color problem was presented in the "experiment and discovery" style at 3rd and 5th grades. V - E + F on an island would make a nice lecture (no proof). Exercises 27-30, 41, 41, 46, provide classroom resources.

MATH STUDENT. Warn him that the 4- color problem has led to at least one suicide. He may read or report on some of the topics presented in the later exercises.

Chapter 16: Types of numbers

Don't be put off by the length or relative "messiness" of this chapter. It consists of several self-contained parts:

(1) one part develops the complex numbers geometrically. It is easy. Point out the discussion of Steinmetz. Note the similarity to some modular fields, e.g. (mod 5), where -1 has a square root.

(2) the part before the complex numbers proves that a polynomial has no more real roots than its degree. This fact is used in the next chapter when showing that there are transcendental numbers. The proof here could be omitted.

(3) the argument after the complex numbers shows that real polynomials of degree 3 can be factored over the reals. It uses the factor theorem proved earlier in the chapter.

In a class where there is lots of time and interest, this chapter, if covered completely, could review (or introduce) a healthy amount of algebra.

The complex numbers are used in Chapter 18 to study constructions by straight edge and compass.

GENERAL STUDENT. The part on complex numbers has general appeal. The applications of complex numbers can be discussed without going into full detail.

EDUCATION STUDENT. The prospective secondary teacher could benefit from this chapter, especially the intuitive development of the complex numbers. (That approach could be used even in junior high school).

MATH STUDENT. There is a variety of topics presented in the later exercises. Exercise 91, for example concerns a problem that is still not solved. In 1974 R. R. Hall, T. H. Jackson, A. Sudbury, and K. Wild proved that for any $\varepsilon > o$ and sufficiently large n there is in an n by n array a subset of at least (3/2 -c)n points of which no three are in a line. This result was submitted for publication under the title, Some advances in the no three in a line problem. (No example is known for which 2n has been shown to be inaccessible.)

Chapter 17: Construction

This chapter is a reinforcement of the complex numbers and high school geometry. The earliest natural stopping place is at Theorem 7, which uses the complex numbers to construct the 5- gon. (An alternative approach, which does not use the complex numbers, is described in Exercise 17.)

Exercises 27-51, independent of the chapter, constitute a substantial self-contained course in trigonometry. It uses the distributive law for complex numbers to obtain the fundamental identities.

The approach in E 10 bypasses geometric series.

GENERAL STUDENT. Bring out the positive aspect of "negative" impossibility theorems. Perhaps the notion of algorithmic unsolvability (in Chapter 19) and impossibility of counting the real numbers (in Chapter 18) might be mentioned. See also E 24-26.

EDUCATION STUDENT. All the chapter could be of value to a high school teacher. An elementary teacher could utilize the constructions, including that of the 5-gon.

MATH STUDENT. A student might demonstrate E 26 in class, flashlight and all, and discuss its implications.

Chapter 18: Infinite sets

Note that the empty set is not included as a set. That is, a "set" has at least one element. Also, in this chapter 0 is not included in the set N, which throughout the chapter denotes the set of positive integers.

This mind-boggling chapter is much easier to cover in the informal style of a classroom than in the necessarily more precise style of a book. This is what I do:

1. While urging the student to read Galileo's dialog, I go directly to the idea of conumerous sets.
2. I list all sorts of infinite sets, and ask for further examples. Eventually, they tend to break into two types, those conumerous with the reals, and those that are denumerable.
3. I allow ample time for discussion of the next step: presumably to show that all the examples are conumerous. But we fail.
4. Then I prove Cantor's theorem (Theorem 4), using no letters (such as n or d_{n1}, d_{n2}, . . .) but specific, randomly chosen digits. I put boxes around the diagonal entries and indicate the rule "choose a digit different from the digit in the box".

Usually, due to time limitations, I don't get to the detailed proof of the existence of transcendentals. A quick sketch of the argument is convincing.

GENERAL STUDENT. The student is astounded by the existence of a scale of infinities. Thus Cantor's theorem is the key to the chapter.

EDUCATION STUDENT. The notion of 1 - 1 correspondence is used in the early grades. Many of the exercises and ideas would be easy to do at the secondary level, since little arithmetic, algebra, or geometry is needed.

MATH STUDENT. A report on Theorems 7 and 8 and some later exercises could be presented to the class by a student.

Chapter 19: General View

The various parts of this chapter are independent. The treatment of geometry for example, with the related Exercises 1-9 and 22, can be extracted.

This chapter could be assigned as reading after a few chapters are covered.

GENERAL STUDENT. Call the chapter to his attention.

EDUCATION STUDENT. Good background, especially for secondary level.

MATH STUDENT. A critical chapter for developing perspective. Many of the references can be read in their entirety.

ANSWERS AND COMMENTS FOR SELECTED EXERCISES

Chapter 1: Questions on Weighing

1-19. Since 4 sevens can be replaced by 7 fours, any amount can be weighed. (Repeatedly replace 4 sevens in a weighing.)

1-20 (a) Add a "large multiple of 3" sevens to the pan with the potato, and the same weight in the 3's to the other side. Then remove the 3's that are with the potato and the same number of 3's from the other pan.

(b) Yes, same idea as in (a).

1-23.
1-24.
1-25. Students will guess that if (A,B) = 1, then A¢ and B¢ stamps can make any postage greater than (AB-A-B)¢ but not (AB-A-B)¢. (A proof can be based on the results in Chapter 3, but is rather technical.)

1-27. All. This is a consequence of the base 3 notation, Chapter 9.

1-28. An interpretation of base ten.

1-29. An interpretation of base 5.

1-30. (Note E 20).

1-32. Only if A and B have no common divisor larger than 1.

1-33. This anticipates Chapter 5. Say there is an A-jug and B-jug, with greatest common divisor 1. By Chapter 3 there are positive integers M and N, MA - NB = 1 (and also m and n such that mB - nA = 1.) Pour the A-jug N times into the B-jug, emptying the latter N times. This shows that 1 quart can be obtained.

