SOLUTIONS MANUAL

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The Finite Element Method

Linear Static and Dynamic Finite Element Analysis

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PREFACE

Notational Conventions

As a rule, we have adopted notations consistent with those that appear in the text. However, the word processor used to produce the solution manual did not possess all the fonts used in typesetting the text. In particular, we wish to point out that the notation for the specified natural boundary condition used herein is "h" in contrast to the script notation used in the text. (The roman font was employed to distinguish from the mesh parameter, denoted by h.)

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CHAPTER 1

1.4 Exercise 1, p. 7

Symmetry:

$$a(w\,,\,u) = \int_0^1 w_{,x} u_{,x}\,dx$$

$$= \int_0^1 u_{,x} w_{,x}\,dx = a(u\,,\,w)$$

Linearity:

$$a(c_1 w_1 + c_2 w_2\,,\,u) = \int_0^1 (c_1 w_{1,x} + c_2 w_{2,x}) u_{,x}\,dx$$

$$= c_1 \int_0^1 w_{1,x} u_{,x}\,dx + c_2 \int_0^1 w_{2,x} u_{,x}\,dx$$

$$= c_1 a(w_1\,,\,u) + c_2 a(w_2\,,\,u)$$

Linearity in the second argument is immediate from symmetry and linearity of the first argument, hence $a(\cdot\,,\,\cdot)$ is bilinear. Symmetry and bilinearity of the inner product $(\cdot\,,\,\cdot)$ may be proved analogously.

1.7 Exercise 1, p. 20

This problem is straightforward and only requires repeating the calculations given in the text while filling in the missing details.

1.8 Exercise 1, p. 22

The weak form of the model problem is

$$\int_0^1 w_{,x} u_{,x}\,dx = \int_0^1 w\,f\,dx + w(0)\mathrm{h}$$

The integrals may be written as sums over element domains,

$$\sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w_{,x} u_{,x} \, dx = \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w \, f \, dx + w(0)\mathrm{h}$$

so that while $\mathcal{S}, \mathcal{V} \subset C^0$, $w_{,x}$ and $u_{,x}$ will be C^1 on element interiors. Integrate by parts:

$$\sum_{A=1}^{n} \Big\{ - \int_{x_A}^{x_{A+1}} w u_{,xx} \, dx + w u_{,x} \Big|_{x_A}^{x_{A+1}} \Big\} = \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w \, f \, dx + w(0)\mathrm{h}$$

Collecting terms:

$$\begin{aligned}
0 = & \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w(u_{,xx} + f) \, dx \; + \; w(0)\mathrm{h} \\
& -\Big\{ w(x_2^-)u_{,x}(x_2^-) - w(x_1^+)u_{,x}(x_1^+) \\
& + w(x_3^-)u_{,x}(x_3^-) - w(x_2^+)u_{,x}(x_2^+) \\
& + w(x_4^-)u_{,x}(x_3^-) - w(x_3^+)u_{,x}(x_3^+) \\
& \qquad \vdots \\
& + w(x_{n+1}^-)u_{,x}(x_{n+1}^-) - w(x_n^+)u_{,x}(x_n^+) \Big\}
\end{aligned}$$

Then noting that $w(x_i^-) = w(x_i^+)$, $w(x_1^+) = w(0)$ and $w(x_{n+1}^-) = w(1) = 0$:

$$\begin{aligned}
0 = & \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w(u_{,xx} + f) \, dx \quad + \quad w(0)\{\mathrm{h} + u_{,x}(0^+)\} \\
& + \sum_{A=2}^{n} w(x_A) \Big\{ u_{,x}(x_A^+) - u_{,x}(x_A^-) \Big\}
\end{aligned}$$

The associated Euler-Lagrange equations may be directly observed from this final expression.

1.10 Exercise 1, p. 31

a) Using a central difference approximation for $u_{,xx}$ in the strong form of our model problem, with uniform nodal spacing $h = x_A - x_{A-1}$, leads to the finite difference equation

$$\frac{d_{A-1} - 2d_A + d_{A+1}}{h^2} + f_A = 0 \tag{1}$$

for each internal node A, i.e. $1 < A < n+1$, where $n+1$ equals the total number of nodes throughout the domain. The corresponding finite element equation is

$$\sum_{B=A-1}^{A+1} a(N_A\,,\,N_B)d_B = (N_A\,,\,f) \tag{2}$$

Assume $f = \sum_{A=1}^{n+1} f_A N_A$ and use the change of variable $\xi = (x - x_A)/(x_{A+1} - x_{A-1})$ to evaluate the integrals:

$$a(N_A\,,\,N_{A-1}) = a(N_A\,,\,N_{A+1}) = -\frac{1}{h}$$

$$a(N_A\,,\,N_A) = \frac{2}{h}$$

$$(N_A\,,\,f) = \frac{h}{6}(f_{A-1} + 4f_A + f_{A+1})$$

Now (2) simplifies to

$$\frac{-d_{A-1} + 2d_A - d_{A+1}}{h} = \frac{h}{6}(f_{A-1} + 4f_A + f_{A+1})$$

or

$$\frac{d_{A-1} - 2d_A + d_{A+1}}{h^2} + \frac{1}{6}(f_{A-1} + 4f_A + f_{A+1}) = 0 \tag{3}$$

Comparison of (1) and (3) shows that the FD method will replicate the superconvergence of the FE solution if

$$f_A = \frac{1}{6}(f_{A-1} + 4f_A + f_{A+1})$$

or, equivalently,

$$f_A = \frac{1}{2}(f_{A-1} + f_{A+1}) \tag{4}$$

We have assumed f is piecewise linear, but (4) requires f to be linear throughout the *entire* domain Ω.

b) We employ text equations (1.6.11) and (1.6.12) to evaluate the coefficients of the FE matrix equation associated with node 1:

$$K_{11} = \frac{1}{h}, \qquad K_{12} = -\frac{1}{h},$$

$$F_1 = (N_1\,,\, f) + N_1(0)\mathrm{h} - a(N_1\,,\, N_{n+1})g$$

$$= f_1 \int_0^h N_1^2 dx + f_2 \int_0^h N_1 N_2 dx + \mathrm{h}$$

$$= \frac{h}{6}(2f_1 + f_2) + \mathrm{h}$$

Rearranging leads to the equation

$$\frac{-d_1 + d_2}{h^2} + \frac{2f_1 + f_2}{6} = -\frac{\mathrm{h}}{h} \tag{5}$$

The corresponding FD approximation might use (1) evaluated at $A = 1$,

$$\frac{d_0 - 2d_1 + d_2}{h^2} + f_1 = 0 \tag{6}$$

where d_0 is a "phantom point" outside of the domain, and enforce the boundary condition by employing the finite difference approximation

$$\frac{d_2 - d_0}{2h} \approx u_{,x}(0) = -\mathrm{h} \tag{7}$$

Combining these equations leads to

$$\frac{-d_1 + d_2}{h^2} + \frac{f_1}{2} = -\frac{\mathrm{h}}{h} \tag{8}$$

Comparing (5) and (8) reveals that the differences between the FD and FE approximations (in this example) arise from the treatment of the forcing function f. Note also that the FE methodology provides a precise recipe for the treatment of the Neumann boundary condition, whereas additional *ad hoc* assumptions (i.e., (6) and (7)) are required in a FD treatment.

1.11 Exercise 1, p. 36

a) Discretizing the domain $\Omega =]0,1[$ with four equal-length linear elements leads to the matrix equation

$$\boldsymbol{K}\boldsymbol{d} = \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \boldsymbol{F} = \frac{q}{96} \begin{pmatrix} 1 \\ 6 \\ 12 \\ 18 \end{pmatrix}$$

which can be solved to find

$$\boldsymbol{d} = \frac{q}{384} \begin{pmatrix} 64 \\ 63 \\ 56 \\ 37 \end{pmatrix}$$

Note that the nodal values u^h_{part} agree with the exact solution, i.e.,

$$u^h_{\text{part}}(x_A) = \tilde{u}_{\text{part}}(x_A) = \frac{q}{6}(1 - x_A^3)$$

b) Using the definition

$$u^h(x) = \sum_{A=1}^{4} N_A(x) d_A + N_5(x) g$$

implies

$$u^h_{,x}(x) = \sum_{A=1}^{4} N_{A,x}(x) d_A + N_{5,x}(x) g$$

Thus, for example,

$$\begin{aligned} u^h_{,x}(\frac{1}{8}) &= N_{1,x}(\frac{1}{8}) d_1 + N_{2,x}(\frac{1}{8}) d_2 \\ &= -4(\frac{q}{6}) + 4(\frac{63q}{64}) = -\frac{q}{96} \end{aligned}$$

Similar calculations lead to the following table.

x	$u^h_{,x}(x)$	$u_{,x}(x)$	$re_{,x} = \frac{2}{q}\lvert u^h_{,x}(x) - u_{,x}(x)\rvert$
$\frac{1}{8}$	$-\frac{1}{96}q$	$-\frac{1}{128}q$	$\frac{1}{192}$
$\frac{3}{8}$	$-\frac{7}{96}q$	$-\frac{9}{128}q$	$\frac{1}{192}$
$\frac{5}{8}$	$-\frac{19}{96}q$	$-\frac{25}{128}q$	$\frac{1}{192}$
$\frac{7}{8}$	$-\frac{37}{96}q$	$-\frac{49}{128}q$	$\frac{1}{192}$

c) One can also find that $h = 1 \Rightarrow re_{,x} = \frac{1}{12}$ and $h = \frac{1}{2} \Rightarrow re_{,x} = \frac{1}{48}$. These results are plotted below.

d) We assume the error $re_{,x}$ is proportional to some power of the mesh parameter h, viz.

$$re_{,x} = Ch^m$$

where C and m are constants. Taking the natural logarithm of this equation yields

$$\ln re_{,x} = m \ln h + \ln C$$

(Recall the standard form of the equation of a straight line $y = mx + b$.) Thus m is the slope of the $\ln re_{,x}$ versus $\ln h$ plot, and $b = \ln C$ is the y-intercept.

(i) The slope of the $\ln re_{,x}$ versus $\ln h$ graph is 2.0, which reflects second-order convergence of the relative error of the derivative at the midpoints.

(ii) The y-intercept is the logarithm of C, which is the error in the case of a one-element mesh ($h = 1$).

1.15 Exercise 1, p. 46

Proof of nodal exactness in Sec. 1.10 relies in part upon Lemma 1, p. 26, which was derived by using a weighting function $w^h \in \mathcal{V}^h \subset \mathcal{V}$ in the weak form

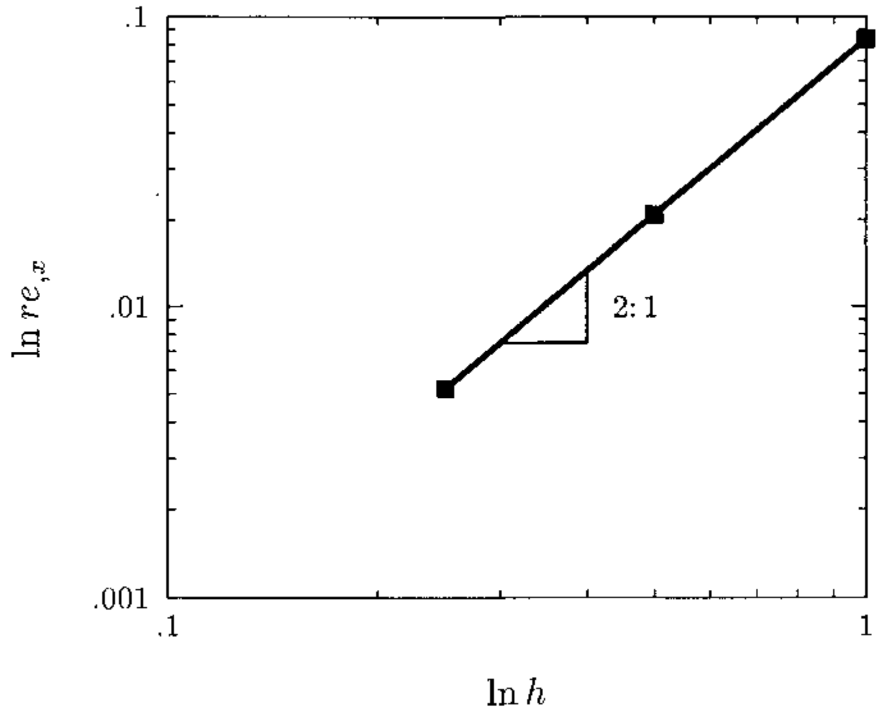
$$(W) \qquad a(w^h\,,\, u) = (w^h\,,\, f) + w^h(0)\text{h}$$

and subtracting the Galerkin equation

$$(G) \qquad a(w^h\,,\, u^h) = (w^h\,,\, f) + w^h(0)\text{h}$$

to find

$$a(w^h\,,\, u - u^h) = 0$$



i.e., the error is a-orthogonal to the weighting function. However, if f is approximated in (G), this result will not follow unless

$$(w^h\,,\,f) = (w^h\,,\,f^h)$$

This condition can be met by enforcing equality at the element level:

$$(w^h\,,\,f)^e = (w^h\,,\,f^h)^e$$

Remember that we have assumed f to be quadratic and that two-point Gauss quadrature can exactly integrate third-order polynomials. Thus f^h can be chosen linear on each element interior if it has the same values as f at the Gauss points of the two-point rule ($\tilde{\xi} = \pm 1/\sqrt{3}$):

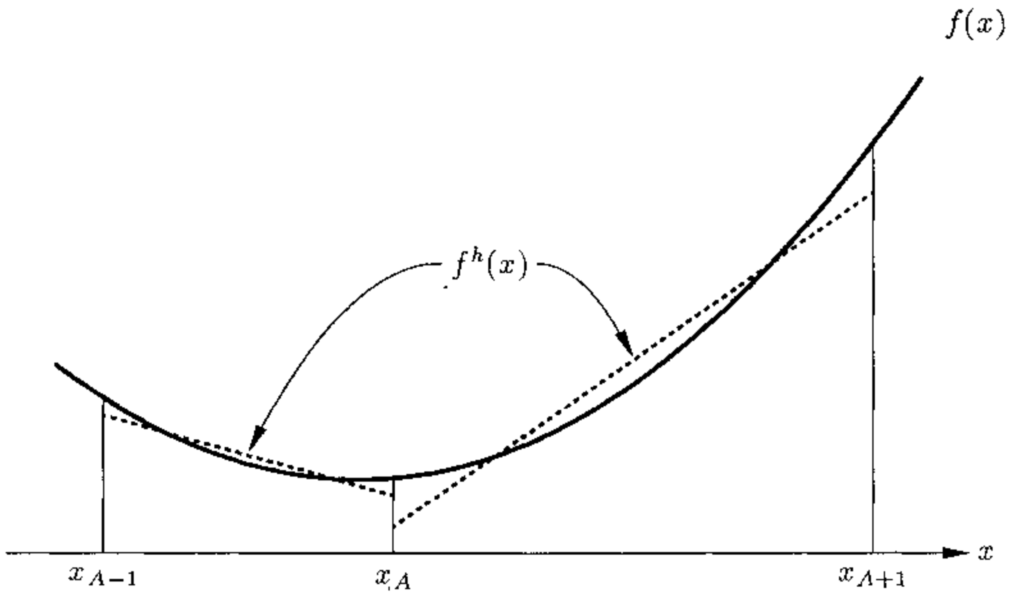
$$f^h\left(\frac{1}{2}\left[(x_A + x_{A+1}) \pm \frac{1}{\sqrt{3}}(x_{A+1} - x_A)\right]\right) = f\left(\frac{1}{2}\left[(x_A + x_{A+1} \pm \frac{1}{\sqrt{3}}(x_{A+1} - x_A)\right]\right)$$

However, in general this will lead to f^h being discontinuous (see sketch below). Note: algebraic proof of this result is straightforward (but mildly tedious), and reveals that if the function is expressed w.r.t. a local variable ξ as

$$f(\xi) = p_0 + p_1\xi + p_2\xi^2\,, \qquad -1 \leq \xi \leq 1$$

then the linear approximation which coincides with f at the Gauss points is

$$f^h(\xi) = p_o + p_1\xi + \frac{1}{3}p_2$$



1.15 Exercise 2, p. 46

(i) Galerkin approximation:

$$a(w^h\,,\, v^h) + \boxed{(w^h\,,\, \lambda v^h)} = (w^h\,,\, f) + w^h(0)\mathrm{h} - a(w^h\,,\, g^h) - \boxed{(w^h\,,\, \lambda g^h)}$$

(ii) Stiffness coefficients:

$$K_{AB} = a(N_A\,,\, N_B) + \boxed{(N_A\,,\, \lambda N_B)}$$
$$k^e_{ab} = a(N_a\,,\, N_b)^e + \boxed{(N_a\,,\, \lambda N_b)^e}$$

(iii) Stiffness matrix for linear element:

$$\boldsymbol{k}^e = [k^e_{ab}] = \frac{1}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\lambda h^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(iv) Symmetry of $\boldsymbol{K} = [K_{AB}]$ follows directly from the symmetry of $a(\cdot\,,\,\cdot)$ which was established in Sec. 1.4, Exercise 1, p. 7, and the fact that

$$\begin{aligned}(N_A, \lambda N_B) &= \int_\Omega N_A \lambda N_B \, dx \\ &= \int_\Omega N_B \lambda N_A \, dx \\ &= (N_B, \lambda N_A)\end{aligned}$$

(v) Positive definiteness of $\boldsymbol{K}$:

$$\begin{aligned}\boldsymbol{c}^T \boldsymbol{K} \boldsymbol{c} &= \sum_{A,B=1}^{n} c_A K_{AB} c_B \\ &= \sum_{A,B=1}^{n} \left\{ c_A a(N_A\,,\, N_B) c_B + c_A (N_A\,,\, \lambda N_B) c_B \right\} \\ &= a\Big(\sum_{A=1}^{n} c_A N_A\,,\, \sum_{B=1}^{n} c_B N_B\Big) + \Big(\sum_{A=1}^{n} c_A N_A\,,\, \lambda \sum_{B=1}^{n} c_B N_B\Big)\end{aligned}$$

The N_A's are linearly independent and form a basis for $\mathcal{V}^h$, hence for any $\{c_A\}$ we can identify a corresponding $w^h \in \mathcal{V}^h$ by

$$\sum_{A=1}^{n} c_A N_A \equiv w^h \in \mathcal{V}^h$$

and thus

$$\begin{aligned}\boldsymbol{c}^T \boldsymbol{K} \boldsymbol{c} &= a(w^h\,,\, w^h) + (w^h\,,\, \lambda w^h) \\ &= \int_\Omega (w^h_{,x})^2 \, d\Omega + \lambda \int_\Omega (w^h)^2 \, d\Omega \\ &\geq 0\end{aligned}$$

so $\boldsymbol{K}$ is at least positive semi-definite. Now assume $\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = 0$, which implies

$$\int_\Omega (w^h_{,x})^2 \, d\Omega = 0 \tag{1}$$

and

$$\lambda \int_\Omega (w^h)^2 \, d\Omega = 0 \tag{2}$$

Eq. (2) requires that $w^h = 0$ in Ω. (1) is therefore also satisfied by this choice. Furthermore, linear independence of the N_A's means $w^h = 0 \iff c_A = 0$ for $A = 1, 2, \ldots n$. Therefore

$$\boldsymbol{c}^T\boldsymbol{K}\boldsymbol{c} = 0 \implies \boldsymbol{c} = \boldsymbol{0}$$

and so $\boldsymbol{K}$ is positive definite. Note that we did not use the essential boundary condition $w^h(1) = 0$ in arriving at this conclusion. The presence of the λ-term enables us to prove positive definiteness without using the essential boundary condition.

(vi) (Note: review Sec. 1.10, pp. 24–27.) Piecewise linear shape functions cannot replicate the exponential form of the Green's function, g, even when $y = x_A$. So the nodal Green's function, $g \notin \mathcal{V}^h$ which is an essential ingredient in the proof of nodal exactness.

(vii) Exponential shape functions:

Evaluating $u^h(x) = c_1 e^{px} + c_2 e^{-px}$ at x_1 and x_2 leads to a system of simultaneous equations for c_1 and c_2:

$$\begin{Bmatrix} u^h(x_1) \\ u^h(x_2) \end{Bmatrix} = \begin{bmatrix} e^{px_1} & e^{-px_1} \\ e^{px_2} & e^{-px_2} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$\implies \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{1}{e^{p(x_1-x_2)} - e^{-p(x_1-x_2)}} \begin{bmatrix} e^{-px_2} & -e^{-px_1} \\ -e^{px_2} & e^{px_1} \end{bmatrix} \begin{Bmatrix} u^h(x_1) \\ u^h(x_2) \end{Bmatrix}$$

Thus

$$u^h(x) = \frac{[u^h(x_1)e^{-px_2} - u^h(x_2)e^{-px_1}]e^{px}}{e^{p(x_1-x_2)} - e^{-p(x_1-x_2)}} + \frac{[-u^h(x_1)e^{px_2} + u^h(x_2)e^{px_1}]e^{-px}}{e^{p(x_1-x_2)} - e^{-p(x_1-x_2)}}$$

or, equivalently,

$$u^h(x) = N_1(x)u^h(x_1) + N_2(x)u^h(x_2)$$

where

$$N_1(x) = \frac{e^{p(x-x_2)} - e^{-p(x-x_2)}}{e^{p(x_1-x_2)} - e^{-p(x_1-x_2)}}$$

$$N_2(x) = \frac{-e^{p(x-x_1)} + e^{-p(x-x_1)}}{e^{p(x_1-x_2)} - e^{-p(x_1-x_2)}}$$

With this choice of shape functions the nodal Green's function $g \in \mathcal{V}^h$, therefore the FE solution will attain nodal exactness by the same proof as for the model problem (see pp. 24–27).

1.16 Exercise 1, p. 48

a) Assume u is a solution of (S). Then, in particular, it satisfies the essential boundary conditions $u(1) = 0$, $u_{,x}(1) = 0$, hence $u \in \mathcal{S}$. Thus for any $w \in \mathcal{V}$ one can write

$$\int_0^1 w(EIu_{,xxxx} - f)\, dx = 0$$

Integrating by parts:

$$-\int_0^1 w_{,x} EIu_{,xxx}\, dx + wEIu_{,xxx}\Big|_0^1 - \int_0^1 wf\, dx = 0$$

$$\int_0^1 w_{,xx} EIu_{,xx}\, dx - w_{,x}EIu_{,xx}\Big|_0^1 + wEIu_{,xxx}\Big|_0^1 = \int_0^1 wf\, dx$$

Noting the essential boundary conditions on $w \in \mathcal{V}$,

$$\int_0^1 w_{,xx} EIu_{,xx}\, dx = \int_0^1 wf\, dx - w_{,x}(0)EIu_{,xx}(0) + w(0)EIu_{,xxx}(0)$$

and the boundary conditions on u:

$$\int_0^1 w_{,xx} EIu_{,xx}\, dx = \int_0^1 wf\, dx - w_{,x}(0)M + w(0)Q \tag{1}$$

We have shown u is a member of $\mathcal{S}$ and that it satisfies (1) for all $w \in \mathcal{V}$. Therefore u is a solution of (W).

Now assume u is a solution of (W). Then $u \in \mathcal{S}$ and hence $u(1) = 0$ and $u_{,x}(1) = 0$. From (W),

$$\int_0^1 w_{,xx} EIu_{,xx}\, dx = \int_0^1 wf\, dx - w_{,x}(0)M + w(0)Q$$

for all $w \in \mathcal{V}$. Integrating by parts,

$$-\int_0^1 w_{,x} EIu_{,xxx}\, dx + w_{,x} EIu_{,xx}\Big|_0^1 = \int_0^1 wf\, dx - w_{,x}(0)M + w(0)Q$$

$$\int_0^1 wEIu_{,xxxx}\, dx - wEIu_{,xxx}\Big|_0^1 + w_{,x} EIu_{,xx}\Big|_0^1 = \int_0^1 wf\, dx - w_{,x}(0)M + w(0)Q$$

Regroup and use the essential boundary conditions $w(1) = 0$, $w_{,x}(1) = 0$:

$$\int_0^1 w(EIu_{,xxxx} - f)\, dx - w_{,x}(0)\{EIu_{,xx}(0) - M\} + w(0)\{EIu_{,xxx}(0) - Q\} = 0 \tag{2}$$

To prove that u satisfies (S) we must show:

$$\begin{aligned} EIu_{,xxxx} &= f \qquad \text{in } \Omega =]0,1[\\ EIu_{,xx}(0) &= M \\ EIu_{,xxx}(0) &= Q \end{aligned}$$

First assume $w = \phi(u_{,xxxx} - f)$, where $\phi(0) = \phi(1) = 0$, $\phi_{,x}(0) = \phi_{,x}(1) = 0$ and $\phi > 0$ on Ω. Hence $w \in \mathcal{V}$. For example, we could choose $\phi = 1 - \cos 2\pi x$. Substituting into (2):

$$\int_0^1 \phi(EIu_{,xxxx} - f)^2\, dx - 0 \cdot M + 0 \cdot Q = 0$$

Having specified $\phi > 0$ on Ω and knowing $(EIu_{,xxxx} - f)^2 \geq 0$, it follows that $(EIu_{,xxxx} - f)^2 = 0$, implying $EIu_{,xxxx} = f$ on Ω.

Now (2) is reduced to

$$-w_{,x}(0)\{EIu_{,xx}(0) - M\} + w(0)\{EIu_{,xxx}(0) - Q\} = 0 \tag{3}$$

for all $w \in \mathcal{V}$. There are no essential boundary conditions at $x = 0$. So select, in turn, a w such that $w(0) = 1$ and $w_{,x}(0) = 0$, and then another w such that $w(0) = 0$ and $w_{,x}(0) = 1$. It follows that $EIu_{,xx}(0) = M$ and $EIu_{,xxx}(0) = Q$. Therefore, the u which satisfies (W) is also a solution of (S).

b) Choosing $\mathcal{V}^h$ to be the space of piecewise-cubic Hermite polynomials, assume w^h has the form

$$w^h = c_1 + c_2 x + c_3 x^2 + c_4 x^3, \quad x_1 \leq x \leq x_2$$

Denoting $w^h(x_1) = \delta_1$, $w^h(x_2) = \delta_2$, $w^h_{,x}(x_1) = \theta_1$, and $w^h_{,x}(x_2) = \theta_2$, a system of four equations may be obtained for determining the c_i's:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \theta_1 \\ \theta_2 \end{pmatrix}$$

Upon forward reduction

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & x_1 + x_2 & x_1^2 + x_1 x_2 + x_2^2 \\ 0 & 0 & 1 & 2x_1 + x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{c} = \begin{pmatrix} \delta_1 \\ \frac{\delta_2 - \delta_1}{h} \\ -\frac{\theta_1}{h} + \frac{\delta_2 - \delta_1}{h^2} \\ \frac{\theta_1 + \theta_2}{h^2} - \frac{2(\delta_2 - \delta_1)}{h^3} \end{pmatrix}$$

and thus

$$
\boldsymbol{c} = \begin{bmatrix} -2x_1x_2^2 + hx_2^2 & 2x_1^2x_2 + hx_1^2 & -x_1x_2^2 & -x_1^2x_2 \\ 6x_1x_2 & -6x_1x_2 & 2x_1x_2 + x_2^2 & x_1^2 + 2x_1x_2 \\ -3(x_1 + x_2) & 3(x_1 + x_2) & -(x_1 + 2x_2) & -(2x_1 + x_2) \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{pmatrix} \frac{\delta_1}{h^3} \\ \frac{\delta_2}{h^3} \\ \frac{\theta_1}{h^2} \\ \frac{\theta_2}{h^2} \end{pmatrix}
$$

Substituting into the expression for w^h,

$$
w^h(x) = N_1(x)w^h(x_1) + N_3(x)w^h(x_2) + N_2(x)w^h_{,x}(x_1) + N_4(x)w^h_{,x}(x_2)
$$

we conclude

$$
\begin{aligned} N_1(x) &= \frac{1}{h^3}\{-2x_1x_2^2 + hx_2^2 + (6x_1x_2)x - 3(x_1 + x_2)x^2 + 2x^3\} \\ &= \frac{-(x - x_2)^2\{-h + 2(x_1 - x)\}}{h^3} \end{aligned}
$$

and similarly,

$$
\begin{aligned} N_3(x) &= \frac{(x - x_1)^2\{h + 2(x_2 - x)\}}{h^3} \\ N_2(x) &= \frac{1}{h^2}\{-x_1x_2^2 + (2x_1x_2 + x_2^2)x - (x_1 + 2x_2)x^2 + x^3\} \\ &= \frac{(x - x_1)(x - x_2)^2}{h^2} \\ N_4(x) &= \frac{(x - x_1)^2(x - x_2)}{h^2} \end{aligned}
$$

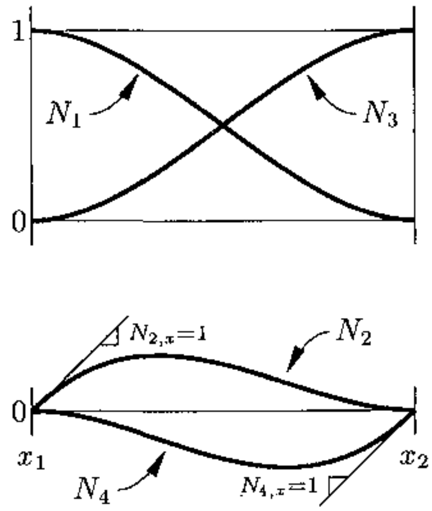
These shape functions are plotted below.