Chapter 2: The Primes

2-3. (b) No one knows.

2-4. (a) No.
(b) No more.
(c) $N^2 - 1 = (N - 1)(N + 1)$. If algebra of students is weak, demonstrate this with dots in a square array N by N, rearranged by moving one row to an N - 1 by N + 1 rectangle.

2-5. (a) $5 = 1 + 2^2$, $257 = 1 + 2^8$.
(b) It is not known whether these "Fermat primes" are infinite.

2-6. (b) The "Mersenne primes". It is not known whether they are infinite. Incidentally, $2^{11213} - 1$ is prime.

2-7. (b) This is Bertrand's theorem, proved in the 19th century. It is also, for large N, a consequence of the Prime number theorem, in the form $P_n \sim N\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{N}\right)$, for this implies that the ratio between consecutive primes approaches 1 as N gets large.

2-8. (b) No one knows.

2-13. (c) Same argument as text.

2-15. (a) No
(b) No more
(c) $N^2 - 4 = (N + 2)(N - 2)$.

2-16. (c) $721 = 7 \cdot 103$, No.

2-19. (a) $2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 5$

2-21. $3 \cdot 19$ and $3 \cdot 3 \cdot 13$

2-22. Take a prime P that is of the form 2^N-1. Then $P \cdot 2^{N-1}$ will be such a number. For instance, $7 = 2^3 - 1$ and $28 = 7 \cdot 2^2$ is such a number, as is $6 = (2^2 - 1)(2^1)$. 496 is next.

2-27. (b) The product of numbers ending in 1, 3, 5, 7, 9 does not end in 0, 2, 4, 6, 8.

2-28. (a) Note that as you keep adding 5, the last digit of your sum keeps switching from 0 to 5 to 0 and so on.
(b) Show that the product of two numbers, each of which ends in 1, 2, 3, 4, 6, 7, 8, or 9, also ends in one of those digits.

2-29. Yes. Reserve proof till Chapter 3.

2-30. (a) $220 = 10 \cdot 22 = 4 \cdot 55$
(b) $484 = 4 \cdot 121 = 22 \cdot 22$

2-32. (a) $N - 3$ = sum of 2 primes, so $N = 3$ + sum of 2 primes.
(b) 11

2-35. No more.

2-36. Since there are three odd and three even numbers among the six, one at least of the even numbers would have to be prime, but 2 is the only even prime.

2-37. No. Such an interval would contain at least 6 primes. Since 2 is not in the interval, the 6 odd numbers in the interval would be prime; hence, would contain the unique prime triplet (3, 5, 7). (See E 35.)

2-38. (a) 4
(b) 8

2-39. (a) $2m \cdot 2m = 4m^2$
(b) $(2m + 1)^2 - 1 = 4m(m + 1)$ and $m(m + 1)$ is even, since m and m + 1 are adjacent (one is odd, the other even)

2-40. (a) Yes $23 = 5^2 - 2$, $47 = 7^2 - 2$, for instance.
(b) There is strong evidence that there is no end to such primes. (Note: $11^2 - 2 = 119 = 7 \cdot 17$ is not prime.)

2-41. If 197 were not prime it would have a divisor (hence a prime divisor) less than its square root, which is about 14.

2-44. (b) Pair each D with N/D (which is different from D).
(c) Like (b) but one number is paired with itself.

2-47. (a) 5, 17, 29, 41, 53

2-48. (a) First case: N = s x s. Then s and 2s both appear as factors in (N - 1)!. Thus, s x 2s divides (N - 1)!; so must s x s, which is N.
Second case: N = a x b, a not b, neither a nor b = 1. Then a and b both appear as terms in (N - 1) (N - 2) ... (2) (1).
(b) Consequence of (a) and the special case N = 4.

2-49. If $A^2 + B^2 = 4N + 3$, A is odd and B is even, say. But then, by algebra or a diagram of dots, $A^2 + B^2$ is of form 4M + 1.

2-50. $10^3 + 7^3 = 12^3 + 1^3$

2-56. Put 2, 3, ..., N into the Prime-manufacturing machine, deleting "4", say, to keep product less than N! - 1.

2-57. Another problem on triangular numbers is suggested by the game of pool. The 15 balls and cue ball are stored in a 4 by 4 square. When the game starts, the 15 balls form a triangle. (There are an infinite number of triangular numbers that are 1 less than a square.)

Chapter 3: The Fundamental Theorem of Arithmetic

3-14. (a) $2^a 5^b$, $0 \leq a \leq 3$, $0 \leq b \leq 4$.
(b) If $DQ = 2^3 \cdot 5^4$, then D must be of form $2^a 5^b$.

3-21. (Important in adding fractions.)

3-22. The product is AB. To prove this, count primes.

3-23. (b) Factorization into primes is unique, except for the number of 1's.

3-24. (a) Sum of 2's; sum of two equal integers.
(b) Powers of 2; squares. [Depends on answer to (a).]

3-35. (b) Any common divisor of A and B would divide MA + NB, which is 1.

3-36. (a) 1, 2, or 4.
(b) If D divides A and B, D divides 4.

3-40. Use Fundamental theorem of arithmetic.

3-42. Use Fundamental theorem of arithmetic.

3-44. (a) Numbers of form 2 · odd (double an odd).
(b) 180 = 2 · 90 = 10 · 18 = 6 · 30.
(c) Each p_i or q_j contributes one "2" to the factorization of the number in question.

3-47. 2[=(14,32)]. Indeed (A,B) can be defined as the smallest positive number of the form MA + NB.

3-48. Multiples of 3.

3-49. First proof: (A,B) = 1, hence $(A^2,B^2) = 1$; then use Lemma 4. Second proof: (More direct, but sly)· Cube MA + NB, obtaining $(M^3A + 3M^2NB) A^2 + (3MN^2A + N^3B)B^2 = 1^3 = 1$.

3-50. Counter example is 6 and 8.