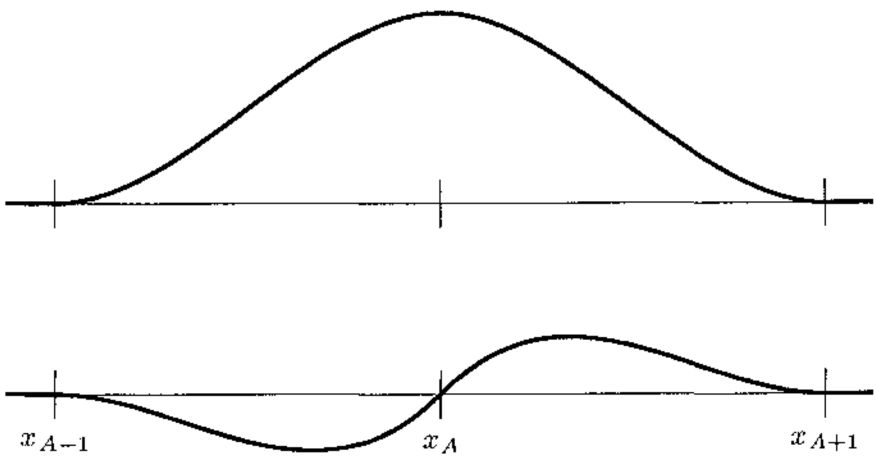
c) The error in curvature is $e_{,xx} = u^h_{,xx} - u_{,xx}$. Let

$$
x(\xi) = \frac{h\xi + x_A + x_{A+1}}{2}
$$

where $x_A = x(-1)$ and $x_{A+1} = x(1)$. By the chain rule

$$
\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{2}{h}\frac{\partial}{\partial \xi}
$$





$$\frac{\partial^2}{\partial x^2} = \frac{2}{h}\frac{\partial}{\partial \xi}\left(\frac{2}{h}\frac{\partial}{\partial \xi}\right) = \left(\frac{2}{h}\right)^2 \frac{\partial^2}{\partial \xi^2}$$

Therefore

$$e_{,xx} = \left(\frac{2}{h}\right)^2 e_{,\xi\xi} = \left(\frac{2}{h}\right)^2 \left[u^h_{,\xi\xi} - u_{,\xi\xi}\right] \tag{1}$$

The shape functions in terms of ξ are

$$N_1(x(\xi)) = N_1(\xi) = \frac{1}{4}(\xi - 1)^2(\xi + 2)$$

$$N_2(x(\xi)) = N_2(\xi) = \frac{h}{8}(\xi + 1)(\xi - 1)^2$$

$$N_3(x(\xi)) = N_3(\xi) = \frac{1}{4}(\xi+1)^2(2-\xi)$$

$$N_4(x(\xi)) = N_4(\xi) = \frac{h}{8}(\xi+1)^2(\xi-1)$$

This enables us to derive

$$\begin{aligned} u^h_{,\xi\xi}(\xi) &= N_{1,\xi\xi}(\xi)u^h(-1) + N_{3,\xi\xi}(\xi)u^h(1) \\ &\quad + N_{2,\xi\xi}(\xi)\xi_{,x}u^h_{,\xi}(-1) + N_{4,\xi\xi}(\xi)\xi_{,x}u^h_{,\xi}(1) \\ &= \frac{3}{2}\xi u^h(-1) - \frac{3}{2}\xi u^h(1) \\ &\quad + \frac{1}{2}(3\xi-1)u^h_{,\xi}(-1) + \frac{1}{2}(3\xi+1)u^h_{,\xi}(1) \end{aligned} \tag{2}$$

Express the *exact values*, $u(-1)$, $u(+1)$, $u_{,\xi}(-1)$ and $u_{,\xi}(+1)$ in finite Taylor expansions about some arbitrary point $\alpha \in \,]-1,+1[$:

$$\begin{aligned} u(-1) &= u(\alpha) - (1+\alpha)u_{,\xi}(\alpha) + \frac{1}{2}(1+\alpha)^2 u_{,\xi\xi}(\alpha) \\ &\quad - \frac{1}{6}(1+\alpha)^3 u_{,\xi\xi\xi}(\alpha) + \frac{1}{24}(1+\alpha)^4 u_{,\xi\xi\xi\xi}(\alpha) - \frac{1}{120}(1+\alpha)^5 u_{,\xi\xi\xi\xi\xi}(C_1) \end{aligned} \tag{3}$$

$$\begin{aligned} u(+1) &= u(\alpha) + (1-\alpha)u_{,\xi}(\alpha) + \frac{1}{2}(1-\alpha)^2 u_{,\xi\xi}(\alpha) \\ &\quad + \frac{1}{6}(1-\alpha)^3 u_{,\xi\xi\xi}(\alpha) + \frac{1}{24}(1-\alpha)^4 u_{,\xi\xi\xi\xi}(\alpha) + \frac{1}{120}(1-\alpha)^5 u_{,\xi\xi\xi\xi\xi}(C_2) \end{aligned} \tag{4}$$

$$\begin{aligned} u_{,\xi}(-1) &= u_{,\xi}(\alpha) - (1+\alpha)u_{,\xi\xi}(\alpha) + \frac{1}{2}(1+\alpha)^2 u_{,\xi\xi\xi}(\alpha) \\ &\quad - \frac{1}{6}(1+\alpha)^3 u_{,\xi\xi\xi\xi}(\alpha) + \frac{1}{24}(1+\alpha)^4 u_{,\xi\xi\xi\xi\xi}(C_3) \end{aligned} \tag{5}$$

$$\begin{aligned} u_{,\xi}(+1) &= u_{,\xi}(\alpha) + (1-\alpha)u_{,\xi\xi}(\alpha) + \frac{1}{2}(1-\alpha)^2 u_{,\xi\xi\xi}(\alpha) \\ &\quad + \frac{1}{6}(1-\alpha)^3 u_{,\xi\xi\xi\xi}(\alpha) + \frac{1}{24}(1-\alpha)^4 u_{,\xi\xi\xi\xi\xi}(C_4) \end{aligned} \tag{6}$$

We presume nodal exactness of the FE solution and its derivative, i.e., $u^h(1) = u(1)$, $u^h_{,\xi}(1) = u_{,\xi}(1)$, etc., which will be confirmed in part (g). Therefore, the left-hand sides of eqs. (3)–(6) may be replaced with $u^h(\pm 1)$ and $u^h_{,\xi}(\pm 1)$ and substituted into (2). This permits the curvature error (1) to be evaluated in terms of the exact solution. After some tedious algebra:

$$\begin{aligned} e_{,\xi\xi}(\alpha) &= (-3\alpha + 3\alpha)u_{,\xi}(\alpha) + (3\alpha^2 - 3\alpha^2 + 1)u_{,\xi\xi}(\alpha) \\ &+ [-\frac{1}{2}(3\alpha^2+1)\alpha + \frac{1}{2}(3\alpha^2+1)\alpha]u_{,\xi\xi\xi}(\alpha) \\ &+ [\frac{1}{2}(\alpha^2+1)\alpha^2 - \frac{1}{2}(\alpha^2+3)\alpha^2 + \frac{1}{6}(3\alpha^2+1)]u_{,\xi\xi\xi\xi}(\alpha) \\ &+ \sum_{i=1}^{4} p_i(\alpha)u_{,\xi\xi\xi\xi\xi}(C_i) - u_{,\xi\xi}(\alpha) \end{aligned}$$

where each $p_i(\alpha)$ is a polynomial in α. Simplifying further:

$$e_{,\xi\xi}(\alpha) = \frac{1}{6}(-3\alpha^2 + 1)u_{,\xi\xi\xi\xi}(\alpha) + \sum_{i=1}^{4} p_i(\alpha)u_{,\xi\xi\xi\xi\xi}(C_i)$$

or in the global coordinate system:

$$\left(\frac{h}{2}\right)^2 e_{,xx}(x(\alpha)) = \frac{1}{6}\left(\frac{h}{2}\right)^4(-3\alpha^2+1)u_{,xxxx}(x(\alpha)) + \left(\frac{h}{2}\right)^5 \sum_{i=1}^{4} p_i(\alpha)u_{,xxxxx}(x(C_i))$$

$$\Longrightarrow e_{,xx}(x(\alpha)) = \frac{h^2}{24}(-3\alpha^2+1)u_{,xxxx}(x(\alpha)) + O(h^3) \tag{7}$$

The curvature is in general second-order accurate, but superconvergence occurs at the two Barlow points $\alpha = \pm\frac{1}{\sqrt{3}} \in \,]-1, +1[$.

d) We have third-order convergence in curvature at the Barlow points.

e) If $f = EIu_{,xxxx} = 0$ throughout the element, then the curvature is exact throughout the element by part (c).

f) Evaluating the global coefficients from the definition

$$K_{AB} = a(N_A\,,\, N_B) = \int_{x_1}^{x_2} N_{A,xx} EI\, N_{B,xx}\, dx$$

with $x_1 = 0$ and $x_2 = 1$:

$$K_{11} = \int_0^1 EI(12x - 6)^2 \, dx$$

$$= \frac{EI}{36}(12x - 6)^3 \Big|_0^1 = 12EI$$

$$K_{22} = \int_0^1 EI(6x - 4)^2 \, dx$$

$$= \frac{EI}{18}(6x - 4)^3 \Big|_0^1 = 4EI$$

$$K_{12} = \int_0^1 EI(12x - 6)(6x - 4) \, dx$$

$$= 6EI(4x^3 - 7x^2 + 4x) \Big|_0^1 = 6EI$$

With $f(x) = c$,

$$F_A = \int_0^1 N_A c \, dx - N_{A,x}(0)M + N_A(0)Q$$

therefore,

$$F_1 = \int_0^1 (x - 1)^2(1 + 2x)c \, dx - 0 \cdot M + 1 \cdot Q$$

$$= \frac{1}{2}c + Q$$

$$F_2 = \int_0^1 x(x - 1)^2 c \, dx - 1 \cdot M + 0 \cdot Q$$

$$= \frac{1}{12}c - M$$

Thus the Galerkin FEM leads to the matrix problem

$$EI\begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix}\begin{pmatrix} u^h(0) \\ u^h_{,x}(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c + Q \\ \frac{1}{12}c - M \end{pmatrix}$$

Solving this 2×2 system:

$$u^h(x) = N_1(x)u^h(0) + N_2(x)u^h_{,x}(0)$$

where

$$u^h(0) = \frac{1}{EI}\Big(\frac{1}{8}c + \frac{1}{2}M + \frac{1}{3}Q\Big)$$

$$u^h_{,x}(0) = -\frac{1}{EI}\Big(\frac{1}{6}c + M + \frac{1}{2}Q\Big)$$

This result is compared with the exact solution, namely,

$$u(x) = \frac{1}{EI}\Big[\frac{1}{24}cx^4 + \frac{1}{6}Qx^3 + \frac{1}{2}Mx^2 - (\frac{1}{6}c + \frac{1}{2}Q + M)x + \frac{1}{8}c + \frac{1}{3}Q + \frac{1}{2}M\Big]$$

on the following page (with $c = 48EI$, $Q = -12EI$, and $M = 0$).

g) To prove nodal exactness of the displacement requires the Green's function solution for a concentrated force (i.e., a delta function) at position $x = y$.

$$EIg_{,xxxx} = \delta(x - y)$$

$$g(1) = 0,\ g_{,x}(1) = 0$$

$$EIg_{,xx}(0) = 0,\ EIg_{,xxx}(0) = 0$$

Integrating:

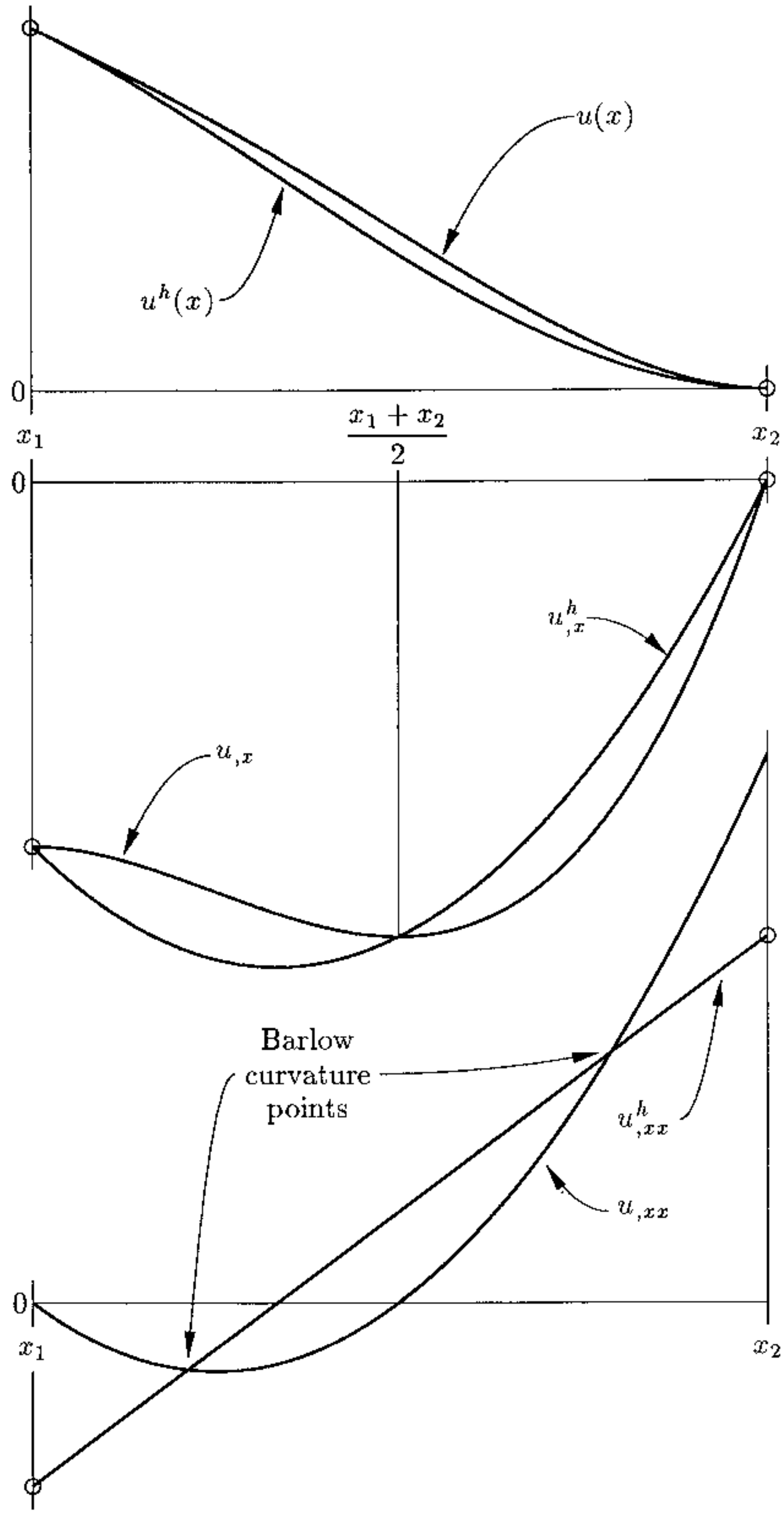
$$EIg_{,xxx} = H(x - y) + c_1$$

$$EIg_{,xx} = \langle x - y\rangle + c_1x + c_2$$

$$EIg_{,x} = \frac{1}{2}\langle x - y\rangle^2 + \frac{1}{2}c_1x^2 + c_2x + c_3$$

$$EIg = \frac{1}{6}\langle x - y\rangle^3 + \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4$$

Applying the boundary conditions



$u(x)$
$u^h(x)$
0
x_1
$\frac{x_1+x_2}{2}$
x_2
0
$u^h_{,x}$
$u_{,x}$
Barlow
curvature
points
$u^h_{,xx}$
$u_{,xx}$
0
x_1
x_2

$$0 = EIg_{,xxx}(0) = c_1$$

$$0 = EIg_{,xx}(0) = \langle 0 - y \rangle + c_2 = c_2$$

$$0 = EIg_{,x}(1) = \frac{1}{2}\langle 1 - y \rangle^2 + c_3 = \frac{1}{2}(1 - y)^2 + c_3$$

$$0 = EIg(1) = \frac{1}{6}\langle 1 - y \rangle^3 - \frac{1}{2}\langle 1 - y \rangle^2 + c_4 = \frac{1}{6}(1 - y)^3 - \frac{1}{2}(1 - y)^2 + c_4$$

leads to the solution

$$EIg = \frac{1}{6}\langle x - y \rangle^3 - \frac{1}{2}(1 - y)^2 x - \frac{1}{6}(1 - y)^3 + \frac{1}{2}(1 - y)^2$$

Note that g is smooth except at y, where $g_{,xxx}$ is discontinuous. Nevertheless, if $y = x_A$, then $g \in \mathcal{V}^h$. This can be seen as follows: $\mathcal{V}^h$ consists of piecewise cubics that satisfy certain homogeneous boundary conditions, are at least globally continuous and possess globally continuous first derivatives. The function g satisfies these conditions for $y = x_A$. Our space of weighting functions has the property $\mathcal{V}^h \subset \mathcal{V}$, so from the weak form of problem

$$a(w^h\,,\,u) = (w^h\,,\,f) - w^h_{,x}(0)M + w^h(0)Q$$

and the Galerkin approximation

$$a(w^h\,,\,u^h) = (w^h\,,\,f) - w^h_{,x}(0)M + w^h(0)Q$$

we find

$$a(w^h\,,\,u) - a(w^h\,,\,u^h) = a(w^h\,,\,u - u^h) = 0$$

that is, the error is a-orthogonal to *all* weighting functions in $\mathcal{V}^h$. The logic of Lemma 2 of Sec. 1.10, p. 27, can be reproduced to obtain

$$u(y) - u^h(y) = (u - u^h, \delta_y) = a(u - u^h\,,\,g)$$

Recalling from above that $y = x_A$ implies $g \in \mathcal{V}^h$, then

$$u(x_A) - u^h(x_A) = a(u - u^h\,,\,g) = 0$$

so the nodal values of u^h are exact.

To prove nodal exactness of the rotations requires the Green's function solution for a concentrated moment (i.e., a dipole which is the generalized derivative of the Dirac delta function) at position $x = y$.

$$EIg_{,xxxx} = \delta_{,x}(x-y)$$

$$g(1) = 0,\ g_{,x}(1) = 0$$

$$EIg_{,xx}(0) = 0,\ EIg_{,xxx}(0) = 0$$

Integrating:

$$EIg_{,xxx} = \delta(x-y) + c_1$$

$$EIg_{,xx} = H(x-y) + c_1 x + c_2$$

$$EIg_{,x} = \langle x-y\rangle + \frac{1}{2}c_1 x^2 + c_2 x + c_3$$

$$EIg = \frac{1}{2}\langle x-y\rangle^2 + \frac{1}{6}c_1 x^3 + \frac{1}{2}c_2 x^2 + c_3 x + c_4$$

Applying the boundary conditions

$$0 = EIg_{,xxx}(0) = c_1$$

$$0 = EIg_{,xx}(0) = c_2$$

$$0 = EIg_{,x}(1) = \langle 1-y\rangle + c_3 = (1-y) + c_3$$

$$0 = EIg(1) = \frac{1}{2}\langle 1-y\rangle^2 - \langle 1-y\rangle + c_4 = \frac{1}{2}(1-y)^2 - (1-y) + c_4$$

leads to the solution

$$EIg = \frac{1}{2}\langle x-y\rangle^2 - (1-y)x - \frac{1}{2}(1-y)^2 + (1-y)$$

Note that g is smooth except at y, where $g_{,xx}$ is discontinuous. Nevertheless, if $y = x_A$, then $g(x) \in \mathcal{V}^h$.

The weak form of this Green's function problem is

$$a(w\,,\,g) = \int_0^1 w(x)\delta_{,x}(x-y)\,dy$$

for all $w \in \mathcal{V}$. Proceeding as before:

$$a(u-u^h\,,\,g) = \int_0^1 (u-u^h)\delta_{,x}(x-y)\,dy$$

$$= -\big(u_{,x}(y) - u^h_{,x}(y)\big)$$

For $y = x_A$, $g \in \mathcal{V}^h$ and therefore

$$a(u - u^h\,,\,g) = 0$$

implying the result,

$$u_{,x}(x_A) = u^h_{,x}(x_A)$$

h) Given $EIu_{,xxxx} = f(x) = c$, a constant, then $u_{,xxxxx} = 0$, so by the result immediately preceding (7), the error in the curvature is zero at the previously identified Barlow points.

i) $\mathcal{V}^h$ needs to be contained in $H^2(\Omega)$. If $\mathcal{V}^h$ was simply continuous, as for the piecewise linear finite element space discussed in Sec. 1.8, then $\mathcal{V}^h \subset H^1(\Omega)$, but not $H^2(\Omega)$, as Dirac delta functions would exist in the generalized second derivatives. Functions need to be at least C^1 in order that the generalized second derivatives have no worse than step discontinuities.

j) Writing the stiffness coefficients in the local coordinate system:

$$k^e_{pq} = a(N_p\,,\,N_q) = \int_{-1}^{1} N_{p,\xi\xi}\,(\xi_{,x})^2 EI N_{q,\xi\xi}\,(\xi_{,x})^2(\xi_{,x}^{-1})d\xi$$

$$= \frac{8EI}{h^3}\int_{-1}^{1} N_{p,\xi\xi}N_{q,\xi\xi}d\xi$$

For example,

$$k^e_{11} = \frac{8EI}{h^3}\int_{-1}^{1}\left(\frac{3}{2}\xi\right)^2 d\xi$$

$$= \frac{12EI}{h^3}$$

Similar calculations lead to the element stiffness matrix

$$\boldsymbol{k}^e = \frac{EI}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ & 4h^2 & -6h & 2h^2 \\ & & 12 & -6h \\ \text{Symm.} & & & 4h^2 \end{bmatrix}$$

k) Given the weak form of beam problem

$$\int_0^1 EIw_{,xx}u_{,xx}\,dx = \int_0^1 wf\,dx - w_{,x}(0)M + w(0)Q$$

write the integrals as sums over element domains,

$$\sum_{A=1}^{n}\int_{x_A}^{x_{A+1}} EIw_{,xx}u_{,xx}\,dx = \sum_{A=1}^{n}\int_{x_A}^{x_{A+1}} wf\,dx - w_{,x}(0)M + w(0)Q$$

Integrate by parts:

$$\sum_{A=1}^{n}\Big\{-\int_{x_A}^{x_{A+1}} EIw_{,x}u_{,xxx}\,dx + EIw_{,x}u_{,xx}\Big|_{x_A}^{x_{A+1}}\Big\} = \sum_{A=1}^{n}\int_{x_A}^{x_{A+1}} wf\,dx - w_{,x}(0)M + w(0)Q$$

Integrate by parts again:

$$\sum_{A=1}^{n}\Big\{\int_{x_A}^{x_{A+1}} EIwu_{,xxxx}\,dx + EIw_{,x}u_{,xx}\Big|_{x_A}^{x_{A+1}} - EIwu_{,xxx}\Big|_{x_A}^{x_{A+1}}\Big\}$$

$$= \sum_{A=1}^{n}\int_{x_A}^{x_{A+1}} wf\,dx - w_{,x}(0)M + w(0)Q$$

Collecting terms:

$$0 = \sum_{A=1}^{n}\int_{x_A}^{x_{A+1}} w(EIu_{,xxxx} - f)\,dx \;+\; w_{,x}(0)M \;-\; w(0)Q$$

$$- EI\Big\{w(x_2^-)u_{,xxx}(x_2^-) - w(x_1^+)u_{,xxx}(x_1^+)$$

$$+ w(x_3^-)u_{,xxx}(x_3^-) - w(x_2^+)u_{,xxx}(x_2^+)$$

$$+ w(x_4^-)u_{,xxx}(x_3^-) - w(x_3^+)u_{,xxx}(x_3^+)$$

$$\vdots$$

$$+ w(x_{n+1}^-)u_{,xxx}(x_{n+1}^-) - w(x_n^+)u_{,xxx}(x_n^+)\Big\}$$

$$+ EI\Big\{w_{,x}(x_2^-)u_{,xx}(x_2^-) - w_{,x}(x_1^+)u_{,xx}(x_1^+)$$

$$+ w_{,x}(x_3^-)u_{,xx}(x_3^-) - w_{,x}(x_2^+)u_{,xx}(x_2^+)$$

$$+ w_{,x}(x_4^-)u_{,xx}(x_3^-) - w_{,x}(x_3^+)u_{,xx}(x_3^+)$$

$$\vdots$$

$$+ w_{,x}(x_{n+1}^-)u_{,xx}(x_{n+1}^-) - w_{,x}(x_n^+)u_{,xx}(x_n^+)\Big\}$$

Then noting $w(x_i^-) = w(x_i^+)$, $w_{,x}(x_i^-) = w_{,x}(x_i^+)$, $w(x_1^+) = w(0)$, $w_{,x}(x_1^+) = w_{,x}(0)$, $w(x_{n+1}^-) = w(1) = 0$, and $w_{,x}(x_{n+1}^-) = w_{,x}(1) = 0$:

$$0 = \sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w(EIu_{,xxxx} - f)\, dx$$

$$+ w(0)\{EIu_{,xxx}(0^+) - Q\}$$

$$- w_{,x}(0)\{EIu_{,xx}(0^+) - M\}$$

$$+ \sum_{A=2}^{n} w(x_A)EI\Big\{u_{,xxx}(x_A^+) - u_{,xxx}(x_A^-)\Big\}$$

$$- \sum_{A=2}^{n} w_{,x}(x_A)EI\Big\{u_{,xx}(x_A^+) - u_{,xx}(x_A^-)\Big\}$$

The associated Euler-Lagrange equations may be inferred from this final expression.

CHAPTER 2

2.3 Exercise 1, p. 63

From the definition of $a(\cdot\,,\cdot)$

$$a(w,u) = \int_{\Omega} w_{,i}\kappa_{ij}u_{,j}\,d\Omega$$

and the symmetry of the conductivities ($\kappa_{ij} = \kappa_{ji}$),

$$\begin{aligned} a(w,u) &= \int_{\Omega} w_{,i}\kappa_{ji}u_{,j}\,d\Omega \\ &= \int_{\Omega} u_{,j}\kappa_{ji}w_{,i}\,d\Omega \\ &= a(u,w) \end{aligned}$$

Therefore, $a(\cdot\,,\cdot)$ is symmetric.

To prove bilinearity, one must show

$$a(c_1u + c_2v, w) = c_1a(u,w) + c_2a(v,w)$$

Substituting into the definition of $a(\cdot,\cdot)$:

$$\begin{aligned} a(c_1u + c_2v, w) &= \int_{\Omega} (c_1u + c_2v)_{,i}\kappa_{ij}w_{,j}\,d\Omega \\ &= \int_{\Omega} (c_1u_{,i} + c_2v_{,i})\kappa_{ij}w_{,j}\,d\Omega \\ &= c_1\int_{\Omega} u_{,i}\kappa_{ij}w_{,j}\,d\Omega + c_2\int_{\Omega} v_{,i}\kappa_{ij}w_{,j}\,d\Omega \\ &= c_1a(u,w) + c_2a(v,w) \end{aligned}$$

The symmetry of $(\cdot\,,\cdot)$ and $(\cdot\,,\cdot)_{\Gamma}$ is analogous.

2.3 Exercise 2, p. 64

For the two-dimensional case ($n_{sd} = 2$)

$$\nabla \boldsymbol{w}^T \boldsymbol{\kappa} \nabla \boldsymbol{u} = \langle\, w_{,1} \quad w_{,2}\,\rangle \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \begin{Bmatrix} u_{,1} \\ u_{,2} \end{Bmatrix}$$

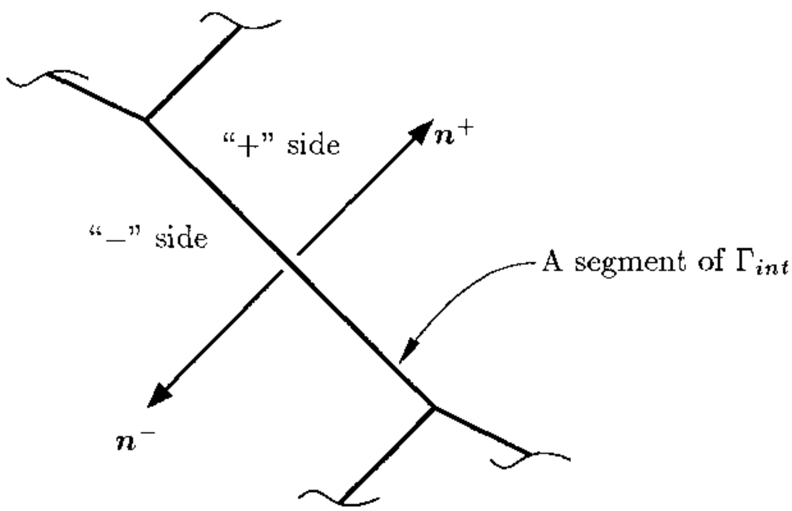
$$= \langle\, w_{,1} \quad w_{,2}\,\rangle \begin{Bmatrix} \kappa_{11} u_{,1} + \kappa_{12} u_{,2} \\ \kappa_{21} u_{,1} + \kappa_{22} u_{,2} \end{Bmatrix}$$

$$= w_{,1}\kappa_{11}u_{,1} + w_{,1}\kappa_{12}u_{,2} + w_{,2}\kappa_{21}u_{,1} + w_{,2}\kappa_{22}u_{,2}$$

$$= w_{,i}\kappa_{ij}u_{,j}$$

Identical steps may be taken for the $n_{sd} = 3$ case.

2.4 Exercise 1, p. 68



Write the weak form as a sum of element integrals

$$0 = \int_{\Omega} w_{,i} q_i \, d\Omega + \int_{\Omega} w f \, d\Omega + \int_{\Gamma_h} w \mathrm{h} \, d\Gamma$$

$$= \sum_{e=1}^{n_{el}} \int_{\Omega^e} (w_{,i} q_i + w f) \, d\Omega + \int_{\Gamma_h} w \mathrm{h} \, d\Gamma$$

Now integrate by parts on each Ω^e:

$$0 = \sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} (-w q_{i,i} + w f) \, d\Omega + \int_{\Gamma^e} w q_i n_i \, d\Gamma \right\} + \int_{\Gamma_h} w \mathrm{h} \, d\Gamma$$

Reorganizing,

$$0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w(q_{i,i} - f) \, d\Omega - \int_{\Gamma_h} w(q_i n_i + \mathrm{h}) \, d\Gamma - \int_{\Gamma_{int}} w(q_i^+ n_i^- + q_i^- n_i^+) \, d\Gamma$$

Note that the last integrand equals $-w(q_i^+ n_i^+ + q_i^- n_i^-)$, so that

$$0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w(q_{i,i} - f) \, d\Omega - \int_{\Gamma_h} w(q_n + \mathrm{h}) \, d\Gamma + \int_{\Gamma_{int}} w[q_n] \, d\Gamma$$

2.5 Exercise 1, p. 71

By the Fourier law

$$q_i = -\kappa_{ij} u_{,j}$$

or in vector form

$$\boldsymbol{q} = -\boldsymbol{\kappa} \nabla u$$

so

$$\boldsymbol{q}^h(\boldsymbol{x}) = -\boldsymbol{\kappa}(\boldsymbol{x}) \nabla u^h(\boldsymbol{x})$$

For all $\boldsymbol{x} \in \Omega^e$

$$u^h(\boldsymbol{x}) = \sum_{a=1}^{n_{en}} N_a(\boldsymbol{x}) d_a^e$$

where $d_a^e = u^h(\boldsymbol{x}_a)$. It follows that

$$\begin{aligned} \nabla u^h(\boldsymbol{x}) &= \sum_{a=1}^{n_{en}} \nabla N_a(\boldsymbol{x}) d_a^e \\ &= \sum_{a=1}^{n_{en}} \boldsymbol{B}_a(\boldsymbol{x}) d_a^e \\ &= \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{d}^e \end{aligned}$$

Then noting $\boldsymbol{D}(\boldsymbol{x}) = \boldsymbol{\kappa}(\boldsymbol{x})$

$$\boldsymbol{q}^h(\boldsymbol{x}) = -\boldsymbol{D}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{d}^e \text{ for all } \boldsymbol{x} \in \Omega^e$$

2.5 Exercise 2, p. 71

(S') : Given f, g, h as in (S) in the text, p. 61,

$$\begin{aligned} q_{i,i} &= f \qquad \text{in} \quad \Omega \\ u &= g \qquad \text{on} \quad \Gamma_g \\ \lambda u - q_i n_i &= \mathrm{h} \qquad \text{on} \quad \Gamma_h \end{aligned} \tag{1}$$

with $\lambda(\boldsymbol{x}) \geq 0$ on Γ_h.

Let $w \in \mathcal{V}$. From (1)

$$\begin{aligned} 0 &= \int_\Omega w(q_{i,i} - f)\, d\Omega \\ &= -\int_\Omega w_{,i} q_i \, d\Omega + \int_{\Gamma_h} w q_i n_i \, d\Gamma - \int_\Omega w f \, d\Omega \\ &= -\int_\Omega w_{,i} q_i \, d\Omega + \int_{\Gamma_h} w(\lambda u - \mathrm{h})\, d\Gamma - \int_\Omega w f \, d\Omega \end{aligned}$$

which suggests that the corresponding weak form is

(W'): Given f, g, h as before, find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$

$$-\int_\Omega w_{,i} q_i \, d\Omega + \int_{\Gamma_h} w \lambda u \, d\Gamma = \int_{\Gamma_h} w \mathrm{h} \, d\Gamma + \int_\Omega w f \, d\Omega$$

To construct the stiffness, use the Galekin approximation

$$\begin{aligned} w^h(\boldsymbol{x}) &= \sum_{A \in \eta - \eta_g} N_A(\boldsymbol{x})\, c_A \\ u^h(\boldsymbol{x}) &= v^h(\boldsymbol{x}) + g^h(\boldsymbol{x}) \\ &= \sum_{A \in \eta - \eta_g} N_A(\boldsymbol{x})\, d_A + \sum_{A \in \eta_g} N_A(\boldsymbol{x})\, g_A \end{aligned}$$

where we take the N_A's to be linearly independent.

The stiffness matrix will then take the form

$$K_{PQ} = \int_{\Omega} \nabla N_A^T(\boldsymbol{x})\boldsymbol{\kappa}(\boldsymbol{x})\nabla N_B(\boldsymbol{x})\, d\Omega + \int_{\Gamma_h} N_A(\boldsymbol{x})\lambda(\boldsymbol{x})N_B(\boldsymbol{x})\, d\Gamma$$

so the added contribution to k_{ab}^e is

$$\int_{\Gamma_h^e} N_a(\boldsymbol{x})\lambda(\boldsymbol{x})N_b(\boldsymbol{x})\, d\Gamma$$

To prove positive definiteness, it suffices to show that the new term is positive semidefinite. Let

$$\Delta K_{PQ} = \int_{\Gamma_h} N_A \lambda N_B \, d\Gamma$$

Then, mimicking the steps on the bottom of p. 67 in the text,

$$\begin{aligned} \boldsymbol{c}^T \Delta \boldsymbol{K} \boldsymbol{c} &= \sum_{P,Q=1}^{n_{eq}} c_P \Delta K_{PQ} c_Q \\ &= \sum_{A,B \in \eta - \eta_g} \breve{c}_A \Delta K_{AB} \breve{c}_B \\ &= \int_{\Gamma_h} \sum_{A,B \in \eta - \eta_g} \breve{c}_A N_A \lambda(\boldsymbol{x}) N_B \breve{c}_B \, d\Gamma \end{aligned}$$

We can define a $w^h \in \mathcal{V}^h$ as

$$w^h(\boldsymbol{x}) = \sum_{A \in \eta - \eta_g} \breve{c}_A N_A(\boldsymbol{x})$$

so

$$\begin{aligned} \boldsymbol{c}^T \Delta \boldsymbol{K} \boldsymbol{c} &= \int_{\Gamma_h} \lambda(\boldsymbol{x}) \{w^h(\boldsymbol{x})\}^2 \, d\Gamma \\ &\geq 0 \end{aligned}$$

2.6 Exercise 1, p. 75

ID Array:

Global node numbers (A)											
1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	0	0	0	0	5	6	7	8

IEN Array:

Local node numbers	Element numbers (e)				
	1	2	3	4	5
1	4	6	5	7	9
2	3	5	7	8	10
3	1	3	9	10	12
4	2	4	3	9	11

LM Array:

Local node numbers	Element numbers (e)				
	1	2	3	4	5
1	4	0	0	0	5
2	3	0	0	0	6
3	1	3	5	6	8
4	2	4	3	5	7

2.7 Exercise 1, p. 81

The proof of symmetry and bilinearity for $(\cdot , \cdot)$ and $(\cdot , \cdot)_\Gamma$ is trivial.

To prove the symmetry of

$$a(\boldsymbol{w}, \boldsymbol{u}) = \int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)} \, d\Omega$$

all that is needed is to recall

$$c_{ijkl} = c_{klij}$$

i.e., the major symmetry given by equation (2.7.3), viz.

$$\begin{aligned} a(\boldsymbol{w}, \boldsymbol{u}) &= \int_{\Omega} w_{(i,j)} c_{ijkl} u_{(k,l)} \, d\Omega \\ &= \int_{\Omega} w_{(i,j)} c_{klij} u_{(k,l)} \, d\Omega \\ &= \int_{\Omega} u_{(k,l)} c_{klij} w_{(i,j)} \, d\Omega \\ &= a(\boldsymbol{u}, \boldsymbol{w}) \end{aligned}$$

Bilinearity is proved in the same manner as is done in Exercise 1 of Sec. 2.3.

2.7 Exercise 2, p. 82

$$\boldsymbol{D} = [D_{IJ}] \qquad D_{IJ} = c_{ijkl}$$

$$\begin{aligned} \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}) &= \begin{Bmatrix} D_{11} u_{1,1} + D_{12} u_{2,2} + D_{13}(u_{1,2} + u_{2,1}) \\ D_{21} u_{1,1} + D_{22} u_{2,2} + D_{23}(u_{1,2} + u_{2,1}) \\ D_{31} u_{1,1} + D_{32} u_{2,2} + D_{33}(u_{1,2} + u_{2,1}) \end{Bmatrix} \\ &= \begin{Bmatrix} c_{1111} u_{1,1} + c_{1122} u_{2,2} + c_{1112}(u_{1,2} + u_{2,1}) \\ c_{2211} u_{1,1} + c_{2222} u_{2,2} + c_{2212}(u_{1,2} + u_{2,1}) \\ c_{1211} u_{1,1} + c_{1222} u_{2,2} + c_{1212}(u_{1,2} + u_{2,1}) \end{Bmatrix} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{w})^T \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}) &= w_{1,1} c_{1111} u_{1,1} + w_{1,1} c_{1122} u_{2,2} + w_{1,1} c_{1112} u_{1,2} + w_{1,1} c_{1121} u_{2,1} \\ &+ w_{2,2} c_{2211} u_{1,1} + w_{2,2} c_{2222} u_{2,2} + w_{2,2} c_{2212} u_{1,2} + w_{2,2} c_{2221} u_{2,1} \\ &+ w_{1,2} c_{1211} u_{1,1} + w_{1,2} c_{1222} u_{2,2} + w_{1,2} c_{1212} u_{1,2} + w_{1,2} c_{1221} u_{2,1} \\ &+ w_{2,1} c_{2111} u_{1,1} + w_{2,1} c_{2122} u_{2,2} + w_{2,1} c_{2112} u_{1,2} + w_{2,1} c_{2121} u_{2,1} \\ &= w_{i,j} c_{ijkl} u_{k,l} \\ &= w_{(i,j)} c_{ijkl} u_{(k,l)} \end{aligned}$$

where the symmetries of c_{ijkl} have been employed.

2.7 Exercise 3, p. 82

From the definition of $\boldsymbol{\varepsilon}$ given in the problem statement, one may directly infer the analog of Table 2.7.1.

I / J	i / k	j / l
1	1	1
2	2	2
3	3	3
4	2	3
4	3	2
5	1	3
5	3	1
6	1	2
6	2	1

2.7 Exercise 4, p. 82

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}) \\ &= \begin{bmatrix} c_{1111} & c_{1122} & c_{1112} \\ & c_{2222} & c_{2212} \\ \text{Symm.} & & c_{1212} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix}\end{aligned}$$

Evaluating the matrix-vector product

$$\begin{aligned}\sigma_{11} &= c_{1111}u_{1,1} + c_{1122}u_{2,2} + c_{1112}(u_{1,2} + u_{2,1}) \\ &= c_{1111}u_{1,1} + c_{1122}u_{2,2} + \frac{1}{2}(c_{1112} + c_{1121})(u_{1,2} + u_{2,1}) \\ &= c_{1111}\varepsilon_{11} + c_{1122}\varepsilon_{22} + (c_{1112} + c_{1121})\varepsilon_{12} \\ &= c_{1111}\varepsilon_{11} + c_{1122}\varepsilon_{22} + c_{1112}\varepsilon_{12} + c_{1121}\varepsilon_{21}\end{aligned}$$

therefore

$$\sigma_{11} = c_{11kl}\varepsilon_{kl}$$

and so forth for the other components of the stress tensor.

2.7 Exercise 6, p. 84

Define the jump in "traction" at an internal boundary:

$$
\begin{aligned}
[\boldsymbol{\sigma}_{in}] &= \sigma_{ij}^{+} n_j^{+} - \sigma_{ij}^{-} n_j^{+} \\
&= \sigma_{ij}^{+} n_j^{+} + \sigma_{ij}^{-} n_j^{-}
\end{aligned}
$$

The given weak form is:

$$
\begin{aligned}
0 &= \int_{\Omega} w_{(i,j)} \sigma_{ij} \, d\Omega - \int_{\Omega} w_i f_i \, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i \mathrm{h}_i \, d\Gamma \\
&= \sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} w_{(i,j)} \sigma_{ij} \, d\Omega - \int_{\Omega^e} w_i f_i \, d\Omega \right\} - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i \mathrm{h}_i \, d\Gamma
\end{aligned}
$$

Note that $w_{(i,j)}\sigma_{ij} = w_{i,j}\sigma_{ij}$ and integrate by parts:

$$
0 = \sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} w_i(-\sigma_{ij,j} - f_i) \, d\Omega + \int_{\Gamma^e} w_i \sigma_{ij} n_j \, d\Gamma \right\} - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i \mathrm{h}_i \, d\Gamma
$$

Reorganizing,

$$
\begin{aligned}
0 = &\sum_{e=1}^{n_{el}} \int_{\Omega^e} w_i(\sigma_{ij,j} + f_i) \, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i(\sigma_{ij} n_j - \mathrm{h}_i) \, d\Gamma \\
&- \int_{\Gamma_{int}} w_i(\sigma_{ij}^{+} n_j^{-} + \sigma_{ij}^{-} n_j^{+}) \, d\Gamma
\end{aligned}
$$

Note that the last integrand equals $-w_i(\sigma_{ij}^{+} n_j^{+} + \sigma_{ij}^{-} n_j^{-})$, so that

$$
0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w_i(\sigma_{ij,j} + f_i) \, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i(\sigma_{in} - \mathrm{h}_i) \, d\Gamma + \int_{\Gamma_{int}} w_i[\sigma_{in}] \, d\Gamma
$$

The extension to $n_{sd} = 3$ is analogous.

2.7 Exercise 5, p. 83

$$c_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}; \qquad i,j,k,l = 1,\ldots,n_{sd}$$

For $n_{sd} = 2$:

$$\boldsymbol{D} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1112} \\ & c_{2222} & c_{2212} \\ \text{Symm.} & & c_{1212} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ & \lambda + 2\mu & 0 \\ \text{Symm.} & & \mu \end{bmatrix}$$

Similarly for $n_{sd} = 3$:

$$\boldsymbol{D} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & \cdot & \cdot & \cdot \\ & c_{2222} & c_{2233} & \cdot & \cdot & \cdot \\ & & c_{3333} & \cdot & \cdot & \cdot \\ & & & c_{2323} & \cdot & \cdot \\ \text{Symmetric} & & & & c_{1313} & \cdot \\ & & & & & c_{1212} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & \cdot & \cdot & \cdot \\ & \lambda + 2\mu & \lambda & \cdot & \cdot & \cdot \\ & & \lambda + 2\mu & \cdot & \cdot & \cdot \\ & & & \mu & \cdot & \cdot \\ \text{Symmetric} & & & & \mu & \cdot \\ & & & & & \mu \end{bmatrix}$$

2.8 Exercise 1, p. 87

Recall for $n_{sd} = 2$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix}$$

$$\boldsymbol{\varepsilon}(N_A \boldsymbol{e}_1) = \begin{Bmatrix} N_{A,1} \\ 0 \\ N_{A,2} \end{Bmatrix}$$

$$\boldsymbol{\varepsilon}(N_A \boldsymbol{e}_2) = \begin{Bmatrix} 0 \\ N_{A,2} \\ N_{A,1} \end{Bmatrix}$$

Thus

$$\boldsymbol{\varepsilon}(N_A \boldsymbol{e}_i) = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix} \boldsymbol{e}_i = \boldsymbol{B}_A \boldsymbol{e}_i$$

The $n_{sd} = 3$ case follows analogously.

2.9 Exercise 1, p. 91

From the finite element approximation

$$u_i^h(\boldsymbol{x}) = \sum_{a=1}^{n_{en}} N_a(\boldsymbol{x}) d_{ia} \qquad i = 1, 2, \ldots, n_{sd}$$

it follows that

$$u_{(i,j)}^h(\boldsymbol{x}) = \sum_{a=1}^{n_{en}} \frac{N_{a,i}(\boldsymbol{x}) d_{ja} + N_{a,j}(\boldsymbol{x}) d_{ia}}{2} \qquad i, j = 1, 2, \ldots, n_{sd}$$

We wish to write the strain-displacement relation in the vector form

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^h) = \sum_{a=1}^{n_{en}} \boldsymbol{B}_a \boldsymbol{d}_a$$

where for $n_{sd} = 3$ it can be verified that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^h) = \begin{Bmatrix} u^h_{1,1} \\ u^h_{2,2} \\ u^h_{3,3} \\ u^h_{2,3} + u^h_{3,2} \\ u^h_{3,1} + u^h_{1,3} \\ u^h_{1,2} + u^h_{2,1} \end{Bmatrix} \quad \boldsymbol{d}_a = \begin{Bmatrix} u_{1a} \\ u_{2a} \\ u_{3a} \end{Bmatrix} \quad \boldsymbol{B}_a = \begin{bmatrix} N_{a,1} & \cdot & \cdot \\ \cdot & N_{a,2} & \cdot \\ \cdot & \cdot & N_{a,3} \\ \cdot & N_{a,3} & N_{a,2} \\ N_{a,3} & \cdot & N_{a,1} \\ N_{a,2} & N_{a,1} & \cdot \end{bmatrix}$$

Use the constitutive relationship verified in Sec. 2.7, Exercise 4,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) \sum_{a=1}^{n_{en}} \boldsymbol{B}_a(\boldsymbol{x}) \boldsymbol{d}^e_a$$

or collect displacements into an element vector instead of nodal vectors, i.e.,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{d}^e$$

where

$$\boldsymbol{B} = [\boldsymbol{B}_1 \ \boldsymbol{B}_2 \ \ldots \ \boldsymbol{B}_{n_{en}}]$$

The case $n_{sd} = 2$ proceeds analogously.

2.10 Exercise 1, p. 98

ID Array:

Global degrees of freedom	Global node numbers (A) 1	2	3	4	5	6	7	8
1	1	3	0	5	7	9	0	12
2	2	4	0	6	8	10	11	13

IEN Array:

Local node numbers	Element numbers (e) 1	2	3	4
1	1	1	3	5
2	2	3	5	7
3	8	4	6	8
4	7	2	4	6

LM Array:

Local equation numbers	Element numbers (e) 1	2	3	4
1	1	1	0	7
2	2	2	0	8
3	3	0	7	0
4	4	0	8	11
5	12	5	9	12
6	13	6	10	13
7	0	3	5	9
8	11	4	6	10

2.12 Exercise 1, p. 103

The first four rows of the extended table in the text, p. 102 are identical to Table 2.7.1, so from Exercise 5, Sec. 2.7

$$\boldsymbol{D}_{33} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ & \lambda + 2\mu & 0 \\ \text{Symm.} & & \mu \end{bmatrix}$$

The remaining entries of the axisymmetric constitutive matrix can be evaluated from Eq. (2.7.31):

$$D_{44} = c_{3333} = \mu(1+1) + \lambda = 2\mu + \lambda$$

$$\boldsymbol{D}_3 = \begin{Bmatrix} c_{1133} \\ c_{2233} \\ c_{1233} \end{Bmatrix} = \begin{Bmatrix} \lambda \\ \lambda \\ 0 \end{Bmatrix}$$

2.12 Exercise 2, p. 103

From the partitioned form of the constitutive equation

$$\begin{Bmatrix} \boldsymbol{\sigma}_{2D} \\ \sigma_{33} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{D}_{33} & \boldsymbol{D}_{3} \\ \boldsymbol{D}_{3}^{T} & D_{44} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_{2D} \\ \varepsilon_{33} \end{Bmatrix}$$

it follows that

$$\sigma_{33} = \boldsymbol{D}_3^T \boldsymbol{\varepsilon}_{2D} + D_{44}\varepsilon_{33}$$

The plane stress condition $\sigma_{33} = 0$ implies

$$\varepsilon_{33} = -D_{44}^{-1}\boldsymbol{D}_3^T \boldsymbol{\varepsilon}_{2D}$$

so then

$$\boldsymbol{\sigma}_{2D} = (\boldsymbol{D}_{33} - \boldsymbol{D}_3 D_{44}^{-1}\boldsymbol{D}_3^T)\boldsymbol{\varepsilon}_{2D}$$

Evaluating the second term

$$\boldsymbol{D}_3 D_{44}^{-1}\boldsymbol{D}_3^T = \frac{\lambda^2}{\lambda + 2\mu}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, under the plane stress assumption

$$\boldsymbol{\sigma}_{2D} = \begin{bmatrix} \frac{4\lambda\mu+4\mu^2}{\lambda+2\mu} & \frac{2\lambda\mu}{\lambda+2\mu} & 0 \\ \frac{2\lambda\mu}{\lambda+2\mu} & \frac{4\lambda\mu+4\mu^2}{\lambda+2\mu} & 0 \\ 0 & 0 & \mu \end{bmatrix}\boldsymbol{\varepsilon}_{2D}$$

which is consistent with Remark 9 of Sec. 2.7.

2.12 Exercise 3, p. 103

The solution to this problem is given in the text.

2.12 Exercise 4, p. 104

The matrix $\boldsymbol{B}_a$ takes the form:

$$\boldsymbol{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \\ N_{a,2} & -N_{a,1} \end{bmatrix}$$

We must define the components of $\boldsymbol{D}$ such that

$$w_{i,j} d_{ijkl} u_{k,l} = \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \\ w_{1,2} - w_{2,1} \end{Bmatrix}^T \boldsymbol{D} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \\ u_{1,2} - u_{2,1} \end{Bmatrix}$$

Expand the tensor product:

$$\begin{aligned} w_{i,j} d_{ijkl} u_{k,l} = {} & w_{1,1} d_{1111} u_{1,1} + w_{1,1} d_{1112} u_{1,2} + w_{1,1} d_{1121} u_{2,1} + w_{1,1} d_{1122} u_{2,2} \\ & + w_{1,2} d_{1211} u_{1,1} + w_{1,2} d_{1212} u_{1,2} + w_{1,2} d_{1221} u_{2,1} + w_{1,2} d_{1222} u_{2,2} \\ & + w_{2,1} d_{2111} u_{1,1} + w_{2,1} d_{2112} u_{1,2} + w_{2,1} d_{2121} u_{2,1} + w_{2,1} d_{2122} u_{2,2} \\ & + w_{2,2} d_{2211} u_{1,1} + w_{2,2} d_{2212} u_{1,2} + w_{2,2} d_{2221} u_{2,1} + w_{2,2} d_{2222} u_{2,2} \end{aligned}$$

Expand the matrix product accounting for the symmetry of $\boldsymbol{D}$:

$$\left\{\begin{array}{c} w_{1,1} \\ w_{2,2} \\ w_{1,2}+w_{2,1} \\ w_{1,2}-w_{2,1} \end{array}\right\}^T \boldsymbol{D} \left\{\begin{array}{c} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \\ u_{1,2}-u_{2,1} \end{array}\right\}$$

$$\begin{aligned}
&= w_{1,1}D_{11}u_{1,1} + w_{1,1}D_{12}u_{2,2} + w_{1,1}D_{13}(u_{1,2}+u_{2,1}) + w_{1,1}D_{14}(u_{1,2}-u_{2,1}) \\
&+ w_{2,2}D_{12}u_{1,1} + w_{2,2}D_{22}u_{2,2} + w_{2,2}D_{23}(u_{1,2}+u_{2,1}) + w_{2,2}D_{24}(u_{1,2}-u_{2,1}) \\
&+ (w_{1,2}+w_{2,1})D_{13}u_{1,1} + (w_{1,2}+w_{2,1})D_{23}u_{2,2} \\
&+ (w_{1,2}+w_{2,1})D_{33}(u_{1,2}+u_{2,1}) + (w_{1,2}+w_{2,1})D_{34}(u_{1,2}-u_{2,1}) \\
&+ (w_{1,2}-w_{2,1})D_{14}u_{1,1} + (w_{1,2}-w_{2,1})D_{24}u_{2,2} \\
&+ (w_{1,2}-w_{2,1})D_{34}(u_{1,2}+u_{2,1}) + (w_{1,2}-w_{2,1})D_{44}(u_{1,2}-u_{2,1})
\end{aligned}$$

$$\begin{aligned}
&= w_{1,1}D_{11}u_{1,1} + w_{1,1}(D_{13}+D_{14})u_{1,2} + w_{1,1}(D_{13}-D_{14})u_{2,1} + w_{1,1}D_{12}u_{2,2} \\
&+ w_{1,2}(D_{13}+D_{14})u_{1,1} + w_{1,2}(D_{33}+2D_{34}+D_{44})u_{1,2} + w_{1,2}(D_{33}-D_{44})u_{2,1} \\
&+ w_{1,2}(D_{23}+D_{24})u_{2,2} + w_{2,1}(D_{13}-D_{14})u_{1,1} + w_{2,1}(D_{33}-D_{44})u_{1,2} \\
&+ w_{2,1}(D_{33}-2D_{34}+D_{44})u_{2,1} + w_{2,1}(D_{23}-D_{24})u_{2,2} + w_{2,2}D_{12}u_{1,1} \\
&+ w_{2,2}(D_{23}+D_{24})u_{1,2} + w_{2,2}(D_{23}-D_{24})u_{2,1} + w_{2,2}D_{22}u_{2,2}
\end{aligned}$$

Equating coefficients, by inspection

$$D_{11} = d_{1111} \qquad D_{22} = d_{2222} \qquad D_{12} = d_{1122}$$

The other coefficients of $\boldsymbol{D}$ lead to coupled equations, e.g.,

$$w_{1,1}d_{1112}u_{1,2} = w_{1,1}(D_{13}+D_{14})u_{1,2}$$
$$w_{1,1}d_{1121}u_{2,1} = w_{1,1}(D_{13}-D_{14})u_{2,1}$$

which implies

$$D_{13} = \frac{1}{2}(d_{1112} + d_{1121})$$

$$D_{14} = \frac{1}{2}(d_{1112} - d_{1121})$$

Similarly,

$$D_{23} = \frac{1}{2}(d_{2212} + d_{2221})$$

$$D_{24} = \frac{1}{2}(d_{2212} - d_{2221})$$

The final three coefficients are found from the relations:

$$\begin{aligned}
w_{1,2}d_{1212}u_{1,2} &= w_{1,2}(D_{33} + 2D_{34} + D_{44})u_{1,2} \\
w_{1,2}d_{1221}u_{2,1} &= w_{1,2}(D_{33} - D_{44})u_{2,1} \\
w_{2,1}d_{2112}u_{1,2} &= w_{2,1}(D_{33} - D_{44})u_{1,2} \\
w_{2,1}d_{2121}u_{2,1} &= w_{2,1}(D_{33} - 2D_{34} + D_{44})u_{2,1}
\end{aligned}$$

This system is not over-determined because $d_{1221} = c_{1221} = c_{2112} = d_{2112}$, therefore

$$D_{33} = \frac{1}{4}(d_{1212} + 2d_{1221} + d_{2121})$$

$$D_{44} = \frac{1}{4}(d_{1212} - 2d_{1221} + d_{2121})$$

$$D_{34} = \frac{1}{4}(d_{1212} - d_{2121})$$

Employing $d_{ijkl} = c_{ijkl} + \delta_{ik}\sigma^0_{jl}$, we arrive at the final result:

$$\boldsymbol{D} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1112} & 0 \\ & c_{2222} & c_{2212} & 0 \\ & & c_{1212} & 0 \\ \text{Symm.} & & & 0 \end{bmatrix} + \begin{bmatrix} \sigma^0_{11} & 0 & \frac{\sigma^0_{12}}{2} & \frac{\sigma^0_{12}}{2} \\ & \sigma^0_{22} & \frac{\sigma^0_{12}}{2} & -\frac{\sigma^0_{12}}{2} \\ & & \frac{\sigma^0_{11}+\sigma^0_{22}}{4} & \frac{\sigma^0_{22}-\sigma^0_{11}}{4} \\ \text{Symm.} & & & \frac{\sigma^0_{11}+\sigma^0_{22}}{4} \end{bmatrix}$$

2.12 Exercise 5, p. 105

On the boundary $\boldsymbol{w}$ can be decomposed as

$$\boldsymbol{w} = w_n \boldsymbol{n} + w_s \boldsymbol{s}$$

where

$$w_n = w_j n_j$$

$$w_s = w_j s_j$$

Build all essential boundary conditions into the function spaces:

$$\mathcal{S} = \{\boldsymbol{u} \mid \boldsymbol{u} \in H^1(\Omega); \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma_1,\ u_n = g_n \text{ on } \Gamma_3,\ u_s = g_s \text{ on } \Gamma_4\}$$

$$\mathcal{V} = \{\boldsymbol{w} \mid \boldsymbol{w} \in H^1(\Omega); \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_1,\ w_n = 0 \text{ on } \Gamma_3,\ w_s = 0 \text{ on } \Gamma_4\}$$

It follows from the strong form that $\int_\Omega w_i(\sigma_{ij,j} + f_i)\, d\Omega = 0$, so integrating by parts

$$0 = -\int_\Omega w_{i,j}\sigma_{ij}\, d\Omega + \int_\Omega w_i f_i\, d\Omega + \int_\Gamma w_i \sigma_{ij} n_j\, d\Gamma$$

Noting

$$\Gamma = \overline{\Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3 \bigcup \Gamma_4}$$

and the boundary conditions on w_i

$$\int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)}\, d\Omega = \int_\Omega w_i f_i\, d\Omega + \int_{\Gamma_2} w_i \sigma_{ij} n_j\, d\Gamma$$

$$+ \int_{\Gamma_3} w_s \sigma_{ij} n_j s_i\, d\Gamma + \int_{\Gamma_4} w_n \sigma_{ij} n_j n_i\, d\Gamma$$

Thus the weak form is

$$\int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)}\, d\Omega = \int_\Omega w_i f_i\, d\Omega + \int_{\Gamma_2} w_i \mathrm{h}_i\, d\Gamma + \int_{\Gamma_3} w_s \mathrm{h}_s\, d\Gamma + \int_{\Gamma_4} w_n \mathrm{h}_n\, d\Gamma$$

(No summation is implied on n or s.)

2.12 Exercise 6, p. 105

The solution to this problem is given in the text.

2.12 Exercise 7, p. 106

This is the one-dimensional boundary value problem of Chapter 1, with

$$f = -p(p-1)x^{p-2}$$

$$g = 1$$

$$\mathrm{h} = 0$$

i) Integrating,

$$u_{,x} = px^{p-1} + c_1$$

$$u = x^p + c_1 x + c_2$$

and employing the boundary conditions,

$$c_1 = 0$$

$$c_2 = 0$$

leads to the exact solution

$$u(x) = x^p$$

(See figure on page 47.)

ii) Given f, g, and h as above, and the function spaces

$$\mathcal{S} = \{u \,|\, u \in H^1 \,,\, u(1) = 1\}$$

$$\mathcal{V} = \{w \,|\, w \in H^1 \,,\, w(1) = 0\}$$

find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$

$$\int_0^1 w_{,x} u_{,x} \, dx = \int_0^1 w f \, dx$$

$$= -p(p-1) \int_0^1 w x^{p-2} \, dx$$

iii) $\mathcal{V}^h \subset \mathcal{V}$, $u^h = v^h + g^h$, $v^h \in \mathcal{V}^h$, $g^h(1) = 1$

Given f and h as in (i), and the finite-dimensional spaces

$$\mathcal{S}^h = \left\{ u \,|\, u^h = v^h + g^h , v^h \in \mathcal{V}^h , g^h(1) = 1 \right\}$$
$$\mathcal{V}^h \subset \mathcal{V}$$

find $u^h \in \mathcal{S}^h$ such that for all $w^h \in \mathcal{V}^h$

$$\int_0^1 w^h_{,x} v^h_{,x} \, dx = \int_0^1 w^h f \, dx - \int_0^1 w^h_{,x} g^h_{,x} \, dx$$

iv) Find $\boldsymbol{d}$ such that

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$$

where

$$\boldsymbol{d} = \left\{ d_B \right\} \qquad B = 1, 2, ..., n$$
$$\boldsymbol{K} = \left[K_{AB} \right] \qquad A, B = 1, 2, ..., n$$
$$K_{AB} = a(N_A \, , \, N_B) = \int_0^1 N_{A,x} N_{B,x} \, dx$$
$$\boldsymbol{F} = \left\{ F_A \right\} \qquad A = 1, 2, ..., n$$
$$F_A = (N_A \, , \, f) - a(N_A \, , \, N_{n+1})$$
$$= \int_0^1 N_A f \, dx - a(N_A \, , \, N_{n+1})$$

v)

(a) one element:

$$K_{11} d_1 = F_1$$
$$K_{11} = 1$$

$$F_1 = (N_1\,,\, f) \;-\; a(N_1\,,\, N_2)$$

$$= -p(p-1)\int_0^1 (1-x)x^{p-2}\,dx \;+\; 1$$

For $p \;=\; 5$, $F_1 \;=\; -1+1 \;=\; 0$, therefore $d_1 \;=\; 0$.

(b) two elements:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Using the results from Example 2 of Sec. 1.7,

$$\boldsymbol{K} = 2\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$F_1 = \int_0^{\frac{1}{2}} (1-2x)f\,dx$$

$$= 2(p-1)(\frac{1}{2})^p - p(\frac{1}{2})^{p-1}$$

$$F_2 = 2\int_0^{\frac{1}{2}} xf\,dx \;+\; 2\int_{\frac{1}{2}}^{1} (1-x)f\,dx \;+\; 2$$

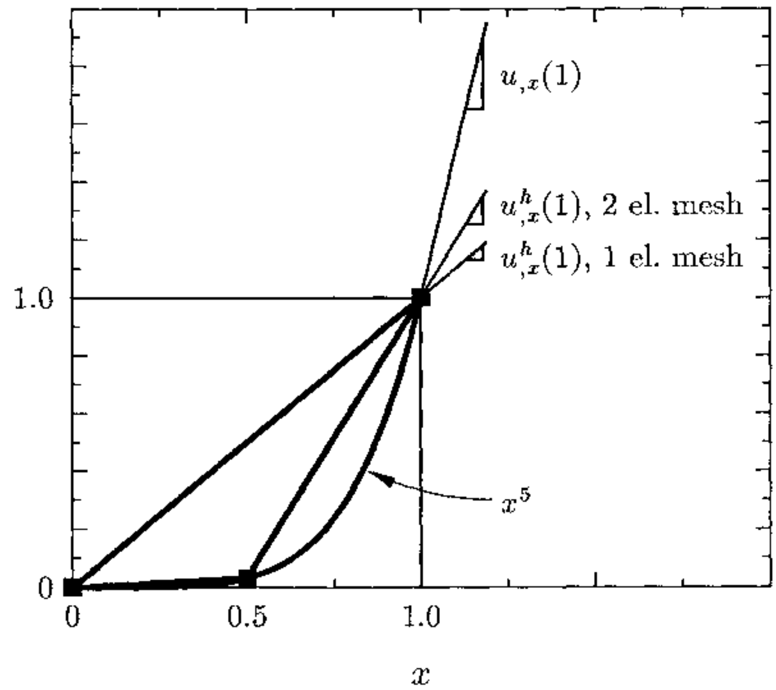
$$= -2(p-1)(\frac{1}{2})^p - 2p + 2(p-1) + 2p(\frac{1}{2})^{p-1}$$

$$- \, 2(p-1)(\frac{1}{2})^p + 2$$

For $p \;=\; 5\,,\; F_1 \;=\; -\frac{1}{16}\,,\;\; F_2 \;=\; \frac{1}{8}$

$$\boldsymbol{d} = \boldsymbol{K}^{-1}\boldsymbol{F} = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} -\frac{1}{16} \\ \frac{1}{8} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ \frac{1}{32} \end{Bmatrix}$$



(vi) Exact solution

$$u_{,x}(1) = 5x^4\Big|_{x=1} = 5$$

One-element approximation:

$$u^h_{,x}(1) = N_{1,x}(1)d_1 + N_{2,x}(1) = 1$$

Two-elements approximation:

$$\begin{aligned} u^h_{,x}(1) &= N_{2,x}(1)d_2 + N_{3,x}(1) \\ &= -2(\frac{1}{32}) + 2(1) = \frac{31}{16} \end{aligned}$$

The finite element solution is exact at the nodes and linear between nodes. $u^h_{,x}(1)$ is thus a secant approximation of the exact value. Therefore, $u^h_{,x}(1)$ will not be accurate if the mesh is crude and the exact solution has a steep slope, as in the situation here (see the figure).

2.12 Exercise 8, p. 107

N.B. We use h to denote the unknown heat flux on Γ_g and, as in all of this manual, h denotes the given natural boundary condition on Γ_h.

i) Considering the left-hand side of the revised weak form, integrate by parts

$$\int_\Omega -w_{,i} q_i \, d\Omega = \int_\Omega w q_{i,i} \, d\Omega - \int_\Gamma w q_i n_i d\Gamma$$

Then substituting into the weak form

$$\int_\Omega w(q_{i,i} - f) \, d\Omega - \int_{\Gamma_h} w(q_i n_i + \mathrm{h}) d\Gamma - \int_{\Gamma_g} w(q_i n_i + h) d\Gamma = 0$$

The standard localization argument leads to the Euler-Lagrange equations

$$\begin{aligned} q_{i,i} &= \ \ f \quad \text{in } \Omega \\ q_i n_i &= -\mathrm{h} \quad \text{on } \Gamma_h \\ q_i n_i &= -h \quad \text{on } \Gamma_g \end{aligned}$$

ii) Define the function spaces

$$\begin{aligned} \mathcal{V}^h \subset \mathcal{V} &= \{ w \,|\, w \in H^1(\Omega) \} \\ \mathcal{V}_0^h \subset \mathcal{V}_0 &= \{ w \,|\, w \in H^1(\Omega), w = 0 \ on \ \Gamma_g \} \\ \mathcal{T}^h \subset \mathcal{T} &= \{ h \,|\, h \in L_2(\Gamma_g) \} \end{aligned}$$

We can then define the Galerkin problem: (G): Given f, g and h as in Sec. 2.2, find $u^h = v^h + g^h$ where $v^h \in \mathcal{V}_0^h$ and $h^h \in \mathcal{T}^h$, such that for all $w^h \in \mathcal{V}^h$

$$a(w^h, v^h) - b(w^h, h^h) = (w^h, f) + (w^h, \mathrm{h})_{\Gamma_h} - a(w^h, g^h) \tag{1}$$

where

$$a(w^h, v^h) = \int_\Omega w^h_{,i} \kappa_{ij} v^h_{,j} \, d\Omega$$

$$b(w^h, h^h) = \int_{\Gamma_g} w^h h^h \, d\Gamma$$

$$(w^h, f) = \int_\Omega w^h f \, d\Omega$$

$$(w^h, \mathrm{h})_{\Gamma_h} = \int_{\Gamma_h} w^h \mathrm{h} \, d\Gamma$$

If w^h is selected in the subspace $\mathcal{V}_0^h \subset \mathcal{V}^h$, then (1) becomes:

$$a(w^h, v^h) = (w^h, f) + (w^h, \mathrm{h})_{\Gamma_h} - a(w^h, g^h) \tag{2}$$

This is the usual equation for v^h. (Note that $w^h \in \mathcal{V}_0^h$ implies $b(w^h, h^h) = 0$, since $w^h = 0$ on Γ_g.) If, on the other hand, we select $w^h \in \mathcal{V}^h - \mathcal{V}_0^h$, the complement of $\mathcal{V}_0^h$, then (1) becomes

$$b(w^h, h^h) = a(w^h, v^h) + a(w^h, g^h) - (w^h, f) - (w^h, \mathrm{h})_{\Gamma_h} \tag{3}$$

Assume the finite element approximations

$$w^h = \sum_{A \in \eta} N_A \, c_A = \sum_{A \in \eta - \eta_g} N_A \, c_A + \sum_{A \in \eta_g} N_A \, c_A$$

$$v^h = \sum_{B \in \eta - \eta_g} N_B \, d_B \quad \in \mathcal{V}_0^h$$

$$h^h = \sum_{B \in \eta_g} N_B \, h_B \quad \in \mathcal{V}^h - \mathcal{V}_0^h$$

Note that w^h consists of two parts: one in $\mathcal{V}_0^h$, and one in $\mathcal{V}^h - \mathcal{V}_0^h$. Due to these assumptions, the integrals in (3) only involve elements such that $\Gamma_h^e = \Gamma^e \bigcap \Gamma_h \neq \emptyset$. Since the c_A's defining each part are independent, two matrix problems result:

(i) $\quad \boldsymbol{K d} = \boldsymbol{F} \quad$ (usual problem)

(ii) $\quad \hat{\boldsymbol{K}} \boldsymbol{h} = \hat{\boldsymbol{F}} \quad$ ("post-processing") where $\boldsymbol{K}$, $\boldsymbol{d}$ and $\boldsymbol{F}$ are as defined in Sec. 2.4.

Defining the remaining terms requires the auxiliary identification array

$$\widehat{ID}(A) = \begin{cases} \text{R} & \text{if } A \in \eta_g \\ 0 & \text{if } A \in \eta - \eta_g \end{cases}$$

then

$$\hat{\boldsymbol{K}} = [\hat{K}_{RS}] \qquad \boldsymbol{h} = \{h_S\} \qquad \hat{\boldsymbol{F}} = \{\hat{F}_R\} \qquad 1 \leq R, S \leq n_{\text{beq}}$$

where n_{beq} is the number of boundary equations, i.e., the number of nodes in η_g, and

$$\hat{K}_{RS} = b(N_A, N_B) \qquad R = \widehat{ID}(A) \qquad S = \widehat{ID}(B)$$

$$\hat{F}_R = \sum_{B \in \eta - \eta_g} a(N_A, N_B) d_B + \sum_{B \in \eta_g} a(N_A, N_B) g_B^h - (N_A, f) - (N_A, \text{h})_{\Gamma_h}$$

Note: d_B is the solution to (i), which is the standard heat conduction matrix problem presented in Sec. 2.4.

iii) Specialize to the one-dimensional case examined in Exercise 7.

$$\mathcal{V}^h \subset \mathcal{V} = \{w \,|\, w \in H^1(0,1)\}$$
$$\mathcal{V}_0^h \subset \mathcal{V}_0 = \{w \,|\, w \in H^1(0,1), w(0) = 0\}$$

(G): Given f as in Exercise 7, find $u^h = v^h + g^h$ and h^h such that for all $w^h \in \mathcal{V}^h$

$$a(w^h, v^h) - w^h(1) h^h = (w^h, f) - a(w^h, g^h)$$

(M): Find $\boldsymbol{d}$ and h^h such that

(a) $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$ (This is the usual problem for $\boldsymbol{d}$, as in Exercise 7.)