3-55. (b) There are several theorems on this.
1. Any postage larger than AB - A - B can be made.
2. AB - A - B itself cannot.
3. If a postage less than AB - A - B can be made, it can be done only one way.
4. Exactly half the postages from 1 through AB - A - B - 1 can be made. [More specifically, for each p in that range, exactly one of the two postages p and (AB - A - B - 1) - p can be made.]

Proof of 1: Let p be a postage larger than AB - A - B. There are integers M and N such that p = MA + NB, but alas, one of them may be negative. In any case, by E 53 we can insist that M is non-negative and no larger than B - 1. For such a B we show that N cannot be negative. For, were N negative, MA + NB could be no larger than (B - 1) A - B = AB - A - B. This violates the assumption that p is larger than AB - A - B.

Proof of 2: Say AB - A - B = MA + NB, where neither M nor N is negative. Then AB =

(M + 1)A + (N + 1)B. Clearly, A divides N + 1 (and B divides M + 1). Thus, N + 1 is not less than A (nor is M + 1 less than B). Thus (M + 1)A + (N + 1)B is at least as large as BA + AB, which is larger than AB (which it was supposed to equal).

3-56. Observe that an integer is a Lagado prime if it is of one of these two forms:

1. An ordinary prime of the form 3x + 1.
2. The product of two ordinary primes of the form 3x - 1 [i.e., (3x - 1) (3y - 1)]. In any Lagado factorization, therefore, the number of each type is fixed (using the fundamental theorem of arithmetic).

3-57. (b) It is 1 or 2.

(c) Any common divisor of A + B and A - B divides their sum, 2A and their difference, 2B; hence, M(2A) + N(2B) = 2(MA + NB), for any M and N. Choose M and N such that MA + NB = 1.

3-60. (a) 1 = MA + NB; hence, 1/AB = M/B + N/A.

3-64. (c) If p is a prime, p^3 has four divisors: 1, p, p^2, p^3. If p and q are distinct primes, pq has four divisors: 1, p, q, pq.

3-65. It is that positive common divisor of A and B that is divisible by all the other common divisors of A and B.

Chapter 4: Rationals and Irrationals

4-10. (c), for from it the other two follow.

4-18. It is a nonrepeating decimal.

4-20. If R + $\sqrt{2}$ = A/B, solve for $\sqrt{2}$, showing it is rational.

4-22. If R $\neq$ 0 and R $\sqrt{2}$ = A/B, solve for $\sqrt{2}$.

4-23. Its square, 2 + 2 $\sqrt{6}$ + 3, is irrational, so it is.

4-39. (a) Yes, e.g., $\sqrt{1/3} \cdot \sqrt{1/3} = 1/3$.

(b) No, the product of rationals is never irrational.

4-41. Between any two numbers is an infinity of rationals and an infinity of irrationals. (See E 42.)

4-42. (a) $0.71\overline{201}$

(b) $0.712\overline{01}0010001\ldots$

4-43. (a) Yes. The sum of two rationals is rational.

(b) No. $\sqrt{2} + (5/3 - \sqrt{2}) = 5/3$.

4-44. Cancel from A and B (A,B).

4-46. (See E 42.) For instance, to produce a rational between 0.733... and 0.735..., consider 0.734$\underline{532}$ (or use any other block).

4-47. A diagram shows that $(\text{diagonal})^2 = 1^2 + 2^2 + 2^2 = 9$; so diagonal is 3.

4-48. 7

4-59. (a) Draw the smallest rectangle containing the triangle with sides parallel to the lines of given dots. The triangle is obtained from the rectangle by deleting three triangles. Since the areas of the rectangles and these deleted triangles are integers or halves of integers, the original triangle has a rational area.

(b) No. The area of an equilateral triangle of side s is $(\sqrt{3}/4)s^2$. But s^2 is an integer (by Pythagorean theorem); hence, area is irrational.

4-61. This is discussed in Chapter 17.

4-65. (a) Rounded off, 0.61. (If students are dismayed that π appears in number theory, remind them that the number 2 appears in many places, too.)

(c) $6/\pi^2 = 0.61$ (approximately) so $\pi^2 = 6/(0.61)$, and π is about 3.14.

4-71. (c) Consider the prime factorizations of a^2, $c - b$, $c + b$.

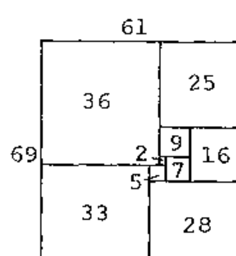
Chapter 5: Tiling

5-2. (b) Since dimensions are $18 \cdot \frac{13}{28}$ by $11 \cdot \frac{13}{28}$ a 13/28 by 13/28 square can be used.

5-3. Use 17 by 17 squares.

5-5. Use 1 by 1 squares.

5-11. Pick two adjacent squares as unknowns. Any answer will be proportional to this:

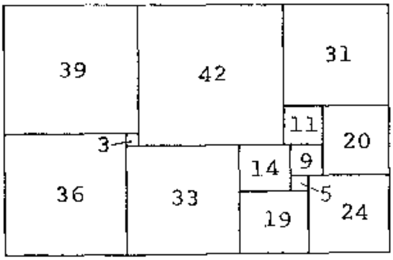


5-19. 5-20. Schematic diagrams, rather than precise procedures, might suggest solutions. A "rigorous" proof might be tedious.

5-21. For example, to show that it can be done for n = 10, simply place a square of the appropriate size alongside a tiling of a rectangle by 9 squares.

5-26. It should be pointed out that "plane" means the endless plane of geometry, that there is no border to worry about. There is only one solution. (Never flip the tile over.)

5-30.