(b) $$h^h = \sum_{B=1}^{n} a(N_{n+1}, N_B) d_B + a(N_{n+1}, N_{n+1})\, 1 - (N_{n+1}, f)$$

For one element $n = 1$, $N_1(x) = 1 - x$ and $N_2(x) = x$. Furthermore, from Exercise 7, $d_1 = 0$, implying:

$$h^h = \int_0^1 (1)^2 \, dx + p(p-1) \int_0^1 x x^{p-2} \, dx$$

$$= 1 + (p-1) x^p \Big|_{x=0}^{1} = p$$

Note that this equals the exact solution for h.

iv) For elasticity define the function spaces

$$\mathcal{S} = \{ \boldsymbol{u} \mid \boldsymbol{u} \in H^1(\Omega), u_i = g_i \quad \text{on } \Gamma_{g_i} \}$$

$$\mathcal{V} = \{ \boldsymbol{w} \mid \boldsymbol{w} \in H^1(\Omega) \}$$

$$\mathcal{T} = \{ \boldsymbol{h} \mid h_i \in L_2(\Gamma_{g_i}) \}$$

(W): Given $\boldsymbol{f}$, $\boldsymbol{g}$ and $\mathbf{h}$, whose components are defined as in Sec. 2.7, find $\boldsymbol{u} \in \mathcal{S}$ and $\boldsymbol{h} \in \mathcal{T}$ such that for all $\boldsymbol{w} \in \mathcal{V}$

$$a(\boldsymbol{w}, \boldsymbol{u}) - b(\boldsymbol{w}, \boldsymbol{h}) = (\boldsymbol{w}, \boldsymbol{f}) + (\boldsymbol{w}, \mathbf{h})_{\Gamma_h} \tag{4}$$

where $a(\boldsymbol{w}, \boldsymbol{u})$, $(\boldsymbol{w}, \boldsymbol{f})$ and $(\boldsymbol{w}, \mathbf{h})_{\Gamma_h}$ are as defined in Sec. 2.7 and

$$b(\boldsymbol{w}, \boldsymbol{h}) = \sum_{i=1}^{n_{sd}} \int_{\Gamma_{g_i}} w_i h_i \, d\Gamma$$

Integration by parts yields the expected Euler-Lagrange equations. Defining

$$\mathcal{V}_0 = \{ \boldsymbol{w} \mid \boldsymbol{w} \in \mathcal{V}, w_i = 0 \quad \text{on } \Gamma_{g_i} \}$$

enables us to state the two problems emanating from (4):

For all $\boldsymbol{w} \in \mathcal{V}_0$, $\qquad a(\boldsymbol{w}, \boldsymbol{u}) = (\boldsymbol{w}, \boldsymbol{f}) + (\boldsymbol{w}, \mathbf{h})_{\Gamma_h}$

For all $\boldsymbol{w} \in \mathcal{V} - \mathcal{V}_0$, $\qquad b(\boldsymbol{w}, \boldsymbol{h}) = a(\boldsymbol{w}, \boldsymbol{u}) - (\boldsymbol{w}, \boldsymbol{f}) - (\boldsymbol{w}, \mathbf{h})_{\Gamma_h}$

v) From part (iii)

$$a(w^h, u^h) - w^h(1)h^h = (w^h, f) \tag{5}$$

$$a(w^h, u) - w^h(1)h = (w^h, f) \tag{6}$$

where u and h are the exact solutions. Subtracting (6) from (5),

$$a(w^h, e) = w^h(1)\{h^h - h\}$$

where

$$e = u^h - u$$

Since the equations governing u^h remain unchanged in the new formulation, the proof of nodal exactness for u^h given in Sec. 1.10 remains valid. We are free to choose $w^h(x) = C$, a positive constant, so $w^h_{,x}(x) = 0$. (Recall, there are no boundary conditions built into $\mathcal{V}^h$ now, thus $w^h(x) = C$ is a legitimate choice.) Then $a(w^h, e) = 0$ and $w^h(1) \neq 0$ implies $h^h - h = 0$. So this method is exact for the one-dimensional model problem, confirming the observation of part (iii).

CHAPTER 3

3.2 Exercise 1, p. 114

$$N_a(\xi,\eta) \quad = \quad \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta) \tag{1}$$

where

$$x(\xi,\eta) \quad = \quad \sum_{a=1}^{4} N_a(\xi,\eta)x_a^e \tag{2}$$

and

$$x(\xi,\eta) \quad = \quad \alpha_0+\alpha_1\xi+\alpha_2\eta+\alpha_3\xi\eta \tag{3}$$

$$x_a^e \quad = \quad x(\xi_a,\eta_a)=\alpha_0+\alpha_1\xi_a+\alpha_2\eta_a+\alpha_3\xi_a\eta_a \tag{4}$$

Equation (3.2.12) is obtained using the above and Table 3.2.1, i.e.,

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad = \quad \begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix} \tag{5}$$

General idea: Solve for α_i's, and substitute into (3). Then collect coefficients of x_a^e's such that

$$x(\xi,\eta) \quad = \quad f_1(\xi,\eta)x_1^e+f_2(\xi,\eta)x_2^e+f_3(\xi,\eta)x_3^e+f_4(\xi,\eta)x_4^e \tag{6}$$

By (2) and (6)

$$N_a(\xi,\eta) \quad = \quad f_a(\xi,\eta) \tag{7}$$

Solving (5) by Gauss elimination

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \overset{(5)}{=} \quad \begin{Bmatrix} x_1^e \\ (x_2^e-x_1^e)/2 \\ (x_3^e-x_2^e)/2 \\ (x_1^e-x_2^e+x_3^e-x_4^e)/4 \end{Bmatrix} \tag{8}$$

$$
\begin{aligned}
\alpha_0 &= \tfrac{1}{4}(x_1^e + x_2^e + x_3^e + x_4^e) & \alpha_1 &= \tfrac{1}{4}(x_2^e - x_1^e + x_3^e - x_4^e) \\
\alpha_2 &= \tfrac{1}{4}(-x_1^e - x_2^e + x_3^e + x_4^e) & \alpha_3 &= \tfrac{1}{4}(x_1^e - x_2^e + x_3^e - x_4^e)
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
x(\xi,\eta) &\overset{(3,9)}{=} \frac{1}{4}(1-\xi-\eta+\xi\eta)x_1^e + \frac{1}{4}(1+\xi-\eta-\xi\eta)x_2^e \\
&\quad + \frac{1}{4}(1+\xi+\eta+\xi\eta)x_3^e + \frac{1}{4}(1-\xi+\eta-\xi\eta)x_4^e \\
&\overset{(2)}{=} \sum_{a=1}^{4} N_a(\xi,\eta)x_a^e
\end{aligned} \tag{10}
$$

3.4 Exercise 1, p. 123

From (3.3.3):

$$
j = \det\begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix} = (x_{,\xi}y_{,\eta} - y_{,\xi}x_{,\eta}) \tag{1}
$$

From (3.4.1):

$$
\boldsymbol{x} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_{a=1}^{3} N'_a \boldsymbol{x}_a^e \tag{2}
$$

where

$$
N'_a = \begin{cases} \frac{1}{4}(1+\xi_a\xi)(1-\eta) & a = 1,2 \\ \frac{1}{2}(1+\eta) & a = 3 \end{cases} \tag{3}
$$

a	$N_{a,\xi}$	$N_{a,\eta}$
1	$-(1-\eta)/4$	$-(1-\xi)/4$
2	$(1-\eta)/4$	$-(1+\xi)/4$
3	0	$1/2$

$$
\begin{aligned}
x_{,\xi} &\overset{(2,3)}{=} -\frac{1}{4}(1-\eta)x_1^e + \frac{1}{4}(1-\eta)x_2^e + (0)x_3^e \\
&= -\frac{1}{4}(1-\eta)x_1^e + \frac{1}{4}(1-\eta)x_2^e
\end{aligned} \tag{4}
$$

$$
x_{,\eta} \overset{(2,3)}{=} -\frac{1}{4}(1-\xi)x_1^e - \frac{1}{4}(1+\xi)x_2^e + \frac{1}{2}x_3^e \tag{5}
$$

$$
y_{,\xi} \overset{(2,3)}{=} -\frac{1}{4}(1-\eta)y_1^e + \frac{1}{4}(1-\eta)y_2^e \tag{6}
$$

$$
y_{,\eta} \overset{(2,3)}{=} -\frac{1}{4}(1-\xi)y_1^e - \frac{1}{4}(1+\xi)y_2^e + \frac{1}{2}y_3^e \tag{7}
$$

$$
\begin{aligned}
j \overset{(1,4-7)}{=} & [\{-\frac{1}{4}(1-\eta)(x_1^e - x_2^e)\}\{-\frac{1}{4}((1-\xi)y_1^e + (1+\xi)y_2^e - 2y_3^e)\}] \\
& -[\{-\frac{1}{4}(1-\eta)(y_1^e - y_2^e)\}\{-\frac{1}{4}((1-\xi)x_1^e + (1+\xi)x_2^e - 2x_3^e)\}]
\end{aligned} \tag{8}
$$

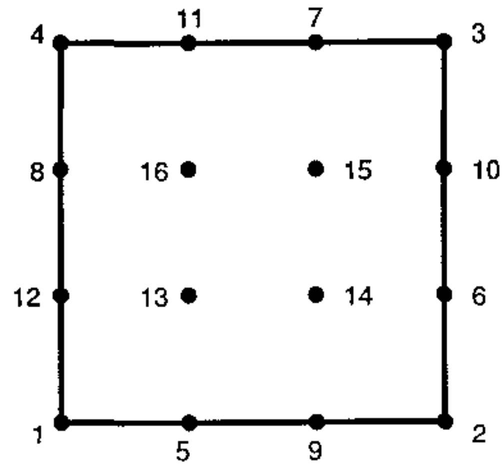
For $\xi = \eta = 0$, $x_2^e = x_3^e = y_1^e = y_3^e = 0$, $y_2^e = 1$,

$$
j \overset{(8)}{=} \frac{1}{16}x_1^e - [-\frac{1}{16}x_1^e] = \frac{1}{8}x_1^e \tag{9}
$$

3.6 Exercise 1, p. 128

$$
\begin{aligned}
N_1(\xi) &= \ell_1^3 = \frac{(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1)}{(\frac{-2}{3})(\frac{-4}{3})(-2)} = \frac{1}{16}(-9\xi^3 + 9\xi^2 + \xi - 1) \\
N_2(\xi) &= \ell_2^3 = \frac{(\xi+1)(\xi-\frac{1}{3})(\xi-1)}{(\frac{2}{3})(\frac{-2}{3})(\frac{-4}{3})} = \frac{1}{16}(27\xi^3 - 9\xi^2 - 27\xi + 9) \\
N_3(\xi) &= \ell_3^3 = \frac{(\xi+1)(\xi+\frac{1}{3})(\xi-1)}{(\frac{4}{3})(\frac{2}{3})(\frac{-2}{3})} = \frac{-1}{16}(27\xi^3 + 9\xi^2 - 27\xi - 9) \\
N_4(\xi) &= \ell_4^3 = \frac{(\xi+1)(\xi+\frac{1}{3})(\xi-\frac{1}{3})}{(2)(\frac{4}{3})(\frac{2}{3})} = \frac{1}{16}(9\xi^3 + 9\xi^2 - \xi - 1)
\end{aligned}
$$

3.6 Exercise 2, p. 130



$$N_a(\xi, \eta) \quad = \quad \ell_b^3(\xi)\, \ell_c^3(\eta)$$

where indices are related by

a	b	c		a	b	c
1	1	1		9	3	1
2	4	1		10	4	3
3	4	4		11	2	4
4	1	4		12	1	2
5	2	1		13	2	2
6	4	2		14	3	2
7	3	4		15	3	3
8	1	3		16	2	3

and the Lagrange polynomials are given in Exercise 1 of this section. The shape functions are

$$N_1(\xi,\eta) = \ell_1^3(\xi)\,\ell_1^3(\eta) = \frac{1}{256}(-9\xi^3 + 9\xi^2 + \xi - 1)(-9\eta^3 + 9\eta^2 + \eta - 1)$$

$$N_2(\xi,\eta) = \ell_4^3(\xi)\,\ell_1^3(\eta) = \frac{1}{256}(9\xi^3 + 9\xi^2 - \xi - 1)(-9\eta^3 + 9\eta^2 + \eta - 1)$$

$$N_3(\xi,\eta) = \ell_4^3(\xi)\,\ell_4^3(\eta) = \frac{1}{256}(9\xi^3 + 9\xi^2 - \xi - 1)(9\eta^3 + 9\eta^2 - \eta - 1)$$

$$N_4(\xi,\eta) = \ell_1^3(\xi)\,\ell_4^3(\eta) = \frac{1}{256}(-9\xi^3 + 9\xi^2 + \xi - 1)(9\eta^3 + 9\eta^2 - \eta - 1)$$

$$N_5(\xi,\eta) = \ell_2^3(\xi)\,\ell_1^3(\eta) = \frac{1}{256}(27\xi^3 - 9\xi^2 - 27\xi + 9)(-9\eta^3 + 9\eta^2 + \eta - 1)$$

$$N_6(\xi,\eta) = \ell_4^3(\xi)\,\ell_2^3(\eta) = \frac{1}{256}(9\xi^3 + 9\xi^2 - \xi - 1)(27\eta^3 - 9\eta^2 - 27\eta + 9)$$

$$N_7(\xi,\eta) = \ell_3^3(\xi)\,\ell_4^3(\eta) = \frac{-1}{256}(27\xi^3 + 9\xi^2 - 27\xi - 9)(9\eta^3 + 9\eta^2 - \eta - 1)$$

$$N_8(\xi,\eta) = \ell_1^3(\xi)\,\ell_3^3(\eta) = \frac{-1}{256}(-9\xi^3 + 9\xi^2 + \xi - 1)(27\eta^3 + 9\eta^2 - 27\eta - 9)$$

$$N_9(\xi,\eta) = \ell_3^3(\xi)\,\ell_1^3(\eta) = \frac{-1}{256}(27\xi^3 + 9\xi^2 - 27\xi - 9)(-9\eta^3 + 9\eta^2 + \eta - 1)$$

$$N_{10}(\xi,\eta) = \ell_4^3(\xi)\,\ell_3^3(\eta) = \frac{-1}{256}(9\xi^3 + 9\xi^2 - \xi - 1)(27\eta^3 + 9\eta^2 - 27\eta - 9)$$

$$N_{11}(\xi,\eta) = \ell_2^3(\xi)\,\ell_4^3(\eta) = \frac{1}{256}(27\xi^3 - 9\xi^2 - 27\xi + 9)(9\eta^3 + 9\eta^2 - \eta - 1)$$

$$N_{12}(\xi,\eta) = \ell_1^3(\xi)\,\ell_2^3(\eta) = \frac{1}{256}(-9\xi^3 + 9\xi^2 + \xi - 1)(27\eta^3 - 9\eta^2 - 27\eta + 9)$$

$$N_{13}(\xi,\eta) = \ell_2^3(\xi)\,\ell_2^3(\eta) = \frac{1}{256}(27\xi^3 - 9\xi^2 - 27\xi + 9)(27\eta^3 - 9\eta^2 - 27\eta + 9)$$

$$N_{14}(\xi,\eta) = \ell_3^3(\xi)\,\ell_2^3(\eta) = \frac{-1}{256}(27\xi^3 + 9\xi^2 - 27\xi - 9)(27\eta^3 - 9\eta^2 - 27\eta + 9)$$

$$N_{15}(\xi,\eta) = \ell_3^3(\xi)\,\ell_3^3(\eta) = \frac{1}{256}(27\xi^3 + 9\xi^2 - 27\xi - 9)(27\eta^3 + 9\eta^2 - 27\eta - 9)$$

$$N_{16}(\xi,\eta) = \ell_2^3(\xi)\,\ell_3^3(\eta) = \frac{-1}{256}(27\xi^3 - 9\xi^2 - 27\xi + 9)(27\eta^3 + 9\eta^2 - 27\eta - 9)$$

3.6 Exercise 3, p. 130

Using the node numbering in Figure 3.7.6

$$N_a(\xi,\eta,\zeta) \quad = \quad \ell_b^2(\xi)\,\ell_c^2(\eta)\,\ell_d^2(\zeta)$$

where indices are related by

a	b	c	d		a	b	c	d		a	b	c	d
1	1	1	1		10	3	2	1		19	3	3	2
2	3	1	1		11	2	3	1		20	1	3	2
3	3	3	1		12	1	2	1		21	2	2	1
4	1	3	1		13	2	1	3		22	2	2	3
5	1	1	3		14	3	2	3		23	2	1	2
6	3	1	3		15	2	3	3		24	2	3	2
7	3	3	3		16	1	2	3		25	1	2	2
8	1	3	3		17	1	1	2		26	3	2	2
9	2	1	1		18	3	1	2		27	2	2	2

Typical results:

Corner node

$$N_1(\xi,\eta,\zeta) \quad = \quad \frac{1}{8}\xi\eta\zeta(\xi-1)(\eta-1)(\zeta-1)$$

Mid-edge node

$$N_9(\xi,\eta,\zeta) \quad = \quad \frac{1}{4}\eta\zeta(1-\xi^2)(\eta-1)(\zeta-1)$$

Mid-face node

$$N_{21}(\xi,\eta,\zeta) \quad = \quad \frac{1}{2}\zeta(1-\xi^2)(1-\eta^2)(\zeta-1)$$

Center node

$$N_{27}(\xi,\eta,\zeta) \quad = \quad (1-\xi^2)(1-\eta^2)(1-\zeta^2)$$

3.6 Exercise 4, p. 130

$$N_a(\xi,\eta) = \ell_b^2(\xi)\,\ell_c^1(\eta)$$

a	b	c
1	1	1
2	3	1
3	3	2
4	1	2
5	2	1
6	2	2

e.g.,

$$N_1(\xi,\eta) = \frac{1}{4}\xi(\xi-1)(1-\eta)$$

$$N_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

3.7 Exercise 1, p. 135

$$N_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta) \qquad N_6(\xi,\eta) = \frac{1}{2}(1-\eta^2)(1+\xi)$$

$$N_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta) \qquad N_8(\xi,\eta) = \frac{1}{2}(1-\eta^2)(1-\xi)$$

$$\begin{aligned} N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}\{1-\eta-\xi^2+\xi^2\eta+1-\xi-\eta^2+\xi\eta^2\} \\ &= \frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta) \end{aligned}$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta)$$

$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)(-1-\xi+\eta)$$

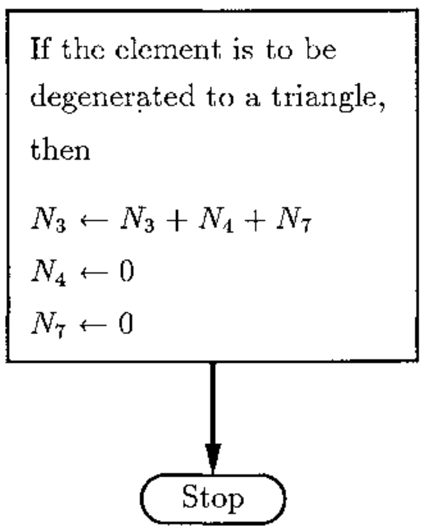
3.7 Exercise 2, p. 135

$$N'_a = N_a \qquad a = 1,2,5,6,8$$

$$N'_3 = N_3 + N_4 + N_7 = \frac{1}{2}\eta(1+\eta)$$

3.7 Exercise 3, p. 135

Replace "Stop" with the following:



3.7 Exercise 4, p. 136

With reference to the node numbering of Figure 3.7.6:

1. Vertex nodes:

   Define ξ_a, η_a, ζ_a $(a = 1, 2, \cdots, 8)$ as in Table 3.5.1, p. 124.

$$N_a = \frac{1}{8}(1+\xi_a\xi)(1+\eta_a\eta)(1+\zeta_a\zeta)$$
$$a \leftarrow 1, 2, \cdots, 8$$

2. Center Node:

$$N_{27} = 0$$

If node 27 is present, then

$$N_{27} \quad = \quad (1-\xi^2)(1-\eta^2)(1-\zeta^2)$$

3. Mid-face nodes:

$$N_{21} = N_{22} = \ldots = N_{26} = 0$$

If node 21 is present, then

$$N_{21} \quad = \quad \frac{1}{2}(1-\xi^2)(1-\eta^2)(1-\zeta) - \frac{1}{2}N_{27}$$

If node 22 is present, then

$$N_{22} \quad = \quad \frac{1}{2}(1-\xi^2)(1-\eta^2)(1+\zeta) - \frac{1}{2}N_{27}$$

If node 23 is present, then

$$N_{23} \quad = \quad \frac{1}{2}(1-\xi^2)(1-\eta)(1-\zeta^2) - \frac{1}{2}N_{27}$$

If node 24 is present, then

$$N_{24} \quad = \quad \frac{1}{2}(1-\xi^2)(1+\eta)(1-\zeta^2) - \frac{1}{2}N_{27}$$

If node 25 is present, then

$$N_{25} \quad = \quad \frac{1}{2}(1-\xi)(1-\eta^2)(1-\zeta^2) - \frac{1}{2}N_{27}$$

If node 26 is present, then

$$N_{26} \quad = \quad \frac{1}{2}(1+\xi)(1-\eta^2)(1-\zeta^2) - \frac{1}{2}N_{27}$$

4. Mid-edge nodes:

$$N_9 = N_{10} = N_{11} = \ldots = N_{20} = 0$$

If node 9 is present, then

$$N_9 \quad = \quad \frac{1}{4}(1-\xi^2)(1-\eta)(1-\zeta) - \frac{1}{2}(N_{21}+N_{23}) - \frac{1}{4}N_{27}$$

If node 10 is present, then

$$N_{10} = \frac{1}{4}(1+\xi)(1-\eta^2)(1-\zeta) - \frac{1}{2}(N_{21}+N_{26}) - \frac{1}{4}N_{27}$$

If node 11 is present, then

$$N_{11} = \frac{1}{4}(1-\xi^2)(1+\eta)(1-\zeta) - \frac{1}{2}(N_{21}+N_{24}) - \frac{1}{4}N_{27}$$

If node 12 is present, then

$$N_{12} = \frac{1}{4}(1-\xi)(1-\eta^2)(1-\zeta) - \frac{1}{2}(N_{21}+N_{25}) - \frac{1}{4}N_{27}$$

If node 13 is present, then

$$N_{13} = \frac{1}{4}(1-\xi^2)(1-\eta)(1+\zeta) - \frac{1}{2}(N_{22}+N_{23}) - \frac{1}{4}N_{27}$$

If node 14 is present, then

$$N_{14} = \frac{1}{4}(1+\xi)(1-\eta^2)(1+\zeta) - \frac{1}{2}(N_{22}+N_{26}) - \frac{1}{4}N_{27}$$

If node 15 is present, then

$$N_{15} = \frac{1}{4}(1-\xi^2)(1+\eta)(1+\zeta) - \frac{1}{2}(N_{22}+N_{24}) - \frac{1}{4}N_{27}$$

If node 16 is present, then

$$N_{16} = \frac{1}{4}(1-\xi)(1-\eta^2)(1+\zeta) - \frac{1}{2}(N_{22}+N_{25}) - \frac{1}{4}N_{27}$$

If node 17 is present, then

$$N_{17} = \frac{1}{4}(1-\xi)(1-\eta)(1-\zeta^2) - \frac{1}{2}(N_{23}+N_{25}) - \frac{1}{4}N_{27}$$

If node 18 is present, then

$$N_{18} = \frac{1}{4}(1+\xi)(1-\eta)(1-\zeta^2) - \frac{1}{2}(N_{23}+N_{26}) - \frac{1}{4}N_{27}$$

If node 19 is present, then

$$N_{19} = \frac{1}{4}(1+\xi)(1+\eta)(1-\zeta^2) - \frac{1}{2}(N_{24}+N_{26}) - \frac{1}{4}N_{27}$$

If node 20 is present, then

$$N_{20} = \frac{1}{4}(1-\xi)(1+\eta)(1-\zeta^2) - \frac{1}{2}(N_{24}+N_{25}) - \frac{1}{4}N_{27}$$

5. Adjust vertex nodes:

$$N_1 \leftarrow N_1 - \frac{1}{2}(N_9+N_{12}+N_{17}) - \frac{1}{4}(N_{21}+N_{23}+N_{25}) - \frac{1}{8}N_{27}$$

$$N_2 \leftarrow N_2 - \frac{1}{2}(N_9+N_{10}+N_{18}) - \frac{1}{4}(N_{21}+N_{23}+N_{26}) - \frac{1}{8}N_{27}$$

$$N_3 \leftarrow N_3 - \frac{1}{2}(N_{10}+N_{11}+N_{19}) - \frac{1}{4}(N_{21}+N_{24}+N_{26}) - \frac{1}{8}N_{27}$$

$$N_4 \leftarrow N_4 - \frac{1}{2}(N_{11}+N_{12}+N_{20}) - \frac{1}{4}(N_{21}+N_{24}+N_{25}) - \frac{1}{8}N_{27}$$

$$N_5 \leftarrow N_5 - \frac{1}{2}(N_{13}+N_{16}+N_{17}) - \frac{1}{4}(N_{22}+N_{23}+N_{25}) - \frac{1}{8}N_{27}$$

$$N_6 \leftarrow N_6 - \frac{1}{2}(N_{13}+N_{14}+N_{18}) - \frac{1}{4}(N_{22}+N_{23}+N_{26}) - \frac{1}{8}N_{27}$$

$$N_7 \leftarrow N_7 - \frac{1}{2}(N_{14}+N_{15}+N_{19}) - \frac{1}{4}(N_{22}+N_{24}+N_{26}) - \frac{1}{8}N_{27}$$

$$N_8 \leftarrow N_8 - \frac{1}{2}(N_{15}+N_{16}+N_{20}) - \frac{1}{4}(N_{22}+N_{24}+N_{25}) - \frac{1}{8}N_{27}$$

3.7 Exercise 5, p. 137

To Exercise 4 add the following step:

6. If the element is to be degenerated to a wedge, then

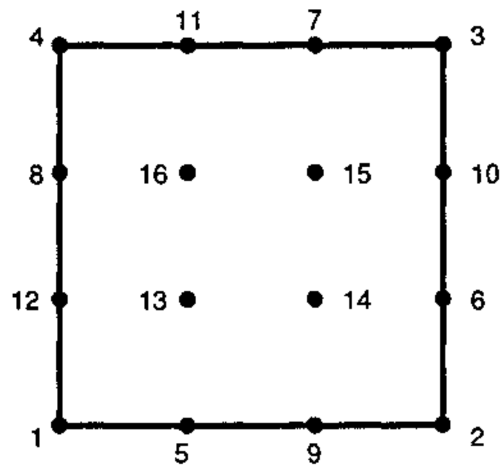
$$N_3 \leftarrow N_3 + N_4 + N_{11}$$

$$N_{19} \leftarrow N_{19} + N_{20} + N_{24}$$

$$N_7 \leftarrow N_7 + N_8 + N_{15}$$

$$N_4 = N_8 = N_{11} = N_{15} = N_{20} = N_{24} = 0$$

3.7 Exercise 6, p. 137



1. Vertex nodes:

$$
\begin{aligned}
N_a &= \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta) \\
a &\leftarrow 1,2,3,4
\end{aligned}
$$

2. Interior nodes (in this implementation, all interior nodes are either present or absent):

$$
\begin{aligned}
N_a &= 0 \\
a &\leftarrow 13,14,15,16
\end{aligned}
$$

If internal nodes are present, then (see Exercise 2, Sec. 3.6):

$$
\begin{aligned}
N_{13}(\xi,\eta) &= \frac{1}{256}(27\xi^3-9\xi^2-27\xi+9)(27\eta^3-9\eta^2-27\eta+9) \\
N_{14}(\xi,\eta) &= \frac{-1}{256}(27\xi^3+9\xi^2-27\xi-9)(27\eta^3-9\eta^2-27\eta+9) \\
N_{15}(\xi,\eta) &= \frac{1}{256}(27\xi^3+9\xi^2-27\xi-9)(27\eta^3+9\eta^2-27\eta-9) \\
N_{16}(\xi,\eta) &= \frac{-1}{256}(27\xi^3-9\xi^2-27\xi+9)(27\eta^3+9\eta^2-27\eta-9)
\end{aligned}
$$

3. Side nodes (in this implementation, side nodes are specified in pairs, e.g., nodes 5 and 9 are either both present or both absent):

$$
\begin{aligned}
N_a &= 0 \\
a &\leftarrow 5, 6, \cdots, 12
\end{aligned}
$$

If nodes 5 and 9 are present, then

$$
\begin{aligned}
N_5 &= \frac{27}{32}(1-\xi^2)(\frac{1}{3}-\xi)(1-\eta) - \frac{2}{3}N_{13} - \frac{1}{3}N_{16} \\
N_9 &= \frac{27}{32}(1-\xi^2)(\frac{1}{3}+\xi)(1-\eta) - \frac{2}{3}N_{14} - \frac{1}{3}N_{15}
\end{aligned}
$$

If nodes 6 and 10 are present, then

$$
\begin{aligned}
N_6 &= \frac{27}{32}(1+\xi)(1-\eta^2)(\frac{1}{3}-\eta) - \frac{2}{3}N_{14} - \frac{1}{3}N_{13} \\
N_{10} &= \frac{27}{32}(1+\xi)(1-\eta^2)(\frac{1}{3}+\eta) - \frac{2}{3}N_{15} - \frac{1}{3}N_{16}
\end{aligned}
$$

If nodes 7 and 11 are present, then

$$
\begin{aligned}
N_7 &= \frac{27}{32}(1-\xi^2)(\frac{1}{3}+\xi)(1+\eta) - \frac{2}{3}N_{15} - \frac{1}{3}N_{14} \\
N_{11} &= \frac{27}{32}(1-\xi^2)(\frac{1}{3}-\xi)(1+\eta) - \frac{2}{3}N_{16} - \frac{1}{3}N_{13}
\end{aligned}
$$

If nodes 8 and 12 are present, then

$$
\begin{aligned}
N_8 &= \frac{27}{32}(1-\xi)(1-\eta^2)(\frac{1}{3}+\eta) - \frac{2}{3}N_{16} - \frac{1}{3}N_{15} \\
N_{12} &= \frac{27}{32}(1-\xi)(1-\eta^2)(\frac{1}{3}-\eta) - \frac{2}{3}N_{13} - \frac{1}{3}N_{14}
\end{aligned}
$$

4. Adjust vertex nodes:

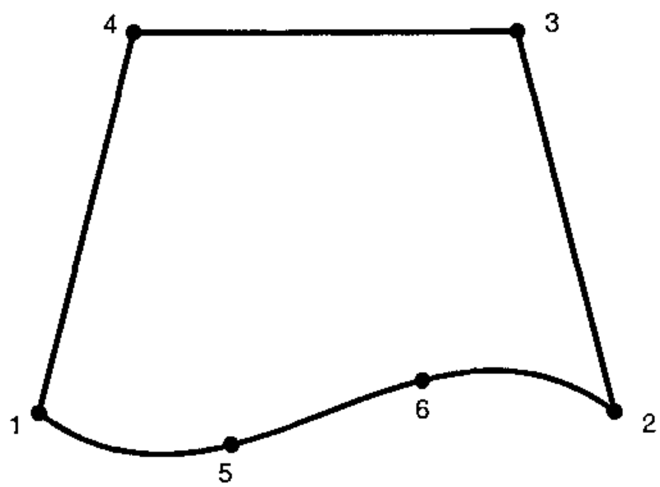
$$N_1 \leftarrow N_1 - \frac{2}{3}(N_5 + N_{12}) - \frac{1}{3}(N_8 + N_9) - \frac{4}{9}N_{13} - \frac{2}{9}(N_{14} + N_{16}) - \frac{1}{9}N_{15}$$

$$N_2 \leftarrow N_2 - \frac{2}{3}(N_6 + N_9) - \frac{1}{3}(N_5 + N_{10}) - \frac{4}{9}N_{14} - \frac{2}{9}(N_{13} + N_{15}) - \frac{1}{9}N_{16}$$

$$N_3 \leftarrow N_3 - \frac{2}{3}(N_7 + N_{10}) - \frac{1}{3}(N_6 + N_{11}) - \frac{4}{9}N_{15} - \frac{2}{9}(N_{14} + N_{16}) - \frac{1}{9}N_{13}$$

$$N_4 \leftarrow N_4 - \frac{2}{3}(N_8 + N_{11}) - \frac{1}{3}(N_7 + N_{12}) - \frac{4}{9}N_{16} - \frac{2}{9}(N_{13} + N_{15}) - \frac{1}{9}N_{14}$$

3.7 Exercise 7, p. 137



$$N_a = \begin{cases} \ell_b^3(\xi)\,\ell_c^1(\eta) & a = 1,2,5,6 \\ \ell_b^1(\xi)\,\ell_c^1(\eta) & a = 3,4 \end{cases}$$

a	b	c
1	1	1
2	4	1
3	2	2
4	1	2
5	2	1
6	3	1

$$
\begin{aligned}
N_1 &= \tfrac{1}{32}(1-9\xi^2)(\xi-1)(1-\eta) & N_2 &= \tfrac{1}{32}(9\xi^2-1)(\xi+1)(1-\eta) \\
N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) & N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \\
N_5 &= \tfrac{1}{32}(\xi^2-1)(27\xi-9)(1-\eta) & N_6 &= \tfrac{1}{32}(1-\xi^2)(27\xi+9)(1-\eta)
\end{aligned}
$$

3.8 Exercise 1, p. 142

This exercise is straightforward, so the solution is omitted.

3.8 Exercise 2, p. 143

Gauss quadrature, $n_{\text{int}} = 3$

Assume

$$\tilde{\xi}_1 = -\tilde{\xi}_3 \quad \tilde{\xi}_2 = 0 \quad W_1 = W_3$$

Consider a general fifth-order polynomial

$$g(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \alpha_4\xi^4 + \alpha_5\xi^5$$

The exact integral of $g(\xi)$ is

$$\int_{-1}^{1}(\alpha_0+\alpha_1\xi+\alpha_2\xi^2+\alpha_3\xi^3+\alpha_4\xi^4+\alpha_5\xi^5)\,d\xi = 2\alpha_0+\frac{2}{3}\alpha_2+\frac{2}{5}\alpha_4 \qquad \text{(odd functions drop out)}$$

This is to be equal to

$$W_1(g(\tilde{\xi}_1)+g(\tilde{\xi}_3)) + W_2 g(0) = W_1(2\alpha_0 + 2\alpha_2\tilde{\xi}_1^2 + 2\alpha_4\tilde{\xi}_1^4) + W_2\alpha_0$$

Since α_0, α_2 and α_4 are arbitrary,

$$
\begin{aligned}
2W_1 + W_2 &= 2 \\
2W_1\tilde{\xi}_1^2 &= \frac{2}{3} \\
2W_1\tilde{\xi}_1^4 &= \frac{2}{5}
\end{aligned}
$$

Solving the last two equations for $\tilde{\xi}_1^2$ yields

$$\frac{W_1 \tilde{\xi}_1^4}{W_1 \tilde{\xi}_1^2} = \frac{1/5}{1/3} \Rightarrow \tilde{\xi}_1^2 = \frac{3}{5}$$

So,

$$\tilde{\xi}_1 = -\sqrt{\frac{3}{5}} = -\tilde{\xi}_3$$

Then $W_1 = \frac{5}{9}$, and $W_2 = \frac{8}{9}$.

3.8 Exercise 3, p. 145

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\, d\xi\, d\eta \approx \sum_{l=1}^{n_{\text{int}}} g(\tilde{\xi}_l, \tilde{\eta}_l) W_l$$

l	$l^{(1)}$	$l^{(2)}$	$\tilde{\xi}_l$	$\tilde{\eta}_l$	W_l
1	1	1	$-\alpha$	$-\alpha$	25/81
2	3	1	α	$-\alpha$	25/81
3	3	3	α	α	25/81
4	1	3	$-\alpha$	α	25/81
5	2	1	0	$-\alpha$	40/81
6	3	2	α	0	40/81
7	2	3	0	α	40/81
8	1	2	$-\alpha$	0	40/81
9	2	2	0	0	64/81

where $\alpha = \sqrt{\frac{3}{5}}$

3.8 Exercise 4, p. 145

One-point Gaussian rule:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g\, d\xi\, d\eta\, d\zeta \approx 8\, g(0,0,0)$$

Consequently $W_1 = 8$, $\tilde{\xi}_1 = \tilde{\eta}_1 = \tilde{\zeta}_1 = 0$.

$2 \times 2 \times 2$ Gaussian rule:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g\, d\xi\, d\eta\, d\zeta \quad \approx \quad \sum_{l=1}^{8} g(\tilde{\xi}_l, \tilde{\eta}_l, \tilde{\zeta}_l)\, W_l$$

l	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$\tilde{\xi}_l$	$\tilde{\eta}_l$	$\tilde{\zeta}_l$	W_l
1	1	1	1	$-\alpha$	$-\alpha$	$-\alpha$	1
2	2	1	1	α	$-\alpha$	$-\alpha$	1
3	2	2	1	α	α	$-\alpha$	1
4	1	2	1	$-\alpha$	α	$-\alpha$	1
5	1	1	2	$-\alpha$	$-\alpha$	α	1
6	2	1	2	α	$-\alpha$	α	1
7	2	2	2	α	α	α	1
8	1	2	2	$-\alpha$	α	α	1

where $\alpha = \frac{1}{\sqrt{3}}$

3.8 Exercise 5, p. 146

It is clear that the constants and linear functions will be integrated exactly so, checking the quadratic terms

$$\int_{-1}^{1}\int_{-1}^{1} (\alpha_0\xi^2 + \alpha_1\xi\eta + \alpha_2\eta^2)\, d\xi\, d\eta \quad = \quad \frac{4}{3}(\alpha_0 + \alpha_2)$$

The quadrature formula results in $2a^2(\alpha_0 + \alpha_2)$ and thus

$$2a^2 \quad = \quad \frac{4}{3} \quad \Rightarrow \quad a \quad = \quad \sqrt{\frac{2}{3}}$$

The third-order terms will also be integrated exactly by this formula and hence it is fourth-order accurate.

3.10 Exercise 1, p. 152

Assume $\boldsymbol{B}$ and $\boldsymbol{D}$ are already given. Let A and M denote the number of addition and multiplication operations, respectively. For each integration point, the following operations are performed:

Setup of $\tilde{\boldsymbol{D}}$: $\quad \tilde{\boldsymbol{D}} = j(\tilde{\boldsymbol{\xi}}_l) * W_l * \boldsymbol{D}$

$\boldsymbol{D}$ is assumed symmetric; its dimensions are NSTR × NSTR. Let const $= j(\tilde{\boldsymbol{\xi}}_l) * W_l$ for integration point l. The necessary multiplications are performed by employing a do-loop structure accounting for symmetry (see subroutine SETUPD in DLEARN):

$$\begin{aligned}
&\text{For } J = 1, \text{NSTR} \\
&\quad \text{For } I = 1, J \\
&\qquad \tilde{D}_{IJ} = \text{const} * D_{IJ} \\
&\qquad \tilde{D}_{JI} = \tilde{D}_{IJ}
\end{aligned}$$

Note that the inner loop is executed $1 + 2 + \cdots + \text{NSTR} = \frac{1}{2}\text{NSTR} * (\text{NSTR} + 1)$ times. Thus to set up $\tilde{\boldsymbol{D}}$ requires at each integration point:

$$\begin{aligned}
A &= 0 \\
M &= 1 + \frac{1}{2}\text{NSTR} * (\text{NSTR} + 1)
\end{aligned}$$

$\tilde{\boldsymbol{D}} * \boldsymbol{B}$ product:

The dimension of $\boldsymbol{B}$ is NSTR × NEE. (Recall NSTR $\leq$ NROWB, the number of rows in $\boldsymbol{B}$ set aside for storage.) The numbers of operations necessary to perform the matrix product at each integration point are:

$$\begin{aligned}
A &= \text{NEE} * \text{NSTR} ** 2 \\
M &= \text{NEE} * \text{NSTR} ** 2
\end{aligned}$$

(See the calling statement to subroutine MULTAB in QDCK and subroutine MULTAB itself.)

$\boldsymbol{B}^T * (\tilde{\boldsymbol{D}}\boldsymbol{B})$ product and accumulation in $\boldsymbol{k}^e$:

For these operations, see subroutine BTDB and the calling statement to BTDB in subroutine QDCK. The innermost loop of BTDB is called $\frac{1}{2}\text{NEE} * (\text{NEE} + 1)$ times. The dot product executed in function COLDOT requires NSTR additions and NSTR multiplications per call. The total numbers of operations per integration point are thus

$$A = \frac{1}{2}\text{NEE} * (\text{NEE} + 1) * (\text{NSTR} + 1)$$

$$M = \frac{1}{2}\text{NEE} * (\text{NEE} + 1) * \text{NSTR}$$

The extra addition accounts for the accumulation into $\boldsymbol{k}^e$ (i.e., ELSTIF).

Summary:

The operation counts must be multiplied by the number of integration points, NINT, resulting in:

$$A = \text{NINT} * \{\text{NEE} * \text{NSTR} **2 + \frac{1}{2}\text{NEE} * (\text{NEE} + 1) * (\text{NSTR} + 1)\}$$

$$M = \text{NINT} * \{1 + \frac{1}{2}\text{NSTR} * (\text{NSTR} + 1) + \text{NEE} * \text{NSTR} **2 + \frac{1}{2}\text{NEE} * (\text{NEE} + 1) * \text{NSTR}\}$$

3.10 Exercise 2, p. 153

The operation counts for a two-dimensional, isotropic, plane-stress, four-node element are performed with reference to the arrays defined by (3.10.6) and (3.10.7). For each integration point, the following operations are performed:

Setup of $\tilde{\boldsymbol{D}}$: $\quad \tilde{\boldsymbol{D}} = j(\tilde{\boldsymbol{\xi}}_l) * W_l * \boldsymbol{D}$

One multiplication is needed to determine const $= j(\tilde{\boldsymbol{\xi}}_l) * W_l$. Note that in the isotropic case $D_{11} = D_{22}$ (see (2.7.34) and (2.7.35)) and therefore only three additional multiplications are needed to calculate const $* \boldsymbol{D}$ (see (3.10.7)). Thus

$$A = 0$$

$$M = 4$$

$\tilde{\boldsymbol{D}} * \boldsymbol{B}$ **product:**

From the formula given in the box on p. 153 it is clear that six multiplications are required to form $\tilde{\boldsymbol{D}} * \boldsymbol{B}_b$, $b = 1, 2, \cdots, n_{en}$. Therefore

$$\begin{aligned} A &= 0 \\ M &= 6n_{en} \end{aligned}$$

$\boldsymbol{B}^T * (\tilde{\boldsymbol{D}}\boldsymbol{B})$ **product and accumulation in** $\boldsymbol{k}^e$**:**

The formula in the innermost loop in the box on p. 153 requires eight additions and eight multiplications to calculate $\boldsymbol{B}_a^T * (\tilde{\boldsymbol{D}}\boldsymbol{B}_b)$. The innermost loop accounts for symmetry and is seen to be executed only $\frac{1}{2}n_{en}(n_{en}+1)$ times. Therefore

$$\begin{aligned} A &= 4n_{en}(n_{en}+1) \\ M &= 4n_{en}(n_{en}+1) \end{aligned}$$

Summary:

$$\begin{aligned} A &= 4n_{en}(n_{en}+1)n_{\text{int}} \\ M &= (4 + 6n_{en} + 4n_{en}(n_{en}+1))n_{\text{int}} \end{aligned}$$

For a four-node element $n_{en} = 4$. Therefore

$$\begin{aligned} A &= 80n_{\text{int}} \\ M &= 108n_{\text{int}} \end{aligned}$$

Comparison with Implementation 1 (See results of previous exercise.):

For the element under consideration NEE = 8 and NSTR = 3. Therefore

$$\begin{aligned} A &= 216n_{\text{int}} \\ M &= 187n_{\text{int}} \end{aligned}$$

Thus we see that in this case, Implementation 2 is roughly twice as efficient as Implementation 1.

3.10 Exercise 3, p. 156

Note first of all that for the plane-stress option λ in the box on p. 155 needs to be replaced by $\bar{\lambda}$ as defined in (2.7.35). This observation will not effect our operation counts since it is common to all implementations. The operation counts per integration point are given as follows:

First nested loop construction:

There is one multiplication to form const $= j(\tilde{\boldsymbol{\xi}}_l)W_l$. The multiplication temp $=$ const $\cdot N_{b,j}(\tilde{\boldsymbol{\xi}}_l)$ is executed $n_{en}n_{ed}$ times. The addition and multiplication $k^e_{iajb} +$ temp $\cdot N_{a,i}(\tilde{\boldsymbol{\xi}}_l)$ is executed $\frac{1}{2}n_{en}n_{ed}(n_{en}n_{ed}+1)$ times. Therefore

$$A = \frac{1}{2}n_{en}n_{ed}(n_{en}n_{ed}+1)$$

$$M = 1 + n_{en}n_{ed} + \frac{1}{2}n_{en}n_{ed}(n_{en}n_{ed}+1)$$

Second nested loop construction:

The calculation temp $= \sum_{k=1}^{n_{ed}} k^e_{kakb}$ involves $n_{ed}-1$ additions. This calculation is executed $\frac{1}{2}n_{en}(n_{en}+1)$ times, resulting in a total number of additions equal to $\frac{1}{2}n_{en}(n_{en}+1)(n_{ed}-1)$. So

$$A = \frac{1}{2}n_{en}(n_{en}+1)(n_{ed}-1)$$

$$M = 0$$

The if-then-else-endif construction is executed $\frac{1}{2}n_{en}(n_{en}+1)\cdot\frac{1}{2}n_{ed}(n_{ed}+1)$ times. However, the test "$i=j$" will only be passed $\frac{1}{2}n_{en}(n_{en}+1)n_{ed}$ times. Thus the two multiplications and one addition, $c_1k^e_{iaib}+c_2$temp, will be executed $\frac{1}{2}n_{en}(n_{en}+1)n_{ed}$ times, resulting in

$$A = \frac{1}{2}n_{en}(n_{en}+1)n_{ed}$$

$$M = n_{en}(n_{en}+1)n_{ed}$$

The test "$i=j$" will be failed the remaining $\frac{1}{2}n_{en}(n_{en}+1)\{\frac{1}{2}n_{ed}(n_{ed}+1)-n_{ed}\} = \frac{1}{2}n_{en}(n_{en}+1)\cdot\frac{1}{2}n_{ed}(n_{ed}-1)$ times. Now we need to consider the test "$a=b$." This will be passed $n_{en} \cdot \frac{1}{2} n_{ed}(n_{ed}-1)$ times, and failed the remaining $\{\frac{1}{2}n_{en}(n_{en}+1)-n_{en}\}\cdot\frac{1}{2}n_{ed}(n_{ed}-1) = \frac{1}{2}n_{en}(n_{en}-1)\cdot\frac{1}{2}n_{ed}(n_{ed}-1)$ times. Therefore the single multiplication $c_1 k^e_{iaja}$ will be executed $n_{en}\cdot\frac{1}{2}n_{ed}(n_{ed}-1)$ times, whereas the two multiplications and one addition, $c_3 k^e_{iajb} + c_2 k^e_{jaib}$, will be performed $\frac{1}{2}n_{en}(n_{en}-1)\cdot\frac{1}{2}n_{ed}(n_{ed}-1)$ times. Consequently, for the if-then-else-endif construction, we have

$$
\begin{aligned}
A &= \frac{1}{2}n_{en}(n_{en}+1)n_{ed} + \frac{1}{2}n_{en}(n_{en}-1)\cdot\frac{1}{2}n_{ed}(n_{ed}-1) \\
M &= n_{en}(n_{en}+1)n_{ed} + n_{en}\cdot\frac{1}{2}n_{ed}(n_{ed}-1) + \frac{1}{2}n_{en}(n_{en}-1)n_{ed}(n_{ed}-1)
\end{aligned}
$$

Summary:

$$
\begin{aligned}
A &= \{\frac{1}{2}n_{en}n_{ed}(n_{en}n_{ed}+1) + \frac{1}{2}n_{en}(n_{en}+1)(2n_{ed}-1) + \frac{1}{2}n_{en}(n_{en}-1)\cdot\frac{1}{2}n_{ed}(n_{ed}-1)\}n_{\text{int}} \\
M &= \{1 + n_{en}n_{ed} + \frac{1}{2}n_{en}n_{ed}(n_{en}n_{ed}+1) + n_{en}(n_{en}+1)n_{ed} \\
&\quad + n_{en}\cdot\frac{1}{2}n_{ed}(n_{ed}-1) + \frac{1}{2}n_{en}(n_{en}-1)n_{ed}(n_{ed}-1)\}n_{\text{int}}
\end{aligned}
$$

Comparison with the previous exercise:

For the element under consideration $n_{en} = 4$ and $n_{ed} = 2$. Therefore for Implementation 3,

$$
\begin{aligned}
A &= 72 n_{\text{int}} \\
M &= 101 n_{\text{int}}
\end{aligned}
$$

Note that the numbers of additions and multiplications for Implementation 3 are only slightly less than those for Implementation 2. Implementation 3 is superior on this basis but only slightly so. In three dimensions (see the next exercise) the superiority becomes considerable.

3.10 Exercise 4, p. 156

The operation counts for Implementations 1 and 3 are valid for the three-dimensional, isotropic case. We need to generalize the operation counts for Implementation 2, which was restricted to the two-dimensional, isotropic, plane-stress case.

Setup of $\tilde{\boldsymbol{D}}$: $\quad \tilde{\boldsymbol{D}} = j(\tilde{\boldsymbol{\xi}}_l) * W_l * \boldsymbol{D}$

In Exercise 5 of Sec. 2.7, the matrix $\boldsymbol{D}$ was constructed for the three-dimensional case:

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 & 0 \\ & & D_{33} & 0 & 0 & 0 \\ & & & D_{44} & 0 & 0 \\ & \text{Symmetric} & & & D_{55} & 0 \\ & & & & & D_{66} \end{bmatrix}$$

where

$$\left.\begin{array}{ccccccc} D_{11} & = & D_{22} & = & D_{33} & = & \lambda + 2\mu \\ D_{12} & = & D_{23} & = & D_{13} & = & \lambda \\ D_{44} & = & D_{55} & = & D_{66} & = & \mu \end{array}\right\} \tag{1}$$

Note, there are only three distinct coefficients. Thus, the situation is similar to the two-dimensional case (see Exercise 2) in that only four multiplications are required to construct $\tilde{\boldsymbol{D}}$. Thus

$$A = 0$$

$$M = 4$$

$\tilde{\boldsymbol{D}} * \boldsymbol{B}$ product:

The formula for $\tilde{\boldsymbol{D}} * \boldsymbol{B}_b$ in the box on page 153 needs to be generalized to the three-dimensional case. We recall also the definition of the matrix $\boldsymbol{B}_b$ in three dimensions given by (2.9.4). Let $B_i = N_{b,i}$, $i = 1, 2, 3$. Then

$$\tilde{\boldsymbol{D}}\boldsymbol{B}_b = \begin{bmatrix} \tilde{D}B_{11} & \tilde{D}B_{12} & \tilde{D}B_{13} \\ \tilde{D}B_{21} & \tilde{D}B_{22} & \tilde{D}B_{23} \\ \tilde{D}B_{31} & \tilde{D}B_{32} & \tilde{D}B_{33} \\ \tilde{D}B_{41} & \tilde{D}B_{42} & \tilde{D}B_{43} \\ \tilde{D}B_{51} & \tilde{D}B_{52} & \tilde{D}B_{53} \\ \tilde{D}B_{61} & \tilde{D}B_{62} & \tilde{D}B_{63} \end{bmatrix}$$

$$\stackrel{\text{def.}}{=} \begin{bmatrix} \tilde{D}_{11}B_1 & \tilde{D}_{12}B_2 & \tilde{D}_{13}B_3 \\ \tilde{D}_{12}B_1 & \tilde{D}_{22}B_2 & \tilde{D}_{23}B_3 \\ \tilde{D}_{13}B_1 & \tilde{D}_{23}B_2 & \tilde{D}_{33}B_3 \\ 0 & \tilde{D}_{44}B_3 & \tilde{D}_{44}B_2 \\ \tilde{D}_{55}B_3 & 0 & \tilde{D}_{55}B_1 \\ \tilde{D}_{66}B_2 & \tilde{D}_{66}B_1 & 0 \end{bmatrix}$$

Note that, due to the fact,

$$\begin{aligned} \tilde{D}_{11} &= \tilde{D}_{22} = \tilde{D}_{33} \\ \tilde{D}_{12} &= \tilde{D}_{23} = \tilde{D}_{13} \\ \tilde{D}_{44} &= \tilde{D}_{55} = \tilde{D}_{66} \end{aligned}$$

it follows that

$$\left.\begin{aligned} \tilde{D}B_{21} &= \tilde{D}B_{31} \\ \tilde{D}B_{12} &= \tilde{D}B_{32} \\ \tilde{D}B_{13} &= \tilde{D}B_{23} \\ \tilde{D}B_{42} &= \tilde{D}B_{51} \\ \tilde{D}B_{43} &= \tilde{D}B_{61} \\ \tilde{D}B_{53} &= \tilde{D}B_{62} \end{aligned}\right\} \qquad (2)$$

so it is only necessary to perform nine of the indicated fifteen multiplications to calculate $\tilde{\boldsymbol{D}}\boldsymbol{B}_b$. Therefore (cf. Exercise 2)

$$A = 0$$

$$M = 9n_{en}$$

$\boldsymbol{B}^T * (\tilde{\boldsymbol{D}}\boldsymbol{B})$ product and accumulation in $\boldsymbol{k}^e$:

Now let $B_i = N_{a,i}$, $i = 1, 2, 3$. The product $\boldsymbol{B}_a^T * (\tilde{\boldsymbol{D}}\boldsymbol{B}_b)$ has the following form (cf. the box on p. 153):

$$\begin{bmatrix} B_1\tilde{D}B_{11} + B_3\tilde{D}B_{51} + B_2\tilde{D}B_{61} & B_1\tilde{D}B_{12} + B_2\tilde{D}B_{62} & B_1\tilde{D}B_{13} + B_3\tilde{D}B_{53} \\ B_2\tilde{D}B_{21} + B_1\tilde{D}B_{61} & B_2\tilde{D}B_{22} + B_3\tilde{D}B_{42} + B_1\tilde{D}B_{62} & B_2\tilde{D}B_{23} + B_3\tilde{D}B_{43} \\ B_3\tilde{D}B_{31} + B_1\tilde{D}B_{51} & B_3\tilde{D}B_{32} + B_2\tilde{D}B_{42} & B_3\tilde{D}B_{33} + B_2\tilde{D}B_{43} + B_1\tilde{D}B_{53} \end{bmatrix}$$

Accounting for relations (2) above, we conclude that of the twenty-one indicated multiplications, only eighteen are independent. There are twelve additions required to form this product, plus nine more to accumulate into $\boldsymbol{k}_{ab}^e$. Recalling that the innermost loop is executed $\frac{1}{2}n_{en}(n_{en}+1)$ times, we have

$$A = \frac{21}{2}n_{en}(n_{en}+1)$$

$$M = 9n_{en}(n_{en}+1)$$

Summary for Implementation 2:

$$A = \left(\frac{21}{2}n_{en}(n_{en}+1)\right)n_{\text{int}}$$

$$M = (4 + 9n_{en} + 9n_{en}(n_{en}+1))\,n_{\text{int}}$$

Comparison of implementations for the eight-node brick:

Implementation 1

$$\text{NEE} = \text{NEN} * \text{NED} = 8 * 3 = 24$$

$$\text{NSTR} = 6$$

$$A = 2964n_{\text{int}}$$

$$M = 2686n_{\text{int}}$$

Implementation 2

$$A = 756 n_{\text{int}}$$
$$M = 724 n_{\text{int}}$$

Implementation 3

$$A = 564 n_{\text{int}}$$
$$M = 733 n_{\text{int}}$$

Discussion

Note that Implementation 3 reduces additions considerably compared with Implementation 2, however, the number of multiplications is slightly greater.

It is interesting to note that if the identities (1) were *not* employed in Implementation 2, the number of additions would remain the same but the number of multiplications would greatly increase to

$$M = \left(10 + 15 n_{en} + \frac{21}{2} n_{en}(n_{en} + 1)\right) n_{\text{int}}$$

which for the eight-node brick results in

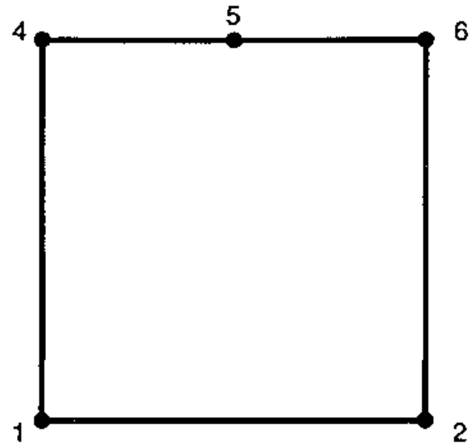
$$M = 886 n_{\text{int}}$$

Often, an elasticity element routine is implemented in this fashion since the form of the coefficient matrix is applicable to the orthotropic case, as well as isotropy.

3.11 Exercise 1, p. 156

Interelement compatibility (C2) is not maintained between elements 1 and 3, nor between elements 2 and 4 because 1 and 2 permit quadratic displacements along their edges, while 3 and 4 are limited to linear edge displacements.

To fix the mesh, elements 3 and 4 could use quadratic shape functions along the appropriate edges. Clearly, this adds no additional nodes to the mesh. Element 3, for example, would have the nodal arrangement shown in the following figure.



3.11 Exercise 2, p. 157

a.

$$f_a^e = \int_{-1}^{1} N_a(\xi) f x_{,\xi} \, d\xi$$

$$x_{,\xi} = \sum_{a=1}^{3} N_{a,\xi} x_a^e = (\xi - \frac{1}{2})x_1^e - 2\xi x_2^e + (\xi + \frac{1}{2})x_3^e = h^e/2$$

So with f constant,

$$f_a^e = \frac{fh^e}{2} \int_{-1}^{1} N_a(\xi) \, d\xi$$

Using exact integration it can be shown that

$$f_a^e = \frac{fh^e}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

b.

$$f_a^e = \int_{-1}^{1} N_a(\xi) \, (x_{,\xi})^{-1} \delta(\xi - \bar{\xi}) \, x_{,\xi} \, d\xi = \int_{-1}^{1} N_a(\xi) \, \delta(\xi - \bar{\xi}) \, d\xi = \begin{Bmatrix} \frac{1}{2}\bar{\xi}(\bar{\xi} - 1) \\ 1 - \bar{\xi}^2 \\ \frac{1}{2}\bar{\xi}(1 + \bar{\xi}) \end{Bmatrix}$$

For $\bar{x} = x_1^e$, $\bar{\xi} = -1$ and

$$f_a^e = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

For $\bar{x} = x_2^e$, $\bar{\xi} = 0$ and

$$f_a^e = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

c. If $x_2^e \neq \frac{1}{2}(x_1^e + x_3^e)$, $x_{,\xi}$ is a linear polynomial in ξ and

$$f_a^e = f \int_{-1}^{1} \underbrace{N_a}_{\text{quadratic}} \underbrace{j}_{\text{linear}} \, d\xi$$

The integrand is third order so $n_{\text{int}} = 2$ is required to attain exact integration.

d. $x_2^e = \frac{1}{2}(x_1^e + x_3^e)$ so $x_{,\xi}$ is a constant.

$$k_{ab}^e = \int_{-1}^{1} (N_{a,\xi}\,(x_{,\xi})^{-1})(N_{b,\xi}\,(x_{,\xi})^{-1})\,x_{,\xi}\,d\xi \quad = \quad (x_{,\xi})^{-1} \int_{-1}^{1} N_{a,\xi} N_{b,\xi}\,d\xi$$

The integrand is quadratic so $n_{\text{int}} = 2$ is needed for exact integration.

e. The solution to this part is given in the text.

f.

$$\begin{aligned} r &= \frac{x - x_1^e}{h^e} \\ &= \Big((\sum_{a=1}^{3} N_a x_a^e) - x_1^e \Big) / h^e \\ &= \frac{1}{h^e} \left(\frac{1}{2}(\xi^2 - \xi) x_1^e + (1 - \xi^2) x_2^e + \frac{1}{2}(\xi^2 + \xi) x_3^e - x_1^e \right) \\ &= \frac{1}{4}(\xi + 1)^2 \end{aligned}$$

Therefore

$$\xi \quad = \quad -1 + 2\sqrt{r}$$

Then $\xi_{,r} = 1/\sqrt{r}$. Using the chain rule,

$$\begin{aligned} u^h_{,r}(r(\xi)) &= u^h_{,\xi}(\xi)\,\xi_{,r} \\ &= \frac{1}{\sqrt{r(\xi)}} u^h_{,\xi}(\xi) \end{aligned}$$

The singularity is of order $\frac{1}{2}$ and corresponds to Westergaard's elasticity solution for the crack-tip strain field.

3.11 Exercise 3, p. 159

We only need to modify N_1 and N_2.

$$N_1' \quad = \quad N_1 - \frac{1}{2} N_5$$

For $\xi \leq 0$,

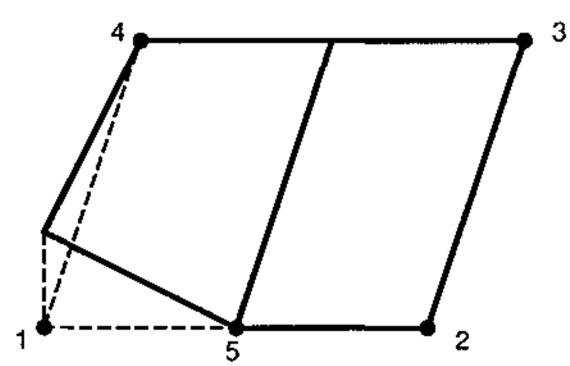
$$N_1' \quad = \quad \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1+\xi)(1-\eta) \quad = \quad -\frac{1}{2}(1-\eta)\xi$$

For $\xi > 0$,

$$N_1' \quad = \quad \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi)(1-\eta) \quad = \quad 0$$

Similarly,

$$N_2' = \begin{cases} 0, & \text{for } \xi \leq 0; \\ \frac{1}{2}\xi(1-\eta), & \text{for } \xi > 0. \end{cases}$$



3.11 Exercise 4, p. 159

We want to obtain

$$r = \sum_{a=1}^{4} N_a(\xi,\eta) r_a^e$$

With only four nodes, the interpolation is of the form

$$r = \alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi\eta$$

Forcing $r(\xi_a,\eta_a) = r_a$,

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

Solution of this system leads to the standard bilinear shape functions (see Sec. 3.2). The situation is similar for θ. Checking conditions (i)–(iv) is a simple exercise, e.g.,

$$\text{(i)} \qquad r(-1,\eta) = r_1[\frac{1}{4}(1-\xi)(1-\eta) + \frac{1}{4}(1-\xi)(1+\eta)] = r_1 \quad \text{, etc.}$$

Notice that the rectilinear parent domain is mapped onto the curvilinear physical domain.

3.11 Exercise 5, p. 160

$$\begin{aligned} x_{,\xi} &= \sum_{a=1}^{4} N_{a,\xi} x_a^e \\ &= \frac{1}{16}(-27\xi^2 + 18\xi + 1)(-h^e/2) + \frac{1}{16}(81\xi^2 - 18\xi - 27)(-h^e/6) \\ &\quad - \frac{1}{16}(81\xi^2 + 18\xi - 27)(h^e/6) + \frac{1}{16}(27\xi^2 + 18\xi - 1)(h^e/2) \\ &= \frac{1}{16}\{(27 - \frac{81}{3})\xi^2 + (0)\xi + (-1 + \frac{27}{3})\}h^e \\ &= h^e/2 \end{aligned}$$

$$e_{,x} \quad = \quad e_{,\xi}\xi_{,x} \quad = \quad 2e_{,\xi}/h^e$$

Using the same strategy as in part (e) of Exercise 2,

$$e_{,\xi} \quad = \quad u^h_{,\xi} - u_{,\xi}$$

If $u(\xi)$ is cubic then $u^h(\xi) = u(\xi)$. Assume $u(\xi) = C\xi^4$ and $u^h(\xi)$ is nodally exact. Then

$$\begin{aligned}
e_{,\xi} &= u^h_{,\xi}(\xi) - u_{,\xi}(\xi) \\
&= \sum_{a=1}^{4} N_{a,\xi} u^h(\xi_a) - 4C\xi^3 \\
&= \frac{1}{16}(-27\xi^2 + 18\xi + 1)C + \frac{1}{16}(81\xi^2 - 18\xi - 27)(C/81) \\
&\quad - \frac{1}{16}(81\xi^2 + 18\xi - 27)(C/81) + \frac{1}{16}(27\xi^2 + 18\xi - 1)C - 4C\xi^3 \\
&= \frac{1}{16}(36 - \frac{4}{9})\xi C - 4C\xi^3
\end{aligned}$$

$e_{,\xi} = 0$ requires $\xi = 0$ or

$$4C\xi^2 \quad = \quad \frac{1}{16}(36 - \frac{4}{9})C \qquad \Rightarrow \qquad \xi^2 \quad = \quad \frac{5}{9}$$

Therefore

$$\xi = 0 \quad \text{or} \quad \xi = \pm\sqrt{5/9}$$

3.11 Exercise 6, p. 161

a. Weak formulation:

$$\begin{aligned}
\int_\Omega w b_i u_{,i}\, d\Omega &= \int_\Omega w\big((\kappa_{ij}u_{,j})_{,i} + f\big)\, d\Omega \\
&= -\int_\Omega w_{,i}\kappa_{ij}u_{,j}\, d\Omega + \int_\Gamma w\kappa_{ij}u_{,j}n_i\, d\Gamma + \int_\Omega wf\, d\Omega
\end{aligned}$$

Define

$$\begin{aligned} \mathcal{S} &= \{u \big| u \in H^1(\Omega),\ u = g\ \forall \boldsymbol{x} \in \Gamma_g\} \\ \mathcal{V} &= \{w \big| w \in H^1(\Omega),\ w = 0\ \forall \boldsymbol{x} \in \Gamma_g\} \end{aligned}$$

Then, given f, g, and h, find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$,

$$\int_\Omega w_{,i}\kappa_{ij}u_{,j}\, d\Omega + \int_\Omega w b_i u_{,i}\, d\Omega \quad = \quad \int_\Omega w f\, d\Omega + \int_{\Gamma_h} w \mathrm{h}\, d\Gamma$$

b. Galerkin formulation:

Let $\mathcal{S}^h \subset \mathcal{S}$ and $\mathcal{V}^h \subset \mathcal{V}$ be defined in the usual way. Then find $u^h \in \mathcal{S}^h$ such that for all $w^h \in \mathcal{V}^h$

$$\int_\Omega w^h_{,i}\kappa_{ij}u^h_{,j}\, d\Omega + \int_\Omega w^h b_i u^h_{,i}\, d\Omega \quad = \quad \int_\Omega w^h f\, d\Omega + \int_{\Gamma_h} w^h \mathrm{h}\, d\Gamma$$

Define w^h and u^h in terms of shape functions in the usual way. Then

$$\sum_{B\in\eta-\eta_g} \{ \int_\Omega N_{A,i}\kappa_{ij}N_{B,j}d_B\, d\Omega + \int_\Omega N_A b_i N_{B,i} d_B\, d\Omega\} \quad =$$

$$\int_\Omega N_A f\, d\Omega + \int_{\Gamma_h} N_A \mathrm{h}\, d\Gamma - \sum_{B\in\eta_g} \{ \int_\Omega N_{A,i}\kappa_{ij}N_{B,j}\, d\Omega + \int_\Omega N_A b_i N_{B,i}\, d\Omega\} g_B \qquad \forall A \in \eta - \eta_g$$

Thus

$$\begin{aligned} K_{PQ} &= a(N_A\, ,\, N_B) + b(N_A\, ,\, N_B) \\ F_P &= (N_A, f) + (N_A, \mathrm{h})_\Gamma - \sum_{B\in\eta_g} [a(N_A\, ,\, N_B) + b(N_A\, ,\, N_B)] g_B \end{aligned}$$

where

$$a(N_A\, ,\, N_B) = \int_\Omega N_{A,i}\kappa_{ij}N_{B,j}\, d\Omega \quad \text{and} \quad b(N_A\, ,\, N_B) = \int_\Omega N_A b_i N_{B,i}\, d\Omega$$

and $P = ID(A)$, $Q = ID(B)$.

c. Consider $\int_\Omega w^h b_i u^h_{,i}\, d\Omega$ where $w^h \in \mathcal{V}^h$ and $u^h \in \mathcal{S}^h$. Integrating by parts,

$$\begin{aligned}
\int_\Omega w^h b_i u^h_{,i}\, d\Omega &= -\int_\Omega (w^h b_i)_{,i} u^h\, d\Omega + \int_{\Gamma_h} w^h b_i u^h n_i\, d\Gamma \\
&= -\int_\Omega \{w^h_{,i} b_i u^h + w^h b_{i,i} u^h\}\, d\Omega + \int_{\Gamma_h} w^h b_i u^h n_i\, d\Gamma
\end{aligned}$$

If $b_{i,i} = 0$ on Ω, and $b_i n_i = 0$ on Γ_h ,

$$\int_\Omega w^h b_i u^h_{,i}\, d\Omega = -\int_\Omega w^h_{,i} b_i u^h\, d\Omega$$

which implies

$$K^b_{PQ} = b(N_A\,,\, N_B) = -b(N_B\,,\, N_A) = -K^b_{QP}$$

Thus this stiffness contribution is skew-symmetric.