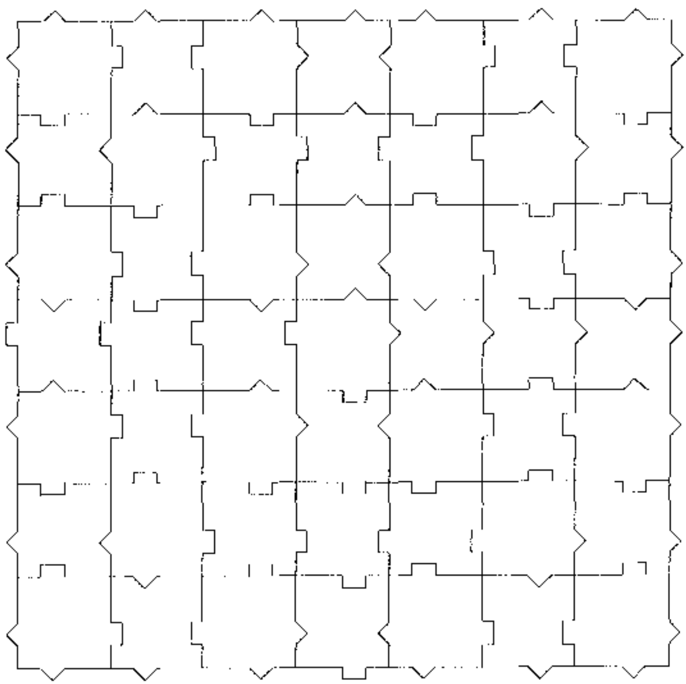
5-31. (c) The known proof uses the 2-adic number field.

5-32. Incidentally, a and b are commensurable if there are positive integers m and n such that ma = nb.

5-36. No. (See hint). The two remaining squares are of the same color; thus 31 dominos are to cover 30 red squares (say) and 32 white squares.

5-37. The accompanying diagram shows a tiling of a 7 by 7 square. Note the similarity of the 3 by 3 subsquares at each corner. It is possible to tile a 15 by 15 square in an analogous fashion, using

7 by 7 patterns. Continuing in this fashion, Robinson tiles the entire plane.

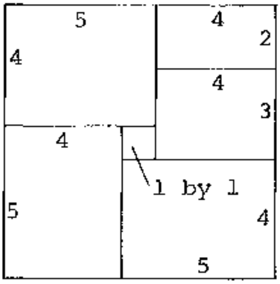


R1 has a fuller description.

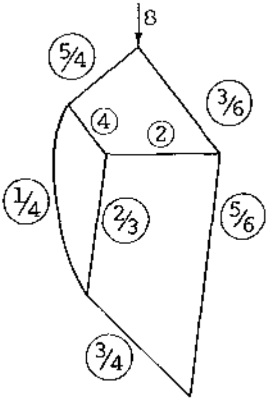
Chapter 6: Tiling and Electricity

6-25. This is a long problem since it has 4 unknown voltages. It could take the student a couple of hours to solve it. The point is that once again rules I and II suffice and that the answers are rational. From top to bottom the voltages are 1350/234, 990/234, 840/234, 750/234. From this data the currents can be calculated.

6-27.



6-28.



6-31. Let t = number of miles in town, r = number of rural miles. Then,

$$t + r = 220 \text{ and } \frac{t}{10} + \frac{r}{15} = 18.$$

Solution: t = 100, r = 120.

6-34. Otherwise it could be tiled with squares (or replace each rational tile by a tiling by congruent squares).

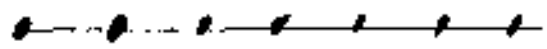
6-35. Let the rectangle have dimensions a by b and the square have side 1. If the tiling by the rectangles involves only rectangles in one direction (not rotated) then ma = 1 and nb = 1 for same integers m and n. If rectangles of both orientations are present, the solution is messier. Starting in a corner tile, consider a largest rectangular area formed by the tiles in that orientation. By comparing total lengths of edges along a border obtain a relation of the type ma = na + pb, where m, n, p are integers.

NOTE: By considering total area, one obtains immediately n · ab = 1 for same integer n.

NOTE: I have not explored the higher dimensional analogs.

Chapter 7: The Highway Inspector and the Salesman

7-12. The number of towns of odd degree is even (in finite systems).

7-14.  ... and so on.

7-16. Let one inspector start at an odd town and travel until stuck, necessarily at another odd town. Delete the edges he inspects. Use Theorem 5 on the remaining system.

7-20. The number of steps is half the number of towns of odd degree.

7-21. $t - s = 1$

7-22. Start with a single town and watch what happens to $t - s$ at each step as towns and edges are added. Or, if you prefer, begin with any tree and delete one edge at a time.

7-25. Let the towns be 0, 1, 2, 3, 4, 5, 6. Join town i to town j by a highway labelled (i,j) (corresponding to domino (i,j)). Thus at 3, for instance, is a loop, corresponding to the domino (3,3). Laying out the dominoes as in (a), but without (0,0), (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), corresponds to a highway inspector route. Since each town has even degree, there is such a route. Then put in the omitted dominoes (0,0), ..., (6,6) easily. The route must form a circle (the inspector's path must end where it starts).

7-27. The number of roadblocks depends only on the system, not on the police.

7-28. $t - s = 1 - b$

7-29 Let the system have t towns and s edges. Putting in b roadblocks (snipping with scissors) creates a system of $t + 2b$ towns and $s + b$ edges, which is a tree. Thus, $(t + 2b) - (s + b) = 1$.

Chapter 8: Memory Wheels

8-6. In the list of quintuplets there are as many 0's as 1's. If you multiply the number of 0's on the memory wheel by 5, you obtain the number of 0's in the list of all quintuplets. Similarly for the number of 1's.

8-11. The first quintuplet of the memory word is the last quintuplet of two sextuplets. One of these two sextuplets must be the final sextuplet in the memory word. Otherwise the quintuplet in question appears 3 times in the memory word, which is impossible

8-12 The mod 2 arithmetic, discussed in Chapter 10, would be useful here. The digit added is simply the sum (mod 2) of the first two digits. You might want to save this exercise to be done during the study of Chapter 10.

Chapter 9: The Representation of Numbers

9-54. Putting a 0 to the right of the decimal representation of a natural number magnifies it by a factor of ten.

9-78. Write the first factor in base two and show that the products kept correspond to the products of the power of 2 appearing in the first factor with the second factor. (e.g., $35 \times 56 = (2^5 + 2 + 1)(56)$. The 1×56 appears because 35 is odd, etc)

9-80. Write B as QA + R, $0 \leq R < A$. The unit fraction to be subtracted is $1/(Q + 1)$, and

$$\frac{A}{B} - \frac{1}{Q + 1} = \frac{A - R}{B(Q + 1)}.$$

The "new" numerator, A - R, is less than the "old" one, A.

Chapter 10. Congruence

10-10. $5^{1001} \equiv (-1)^{1001} = -1 \equiv 5 \pmod 6$.
Answer: 5.

10-11. $3^{100} \equiv 9^{50} \equiv 1^{50} = 1 \pmod 8$.
Answer: 1.

10-22. 6X - 8 is not divisible by 3; hence, not divisible by 15.

10-24. (a) The fact that $X^2 \equiv 1 \pmod 8$ has more than 2 solutions may surprise some students.

10-41. The sums of the digits of N_1 and of N_2 are the same

10-52. (c) For prime modulus, since a prime is special.

10-53. (c) You might want to state the quadratic reciprocity theorem.
(d) This prepares the student for complex numbers and

similar equations there. (See the discussion of complex numbers in Chapter 16, which part could be done next.)

10-66. Yes. Write $A = B + QM$ and note that any common divisor of A and M is a common divisor of B and M, and conversely.

10-71. (a) At least two of the 72 numbers $3^0, 3^1, \ldots, 3^{71}$ have the same remainder when divided by 71. So there are a and b > a, $3^b \equiv 3^a \pmod{71}$; hence, $3^{b-a} \equiv 1 \pmod{71}$.
(b) If $(a,m) = 1$, then some positive power of a has remainder 1 when divided by m.

10-72. The maximum number is not known in general. Clearly in an N by N array no more than 2N can be chosen. For N = 4, 8 indeed can be found, e.g.,

```
. x x .
x . . x
x . . x
. x x .
```

and for N = 3, 6 can be found, e.g.,

```
x x .
x . x
. x x
```

Similarly, for N = 5 and 6, 2N can be found. If N is prime and we choose the dots (x,y) where $x = 1, 2, \ldots, N$ and $y \equiv x^2 \pmod{N}$ and $1 \leq y \leq N$, we obtain a set of N dots, no three in a line. (The proof reduces to the fact that a polynomial of degree 2 whose coefficients lie in a field has at most two roots. This theorem, for the field of real numbers, is found in Chapter 16.)

It has been proved that when N is large at least 3N/2 points can be found. The question remains open.

10-73. For odd A, $A^2 \equiv 1 \pmod 8$; hence, $A^3 \equiv A \pmod 8$. For any A, $A^3 \equiv A \pmod 3$ (for $0^3 \equiv 0$, $1^3 \equiv 1$, $2^3 \equiv 2 \pmod 3$). Thus $A^3 \equiv A \pmod{24}$ for all odd A.

10-77. Take congruence mod 9.

Chapter 11: Strange Algebras

11-19. (b) Associativity.
(c) Associativity.

11-22. Note similarity to proving trigonometric identities.

11-23.
11-24. See remark on 11-22.

11-31. Since each letter appears an even number of times off the diagonal, each letter appears an odd number of times on the diagonal; hence, each letter appears on the diagonal. Since the number of boxes in the diagonal is equal to the number of letters, each letter appears only once. (Warn the students to think about this problem. They feel euphoric when they solve it.)

11-36. If X - Y were equal to an expression involving only the operation +, then when you replace both X and Y by 1, the left side would assume the value 0, while the right side would be greater than 0

11-39. Similar to E 36. This time replace X and Y by 2.

11-40. This is another hard one. You might want to save it until the students are in the midst of another chapter. One solution is. from the first rule we obtain $(X\circ(X\circ Y))\circ(X\circ Y) = Y\circ(X\circ Y)$. Applying the second rule to the left side of this equation, we obtain $X = Y\circ(X\circ Y)$. "Multiplying this equation on the left by Y," we obtain $Y\circ X = Y\circ(Y\circ(X\circ Y))$. Applying the first rule to the right side of this equation, we obtain $Y\circ X = X\circ Y$.

11-41. Assume that row X has a duplication. Then $X\circ Y = X\circ Z$. Thus, $X\circ(X\circ Y) = X\circ(X\circ Z)$. By the rule, $Y\circ X = Z\circ X$. Since there is no duplication in column X, we have a contradiction.

11-42. There are many solutions. For instance Let U and V be letters. Then UU = U, hence $UU\circ V = UV$. By associativity, $U\circ UV = UV$. Since there are no duplications in row U, UV = V. But VV = V. Since there are no duplications in column V, U = V.

11-43. If U and V are letters and $U\circ V = V\circ U$, then $(U\circ V)\circ(V\circ U) = (V\circ U)\circ(U\circ V)$. The rule then tells us that U = V. Students are exasperated when they see how easy the solution is. You might assign the somewhat harder E 44 the next day. If they get gloomy, remind them of the purpose of this chapter: appreciation of ordinary algebra.

11-48. (a) Write X in the form $A\circ Z$. Then $T\circ X = T\circ(A\circ Z)$ $= (T\circ A)\circ Z = A\circ Z = X$.
(b) The same.

11-49. Like E 48.

11-54. (Note that both are the cyclic group of order 4.) In the second table, say, interchange last two columns (and corresponding guide letters). Then reletter 1, 2, 3, 4, as follows, in order, 2, 4, 3, 1. (Motivation: 1 generates first table, while 2 generates second.)

11-55. Each symbol that appears in the diagonal appears at least twice.