3.11 Exercise 7, p. 161

a. For two-dimensional problems,

$$\int_{\Omega^e} d\Omega = \int_{-1}^{1}\int_{-1}^{1} j\, d\xi\, d\eta$$

where

$$j = \det\begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix} = \det\begin{bmatrix} \sum N_{a,\xi} x_a & \sum N_{a,\eta} x_a \\ \sum N_{a,\xi} y_a & \sum N_{a,\eta} y_a \end{bmatrix}$$

For the four-node bilinear element,

$$j = \frac{1}{16}\det\begin{bmatrix} (1-\eta)(x_2 - x_1) + (1+\eta)(x_3 - x_4) & (1-\xi)(x_4 - x_1) + (1+\xi)(x_3 - x_2) \\ (1-\eta)(y_2 - y_1) + (1+\eta)(y_3 - y_4) & (1-\xi)(y_4 - y_1) + (1+\xi)(y_3 - y_2) \end{bmatrix}$$

The only quadratic terms in j are of the form $\xi\eta$. Terms of this form and linear terms are integrated exactly by one-point Gaussian quadrature.

b. By (3.9.3) and (3.9.9)

$$
\begin{aligned}
u^h_{i,i} &= \sum_{a=1}^{4} \langle N_{a,x}\; N_{a,y} \rangle \boldsymbol{d}^e_a \\
&= \sum_{a=1}^{4} \frac{1}{j} \langle N_{a,\xi}\; N_{a,\eta} \rangle \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix} \boldsymbol{d}^e_a
\end{aligned}
$$

$$
\begin{aligned}
\int_{\Omega^e} u^h_{i,i}\, d\Omega &= \int_{-1}^{1}\int_{-1}^{1} u^h_{i,i} j \, d\xi\, d\eta \\
&= \sum_{a=1}^{4} \int_{-1}^{1}\int_{-1}^{1} \langle N_{a,\xi}\; N_{a,\eta} \rangle \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix} d\xi\, d\eta\; \boldsymbol{d}^e_a \\
&= \sum_{a=1}^{4} \int_{-1}^{1}\int_{-1}^{1} \{ (N_{a,\xi} y_{,\eta} - N_{a,\eta} y_{,\xi}) d^e_{1a} - (N_{a,\xi} x_{,\eta} - N_{a,\eta} x_{,\xi}) d^e_{2a} \}\, d\xi\, d\eta
\end{aligned}
$$

Examining this expression, we note that for the four-node bilinear element all derivatives of N_a, x and y with respect to ξ are linear polynomials in η and vice versa (e.g., see the entries in the Jacobian matrix j above). The only quadratic terms in the integrand are thus of the form $\xi\eta$ and are integrated exactly by one-point Gaussian quadrature. Likewise, the linear terms in ξ and η are integrated exactly by one-point Gaussian quadrature. Therefore

$$
\begin{aligned}
\int_{\Omega^e} u^h_{i,i}\, d\Omega &= \int_{-1}^{1}\int_{-1}^{1} u^h_{i,i} j \, d\xi\, d\eta \\
&= 4 u^h_{i,i}(0,0)\, j(0,0)
\end{aligned}
$$

Consequently, if $u^h_{i,i}(0,0) = 0$, then $\int_{\Omega^e} u^h_{i,i}\, d\Omega = 0$.

3.11 Exercise 8, p. 162

The solution to this problem is given in the text.

3.11 Exercise 9, p. 162

$$f_{21}^e = \int_\Omega N_1 f_2 \, d\Omega \,, \qquad f_{25}^e = \int_\Omega N_5 f_2 \, d\Omega$$

where, from Exercise 1, Sec. 3.7,

$$N_1 = N_1(\xi, \eta) = \frac{1}{4}(-1 + \xi\eta + \xi^2(1-\eta) + \eta^2(1-\xi))$$

and

$$N_5(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\begin{aligned}
f_{21}^e &= \int_{-1}^{1}\int_{-1}^{1} N_1(\xi,\eta) f_2 j \, d\xi \, d\eta \\
&= -g\frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1} N_1 \, d\xi \, d\eta \\
&= -g\frac{h^2}{4}\left(-\frac{1}{3}\right) \\
&= +\frac{gh^2}{12} \\
f_{25}^e &= -g\frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1} N_5 \, d\xi \, d\eta \\
&= -g\frac{h^2}{8}\left(\frac{8}{3}\right) \\
&= -\frac{gh^2}{3}
\end{aligned}$$

The nodal forces at the corners point in the positive y-direction whereas the gravitational force is assumed to act in the negative y-direction.

3.11 Exercise 10, p. 163

As in Exercise 8,

$$\int_{\Gamma_{h_i}} N_a \mathrm{h}_i \, d\Gamma \quad = \quad \mathrm{h}_i \int_{-1}^{1} N_a s_{,\xi} \, d\xi$$

In general

$$s_{,\xi} = \sqrt{(x_{,\xi})^2 + (y_{,\xi})^2}$$

Since the three nodes do not lie on a straight line the integrand is generally *not* a polynomial in ξ (due to the square-root), and hence cannot be integrated exactly by Gaussian quadrature.

3.11 Exercise 11, p. 163

Note that

$$\mathrm{h}_i \quad = \quad -p\, n_i(\xi)$$

so in this case

$$\int_{\Gamma_{h_i}} N_a \mathrm{h}_i \, d\Gamma \quad = \quad -p \int_{-1}^{1} N_a n_i s_{,\xi} \, d\xi$$

where

$$s_{,\xi} = \sqrt{(x_{,\xi})^2 + (y_{,\xi})^2}$$

as before, and

$$n_1 = \frac{y_{,\xi}}{s_{,\xi}}, \qquad n_2 = -\frac{x_{,\xi}}{s_{,\xi}}$$

N_a is a second-order polynomial in ξ, while the product $n_i\, s_{,\xi}$ is a first-order polynomial in ξ. Thus the combined result is a third-order polynomial and thus two-point Gaussian quadrature is required for exact integration.

3.11 Exercise 12, p. 163

Consider the pressure applied to the surface $\zeta = 1$. The isoparametric mapping restricted to this surface is

$$\boldsymbol{x} = \sum_{a=1}^{8} N_a(\xi, \eta, 1)\, \boldsymbol{x}_a^e = \sum_{a=1}^{4} N_a^*(\xi, \eta)\, \boldsymbol{x}_a^e$$

where the N_a^*'s are the usual bilinear shape functions in ξ, η. The vectors $\boldsymbol{x}_{,\xi}$ and $\boldsymbol{x}_{,\eta}$ are tangent to the surface and the normal is parallel to, and in the same directions as, their cross product. Hence

$$\boldsymbol{n} = \frac{\boldsymbol{x}_{,\xi} \times \boldsymbol{x}_{,\eta}}{\|\boldsymbol{x}_{,\xi} \times \boldsymbol{x}_{,\eta}\|}$$

The prescribed traction contribution to the force vector is

$$\begin{aligned}
f_{ia}^e &= \int_{\Gamma_{h_i}} N_a^* \mathrm{h}_i \, d\Gamma = -p \int_{\Gamma_{h_i}} N_a^* n_i \, d\Gamma \\
&= -p \int_{-1}^{1} \int_{-1}^{1} N_a^*(\xi, \eta)\, n_i(\xi, \eta)\, \|\boldsymbol{x}_{,\xi}(\xi, \eta) \times \boldsymbol{x}_{,\eta}(\xi, \eta)\| \, d\xi \, d\eta \\
&= -p \int_{-1}^{1} \int_{-1}^{1} N_a^* \, (\boldsymbol{x}_{,\xi} \times \boldsymbol{x}_{,\eta})_i \, d\xi \, d\eta
\end{aligned}$$

For the trilinear brick $\boldsymbol{x}_{,\xi}(\xi, \eta)$ is linear in η, and $\boldsymbol{x}_{,\eta}(\xi, \eta)$ is linear in ξ. The result, combined with N_a^*, is a biquadratic polynomial in ξ, η. Thus 2×2 Gaussian quadrature is needed to exactly integrate constant normal pressure.

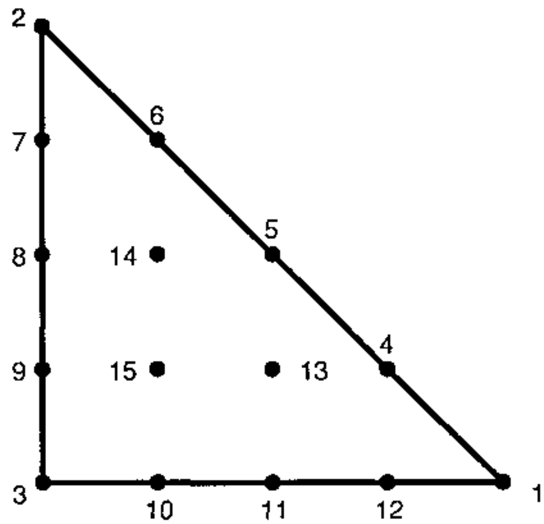
3.11 Exercise 13, p. 163

As in the previous problem,

$$f_{ia}^e = -p \int_{-1}^{1} \int_{-1}^{1} N_a^* \, (\boldsymbol{x}_{,\xi} \times \boldsymbol{x}_{,\eta})_i \, d\xi \, d\eta$$

For the triquadratic brick, $N_a^*(\xi,\eta)$ is biquadratic in ξ and η, $\boldsymbol{x}_{,\xi}(\xi,\eta)$ is linear in ξ and quadratic in η, and $\boldsymbol{x}_{,\eta}(\xi,\eta)$ is linear in η and quadratic in ξ. The combined result contains monomials of order no higher than five in ξ or η. Thus we need 3×3 Gaussian quadrature to exactly integrate the normal pressure.

Appendix 3.I Exercise 1, p. 169



$$T_1(r) = 1$$

$$T_2(r) = \ell_2^1(\frac{2r}{r_2}-1) = \ell_2^1(8r-1) = 4r$$

$$T_3(r) = \ell_3^2(\frac{2r}{r_3}-1) = \ell_3^2(4r-1) = 2r(4r-1)$$

$$T_4(r) = \ell_4^3(\frac{2r}{r_4}-1) = \ell_4^3(\frac{8}{3}r-1) = \frac{4}{3}r(4r-1)(2r-1)$$

$$T_5(r) = \ell_5^4(\frac{2r}{r_5}-1) = \ell_5^4(2r-1) = r(4r-1)(2r-1)(\frac{4}{3}r-1)$$

Then,

$$N_a(r,s,t) = T_I(r)\,T_J(s)\,T_K(t)$$

where the indices are related as follows:

a	I	J	K		a	I	J	K
1	5	1	1		9	1	2	4
2	1	5	1		10	2	1	4
3	1	1	5		11	3	1	3
4	4	2	1		12	4	1	2
5	3	3	1		13	3	2	2
6	2	4	1		14	2	3	2
7	1	4	2		15	2	2	3
8	1	3	3					

Appendix 3.I Exercise 2, p. 169

From the quadratic triangle, $N_4 = 4rs$. Then

$$N_1' = N_1 - \frac{1}{2}N_4 = r - 2rs = r(1-2s)$$

Similarly,

$$N_2' = s(1-2r)$$
$$N_3 = t = 1 - r - s$$

Appendix 3.I Exercise 3, p. 170

The 10 node quadratic tetrahedron:

$$N_a(r,s,t,u) = T_I(r)\,T_J(s)\,T_K(t)\,T_L(u)$$

Using Figure 3.I.9,

$$N_1 = \ell_3^2(\frac{2r}{r_3} - 1) = 2r(r-1)$$

By cyclic symmetry,

$$N_2 = s(2s-1) \qquad N_3 = t(2t-1) \qquad N_4 = u(2u-1)$$

$$N_5 \quad = \quad \ell_2^1(\frac{2r}{r_2} - 1)\, \ell_2^1(\frac{2s}{s_2} - 1) \quad = \quad 4rs$$

Similarly,

$$N_6 = 4st \qquad N_7 = 4tu \qquad N_8 = 4ru \qquad N_9 = 4rt \qquad N_{10} = 4su$$

It is easily verified that these functions satisfy the interpolation property.

CHAPTER 4

4.1 Exercise 1, p. 187

$\forall \boldsymbol{w} \in \mathcal{V}$

$$a(\boldsymbol{w}\,,\,\boldsymbol{u}) \quad = \quad (\boldsymbol{w}, \boldsymbol{f})$$

and $\forall \boldsymbol{w}^h \in \mathcal{V}^h$

$$a(\boldsymbol{w}^h\,,\,\boldsymbol{u}^h) \quad = \quad (\boldsymbol{w}^h, \boldsymbol{f})$$

Since $\boldsymbol{g} \,=\, \boldsymbol{0}\ \boldsymbol{u}^h \in \mathcal{V}^h$ and $\boldsymbol{u} \in \mathcal{V}$

$$a(\boldsymbol{u}\,,\,\boldsymbol{u}) \quad = \quad (\boldsymbol{u}, \boldsymbol{f})$$

$$a(\boldsymbol{u}^h\,,\,\boldsymbol{u}^h) \quad = \quad (\boldsymbol{u}^h, \boldsymbol{f})$$

Thus,

$$(\boldsymbol{u}^h, \boldsymbol{f}) \quad \leq \quad (\boldsymbol{u}, \boldsymbol{f})$$

with $\boldsymbol{f}(\boldsymbol{x}) \,=\, \delta(\boldsymbol{x} - \bar{\boldsymbol{x}})\boldsymbol{e}_i$

$$(\boldsymbol{u}^h, \boldsymbol{e}_i\delta(\boldsymbol{x} - \bar{\boldsymbol{x}})) \quad \leq \quad (\boldsymbol{u}, \boldsymbol{e}_i\delta(\boldsymbol{x} - \bar{\boldsymbol{x}}))$$

$$\Rightarrow \qquad u_i^h(\bar{\boldsymbol{x}}) \quad \leq \quad u_i(\bar{\boldsymbol{x}})$$

4.1 Exercise 2, p. 188

a.

$$\begin{aligned}
\left(\frac{d}{d\epsilon} I(\boldsymbol{U}_\epsilon)\right)\Big|_\epsilon \quad &= \quad \frac{d}{d\epsilon}\Big|_{\epsilon=0}\Big\{\frac{1}{2}a(\boldsymbol{u}+\epsilon\boldsymbol{w}\,,\,\boldsymbol{u}+\epsilon\boldsymbol{w}) \\
&\qquad - (\boldsymbol{u}+\epsilon\boldsymbol{w}, \boldsymbol{f}) - (\boldsymbol{u}+\epsilon\boldsymbol{w}\,,\,\mathbf{h})_\Gamma\Big\} \\
&= \quad \frac{d}{d\epsilon}\Big|_{\epsilon=0}\Big\{\frac{1}{2}a(\boldsymbol{u}\,,\,\boldsymbol{u}) - (\boldsymbol{u}, \boldsymbol{f}) - (\boldsymbol{u}\,,\,\mathbf{h})_\Gamma + \epsilon a(\boldsymbol{w}\,,\,\boldsymbol{u}) \\
&\qquad + \frac{1}{2}\epsilon^2 a(\boldsymbol{w}\,,\,\boldsymbol{w}) - \epsilon(\boldsymbol{w}, \boldsymbol{f}) - \epsilon(\boldsymbol{w}\,,\,\mathbf{h})_\Gamma\Big\} \\
&= \quad a(\boldsymbol{w}\,,\,\boldsymbol{u}) - (\boldsymbol{w}, \boldsymbol{f}) - (\boldsymbol{w}\,,\,\mathbf{h})_\Gamma
\end{aligned}$$

Thus

$$a(\boldsymbol{w}\,,\,\boldsymbol{u}) - (\boldsymbol{w},\boldsymbol{f}) - (\boldsymbol{w}\,,\,\mathbf{h})_\Gamma \quad = \quad 0$$

if and only if

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} I(\boldsymbol{U}_\epsilon) \quad = \quad 0$$

b. For $I(\boldsymbol{U}_\epsilon)$ to attain a minimum at $\boldsymbol{u}$, the following must hold:

$$\text{i.} \quad \frac{d}{d\epsilon}\bigg|_{\epsilon=0} I(\boldsymbol{U}_\epsilon) \quad = \quad 0$$

$$\text{ii.} \quad \frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} I(\boldsymbol{U}_\epsilon) \quad \geq \quad 0$$

From part a, $\boldsymbol{u}$ satisfies i. Also from a,

$$\begin{aligned} \frac{d}{d\epsilon} I(\boldsymbol{U}_\epsilon) \quad &= \quad \frac{d}{d\epsilon}\Big\{\frac{1}{2}a(\boldsymbol{u}\,,\,\boldsymbol{u}) - (\boldsymbol{u},\boldsymbol{f}) - (\boldsymbol{u}\,,\,\mathbf{h})_\Gamma \\ &\qquad + \epsilon\{a(\boldsymbol{w}\,,\,\boldsymbol{u}) - (\boldsymbol{w},\boldsymbol{f}) - (\boldsymbol{w}\,,\,\mathbf{h})_\Gamma\} + \frac{1}{2}\epsilon^2 a(\boldsymbol{w}\,,\,\boldsymbol{w})\Big\} \end{aligned}$$

and thus

$$\begin{aligned} \frac{d^2}{d\epsilon^2} I(\boldsymbol{U}_\epsilon) \quad &= \quad \frac{d}{d\epsilon}\Big\{a(\boldsymbol{w}\,,\,\boldsymbol{u}) - (\boldsymbol{w},\boldsymbol{f}) - (\boldsymbol{w}\,,\,\mathbf{h})_\Gamma + \epsilon a(\boldsymbol{w}\,,\,\boldsymbol{w})\Big\} \\ &= \quad a(\boldsymbol{w}\,,\,\boldsymbol{w}) \end{aligned}$$

Consequently,

$$\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} I(\boldsymbol{U}_\epsilon) \quad = \quad a(\boldsymbol{w}\,,\,\boldsymbol{w})$$

Since $a(\cdot\,,\,\cdot)$ is positive definite,

$$a(\boldsymbol{w}\,,\,\boldsymbol{w}) \quad \geq \quad 0$$

Thus $I(\boldsymbol{U}_\epsilon)$ is minimized at $\boldsymbol{u}$.

c. By part b, I is minimized at $\boldsymbol{u} \in \mathcal{S}$. Since $\boldsymbol{u}^h \in \mathcal{S}^h \subset \mathcal{S}$, it follows that $I(\boldsymbol{u}^h) \geq I(\boldsymbol{u})$.

4.1 Exercise 3, p. 190

$$\| \boldsymbol{e} \|_s \quad \leq \quad ch^{\beta} \| \boldsymbol{u} \|_{k+1}$$

$$\beta = \min\{k+1-s\,,\, 2(k+1-m)\}$$

$$m = 1\,,\, k = 2$$

For L_2, $\quad s = 0 \Rightarrow k+1-s = 3, \quad 2(k+1-m) = 4$. Therefore

$$\| \boldsymbol{e} \|_0 \quad \leq \quad ch^3 \| \boldsymbol{u} \|_3$$

For H^1, $\quad s = 1 \Rightarrow k+1-s = 2, \quad 2(k+1-m) = 4$. Therefore

$$\| \boldsymbol{e} \|_1 \quad \leq \quad ch^2 \| \boldsymbol{u} \|_3$$

4.1 Exercise 4, p. 190

For Bernoulli-Euler beam theory, $m = 2$. For Hermite cubics, $k = 3$.

space	s	$k+1-s$	$2(k+1-m)$	$\| \boldsymbol{e} \|_s$ convergence rate
L_2	0	4	4	h^4
H^1	1	3	4	h^3
H^2	2	2	4	h^2

4.2 Exercise 1, p. 193

$$\boldsymbol{D} \quad = \quad \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

For plane stress

$$\lambda \quad \leftarrow \quad \bar{\lambda} \quad \text{where} \quad \bar{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu} \qquad (2.7.35)$$

Since

$$\lambda = \frac{2\nu\mu}{1-2\nu} \tag{4.2.3}$$

$$\bar{\lambda} = \frac{2\nu\mu}{1-\nu}$$

Hence,

$$\lim_{\nu \to \frac{1}{2}} \bar{\lambda} = 2\mu$$

Thus the coefficients of the isotropic plane stress constitutive equation remain bounded as $\nu \to \frac{1}{2}$.

4.3 Exercise 1, p. 199

We need to show that

$$\int_\Omega w_{(i,j)}\sigma_{ij}\, d\Omega = \bar{a}(\boldsymbol{w}\,,\, \boldsymbol{u}) - (\operatorname{div} \boldsymbol{w}\,, p)$$

By definition,

$$\begin{aligned}
\bar{a}(\boldsymbol{w}\,,\, \boldsymbol{u}) - (\operatorname{div} \boldsymbol{w}\,, p) &= \int_\Omega w_{(i,j)}\mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})u_{(k,l)}\, d\Omega - \int_\Omega w_{i,i}\, p\, d\Omega \\
&= 2\int_\Omega \mu w_{(i,j)}u_{(i,j)}\, d\Omega - \int_\Omega w_{i,i}\, p\, d\Omega
\end{aligned}$$

$$\begin{aligned}
\int_\Omega w_{(i,j)}\sigma_{i,j}\, d\Omega &= \int_\Omega w_{(i,j)}(-p\delta_{ij} + 2\mu u_{(i,j)})\, d\Omega \\
&= 2\int_\Omega \mu w_{(i,j)}u_{(i,j)}\, d\Omega - \int_\Omega w_{i,i}\, p\, d\Omega
\end{aligned}$$

The remaining terms are clearly identical.

4.3 Exercise 2, p. 200

We need to show that (1.4.11) and (1.4.13) are satisfied, namely

$$\begin{aligned} \bar{a}(\boldsymbol{u}\,,\,\boldsymbol{v}) &= \bar{a}(\boldsymbol{v}\,,\,\boldsymbol{u}) \\ \bar{a}(c_1\boldsymbol{u} + c_2\boldsymbol{v}\,,\,\boldsymbol{w}) &= c_1\bar{a}(\boldsymbol{u}\,,\,\boldsymbol{w}) + c_2\bar{a}(\boldsymbol{v}\,,\,\boldsymbol{w}) \end{aligned}$$

where

$$\bar{a}(\boldsymbol{w}\,,\,\boldsymbol{u}) = \int_{\Omega} w_{(i,j)}\bar{c}_{ijkl}u_{(k,l)}\,d\Omega$$

and

$$\bar{c}_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Note that $\bar{c}_{ijkl} = \bar{c}_{klij}$, the so-called major symmetry. Consequently,

$$\begin{aligned} \bar{a}(\boldsymbol{u}\,,\,\boldsymbol{v}) &= \int_{\Omega} u_{(i,j)}\bar{c}_{ijkl}v_{(k,l)}\,d\Omega \\ &= \int_{\Omega} v_{(k,l)}\bar{c}_{klij}u_{(i,j)}\,d\Omega \\ &= \bar{a}(\boldsymbol{v}\,,\,\boldsymbol{u}) \end{aligned}$$

Linearity is straightforward, viz.

$$\begin{aligned} \bar{a}(c_1\boldsymbol{u} + c_2\boldsymbol{v}\,,\,\boldsymbol{w}) &= \int_{\Omega}(c_1u_{(i,j)} + c_2v_{(i,j)})\bar{c}_{ijkl}w_{(k,l)}\,d\Omega \\ &= c_1\int_{\Omega} u_{(i,j)}\bar{c}_{ijkl}w_{(k,l)}\,d\Omega + c_2\int_{\Omega} v_{(i,j)}\bar{c}_{ijkl}w_{(k,l)}\,d\Omega \\ &= c_1\bar{a}(\boldsymbol{u}\,,\,\boldsymbol{w}) + c_2\bar{a}(\boldsymbol{v}\,,\,\boldsymbol{w}) \end{aligned}$$

4.3 Exercise 3, p. 200

Positive definiteness of $\bar{c}_{ijkl}$:

$$\begin{aligned}
\bar{c}_{ijkl}\psi_{ij}\psi_{kl} &= \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\psi_{ij}\psi_{kl} \\
&= \mu(\psi_{ij}\psi_{ij} + \psi_{ij}\psi_{ji}) \\
&= 2\mu\psi_{ij}\psi_{ij} \quad \geq \quad 0
\end{aligned}$$

It follows from this result that if

$$\bar{c}_{ijkl}\psi_{ij}\psi_{kl} = 0$$

then

$$\psi_{ij} = 0$$

Hence $\bar{c}_{ijkl}$ is positive definite.

4.3 Exercise 4, p. 203

From (4.3.22)

$$\boldsymbol{d} = \bar{\boldsymbol{K}}^{-1}(\bar{\boldsymbol{F}} - \boldsymbol{G}\boldsymbol{p})$$

Substituting into (4.3.23),

$$\boldsymbol{G}^T\bar{\boldsymbol{K}}^{-1}(\bar{\boldsymbol{F}} - \boldsymbol{G}\boldsymbol{p}) + \boldsymbol{M}\boldsymbol{p} = \boldsymbol{H}$$

$$(\boldsymbol{M} - \boldsymbol{G}^T\bar{\boldsymbol{K}}^{-1}\boldsymbol{G})\boldsymbol{p} = \boldsymbol{H} - \boldsymbol{G}^T\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{F}}$$

Rearranging yields the generalization of (4.3.26):

$$(\boldsymbol{G}^T\bar{\boldsymbol{K}}^{-1}\boldsymbol{G} - \boldsymbol{M})\boldsymbol{p} = \boldsymbol{G}^T\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{F}} - \boldsymbol{H}$$

4.3 Exercise 5, p. 206

Part 1

ID array:

Global d.o.f. (i)	Global node numbers (A) 1	2	3	4	5	6	7	8	9	10	11	12
1	0	2	5	7	10	13	16	19	22	0	27	30
2	0	3	0	8	11	14	17	20	23	25	28	31
3	1	4	6	9	12	15	18	21	24	26	29	32

The IEN array is unchanged; see Fig. 2.10.2.

LM array:

Local equation numbers (p)	Element numbers (e) 1	2	3	4	5	6
1	0	2	7	10	16	19
2	0	3	8	11	17	20
3	1	4	9	12	18	21
4	2	5	10	13	19	22
5	3	0	11	14	20	23
6	4	6	12	15	21	24
7	10	13	19	22	27	30
8	11	14	20	23	28	31
9	12	15	21	24	29	32
10	7	10	16	19	0	27
11	8	11	17	20	25	28
12	9	12	18	21	26	29

The maximum half-bandwidth is 15.

Part 2

ID array:

Global d.o.f. (i)	Global node numbers (A) 1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	3	4	6	8	10	12	14	0	17	19
2	0	2	0	5	7	9	11	13	15	16	18	20
3	21	22	23	24	25	26	27	28	29	30	31	32

The IEN array is unchanged.

LM array:

Local equation numbers (p)	Element numbers (e) 1	2	3	4	5	6
1	0	1	4	6	10	12
2	0	2	5	7	11	13
3	21	22	24	25	27	28
4	1	3	6	8	12	14
5	2	0	7	9	13	15
6	22	23	25	26	28	29
7	6	8	12	14	17	19
8	7	9	13	15	18	20
9	25	26	28	29	31	32
10	4	6	10	12	0	17
11	5	7	11	13	16	18
12	24	25	27	28	30	31

The maximum half-bandwidth is 26. This illustrates that the segregated form increases the band-profile population of the coefficient matrix. The sketches are straightforward consequences of the LM arrays.

4.3 Exercise 6, p. 206

Part 1

ID array:

Global d.o.f. (i)	Global node numbers (A) 1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	3	4	7	10	12	15	18	0	22	25
2	0	2	0	5	8	11	13	16	19	20	23	26
3	0	0	0	6	9	0	14	17	0	21	24	0

The IEN array is unchanged.

LM array:

Local equation numbers (p)	Element numbers (e) 1	2	3	4	5	6
1	0	1	4	7	12	15
2	0	2	5	8	13	16
3	0	0	0	0	0	0
4	1	3	7	10	15	18
5	2	0	8	11	16	19
6	0	0	0	0	0	0
7	7	10	15	18	22	25
8	8	11	16	19	23	26
9	0	0	0	0	0	0
10	4	7	12	15	0	22
11	5	8	13	16	20	23
12	6	9	14	17	21	24

The maximum half-bandwidth is 13.

Part 2

ID array:

Global d.o.f. (i)	Global node numbers (A) 1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	3	4	6	8	10	12	14	0	17	19
2	0	2	0	5	7	9	11	13	15	16	18	20
3	0	0	0	21	22	0	23	24	0	25	26	0

The IEN array is unchanged.

LM array:

Local equation numbers (p)	Element numbers (e) 1	2	3	4	5	6
1	0	1	4	6	10	12
2	0	2	5	7	11	13
3	0	0	0	0	0	0
4	1	3	6	8	12	14
5	2	0	7	9	13	15
6	0	0	0	0	0	0
7	6	8	12	14	17	19
8	7	9	13	15	18	20
9	0	0	0	0	0	0
10	4	6	10	12	0	17
11	5	7	11	13	16	18
12	21	22	23	24	25	26

The maximum half-bandwidth is 22. The conclusions are the same as for the previous exercise.

4.4 Exercise 1, p. 223

$$\tilde{N}_{\tilde{a}}(\xi,\eta) = \frac{1}{4}(1+3\tilde{\xi}_{\tilde{a}}\xi)(1+3\tilde{\eta}_{\tilde{a}}\eta)$$

For the selectively integrated nine-node quadrilateral, pressure is interpolated at the 2×2 Gauss points,

$$\tilde{\xi}_{\tilde{a}} = (-1)^i \frac{1}{\sqrt{3}}, \qquad \tilde{\eta}_{\tilde{a}} = (-1)^j \frac{1}{\sqrt{3}}$$

$\tilde{a}$	i	j
1	1	1
2	2	1
3	2	2
4	1	2

$$\tilde{N}_{\tilde{1}}(\xi\,,\eta) = \frac{1}{4}(1-\sqrt{3}\xi)(1-\sqrt{3}\eta)$$

$$\tilde{N}_{\tilde{2}}(\xi\,,\eta) = \frac{1}{4}(1+\sqrt{3}\xi)(1-\sqrt{3}\eta)$$

$$\tilde{N}_{\tilde{3}}(\xi\,,\,\eta) \quad = \quad \frac{1}{4}(1+\sqrt{3}\xi)(1+\sqrt{3}\eta)$$

$$\tilde{N}_{\tilde{4}}(\xi\,,\,\eta) \quad = \quad \frac{1}{4}(1-\sqrt{3}\xi)(1+\sqrt{3}\eta)$$

From these expressions, it can be verified that the interpolation property is satisfied.

4.4 Exercise 2, p. 223

The selective integration procedure can be implemented in the following two-loop generalization of the box on p. 151 in the text.

Loop over the number of normal integration points for $\bar{\boldsymbol{k}}^e$, $l = 1, 2, \ldots, \bar{n}_{int}$.

Set up the strain-displacement matrix $\boldsymbol{B}$.

Set up the constitutive matrix $\tilde{\bar{\boldsymbol{D}}}$.

Multiply $\tilde{\bar{\boldsymbol{D}}} * \boldsymbol{B}$.

Multiply $\boldsymbol{B}^T * (\tilde{\bar{\boldsymbol{D}}}\boldsymbol{B})$, taking account of symmetry, and accumulate in $\boldsymbol{k}^e$.

Next integration point

Loop over the number of reduced integration points for $\bar{\bar{\boldsymbol{k}}}^e$, $l = 1, 2, \ldots, \bar{\bar{n}}_{int}$.

Set up the strain-displacement matrix $\boldsymbol{B}$.

Set up the constitutive matrix $\tilde{\bar{\bar{\boldsymbol{D}}}}$.

Multiply $\tilde{\bar{\bar{\boldsymbol{D}}}} * \boldsymbol{B}$.

Multiply $\boldsymbol{B}^T * (\tilde{\bar{\bar{\boldsymbol{D}}}}\boldsymbol{B})$, taking account of symmetry, and accumulate in $\boldsymbol{k}^e$.