11-57. Just evaluate $A\circ(B\circ C)$ and $(A\circ B)\circ C$. However, the construction is based on the following idea. Let $f(x) = x + 1$. Then

$$A\circ B = (A + 1)\ (B + 1) - 1 = f(A)\ f(B) - 1,$$

hence

$$f(A\circ B) = f(A)\ f(B),$$

or

$$A\circ B = f^{-1}\ (f(A)\ f(B)).$$

This exhibits the isomorphism between $\circ$ and ordinary multiplication. Other examples can be obtained by using different one-to-one correspondences f.

Chapter 12. Orthogonal Tables

12-36.

11	3	1	9
1	9	11	3
9	1	3	11
3	11	9	1

5	0	1	4
4	1	0	5
0	5	4	1
1	4	5	0

Not all magic squares are obtainable this way. For instance, there are magic squares of order 6, yet no orthogonal tables of order 6.

12-39.

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	1	2	3	1	2	3	0	2	3	0	1	3	0	1	2

12-40. Two columns (or 4 entries each) do not coincide in more than one place. (See R 41.)

12-41. There do not exist orthogonal squares of order 6.

12-42. (b) Because they are orthogonal.

12-58. In the order 3 case, note first that the main diagonal has all symbols, which can be taken in order, A, B, C. Hence the table is commutative.

12-59. Take a partial sampling system which is not a proper subset of another. It is no loss of generality to assume that it consists of the numbers 1, 2, 3, ..., t and the boxes go down from the top left corner on the main diagonal. Consider the N - t by N - t square in the bottom right. Only the symbols 1, 2, ..., t appear there. Thus, since there are no duplications in a row, $N - t \leq t$. This shows that $t \geq N/2$. Note that more is proved: If there at least N rows or columns without duplication then $t \geq N/2$.

Chapter 13: Chance

13-8. Note that it is a 999 to 1 chance, as odds, or 1 in 1000 chance.

13-12. Odds "for" and "against" may require discussion.

13-14. Note that (c) and (d) require counting three cases, as in the binomial distribution.

13-15. This is a nice occasion to examine the behavior of $(1 - 1/n)^n$. A calculator in the classroom might help.

13-16. (a) If this is the first time the student has been exposed to this problem, probably "yes".
(c) 1/4, 1/2, 1/4 respectively. (The middle one can be obtained by subtracting the outer ones from 1.)

13-20. (a) This may open a discussion
(b) 1/1024.

13-31. (a) Yes
(b) Yes
(c) No.

13-34. This can be developed as a project.

13-36. Point out the relation between (f) and (g). The tickets cost \$500,000. The prize money is \$198,455. This 40%-return is typical of a lottery.

13-37. There are many simplifications here, yet the result is, as they say, "in the ball park", 2.35 as compared to an observed 4.

13-38. Worthy of class discussion.

13-39. Class discussion for (b).

13-41. Independence of the separate years is assumed, etc.

13-44. While Policy 2 costs $20 more than Policy 1 it has only $17.50 more expected "pay off".

13-46. With n letters the probability is $1 - \frac{1}{2!} + \frac{1}{3!} - \dots \pm \frac{1}{n!}$. As n gets larger, this approaches a limit ($1 - 1/e \doteq 0.63$) in an oscillating manner.

13-48. Nice in-class experiment. To show that I is proportional to L, think of a wire as composed of two pieces of equal length. When the whole wire is thrown "randomly" each part is also. So if one wire is twice as long as another, its "I" should be twice as large.

Incidentally, the formula $I/L = 2/\pi$ (which is obvious in the case of the circular wire of diameter 1 inch) has been applied to measure the length of a tangled bacillus on a ruled slide.

It is easy to deduce from the formula $I/L = 2/\pi$ that the circumference of a convex curve is equal to its "average width" divided by 2π. (The width in a given direction is the distance between the parallel tangents perpendicular to that direction.)

13-49. $2/\pi$

13-50. (a) No
(b) Yes
Incidentally, the house usually has a limit on the size of a wager.

13-53. It may be amusing that comparison of the two approaches enables one to "sum" a geometric series.

13-55 Worthy of discussion, if class is interested. In (c) one might mention that

$$\left(1 - \frac{1}{1{,}000{,}000}\right)^{10{,}000} \doteq 1 - \frac{10{,}000}{1{,}000{,}000}$$

13-60. (a) No. No indication of time covered
(b) A topic for debate.

13-61. This raises the general question of quantifying utility, an active field of applied research.

13-62. For independent reports or talks.

13-63.
13-64. In view of the increasing interest in prophets, these exercises may appeal to the sceptic and the believer. It is not easy to get the old predictions of psychics, especially the wrong predictions.

Chapter 14: The Fifteen Puzzle

14-1. Trivial, but needing emphasis.

14-15.

8	7	4	2
3	6	5	10
1	9	///	15
12	13	14	11

14-16. Argument: Color of blank stays same, but B goes from even (0) to odd (15).

14-17. No. May argue as in 1-16 or more briefly, as follows: Color of blank changes, but it is possible to obtain the final from the initial position by 8 (an even number) interchanges of squares (e.g. 9 with 12, etc.)

14-20. See Exercise 38.

14-21. It is even.

14-23. Color stays <u>same</u>, but second differs from first by 3 (an odd number) interchanges. Hence, it is not possible.

14-24. <u>19</u>. With one switch we can put 1 in place. With a second switch, 2 in place...etc. With 19 switches 1,2,..., 19 in place, thus 20 is too.

14-25. (a) 1
(b) 3,5
(c) 6,5,3,1
(Note that the pattern is reversible.)

14-26. (b) 12
(c) 12

14-27. (b) 3
(c) 3

14-30. Clearly, it can be done in 4 switches. Since all 8 numbers must be moved, and a switch moves only 2 numbers, it cannot be done in less than 4 switches.

14-31. (See 30.)

14-32. (a) A sum of 2's (perhaps only one or no summand)

(b) A product of 2 and a natural number
(c) Unit digit is 0,2,4,6,8.

14-37. Seven numbers must move left, so it will take at least 7 switches. But B changes by an even number, so it will take at least 8 switches. It can be done in 8 switches. (Put 1, then 2, ..., then 7, then 8, in place.)