Next integration point

4.4 Exercise 3, p. 223

$$a(\boldsymbol{w}\,,\,\boldsymbol{u}) \quad = \quad \int_\Omega w_{(i,j)}\, c_{ijkl}\, u_{(k,l)}\, d\Omega$$

For isotropic elasticity

$$\begin{aligned}
a(\boldsymbol{w}\,,\,\boldsymbol{u}) &= \int_\Omega w_{(i,j)}\left\{\mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}\right\} u_{(k,l)}\, d\Omega \\
&= \int_\Omega w_{(i,j)}\bar{c}_{ijkl}u_{(k,l)}\, d\Omega + \int_\Omega w_{(i,i)}\lambda u_{(k,k)}\, d\Omega \\
&= \bar{a}(\boldsymbol{w}\,,\,\boldsymbol{u}) + (\operatorname{div}\boldsymbol{w}\,,\,\lambda\operatorname{div}\boldsymbol{u}) \\
&= \tilde{a}(\boldsymbol{w}\,,\,\boldsymbol{u}) + \bar{\bar{a}}(\boldsymbol{w}\,,\,\boldsymbol{u})
\end{aligned}$$

4.4 Exercise 4, p. 225

The two-dimensional case is solved in the text. Here we consider the three-dimensional case.

$$\begin{aligned}
n_{eq} &= 3(n-1)^3 \\
n_c &= n^3 - 1 \qquad \text{(normal integration; exact)} \\
r &= \frac{3(n-1)^3}{n^3-1} = \frac{3(n-1)^2}{n^2+n+1}
\end{aligned}$$

As in the two-dimensional case, r improves as the order of the element increases:

n	r
2	$\frac{3}{7}$
3	$\frac{12}{13}$
4	$\frac{9}{7}$
$\vdots$	$\vdots$
∞	3 $(= n_{sd})$

In the case of reduced integration of the λ-term,

$$\begin{aligned}
n_c &= (n-1)^3 \\
r &= 3 \qquad (= n_{sd}) \qquad \text{independent of } n
\end{aligned}$$

4.5 Exercise 1, p. 233

$$\epsilon^{\text{dil}}_{ij}(\boldsymbol{u}^h) = \frac{1}{3}\delta_{ij}u^h_{k,k} = \frac{1}{3}\delta_{ij}\sum_{a=1}^{n_{en}} N_{a,k}d^e_{ka}$$

Vector form:

$$\boldsymbol{\epsilon}^{\text{dil}}(\boldsymbol{u}^h) = \begin{Bmatrix} \epsilon^{\text{dil}}_{11}(\boldsymbol{u}^h) \\ \epsilon^{\text{dil}}_{22}(\boldsymbol{u}^h) \\ \epsilon^{\text{dil}}_{33}(\boldsymbol{u}^h) \\ \epsilon^{\text{dil}}_{23}(\boldsymbol{u}^h) \\ \epsilon^{\text{dil}}_{31}(\boldsymbol{u}^h) \\ \epsilon^{\text{dil}}_{12}(\boldsymbol{u}^h) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(\frac{1}{3}\sum_{a=1}^{n_{en}} N_{a,k}d^e_{ka} \right)$$

$$= \frac{1}{3}\sum_{a=1}^{n_{en}} \begin{bmatrix} N_{a,1} & N_{a,2} & N_{a,3} \\ N_{a,1} & N_{a,2} & N_{a,3} \\ N_{a,1} & N_{a,2} & N_{a,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d^e_{1a} \\ d^e_{2a} \\ d^e_{3a} \end{Bmatrix} = \sum_{a=1}^{n_{en}} \boldsymbol{B}^{\text{dil}}_a \boldsymbol{d}^e_a$$

Therefore

$$\boldsymbol{B}^{\text{dil}}_a = \frac{1}{3}\begin{bmatrix} B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $B_i = N_{a,i}$.

4.6 Exercise 1, p. 241

Assume an isotropic, homogeneous material in which

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where the Lamé parameters are constants. Thus we have

$$u_{(i,j)}c_{ijkl}u_{(k,l)} = \lambda u_{i,i}u_{j,j} + 2\mu u_{(i,j)}u_{(i,j)}$$

The exact strain energy is given by

$$\frac{1}{2}\int_{\Omega^e} u_{(i,j)} c_{ijkl} u_{(k,l)}\, d\Omega \quad = \quad \frac{\lambda}{2}\int_{\Omega^e} u_{i,i} u_{j,j}\, d\Omega + \mu \int_{\Omega^e} u_{(i,j)} u_{(i,j)}\, d\Omega$$

Given the regular geometry of the two-dimensional element under consideration (i.e., $x = h\xi/2$, $y = h\eta/2$), it is a simple matter to exactly evaluate these integrals. Likewise, it is simple to evaluate the one-point quadrature expression for strain energy

$$2(j\, u_{(i,j)} c_{ijkl} u_{(k,l)})|_{\xi=\eta=0} \quad = \quad h^2 \left(\frac{\lambda}{2} u_{i,i} u_{j,j} + \mu u_{(i,j)} u_{(i,j)}\right)\Big|_{\xi=\eta=0}$$

These results are summarized in the following table:

	Exact	One-point
x-translation	0	0
y-translation	0	0
infinitesimal rotation	0	0
x-hourglass	$2c^2(\frac{\lambda}{3} + \mu)$	0
y-hourglass	$2c^2(\frac{\lambda}{3} + \mu)$	0
uniform x-extension	$2c^2(\lambda + 2\mu)$	$2c^2(\lambda + 2\mu)$
uniform y-extension	$2c^2(\lambda + 2\mu)$	$2c^2(\lambda + 2\mu)$
uniform shear	$8c^2\mu$	$8c^2\mu$

4.7 Exercise 1, p. 250

$$N_1(r,s,t) = 1 - 2r$$

$$N_2(r,s,t) = 1 - 2s$$

$$N_3(r,s,t) = 1 - 2t$$

a. Recall from (3.I.30) that

$$\int_{\Omega} r^{\alpha} s^{\beta} t^{\gamma}\, d\Omega \quad = \quad \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} 2A$$

where A is the element area.

$$\begin{aligned}
\int_{\Omega^e} N_1^2 \, d\Omega &= \int_{\Omega^e} (1-2r)^2 \, d\Omega \\
&= \int_{\Omega^e} (1 - 4r + 4r^2) \, d\Omega \\
&= \int_{\Omega^e} (r^0 s^0 t^0 - 4r^1 s^0 t^0 + 4r^2 s^0 t^0) \, d\Omega \\
&= \left(\frac{1}{2} - \frac{4}{3!} + \frac{4 \cdot 2}{4!} \right) 2A \\
&= \frac{A}{3}
\end{aligned}$$

It is obvious that the same result is obtained for N_2^2 and N_3^2.

$$\begin{aligned}
\int_{\Omega^e} N_2 N_3 \, d\Omega &= \int_{\Omega^e} (1-2s)(1-2t) \, d\Omega \\
&= \int_{\Omega^e} (1 - 2s - 2t + 4st) \, d\Omega \\
&= \int_{\Omega^e} (r^0 s^0 t^0 - 2r^0 s^1 t^0 - 2r^0 s^0 t^1 + 4r^0 s^1 t^1) \, d\Omega \\
&= \left(\frac{1}{2} - \frac{2}{3!} - \frac{2}{3!} + \frac{4}{4!} \right) 2A \\
&= 0
\end{aligned}$$

The calculation of the other cases yields identical results.

b.

$$\begin{aligned}
n_{eq} &= 6n_{es}^2 \\
n_c &= 2n_{es}^2 \\
r &= 3
\end{aligned}$$

c. We want shape functions that take on the value of one at the node they are associated with (i.e., the centroid of one face of the tetrahedron) and vanish at the centroids of the other faces. These requirements result in

$$\begin{aligned} N_1(r,s,t,u) &= 1-3r \qquad & N_2(r,s,t,u) &= 1-3s \\ N_3(r,s,t,u) &= 1-3t \qquad & N_4(r,s,t,u) &= 1-3u \end{aligned}$$

4.8 Exercise 1, p. 252

$$\boldsymbol{f}_a^{\text{int}} = \int_{\Omega^e} \boldsymbol{B}_a^T \boldsymbol{\sigma}\, d\Omega = \sigma_0 \int_{\Omega^e} \left\{ \begin{matrix} yN_{a,x} \\ 0 \end{matrix} \right\} d\Omega$$

For a rectangular element aligned with the global x and y coordinates,

$$N_{a,x} = N_{a,\xi}\xi_{,x} + N_{a,\eta}\eta_{,x} = N_{a,\xi}\xi_{,x}$$

$$\xi_{,x} = \frac{h_2}{2} j^{-1} = \frac{2}{h_1}$$

$$j = \frac{h_1 h_2}{4} \qquad \text{vol}(\Omega^e) = h_1 h_2 t$$

N.B. The element thickness, t, was omitted in early printings of the book.

$$y = \frac{h_2}{2}\eta = \frac{h_1 h_2}{4}\frac{2}{h_1}\eta = \frac{1}{4t}\text{vol}(\Omega^e)\xi_{,x}\eta$$

Thus,

$$yN_{a,x} = \frac{1}{4t}\,\text{vol}(\Omega^e)(\xi_{,x})^2\eta N_{a,\xi} = \frac{1}{16t}\,\text{vol}(\Omega^e)(\xi_{,x})^2\eta\xi_a(1+\eta_a\eta)$$

$$\int_{\Omega^e} yN_{a,x}\, d\Omega = \frac{1}{64t}\,\text{vol}(\Omega^e)^2(\xi_{,x})^2\xi_a \int_{-1}^{+1}\int_{-1}^{+1} \eta(1+\eta_a\eta)\, d\xi d\eta$$

Note that

$$\int_{-1}^{+1} \eta\, d\eta = 0\,, \text{ and } \int_{-1}^{+1} \eta^2\, d\eta = \frac{2}{3}$$

Thus,

$$\int_{\Omega^e} y N_{a,x}\, d\Omega \;=\; \frac{1}{64t}\,\mathrm{vol}(\Omega^e)^2(\xi_{,x})^2\xi_a\eta_a(\frac{4}{3}) \;=\; -\frac{1}{48t}\,\mathrm{vol}(\Omega^e)^2(\xi_{,x})^2\phi_a$$

where $\phi_a = (-1)^a$. With $\sigma_0 = -\frac{ct}{\mathrm{vol}(\Omega^e)}(\sum_{b=1}^4 \phi_b d_{1b})$, we have finally

$$\boldsymbol{f}_a^{\mathrm{int}} \;=\; \left\{ \begin{array}{c} \frac{c\,\mathrm{vol}(\Omega^e)}{48}(\xi_{,x})^2\phi_a(\sum_{b=1}^4 \phi_b d_{1b}) \\ \\ 0 \end{array} \right\}$$

4.8 Exercise 2, p. 252

Let the (x', y') coordinate system be aligned with the (ξ, η) local coordinate system. Then by using (4.8.5) and (4.8.7)

$$\boldsymbol{f}_a^{\mathrm{int}'} \;=\; \frac{c\,\mathrm{vol}(\Omega^e)}{48}\begin{bmatrix} (\xi_{,x'})^2 & 0 \\ 0 & (\eta_{,y'})^2 \end{bmatrix}\left\{ \begin{array}{c} \phi_a \sum_{b=1}^4 d'_{1b} \\ \phi_a \sum_{b=1}^4 d'_{2b} \end{array} \right\}$$

Let

$$\boldsymbol{g} \;=\; \begin{bmatrix} (\xi_{,x'})^2 & 0 \\ 0 & (\eta_{,y'})^2 \end{bmatrix} \;=\; \begin{bmatrix} (\xi_{,x'_1})^2 & 0 \\ 0 & (\eta_{,x'_2})^2 \end{bmatrix}$$

where

$$x'_1 = x', \qquad x'_2 = y'$$

$$\left\{ \begin{array}{c} x'_1 \\ x'_2 \end{array} \right\} \;=\; \boldsymbol{G}\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}$$

and

$$\boldsymbol{G} \;=\; \begin{bmatrix} \dfrac{\partial x'_1}{\partial x_1} & \dfrac{\partial x'_1}{\partial x_2} \\ \\ \dfrac{\partial x'_2}{\partial x_1} & \dfrac{\partial x'_2}{\partial x_2} \end{bmatrix}$$

Since x_1' and x_2' are parallel to ξ and η, respectively,

$$x'_{1,\xi} = (\xi_{,x_1'})^{-1} \qquad x'_{2,\eta} = (\eta_{,x_2'})^{-1} \qquad x'_{1,\eta} = 0 \qquad x'_{2,\xi} = 0$$

and therefore

$$\frac{\partial x_1'}{\partial x_1} = x'_{1,\xi}\xi_{,x_1} + x'_{1,\eta}\eta_{,x_1} = \frac{\xi_{,x_1}}{\xi_{,x_1'}}$$

Thus,

$$\begin{aligned}
\boldsymbol{G} &= \begin{bmatrix} \dfrac{\xi_{,x_1}}{\xi_{,x_1'}} & \dfrac{\xi_{,x_2}}{\xi_{,x_1'}} \\ \dfrac{\eta_{,x_1}}{\eta_{,x_2'}} & \dfrac{\eta_{,x_2}}{\eta_{,x_2'}} \end{bmatrix} \\
\boldsymbol{f}_a^{\text{int}} &= \boldsymbol{G}^T \boldsymbol{f}_a^{\text{int}\prime} \\
\boldsymbol{d}_b' &= \begin{Bmatrix} d_{1b}' \\ d_{2b}' \end{Bmatrix} = \boldsymbol{G}\boldsymbol{d}_b
\end{aligned}$$

and

$$\boldsymbol{f}_a^{\text{int}} = \frac{c\ \text{vol}(\Omega^e)}{48}\phi_a \boldsymbol{G}^T \boldsymbol{g}\boldsymbol{G} \begin{Bmatrix} \sum_{b=1}^4 d_{1b} \\ \sum_{b=1}^4 d_{2b} \end{Bmatrix}$$

where

$$\begin{aligned}
\boldsymbol{g}\boldsymbol{G} &= \begin{bmatrix} \xi_{,x_1'}\xi_{,x_1} & \xi_{,x_1'}\xi_{,x_2} \\ \eta_{,x_1}\eta_{,x_2'} & \eta_{,x_2}\eta_{,x_2'} \end{bmatrix} \\
\boldsymbol{G}^T\boldsymbol{g}\boldsymbol{G} &= \begin{bmatrix} (\xi_{,x_1})^2 + (\eta_{,x_1})^2 & \xi_{,x_1}\xi_{,x_2} + \eta_{,x_1}\eta_{,x_2} \\ \xi_{,x_2}\xi_{,x_1} + \eta_{,x_2}\eta_{,x_1} & (\xi_{,x_2})^2 + (\eta_{,x_2})^2 \end{bmatrix}
\end{aligned}$$

The result can be written compactly in indexed notation as:

$$\begin{aligned} \boldsymbol{f}_a^{\text{int}} &= \left\{ f_{ia}^{\text{int}} \right\} \\ &= s_{ij} \phi_a \sum_{b=1}^{4} \phi_b d_{jb} \qquad (\text{sum on } j = 1, 2) \end{aligned}$$

where

$$s_{ij} = \frac{c \ \text{vol}(\Omega^e)}{48} \left(\xi_{,x_i} \xi_{,x_j} + \eta_{,x_i} \eta_{,x_j} \right)$$

4.8 Exercise 3, p. 254

a. $x_{i,j} = \dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$ and $x_i = \sum_{a=1}^{4} N_a x_{ia}$. Therefore

$$\sum_{a=1}^{4} b_{ja} x_{ia} = \sum_{a=1}^{4} N_{a,j} x_{ia} = x_{i,j} = \delta_{ij}$$

b. This is equivalent to proving

$$a_1 = -\sum_{b=1}^{4} h_b x_{1b} \,, \qquad a_2 = -\sum_{b=1}^{4} h_b x_{2b} \,, \qquad \text{and} \qquad a_3 = 0$$

We recall (4.8.14)–(4.8.16):

$$\sum_{a=1}^{4} \tilde{\phi}_a = 0 \,, \qquad \sum_{a=1}^{4} \tilde{\phi}_a x_{1a} = 0 \,, \qquad \sum_{a=1}^{4} \tilde{\phi}_a x_{2a} = 0$$

Employing the first of these and the definition of $\tilde{\phi}_a$ yields

$$a_1 \sum_{a=1}^{4} b_{1a} + a_2 \sum_{a=1}^{4} b_{2a} + 4a_3 + \sum_{a=1}^{4} h_a = \sum_{a=1}^{4} \tilde{\phi}_a = 0$$

Note that $\sum_{a=1}^{4} h_a = (-1+1-1+1) = 0$. Thus

$$a_1 \sum_{a=1}^{4} b_{1a} + a_2 \sum_{a=1}^{4} b_{2a} + 4a_3 \quad = \quad 0$$

We recall that

$$\sum_{a=1}^{4} N_a \quad = \quad 1 \quad \text{and} \quad \sum_{a=1}^{4} N_{a,x_1} \quad = \quad \sum_{a=1}^{4} N_{a,x_2} \quad = \quad 0$$

and, using the definitions from part a, we get

$$\sum_{a=1}^{4} b_{1a} = \sum_{a=1}^{4} N_{a,x_1} = 0 \qquad \sum_{a=1}^{4} b_{2a} = \sum_{a=1}^{4} N_{a,x_2} = 0$$

Combining these results yields $a_3 = 0$.

Now we employ (4.8.15) and the definition of $\tilde{\phi}_a$:

$$a_1 \sum_{a=1}^{4} b_{1a}x_{1a} + a_2 \sum_{a=1}^{4} b_{2a}x_{1a} + a_3 \sum_{a=1}^{4} x_{1a} + \sum_{a=1}^{4} h_a x_{1a} = \sum_{a=1}^{4} \tilde{\phi}_a x_{1a} = 0$$

By part a,

$$\sum_{a=1}^{4} b_{1a}x_{1a} \quad = \quad 1 \qquad\qquad \sum_{a=1}^{4} b_{2a}x_{1a} \quad = \quad 0$$

Since $a_3 = 0$

$$a_1 \quad = \quad -\sum_{a=1}^{4} h_a x_{1a}$$

Likewise, we employ (4.8.16), the definition of $\tilde{\phi}_a$, and similar reasoning to obtain:

$$\begin{aligned} 0 \quad &= \quad \sum_{a=1}^{4} \tilde{\phi}_a x_{2a} \\ &= \quad a_1 \sum_{a=1}^{4} b_{1a}x_{2a} + a_2 \sum_{a=1}^{4} b_{2a}x_{2a} + a_3 \sum_{a=1}^{4} x_{2a} + \sum_{a=1}^{4} h_a x_{2a} \\ &= \quad a_2 + \sum_{a=1}^{4} h_a x_{2a} \end{aligned}$$

Therefore

$$a_2 = -\sum_{a=1}^{4} h_a x_{2a}$$

Thus we have obtained the desired result:

$$\tilde{\phi}_a = h_a - \sum_{b=1}^{4}[(h_b x_{1b}) b_{1a} + (h_b x_{2b}) b_{2a}]$$

4.9 Exercise 4, p. 258

From Sec. 4.1, p. 190, using results obtained by the Aubin-Nitsche method:

$$\| \boldsymbol{e} \|_s \leq ch^{\beta} \| \boldsymbol{u} \|_{k+1}$$

where $\beta = \min(k + 1 - s, 2(k + 1 - m))$ and $0 \leq s \leq m$. For elasticity, $m = 1$. For complete fifth-order polynomials, $k = 5$. Thus for $s = 0$, $\beta = \min(6, 2(6 - 1)) = 6$. For $s = 1$, $\beta = \min(5, 2(5 - 1)) = 5$. Therefore

$$\| \boldsymbol{e} \|_0 \leq ch^6 \| \boldsymbol{u} \|_6 \quad \text{and} \quad \| \boldsymbol{e} \|_1 \leq ch^5 \| \boldsymbol{u} \|_6$$

4.9 Exercise 5, p. 258

The standard error estimate, from Sec. 4.1, p. 190, is

$$\| \boldsymbol{e} \|_m \leq ch^{k+1-m} \| \boldsymbol{u} \|_{k+1}$$

Here $m = 1$, so,

$$\| \boldsymbol{e} \|_1 \leq ch^k \| \boldsymbol{u} \|_{k+1}$$

Since $\| \boldsymbol{e} \|_1 = O(h^2)$, $k = 2$. Therefore the nine-node Lagrangian element was used.

4.9 Exercise 6, p. 258

a. Lagrange-multiplier method:

Let

$$\mathcal{H}(\boldsymbol{d}, m) \quad = \quad \mathcal{F}(\boldsymbol{d}) + m\mathcal{G}(\boldsymbol{d})$$

where

$$\mathcal{F}(\boldsymbol{d}) \quad = \quad \frac{1}{2}\boldsymbol{d}^T\boldsymbol{K}\boldsymbol{d} - \boldsymbol{d}^T\boldsymbol{F} \qquad \text{and} \qquad \mathcal{G}(\boldsymbol{d}) \quad = \quad d_1 + d_2$$

Minimizing $\mathcal{H}$ results in

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{1} \\ \boldsymbol{1}^T & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{d} \\ m \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F} \\ 0 \end{Bmatrix} \quad \text{where} \quad \boldsymbol{1} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

or, in expanded form,

$$\begin{bmatrix} K & \epsilon K & 1 \\ \epsilon K & K & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ m \end{Bmatrix} \quad = \quad \begin{Bmatrix} F_1 \\ F_2 \\ 0 \end{Bmatrix}$$

For $\quad K = 1\,,$ and $\epsilon = 0\,,$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ m \end{Bmatrix} \quad = \quad \begin{Bmatrix} F_1 \\ F_2 \\ 0 \end{Bmatrix}$$

Solving,

$$m = \frac{1}{2}(F_1 + F_2)\,, \qquad d_1 = \frac{1}{2}(F_1 - F_2) \quad \text{and} \quad d_2 = \frac{1}{2}(F_2 - F_1)$$

b. Penalty method:

Let

$$\mathcal{I}(\boldsymbol{d}) \quad = \quad \mathcal{F}(\boldsymbol{d}) + \frac{k}{2}\mathcal{G}(\boldsymbol{d})^2$$

Minimizing $\mathcal{I}$ results in

$$(\boldsymbol{K} + k\,\boldsymbol{1}\,\boldsymbol{1}^T)\boldsymbol{d} \quad = \quad \boldsymbol{F}$$

$$\begin{bmatrix} K+k & \epsilon K+k \\ \epsilon K+k & K+k \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Setting $\epsilon = 0$ and $K = 1$, we have

$$\begin{bmatrix} 1+k & k \\ k & 1+k \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

The solution is

$$d_1 = \frac{1+k^{-1}}{2+k^{-1}}F_1 - \frac{1}{2+k^{-1}}F_2 \quad \text{and} \quad d_2 = \frac{1+k^{-1}}{2+k^{-1}}F_2 - \frac{1}{2+k^{-1}}F_1$$

Thus,

$$\lim_{k\to\infty} d_1 = \frac{1}{2}(F_1 - F_2) \qquad \text{and} \qquad \lim_{k\to\infty} d_2 = \frac{1}{2}(F_2 - F_1)$$

From (4.2.27)

$$\begin{aligned} m &\cong k\mathcal{G}(\boldsymbol{d}) \\ &= k(d_1 + d_2) \\ &= \frac{1}{2+k^{-1}}(F_1 + F_2) \end{aligned}$$

Note that

$$m = \lim_{k\to\infty} \frac{1}{2+k^{-1}}(F_1 + F_2) = \frac{1}{2}(F_1 + F_2)$$

4.9 Exercise 7, p. 258

a. Lagrange-multiplier method:

$$\boldsymbol{Kd} = \boldsymbol{F} \qquad \alpha d_P + \beta d_Q = \gamma$$

So,

$$\mathcal{G}(\boldsymbol{d}) = (\alpha \boldsymbol{1}_P^T + \beta \boldsymbol{1}_Q^T)\boldsymbol{d} - \gamma \quad \text{and} \quad \mathcal{H}(\boldsymbol{d}, m) = \mathcal{F}(\boldsymbol{d}) + m\mathcal{G}(\boldsymbol{d})$$

Setting

$$\frac{\partial \mathcal{H}}{\partial d_1} = 0\,, \qquad \frac{\partial \mathcal{H}}{\partial d_2} = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial m} = 0$$

results in the matrix equation

$$\begin{bmatrix} \boldsymbol{K} & \alpha\mathbf{1}_P + \beta\mathbf{1}_Q \\ \alpha\mathbf{1}_P^T + \beta\mathbf{1}_Q^T & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{d} \\ m \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F} \\ \gamma \end{Bmatrix}$$

b. Penalty method:

$$m \cong k\mathcal{G}(\boldsymbol{d}) \ , \qquad \mathcal{I}(\boldsymbol{d}) = \mathcal{F}(\boldsymbol{d}) + \frac{1}{2}k\mathcal{G}(\boldsymbol{d})^2$$

Minimization of $\mathcal{I}$ leads to

$$\left[\boldsymbol{K} + k(\alpha\mathbf{1}_P + \beta\mathbf{1}_Q)(\alpha\mathbf{1}_P^T + \beta\mathbf{1}_Q^T)\right]\boldsymbol{d} = \boldsymbol{F} + k(\alpha\mathbf{1}_P + \beta\mathbf{1}_Q)\gamma$$

4.9 Exercise 8, p. 259

a. "Enriched" bilinear displacement–constant pressure element:

$$r = \frac{4}{1} = 4$$

b. "Enriched" trilinear displacement–constant pressure element:

$$r = \frac{6}{1} = 6$$

4.9 Exercise 9, p. 259

$$r = \frac{24}{12} = 2$$

4.9 Exercise 10, p. 259

Problem 2 requires $d_a = x_a$ but $d_5 = 2 \neq 2.1 = x_5$. So the element *fails* the patch test because it does not represent the linear temperature field,

$$u(x,y) = x$$

4.9 Exercise 11, p. 260

See the examples on the bottom of p. 238. The solution at nodes $a = 5, 6, 7, 8$ may be determined from the tables in the examples on p. 238 and the table on the top of p. 261.

Appendix 4.I Exercise 1, p. 265

i. Positive-definiteness immediately follows from property (iv) of the definition of an inner product (p. 264).

ii. $\| \alpha x \| = \langle \alpha x , \alpha x \rangle^{\frac{1}{2}} = (\alpha^2 \langle x , x \rangle)^{\frac{1}{2}} = |\alpha| \|x\|$

iii. $\| x + y \| = \langle x + y , x + y \rangle^{\frac{1}{2}} = (\langle x , x \rangle + 2\langle x , y \rangle + \langle y , y \rangle)^{\frac{1}{2}}$. Using the Schwarz inequality,

$$|\langle x , y \rangle| \quad < \quad \langle x , x \rangle^{\frac{1}{2}} \langle y , y \rangle^{\frac{1}{2}}$$

Therefore

$$\begin{aligned} \| x + y \| &\leq \Big(\langle x , x \rangle + 2\langle x , x \rangle^{\frac{1}{2}} \langle y , y \rangle^{\frac{1}{2}} + \langle y , y \rangle \Big)^{\frac{1}{2}} \\ &= \left[\Big(\langle x , x \rangle^{\frac{1}{2}} + \langle y , y \rangle^{\frac{1}{2}} \Big)^2 \right]^{\frac{1}{2}} = \| x \| + \| y \| \end{aligned}$$

Appendix 4.I Exercise 2, p. 268

It is obvious that $f = \log \left(\log r^{-1} \right)$ is not in $C_b^0(\Omega)$ because it is unbounded at $r = 0$, hence not continuous. The $H^1(\Omega)$ norm can be expressed in terms of the L_2 norm and the H^1 seminorm:

$$\|f\|_1^2 = \|f\|_0^2 + |f|_1^2$$

To calculate the H^1 seminorm of f it is convenient to work in polar coordinates.

$$\begin{aligned}
|f|_1^2 &= \int_0^{2\pi}\int_0^{\frac{1}{2}} \left(\frac{d}{dr}\left(\log(\log r^{-1})\right)\right)^2 r\,dr\,d\theta \\
&= 2\pi \int_0^{\frac{1}{2}} \left(\frac{-1}{r\log r^{-1}}\right)^2 r\,dr \\
&= 2\pi \int_{\log 2}^{\infty} \frac{du}{u^2} \\
&= 2\pi \int_0^{(\log 2)^{-1}} dv \\
&= \frac{2\pi}{\log 2}
\end{aligned}$$

We made use of the substitutions $u = \log r^{-1}$ and $v = u^{-1}$ in the above calculations.

In order to bound the L_2 norm of f we consider an auxiliary function

$$g = f - \log(\log 2) \qquad \text{in } \Omega$$

which was selected to satisfy the following two conditions:

$$\begin{aligned}
|g|_1 &= |f|_1 \\
g &= 0 \qquad \text{on } \partial\Omega
\end{aligned}$$

Satisfaction of the second condition allows us to apply a Poincaré-Friedrichs inequality* which states that there exists a constant $c = c(\Omega)$ such that

$$c\,|g|_1^2 \geq \|g\|_0^2$$

* This is a well-known result in functional analysis with numerous applications in finite element methods; see, e.g., G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, 1973, p. 42 for the derivation of a one-dimensional version of the inequality, and p. 69 for the statement of a two-dimensional version; a proof for the multi-dimensional case is presented by B. Mercier, *Topics in Finite Element Solution of Elliptic Problems*, Springer-Verlag, 1979, pp. 9–10; also see J. Mason, *Methods of Functional Analysis for Application in Solid Mechanics*, Elsevier, 1985, p. 94 and references therein, for generalizations and proofs of inequalities this type.

Therefore

$$\begin{aligned}
\sqrt{c}\,|f|_1 &= \sqrt{c}\,|g|_1 \\
&\geq \|g\|_0 \\
&= \|f - \log(\log 2)\|_0 \\
&\geq \|f\|_0 - \|\log(\log 2)\|_0 \\
&= \|f\|_0 + \sqrt{\frac{\pi}{4}}\log(\log 2)
\end{aligned}$$

In the above calculation (fourth line) we made use of the triangle inequality. Hence

$$\|f\|_0 \quad \leq \quad \sqrt{\frac{2\pi c}{\log 2}} - \sqrt{\frac{\pi}{4}}\log(\log 2)$$

Combining the results verifies that $\log\left(\log r^{-1}\right)$ is a member of $H^1(\Omega)$.

CHAPTER 5

5.2 Exercise 1, p. 319

Using (5.2.29)–(5.2.32), (5.2.37) can be written as

$$\int_A [-\bar{\theta}_{(\alpha,\beta)} m_{\alpha\beta} - \bar{\theta}_\alpha q_\alpha + \bar{w}_{,\alpha} q_\alpha]\, dA + \int_A [\bar{\theta}_\alpha C_\alpha - \bar{w} F]\, dA + \int_{s_h} (\bar{\theta}_\alpha M_\alpha - \bar{w} Q)\, ds = 0$$

Due to the symmetry of tensor $m_{\alpha\beta}$

$$\bar{\theta}_{(\alpha,\beta)} m_{\alpha\beta} = \bar{\theta}_{\alpha,\beta} m_{\alpha\beta}$$

Rearranging the weak form, we have

$$\int_A [-\bar{\theta}_{\alpha,\beta} m_{\alpha\beta} + w_{,\alpha} q_\alpha]\, dA + \int_A [\bar{\theta}_\alpha (C_\alpha - q_\alpha) - \bar{w} F]\, dA + \int_{s_h} (\bar{\theta}_\alpha M_\alpha - \bar{w} Q)\, ds = 0$$

Using integration by parts on the first integral (assuming necessary differentiability) yields,

$$\int_A \bar{\theta}_\alpha (C_\alpha - q_\alpha + m_{\alpha\beta,\beta})\, dA - \int_A \bar{w}(q_{\alpha,\alpha} + F)\, dA + \int_{s_h} \bar{\theta}_\alpha (M_\alpha - m_{\alpha\beta} n_\beta)\, ds + \int_{s_h} \bar{w}(q_\alpha n_\alpha - Q)\, ds = 0$$

To complete the proof select in turn different choices for $\bar{\theta}_\alpha$ and w.

First, let $\bar{w} = 0$, and

$$\bar{\theta}_\alpha = (m_{\alpha\beta,\beta} + C_\alpha - q_\alpha)\phi$$

where $\phi > 0$ in A, $\phi = 0$ on s_h, and ϕ is smooth. This choice implies

$$m_{\alpha\beta,\beta} + C_\alpha - q_\alpha = 0 \qquad \text{in } A$$

Next, let

$$\bar{w} = (q_{\alpha,\alpha} + F)\phi \quad \text{and} \quad \bar{\theta}_\alpha = 0$$

This choice implies

$$q_{\alpha,\alpha} + F = 0 \quad \text{in } A$$

Now, let $\bar{w} = 0$, and

$$\bar{\theta}_\alpha = (M_\alpha - m_{\alpha\beta} n_\beta)\psi$$

where $\psi = 0$ on s_g, $\psi > 0$ on s_h, and ψ is smooth. Then,

$$M_\alpha = m_{\alpha\beta} n_\beta \quad \text{on } s_h$$

Finally, let

$$\bar{w} = (q_\alpha n_\alpha - Q)\psi$$

which leads to

$$Q = q_\alpha n_\alpha \quad \text{on } s_h$$

5.2 Exercise 2, p. 319

$$\begin{aligned}
a(\bar{\boldsymbol{u}}, \boldsymbol{u}) &= \int_A [\bar{\theta}_{(\alpha,\beta)} c_{\alpha\beta\gamma\delta} \theta_{(\gamma,\delta)} + (\bar{w}_{,\alpha} - \bar{\theta}_\alpha) c_{\alpha\beta} (w_{,\beta} - \theta_\beta)]\, dA \\
(\bar{\boldsymbol{u}}, \boldsymbol{f}) &= \int_A (\bar{w} F - \bar{\theta}_\alpha C_\alpha)\, dA = \int_A \boldsymbol{u} \cdot \boldsymbol{f}\, dA \\
(\bar{\boldsymbol{u}}, \mathbf{h})_\Gamma &= \int_{s_h} (\bar{w} Q - \bar{\theta}_\alpha M_\alpha)\, ds = \int_{s_h} \boldsymbol{u} \cdot \mathbf{h}\, ds
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{u} &= \left\langle w \quad \theta_1 \quad \theta_2 \right\rangle^T \\
\boldsymbol{f} &= \left\langle F \quad -C_1 \quad -C_2 \right\rangle^T \\
\mathbf{h} &= \left\langle Q \quad -M_1 \quad -M_2 \right\rangle^T
\end{aligned}$$

To prove symmetry of $a(\cdot\,,\,\cdot)$ recall the major symmetry of $c_{\alpha\beta\gamma\delta}$, (5.2.24), and symmetry of $c_{\alpha\beta}$, (5.2.25). Symmetry and bilinearity are then shown in the same manner as in Exercise 1 of Sec. 2.3 and Exercise 1 of Sec. 2.7. The proofs of symmetry and bilinearity of $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ are trivial.