14-38. Not difficult. With a lecture devoted to this, the class would be quite adept in solving the solvable fifteen puzzles.

14-45. Warning: Both types of dice are manufactured!
(c) Yes. (You then know the opposite two faces also. Then the "clockwise-counterclockwise" determines the other faces.)

14-46. This proof is quite simple, as long as the students are comfortable with the symbolism, 2n - i, etc.

Chapter 15: Map Coloring

15-2. (d) The two colors alternate along the coast. Thus, the number of edges on the coast is even.

15-16. In a regular map there is at least one region with fewer than 6 sides.

15-17. $3R = 2E$ and $100 - E + R = 2$; hence, $100 - R/2 = 2$, thus, $R = 196$.

15-18. Add an ocean to show that $V - E + R + 1 = 2$, hence $V - E + R = 1$.

15-20. (b) Put a pebble on both sides of each edge, and count them in two ways.

15-23. Since $4R_2 + 3R_3 + 2R_4 + R_5 = 12 + 2 + 6 = 20$ we see that

$$4(R_2 + R_3 + R_4 + R_5) \geq 20;$$

hence,

$$R_2 + R_3 + R_4 + R_5 \geq 5.$$

15-24. We have $2 = R_7 + 2R_8 + 3R_9 + \ldots$, whence

$R_9 = R_{10} = \ldots = 0$ and ($R_7 = 2$, $R_8 = 0$) or

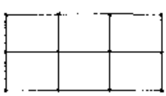
($R_7 = 0$, $R_8 = 1$). The class may wish to see if there is a map satisfying these conditions.

15-25. (a) If R_2 or R_3 not 0, argue as in Case 1 of proof of Lemma 6. If R_4 not 0, merge a 4-sided region with two of its neighbors that do not form a ring (like Case 2 in proof of Lemma 6).

15-37. There must be a quadrilateral present. Remove borders between it and two of its opposite neighbors that together form a region. Note that this large region (of 3 former regions) has an even number of edges. Color reduced map in 3 colors. Then observe that by changing color of original quadrilateral, appropriately, one obtains coloring of original map with 3 colors.

15-39. In one piece use red and blue, in the other yellow and green.

15-45. No. Consider this dressmaker's pattern for 6 regions on inner tube (each vertex has degree 4).



15-47. (This strengthens E 16.) This says that any net the sphere must have a region with few sides. Parallel argument for Lemma 5, but note $3V \leq 2E$ and $2E \geq 6R$. Thus, $V - E + R \leq (2/3)\ E - E + (1/3)E = 0$; but, $V - E + R = 2$.

15-58. Yes. This is the substance of Chapter 14 concerning the relation between arrangements of odd B and even B.

15-60. (a) Just use 4 of the 5 colors of a 5-coloring. (Omitting regions of the color least used.)
(b) If 4-color conjecture were valid, then just use 3 of the 4 colors. (Omitting regions of the color least used.)

15-61. (a) Color in 5 colors. Consider the countries with the most-used color.
(b) If 4-color theorem were valid, use similar argument.

Chapter 16: Types of Numbers

16-18. (c) Argument:

$$X = -3 - \sqrt{7},\ X + 3 = -\sqrt{7},\ \text{so}\ X^2 + 6X + 9 = 7.$$

Thus, $-3 - \sqrt{7}$ is root of $X^2 + 6X + 2 = 0$.

16-34. Odd degree.

16-44. π is transcendental.

16-45. π is transcendental.

16-68. Let $X = 1 + \sqrt{2}$; thus, $(X-1)^2 = 2$ or $X^2 - 2X - 1 = 0$.

16-69. Let $X = \sqrt{2} + \sqrt{3}$; thus, $X^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$. Then $X^2 - 5 = 2\sqrt{6}$; square, $X^4 - 10X^2 + 25 = 24$; hence, $\sqrt{2} + \sqrt{3}$ is root of $X^4 - 10X^2 + 1 = 0$.

16-70. (a) Divide $Ar^3 + Br^2 + Cr + D = 0$ by r^3.

16-76. (a) If such a factoring existed $\sqrt[3]{2}$ would be a root of a polynomial of degree 1 or degree 2 with rational coefficients. The first is out, since $\sqrt[3]{2}$ is irrational. The second is out by showing that no number of the form $a + b\sqrt{n}$, where a, b, n are rational, will have cube equal to 2. (Don't expect this carried out fully, of course.)
(b) $(X - \sqrt[3]{2})(X^2 + \sqrt[3]{2}X + \sqrt[3]{4})$
(c) $(X - \sqrt[3]{2})(X - r)(X - s)$ where r and s are the other two complex roots of $X^3 = 2$.

16-77. (a) No. For coefficient of i to be 0, X = 0 or 1, which are not roots.
(b) Any polynomial of degree at least one does.

16-79. (b) In fact $F + F^4 = 2 \cos 72$; $\cos 72 \doteq 0.309$.

16-80. (a) Slide them all to the left, above the interval [1,2].
(b) Each shaded region is larger than a triangle obtained from the corresponding rectangle by cutting it in two by a diagonal.

16-81. (a) Care must be taken to point out that $X + 1 = (1/2)(2X + 2)$, etc. We shall not consider factors of degree 0.

16-82. (a) It can be proved that $X^n - 2$ is prime for any positive integer n.

16-83. (b) $m = A\overline{A}$, $n = B\overline{B}$. Thus $mn = A\overline{A}B\overline{B} = (AB)(\overline{AB})$. Actually, we are recording the identity, known to Fibonacci,

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

16-87. (a) The little bricks can then be placed parallel.
(b) Yes, as follows. following ideas of E 85, we would have,

$$(1 + X + X^2 + \ldots + X^{a-1})\ (1 + Y + Y^2 + \ldots + Y^{b-1})$$

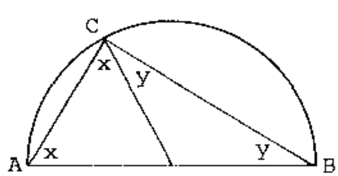
$$= P(1 + X \quad + \ldots + X^{n-1}) + Q(1 + Y \quad + \ldots + Y^{n-1})$$

Then replace X and Y by the complex number of length 1 and angle $(360/n)°$. A coloring argument as in Reference 17 also settles the question. Label the square whose center is (i,j) with the remainder of $i + j \pmod n$. Each brick covers one each of the n labels. But if n divides neither a nor b, the n labels do not appear the same number of times in the rectangle.

16-92. Let A, B, C be the alleged points, considered as complex numbers. Then the complex number $(A - B)/(C - B)$ would have angle 60° and, clearly, rational coefficients. But the ratio of the legs of a 60-30 right triangle is $\sqrt{3}/2$, not rational. The general argument shows that if an angle can be displayed on the "geoboard" one of its sides can be chosen to be horizontal.

Chapter 17: Construction by Straightedge and Compass

17-6. (a)



$2x + 2y = 180$, hence $x + y = 90$

17-10. This argument could be used to replace the use of geometric series in the text.

17-12. Since a 9-gon cannot be constructed, neither can 20°, 10°, 1°. Since a 15-gon can be constructed, 24° can; hence, 12° and 3°, and 1 1/2°. Since 45° can be constructed, so can 22 1/2°. Since 72° can, so can 36°. Since a 24-gon can be obtained from a 3-gon, 15° can be.

17-13. (a) We know that $\sqrt[3]{2}$ is irrational.

17-14. If $8M^3 - 6MN^2 - N^3 = 0$, M divides N^3 (and N divides $8M^3$). Thus, since $(M,N) = 1$, $M = 1$ or -1; since N divides 8, $N = 1, -1, 2,$ or -2. None of these cases provides a root.

17-18. (c) Gauss's theorem must be used.

17-19. (a) Let N = 2M + 1. Then $\sqrt{2M + 1} = \sqrt{(M + 1)^2 - M^2}$.

17-20. Trisection of CD does not trisect the arc. (Draw it when angle AOB is 90°.)

17-21. How was the 1° mark determined?

17-22. (b) We marked the straight edge.

17-25. (b) Imagine the construction recorded in wires on a glass plane. Shine the flashlight on it and consider (a).

17-26. (b) As (a) shows, it will not be.
(d) See discussion of E 25 (b).

17-27. (e) Error should be at most 0.02, if pencil is sharp.

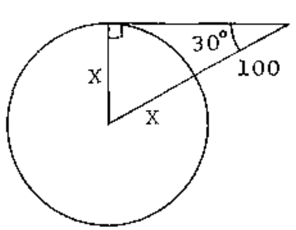
17-39. (a) $8 \cos^4 A - 8 \cos^2 A + 1$.
(b) $128 \cos^8 A - 256 \cos^6 A + 160 \cos^4 A - 32 \cos^2 A + 1$.
(c) Show step by step that sin(nx) = (sin x)·(polynomial in cos x), and, simultaneously, cos(nx) is a polynomial in cos x. Coefficients are integers. The proof is inductive, using identities in E 29 (b) repeatedly.
(d) 0 = cos 90 = "polynomial in cos 1" with integer coefficients.

17-40. (a) Use identity cos 45 = $2(\cos 22\ 1/2)^2 - 1$, from E 30 (a).

17-42. (b) Using $\sin^2 x + \cos^2 x = 1$; the sum equals the sum of ninety 1's.

17-47. (b) Observe that 4 sin A cos A = 2 sin 2A, maximum when 2A = 90°.

17-49.



1/2 = sin 30° = X/(X + 100); whence, 2X = 200.

17-51. (b) 45° (to make sin 2A a maximum).

Chapter 18: Infinite Sets

18-6. The empty set is not referred to in the chapter.

18-10. By Theorem 3.

18-20. Pair n with 2n.

18-23. The number constructed will not be rational (Another proof that there are irrationals.)

18-24. No. See remark on E 23

18-25. No. Use Cantor's argument.

18-26. $\sqrt{2}$ is irrational; nonrepeating decimal is irrational; rationals are denumerable but reals are not.

18-28. (a) $2^0 3^1 5^{-5} 7^0 11^0 13^0 17^0 19^1 = 57/3125$

(b) $400 = 2^4 3^0 5^2$; hence, $4 + 2X^2$.

(c) $5/34 = 2^{-1} 3^0 5^1 7^0 11^0 13^0 17^{-1}$; hence,

$-1 + X^2 - X^6$

18-33. (a) Pair the set $(n_1, n_2, \ldots)$ with the sequence that has 1 at the n_1th place, n_2nd place, etc , and 0's elsewhere.

18-52. Yes. The number constructed has only the digits 7 and 1.

Chapter 19: A General View

19-10 (1) Find (A,B) by Euclidean algorithm.
(2) Find ((A,B),C) by Euclidean algorithm.
(3) Does latter divide D?

19-24. (a) $2 = 2^{2-1}$, $4 = 2^{3-1}$, $16 = 2^{5-1}$

(b) Probably 2^{n-1} in general.

(c) No, false at n = 6.

Appendix E

24. No. Using sum of geometric series, we would have $(3^m - 1)/(3 - 1) = (2^n - 1)/(2 - 1)$. Thus $3^m - 1 = 2(2^n - 1)$ or $2^{n+1} - 1 = 3^m$. If $2^{n+1} - 1 = 3^m$, then $(-1)^{n+1} - 1 \equiv 0 \pmod 3$. Thus n is odd, $n + 1 = 2p$; hence, $2^{n+1} - 1 = 2^{2p} - 1$

$= (2^p-1)(2^p+1) = 3^m$. Thus, $2^p - 1 = 3^a$ by fundamental theorem of arithmetic. Since $a<m$, n would have a smaller case. (So, at beginning, consider smallest nontrivial case.)