5.2 Exercise 3, p. 322

(Consult Fig. 5.2.1 for an illustration of the relationships among the rotations.) The curvature-displacement relation is

$$\boldsymbol{\kappa} \quad = \quad \boldsymbol{B}^b \boldsymbol{d}^e \quad = \quad \sum_{a=1}^{n_{en}} \boldsymbol{B}_a^b \boldsymbol{d}_a^e$$

By defining a new displacement vector, $\hat{\boldsymbol{d}}_a^e$, using,

$$\boldsymbol{d}_a^e \quad = \quad \begin{Bmatrix} w_a^h \\ \theta_{1a}^h \\ \theta_{2a}^h \end{Bmatrix} \quad = \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} w_a^h \\ \hat{\theta}_{1a}^h \\ \hat{\theta}_{2a}^h \end{Bmatrix} \quad = \quad \boldsymbol{R}\hat{\boldsymbol{d}}_a^e$$

the new curvature-displacement relation is

$$\boldsymbol{\kappa} = \sum_{a=1}^{n_{en}} \boldsymbol{B}_a^b \boldsymbol{d}_a^e = \sum_{a=1}^{n_{en}} \hat{\boldsymbol{B}}_a^b \hat{\boldsymbol{d}}_a^e$$

where

$$\begin{aligned} \hat{\boldsymbol{B}}_a^b \quad &= \quad \begin{bmatrix} 0 & N_{a,x} & 0 \\ 0 & 0 & N_{a,y} \\ 0 & N_{a,y} & N_{a,x} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \quad \begin{bmatrix} 0 & 0 & -N_{a,x} \\ 0 & N_{a,y} & 0 \\ 0 & N_{a,x} & -N_{a,y} \end{bmatrix} \end{aligned}$$

Using the same type of argument for the shear strain-displacement relation,

$$\boldsymbol{\gamma} = \boldsymbol{B}^s \boldsymbol{d}^e = \sum_{a=1}^{n_{en}} \boldsymbol{B}^s_a \boldsymbol{d}^e_a = \sum_{a=1}^{n_{en}} \hat{\boldsymbol{B}}^s_a \boldsymbol{R} \hat{\boldsymbol{d}}^e_a$$

where

$$\hat{\boldsymbol{B}}^s_a = \begin{bmatrix} N_{a,x} & -N_a & 0 \\ N_{a,y} & 0 & -N_a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} N_{a,x} & 0 & N_a \\ N_{a,y} & -N_a & 0 \end{bmatrix}$$

To derive the modified source vector it is necessary to recall that in the variational equation the source vector term has the form:

$$\bar{\boldsymbol{d}}^{e\,T}_a \boldsymbol{f}^e_a = \left\langle \bar{w}^h_a \quad \bar{\theta}^h_{1a} \quad \bar{\theta}^h_{2a} \right\rangle \begin{Bmatrix} \int_{A^e} N_a F dA + \int_{s^e \cap s_h} N_a Q ds \\ -\int_{A^e} N_a C_1 dA - \int_{s^e \cap s_h} N_a M_1 ds \\ -\int_{A^e} N_a C_2 dA - \int_{s^e \cap s_h} N_a M_2 ds \end{Bmatrix}$$

Again the change in the displacement selection leads to

$$\bar{\boldsymbol{d}}^{e\,T}_a \boldsymbol{f}^e_a = \hat{\boldsymbol{d}}^{e\,T}_a \boldsymbol{R}^T \boldsymbol{f}^e_a = \hat{\bar{\boldsymbol{d}}}^{e\,T}_a \hat{\boldsymbol{f}}^e_a$$

where

$$\hat{\boldsymbol{f}}^e_a = \boldsymbol{R}^T \boldsymbol{f}^e_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \boldsymbol{f}^e_a$$

$$= \begin{Bmatrix} \int_{A^e} N_a F dA + \int_{s^e \cap s_h} N_a Q ds \\ -\int_{A^e} N_a C_2 dA - \int_{s^e \cap s_h} N_a M_2 ds \\ \int_{A^e} N_a C_1 dA + \int_{s^e \cap s_h} N_a M_1 ds \end{Bmatrix}$$

5.5 Exercise 1, p. 378

We first collect some preliminary results. The isoparametric mapping for a linear element of length h is

$$\begin{aligned} x &= \frac{1}{2}(1-\xi)x_1 + \frac{1}{2}(1+\xi)x_2 \\ &= \frac{1}{2}(x_1+x_2) + \frac{h}{2}\xi \end{aligned}$$

Displacement and rotation interpolations:

$$\begin{aligned} w &= \frac{1}{2}(1-\xi)w_1 + \frac{1}{2}(1+\xi)w_2 \\ \theta &= \frac{1}{2}(1-\xi)\theta_1 + \frac{1}{2}(1+\xi)\theta_2 \end{aligned}$$

Strains:

$$\begin{aligned} \kappa = \theta' = -\frac{\theta_1}{h} + \frac{\theta_2}{h} &= \frac{1}{h}\left\langle 0 \quad -1 \quad 0 \quad 1 \right\rangle \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \\ &= \boldsymbol{B}_b \boldsymbol{d} \end{aligned}$$

$$\begin{aligned} \gamma = w' - \theta &= -\frac{w_1}{h} + \frac{w_2}{h} - \frac{1}{2}(1-\xi)\theta_1 - \frac{1}{2}(1+\xi)\theta_2 \\ &= \left\langle -\frac{1}{h} \quad -\frac{1}{2}(1-\xi) \quad \frac{1}{h} \quad -\frac{1}{2}(1+\xi) \right\rangle \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \\ &= \boldsymbol{B}_s \boldsymbol{d} \end{aligned}$$

$$\boldsymbol{k}_b = \int_0^h \boldsymbol{B}_b^T (EI) \boldsymbol{B}_b dx \qquad \boldsymbol{k}_s = \int_0^h \boldsymbol{B}_s^T (\mu A^s) \boldsymbol{B}_s dx$$

i. Since $\boldsymbol{B}_b$ is a constant matrix,

$$\begin{aligned}
\boldsymbol{k}_b &= \frac{EI}{h}\left\langle 0 \quad -1 \quad 0 \quad 1 \right\rangle^T \left\langle 0 \quad -1 \quad 0 \quad 1 \right\rangle \\
&= \frac{EI}{h}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{aligned}$$

ii. Using one-point Gaussian quadrature,

$$\begin{aligned}
\boldsymbol{k}_s &= \frac{\mu A^s}{h}\left\langle -1 \quad -\tfrac{h}{2} \quad 1 \quad -\tfrac{h}{2} \right\rangle^T \left\langle -1 \quad -\tfrac{h}{2} \quad 1 \quad -\tfrac{h}{2} \right\rangle \\
&= \frac{\mu A^s}{h}\begin{bmatrix} 1 & \frac{h}{2} & -1 & \frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \\ -1 & -\frac{h}{2} & 1 & -\frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \end{bmatrix}
\end{aligned}$$

iii. Using two-point Gaussian quadrature,

$$\begin{aligned}
\boldsymbol{k}_s &= \frac{\mu A^s}{2h}\sum_{\xi=\pm\frac{1}{\sqrt{3}}}\begin{bmatrix} 1 & \frac{1-\xi}{2}h & -1 & \frac{1+\xi}{2}h \\ \frac{1-\xi}{2}h & (\frac{1-\xi}{2}h)^2 & -\frac{1-\xi}{2}h & \frac{1-\xi^2}{4}h^2 \\ -1 & -\frac{1-\xi}{2}h & 1 & -\frac{1+\xi}{2}h \\ \frac{1+\xi}{2}h & \frac{1-\xi^2}{4}h^2 & -\frac{1+\xi}{2}h & (\frac{1+\xi}{2}h)^2 \end{bmatrix} \\
&= \frac{\mu A^s}{h}\begin{bmatrix} 1 & \frac{h}{2} & -1 & \frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{3} & -\frac{h}{2} & \frac{h^2}{6} \\ -1 & -\frac{h}{2} & 1 & -\frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{6} & -\frac{h}{2} & \frac{h^2}{3} \end{bmatrix}
\end{aligned}$$

iv. The one-point quadrature version of $\boldsymbol{k}_s$ is proportional to the outer product of a single non-trivial vector. Consequently, its rank is exactly one. When two-point quadrature is used, $\boldsymbol{k}_s$ is proportional to the sum of two outer products of non-trivial vectors. Consequently, its rank is at most two. To verify that the rank is exactly two, it is necessary to show that the two vectors are linearly independent. This is verified as follows:

$$\left\{\begin{array}{c} -\frac{1}{h} \\ -\frac{1}{2}(1+\frac{1}{\sqrt{3}}) \\ \frac{1}{h} \\ -\frac{1}{2}(1-\frac{1}{\sqrt{3}}) \end{array}\right\} \qquad \left\{\begin{array}{c} -\frac{1}{h} \\ -\frac{1}{2}(1-\frac{1}{\sqrt{3}}) \\ \frac{1}{h} \\ -\frac{1}{2}(1+\frac{1}{\sqrt{3}}) \end{array}\right\}$$

There is no constant of proportionality between these vectors. Hence, they are linearly independent. (There are other ways to solve this problem.)

v. Using the residual flexibility technique, the one-point shear stiffness matrix is replaced by

$$\boldsymbol{k}_s \quad = \quad \frac{1}{h}\left(\frac{1}{\mu A^s}+\frac{h^2}{12EI}\right)^{-1}\begin{bmatrix} 1 & \frac{h}{2} & -1 & \frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \\ -1 & -\frac{h}{2} & 1 & -\frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \end{bmatrix}$$

The total stiffness is formed by adding to this shear matrix the bending stiffness,

$$\boldsymbol{k} \quad = \quad \boldsymbol{k}_b + \boldsymbol{k}_s$$

In the limit $\mu \to \infty$ the resultant stiffness is

$$\frac{EI}{h}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \frac{12EI}{h^3}\begin{bmatrix} 1 & \frac{h}{2} & -1 & \frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \\ -1 & -\frac{h}{2} & 1 & -\frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12EI}{h^3} & \frac{6EI}{h^2} & -\frac{12EI}{h^3} & \frac{6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{4EI}{h} & -\frac{6EI}{h^2} & \frac{2EI}{h} \\ -\frac{12EI}{h^3} & -\frac{6EI}{h^2} & \frac{12EI}{h^3} & -\frac{6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{2EI}{h} & -\frac{6EI}{h^2} & \frac{4EI}{h} \end{bmatrix}$$

which is identical to the one obtained using the Bernoulli-Euler theory with Hermite cubic interpolation functions. (See the solution of Exercise 1, part j, Sec. 1.16.)

5.5 Exercise 2, p. 379

The displacement and rotation interpolations are

$$w_i = \frac{1}{2}(1-\xi)w_{1i} + \frac{1}{2}(1+\xi)w_{2i} \qquad i = 1,2,3$$

$$\theta_i = \frac{1}{2}(1-\xi)\theta_{1i} + \frac{1}{2}(1+\xi)\theta_{2i} \qquad i = 1,2,3$$

The strains are

$$
\begin{aligned}
\gamma_1 &= \left\langle -\tfrac{1}{h} \quad 0 \quad 0 \quad 0 \quad -\tfrac{1}{2}(1-\xi) \quad 0 \quad \tfrac{1}{h} \quad 0 \quad 0 \quad 0 \quad -\tfrac{1}{2}(1+\xi) \quad 0 \right\rangle \boldsymbol{d} \\
\gamma_2 &= \left\langle 0 \quad -\tfrac{1}{h} \quad 0 \quad \tfrac{1}{2}(1-\xi) \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{h} \quad 0 \quad \tfrac{1}{2}(1+\xi) \quad 0 \quad 0 \right\rangle \boldsymbol{d} \\
\epsilon &= \left\langle 0 \quad 0 \quad -\tfrac{1}{h} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{h} \quad 0 \quad 0 \quad 0 \right\rangle \boldsymbol{d} \\
\kappa_1 &= \left\langle 0 \quad 0 \quad 0 \quad -\tfrac{1}{h} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{h} \quad 0 \quad 0 \right\rangle \boldsymbol{d} \\
\kappa_2 &= \left\langle 0 \quad 0 \quad 0 \quad 0 \quad -\tfrac{1}{h} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{h} \quad 0 \right\rangle \boldsymbol{d} \\
\psi &= \left\langle 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\tfrac{1}{h} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{h} \right\rangle \boldsymbol{d}
\end{aligned}
$$

where $\boldsymbol{d}$ is the element displacement vector,

$$
\boldsymbol{d} = \left\langle w_{11} \quad w_{12} \quad w_{13} \quad \theta_{11} \quad \theta_{12} \quad \theta_{13} \quad w_{21} \quad w_{22} \quad w_{23} \quad \theta_{21} \quad \theta_{22} \quad \theta_{23} \right\rangle^T
$$

The element stiffness in the local coordinate system may be inferred from the expression

$$
\bar{\boldsymbol{d}}^T \boldsymbol{k} \boldsymbol{d} = \int_0^h \left(\bar{\gamma}_1 \mu A_1^s \gamma_1 + \bar{\gamma}_2 \mu A_2^s \gamma_2 + \bar{\kappa}_1 E I_1 \kappa_1 + \bar{\kappa}_2 E I_2 \kappa_2 + \bar{\epsilon} E A \epsilon + \bar{\psi} E J \psi \right) dx
$$

Assuming one-point quadrature and carrying out the necessary calculations leads to

$$
\boldsymbol{k} = \begin{bmatrix}
k_1 & 0 & 0 & 0 & \frac{h}{2}k_1 & 0 & -k_1 & 0 & 0 & 0 & \frac{h}{2}k_1 & 0 \\
 & k_2 & 0 & -\frac{h}{2}k_2 & 0 & 0 & 0 & -k_2 & 0 & -\frac{h}{2}k_2 & 0 & 0 \\
 & & k_3 & 0 & 0 & 0 & 0 & 0 & -k_3 & 0 & 0 & 0 \\
 & & & k_4 & 0 & 0 & 0 & \frac{h}{2}k_2 & 0 & k_5 & 0 & 0 \\
 & & & & k_6 & 0 & -\frac{h}{2}k_1 & 0 & 0 & 0 & k_7 & 0 \\
 & & & & & k_8 & 0 & 0 & 0 & 0 & 0 & -k_8 \\
 & & & & & & k_1 & 0 & 0 & 0 & -\frac{h}{2}k_1 & 0 \\
 & & & & & & & k_2 & 0 & \frac{h}{2}k_2 & 0 & 0 \\
 & & & & & & & & k_3 & 0 & 0 & 0 \\
 & & & \text{Symmetric} & & & & & & k_4 & 0 & 0 \\
 & & & & & & & & & & k_6 & 0 \\
 & & & & & & & & & & & k_8
\end{bmatrix}
$$

where

$$k_1 = \frac{\mu A_1^s}{h} \qquad k_2 = \frac{\mu A_2^s}{h} \qquad k_3 = \frac{EA}{h}$$

$$k_4 = \frac{\mu A_2^s}{4}h + \frac{EI_1}{h} \qquad k_5 = \frac{\mu A_2^s}{4}h - \frac{EI_1}{h} \qquad k_6 = \frac{\mu A_1^s}{4}h + \frac{EI_2}{h}$$

$$k_7 = \frac{\mu A_1^s}{4}h - \frac{EI_2}{h} \qquad k_8 = \frac{EJ}{h}$$

An alternative procedure for calculating $\boldsymbol{k}$ is to simply employ equations (5.4.73), (5.4.74), and (5.4.85)–(5.4.97) directly.

CHAPTER 6

6.2 Exercise 1, p. 391

The $\boldsymbol{D}$ matrix for the three-dimensional isotropic case was derived in Exercise 5, Sec. 2.7, p. 83 in terms of Lamé parameters λ and μ, and with respect to the stress and strain component ordering given by (2.7.30) and (2.7.27), respectively. This matrix can be expressed in terms of E and ν by employing (2.7.32) and (2.7.33). Reordering according to that delineated by (6.2.29) yields:

$$\boldsymbol{D}^l = \boldsymbol{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 & \frac{\nu}{1-\nu} \\ & 1 & 0 & 0 & 0 & \frac{\nu}{1-\nu} \\ & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & \text{Symmetric} & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ & & & & & 1 \end{bmatrix}$$

The components of the 5×5 reduced constitutive matrix are defined by (6.2.36) and (6.2.37), namely

$$\begin{aligned} \tilde{\boldsymbol{D}}^l &= [\tilde{D}^l_{IJ}] \qquad 1 \le I, J \le 5 \\ \tilde{D}^l_{IJ} &= D^l_{IJ} - \frac{D^l_{I6} D^l_{6J}}{D^l_{66}} \end{aligned}$$

For example,

$$
\begin{aligned}
\tilde{D}^l_{11} &= D^l_{11} - \frac{D^l_{16} D^l_{61}}{D^l_{66}} \\
&= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(1 - \frac{\nu^2}{(1-\nu)^2} \right) \\
&= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(\frac{(1-\nu)^2 - \nu^2}{(1-\nu)^2} \right) \\
&= \frac{E}{(1+\nu)(1-2\nu)} \left(\frac{1-2\nu}{1-\nu} \right) \\
&= \frac{E}{1-\nu^2}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}^l_{12} &= D^l_{12} - \frac{D^l_{16} D^l_{62}}{D^l_{66}} \\
&= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(\frac{\nu}{1-\nu} - \frac{\nu^2}{(1-\nu)^2} \right) \\
&= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(\frac{\nu(1-\nu) - \nu^2}{(1-\nu)^2} \right) \\
&= \frac{\nu E}{1-\nu^2}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}^l_{33} &= D^l_{33} - \frac{D^l_{36} D^l_{63}}{D^l_{66}} \\
&= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(\frac{1-2\nu}{2(1-\nu)} \right) \\
&= \frac{E}{2(1+\nu)} \\
&= \frac{E}{1-\nu^2} \frac{1-\nu}{2}
\end{aligned}
$$

etc.

The result of these calculations is (6.2.38).

6.2 Exercise 2, p. 395

In attempting to solve this problem it is worthwhile to review the isoparametric concept vis-à-vis completeness (see Sec. 3.3). However, in the case of the shell interpolations the technique is somewhat different. Recall that the geometry of the element is defined by (6.2.1)–(6.2.6) and the kinematics by (6.2.20)–(6.2.24). The representations look deceptively alike:

$$\begin{aligned} \boldsymbol{x} &= \sum_{a=1}^{n_{en}} N_a \bar{\boldsymbol{x}}_a + \sum_{a=1}^{n_{en}} N_a \boldsymbol{X}_a \\ \boldsymbol{u} &= \sum_{a=1}^{n_{en}} N_a \bar{\boldsymbol{u}}_a + \sum_{a=1}^{n_{en}} N_a \boldsymbol{U}_a \end{aligned}$$

but $\boldsymbol{X}_a$ and $\boldsymbol{U}_a$ are perpendicular by construction, viz.

$$\begin{aligned} \boldsymbol{X}_a &= z_a \hat{\boldsymbol{X}}_a \\ \boldsymbol{U}_a &= z_a \hat{\boldsymbol{U}}_a \\ \hat{\boldsymbol{U}}_a &= \theta_{a2} \boldsymbol{e}^f_{a1} - \theta_{a1} \boldsymbol{e}^f_{a2} \\ \boldsymbol{e}^f_{a1} \cdot \hat{\boldsymbol{X}}_a &= 0 \qquad \boldsymbol{e}^f_{a2} \cdot \hat{\boldsymbol{X}}_a = 0 \end{aligned}$$

Thus the geometric and kinematic descriptions are not form-identical, precluding the type of argument used for continuum elements.

What we need to show is that $\boldsymbol{u}$ is capable of exactly representing a general linear polynomial in the global coordinates, namely

$$u_i = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3$$

when the nodal degrees of freedom are appropriately specified. To see that this is the case, set

$$\begin{aligned} \bar{u}_{ia} &= c_0 + c_1 x_{1a} + c_2 \bar{x}_{2a} + c_3 \bar{x}_{3a} + c_1 X_{1a} + c_2 X_{2a} + c_3 X_{3a} \\ \hat{U}_{ia} &= 0 \end{aligned}$$

Note that the components of $\boldsymbol{X}_a$ depend on ζ. Consequently, $\bar{u}_{ia}$ also inherits this dependence. (This is not a problem.) Making use of these specifications, we have that

$$
\begin{aligned}
u_i &= \sum_{a=1}^{n_{en}} N_a(c_0 + c_1\bar{x}_{1a} + c_2\bar{x}_{2a} + c_3\bar{x}_{3a}) \\
&\quad + \sum_{a=1}^{n_{en}} N_a(c_1 X_{1a} + c_2 X_{2a} + c_3 X_{3a}) \\
&= \left(\sum_{a=1}^{n_{en}} N_a\right) c_0 + c_1 \left(\sum_{a=1}^{n_{en}} N_a\bar{x}_{1a} + \sum_{a=1}^{n_{en}} N_a X_{1a}\right) \\
&\quad + c_2 \left(\sum_{a=1}^{n_{en}} N_a\bar{x}_{2a} + \sum_{a=1}^{n_{en}} N_a X_{2a}\right) + c_3 \left(\sum_{a=1}^{n_{en}} N_a\bar{x}_{3a} + \sum_{a=1}^{n_{en}} N_a X_{3a}\right) \\
&= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3
\end{aligned}
$$

which completes the demonstration.

6.3 Exercise 1, p. 408

Corresponding to the ordering determined by (6.3.10), the matrix $\boldsymbol{D}^l$ takes the form

$$
\boldsymbol{D}^l = \left[\begin{array}{ccc|ccc}
1 & 0 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 \\
0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 & 0 & 0 \\
\frac{\nu}{1-\nu} & 0 & 1 & \frac{\nu}{1-\nu} & 0 & 0 \\
\hline
\frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\
0 & 0 & 0 & 0 & & 0\frac{1-2\nu}{2(1-\nu)}
\end{array}\right]
$$

Employing (6.3.13) results in the desired reduced constitutive matrix $\tilde{\boldsymbol{D}}^l$, namely (6.3.17), viz.

$$\tilde{\boldsymbol{D}}^l = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\begin{bmatrix} 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} \frac{\nu}{1-\nu} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & 0 & \nu \\ 0 & \frac{1-\nu}{2} & 0 \\ \nu & 0 & 1 \end{bmatrix}$$

6.3 Exercise 2, p. 409

Apply formula (6.3.20) to (6.3.18) in order to derive (6.3.21), viz.

$$\tilde{D}^l_{11} \leftarrow \tilde{D}^l_{11} - \frac{\tilde{D}^l_{13}\tilde{D}^l_{31}}{\tilde{D}^l_{33}} = \frac{E}{1-\nu^2}(1-\nu^2) = E$$

$$\tilde{D}^l_{12} \leftarrow \tilde{D}^l_{12} - \frac{\tilde{D}^l_{13}\tilde{D}^l_{32}}{\tilde{D}^l_{33}} = 0$$

$$\tilde{D}^l_{13} \leftarrow \tilde{D}^l_{13} - \frac{\tilde{D}^l_{13}\tilde{D}^l_{33}}{\tilde{D}^l_{33}} = 0$$

$$\tilde{D}^l_{22} \leftarrow \tilde{D}^l_{22} - \frac{\tilde{D}^l_{23}\tilde{D}^l_{32}}{\tilde{D}^l_{33}} = \frac{E}{1-\nu^2}\left(\frac{\kappa(1-\nu)}{2}\right) = \frac{\kappa E}{2(1+\nu)}$$

etc.

CHAPTER 7

7.1 Exercise 1, p. 420

Use the same procedure as in Sec. 2.3. The only additional requirement is to show equivalence of (7.1.12) and (7.1.8). Assume u is a solution of (S). Then, using (7.1.8)

$$(w\,,\,\rho c\,u(\boldsymbol{x},0)) \quad = \quad (w\,,\,\rho c\,u_0(\boldsymbol{x})) \qquad\qquad \forall w \in \mathcal{V}$$

or, suppressing the argument $\boldsymbol{x}$,

$$(w\,,\,\rho c\,u(0)) \quad = \quad (w\,,\,\rho c\,u_0)$$

which is identical to (7.1.12).

Now, assume u is a solution of (W). Then, by (7.1.12) and bilinearity

$$(w\,,\,\rho c\,(u(0)-u_0)) \quad = \quad 0$$

or

$$\int_\Omega w\rho c(u(0)-u_0)\,d\Omega \quad = \quad 0$$

Let

$$w \quad = \quad \phi(u(0)-u_0)$$

where $\phi > 0$ on Ω. Then,

$$\int_\Omega \phi\rho c(u(0)-u_0)^2\,d\Omega \quad = \quad 0$$

and therefore $u(0) = u_0$.

7.1 Exercise 2, p. 422

The development of the matrix equations follows the same approach given in Secs. 1.6, 2.4 and 2.5.

7.2 Exercise 1, p. 424

Equivalence of (7.2.6)–(7.2.8) and (7.2.11) is done in the same manner as in Sec. 2.7. To prove equivalence of (7.2.9) and (7.2.12), and (7.2.10) and (7.2.13), use the same argument as given for Exercise 1, Sec. 7.1.

7.2 Exercise 2, p. 426

The development of the matrix equations follows the same approach as given in Secs. 2.8 and 2.9.

7.2 Exercise 3, p. 428

i. Linear element:

$$\boldsymbol{m}^e = [m^e_{ab}] = \left[\int_{\Omega^e} \rho N_a N_b \, d\Omega\right]$$

$$N_1(\xi) = \frac{1}{2}(1-\xi) \qquad N_2(\xi) = \frac{1}{2}(1+\xi)$$

$$m^e_{11} = \rho j \int_{-1}^{1} N_1^2 d\xi = \frac{\rho h}{2}\int_{-1}^{1}\frac{1}{4}(1-\xi)^2 d\xi$$

$$= \frac{\rho h}{3}$$

$$m^e_{12} = \frac{\rho h}{2}\int_{-1}^{1}\frac{1}{4}(1-\xi^2)d\xi = \frac{\rho h}{6} \quad \text{, etc.}$$

Thus,

$$\boldsymbol{m}^e = \frac{\rho h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

ii. Quadratic element:

We assume the Jacobian $j = h/2$.

$$N_1(\xi) = \frac{1}{2}\xi(\xi-1) \qquad N_2(\xi) = 1-\xi^2 \qquad N_3(\xi) = \frac{1}{2}\xi(\xi+1)$$

$$
\begin{aligned}
N_1^2 &= \frac{1}{4}(\xi^4 - 2\xi^3 + \xi^2) \\
N_2^2 &= 1 - 2\xi^2 + \xi^4 \\
N_3^2 &= \frac{1}{4}(\xi^4 + 2\xi^3 + \xi^2) \\
N_1 N_2 &= \frac{1}{2}(-\xi^4 + \xi^3 + \xi^2 - \xi) \\
N_1 N_3 &= \frac{1}{4}(\xi^4 - \xi^2) \\
N_2 N_3 &= \frac{1}{2}(-\xi^4 - \xi^3 + \xi^2 + \xi)
\end{aligned}
$$

In integrating the above expressions over the parent domain, all odd power terms in ξ integrate to zero. Thus we have

$$
m_{ab}^e = \frac{\rho h}{2} \int_{-1}^{1} N_a N_b d\xi
$$

and carrying out the details results in

$$
\boldsymbol{m}^e = \frac{\rho h}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}
$$

7.2 Exercise 4, p. 428

The solution to this problem is given in the text.

7.3 Exercise 1, p. 432

$$
\boldsymbol{m}^e = [m_{pq}^e] \,, \qquad m_{pq}^e = \int_{\Omega^e} N_{p,x} N_{q,x} \, d\Omega
$$

This is the geometric stiffness matrix for the Bernoulli-Euler beam. Useful forms of the shape functions may be obtained by transforming the expressions given in Exercise 1.16, part b, using $x = \xi h/2$, $x_1 = -h/2$, and $x_2 = h/2$, viz.

$$N_1(\xi) = \frac{1}{4}(1-\xi)^2(\xi+2) \qquad N_2(\xi) = \frac{h}{8}(\xi-1)^2(\xi+1)$$

$$N_3(\xi) = \frac{1}{4}(1+\xi)^2(2-\xi) \qquad N_4(\xi) = \frac{h}{8}(\xi+1)^2(\xi-1)$$

So,

$$N_{1,\xi}(\xi) = \frac{3}{4}(\xi^2-1) \qquad N_{2,\xi}(\xi) = \frac{h}{8}(3\xi^2-2\xi-1)$$

$$N_{3,\xi}(\xi) = \frac{3}{4}(1-\xi^2) \qquad N_{4,\xi}(\xi) = \frac{h}{8}(3\xi^2+2\xi-1)$$

$$m^e_{pq} = (\xi_{,x})^2 \int_{\Omega^e} N_{p,\xi}(\xi) N_{q,\xi}(\xi) j \, d\xi = \xi_{,x} \int_{\Omega^e} N_{p,\xi}(\xi) N_{q,\xi}(\xi) \, d\xi$$

where $\xi_{,x} = 2/h$. Clearly, $\boldsymbol{m}^e$ is symmetric, and it can also be easily verified that:

$$m^e_{11} = m^e_{33} = -m^e_{13} \qquad m^2_{22} = m^e_{44}$$

$$m^e_{12} = m^e_{14} = -m^e_{23} = -m^e_{34}$$

Thus, we need to calculate

$$\begin{bmatrix} m^e_{11} & m^e_{12} & -m^e_{11} & m^e_{12} \\ & m^e_{22} & -m^e_{12} & m^e_{24} \\ & & m^e_{11} & -m^e_{12} \\ \text{Symmetric} & & & m^e_{22} \end{bmatrix}$$

which involves only four distinct integrals:

$$\int_{-1}^{1} (N_{1,\xi}(\xi))^2 d\xi = \frac{18}{16}\left(\frac{1}{5}\xi^5 - \frac{2}{3}\xi^3 + \xi\right)\Big|_{-1}^{+1}$$

$$= \frac{18}{16}\left(\frac{1}{5} + \frac{1}{3}\right) = \frac{18}{30}$$

Summarizing the remaining integrations:

$$\int_{-1}^{1} N_{1,\xi}(\xi)N_{2,\xi}(\xi)d\xi = \frac{h}{20}$$

$$\int_{-1}^{1} (N_{2,\xi}(\xi))^2 d\xi = \frac{h^2}{15}$$

$$\int_{-1}^{1} N_{2,\xi}(\xi)N_{4,\xi}(\xi)d\xi = -\frac{h^2}{60}$$

Thus, accounting for the factor $\xi_{,x} = 2/h$,

$$\boldsymbol{m}^e = \frac{1}{30h}\begin{bmatrix} 36 & 3h & -36 & 3h \\ & 4h^2 & -3h & -h^2 \\ & & 36 & -3h \\ \text{Symmetric} & & & 4h^2 \end{bmatrix}$$

7.3 Exercise 2, p. 433

$$\boldsymbol{m}^e = [m^e_{pq}]$$

where

$$m^e_{pq} = \int_{\Omega^e} N_p \rho A N_q \, d\Omega$$

Recall from Exercise 1, Sec. 7.3, that

$$N_1(\xi) = \frac{1}{4}(1-\xi)^2(\xi+2) \qquad N_2(\xi) = \frac{h}{8}(\xi-1)^2(\xi+1)$$

$$N_3(\xi) = \frac{1}{4}(1+\xi)^2(2-\xi) \qquad N_4(\xi) = \frac{h}{8}(\xi+1)^2(\xi-1)$$

$$m^e_{pq} = \int_{-1}^{1} \rho A j N_p N_q \, d\xi = \rho A x_{,\xi} \int_{-1}^{1} N_p N_q \, d\xi$$

where $x_{,\xi} = h/2$, and ρ and A are constants. Clearly $\boldsymbol{m}^e$ is symmetric and it may also be verified that

$$\begin{aligned} m_{11}^e &= m_{33}^e \qquad & m_{22}^e &= m_{44}^e \\ m_{12}^e &= -m_{34}^e \qquad & m_{14}^e &= -m_{23}^e \end{aligned}$$

Therefore there are only six distinct integrals. Straightforward calculations result in:

$$\boldsymbol{m}^e = \frac{\rho A h}{420} \begin{bmatrix} 156 & 22h & 54 & -13h \\ & 4h^2 & 13h & -3h^2 \\ & & 156 & -22h \\ \text{Symmetric} & & & 4h^2 \end{bmatrix}$$

7.3.1 Exercise 3, p. 435

It is immediately obvious that $\lambda_l \leq \lambda_l^h$, $l = 1, 2, ..., n_{eq}$ since the minimization over the subspace E_l also includes the minimization over the subset E_l^h and hence can achieve a smaller minimum because E_l is a larger set.

7.3.1 Exercise 4, p. 435

a. Using (7.3.30),

$$\frac{\omega^h}{\omega} - 1 = O(h^{2(k+1-m)})$$

Thus,

$$\ln\left(\frac{\omega^h}{\omega} - 1\right) \sim 2(k+1-m)\ln h$$

and from the given data,

$$k + 1 - m = 1 \quad \text{and} \quad m = 1$$

Therefore $k = 1$ which indicates the three-node linear triangle was used.

b. From (7.3.27) and (7.3.31),

$$\|\boldsymbol{u}_{(l)}^h - \boldsymbol{u}_{(l)}\|_m \leq ch^{(k+1-m)}\lambda_l^{(k+1)/(2m)}$$

$$\|\boldsymbol{u}_{(l)}^h - \boldsymbol{u}_{(l)}\|_0 \leq ch^{\sigma}\lambda_l^{(k+1)/(2m)}$$

where $\sigma = \min(k+1\,,2(k+1-m))$. Here $m = 1\,, k = 1\,,$ so

$$\|\boldsymbol{u}_{(l)}^h - \boldsymbol{u}_{(l)}\|_1 = O(h^1)$$

$$\|\boldsymbol{u}_{(l)}^h - \boldsymbol{u}_{(l)}\|_0 = O(h^2)$$

7.3.2 Exercise 5, p. 436

The solution to this problem is given in the text.

7.3.2 Exercise 6, p. 440

Employ the Lagrange interpolation formula with $\xi_1 = -1$, $\xi_2 = -1/\sqrt{5}$, $\xi_3 = 1/\sqrt{5}$, and $\xi_4 = 1$:

$$N_1(\xi) = \frac{(\xi + \frac{1}{\sqrt{5}})(\xi - \frac{1}{\sqrt{5}})(\xi - 1)}{(-1 + \frac{1}{\sqrt{5}})(-1 - \frac{1}{\sqrt{5}})(-1 - 1)}$$

$$= -\frac{5}{8}(\xi - 1)(\xi^2 - \frac{1}{5})$$

Similarly,

$$N_2(\xi) = \frac{5\sqrt{5}}{8}(\xi - \frac{1}{\sqrt{5}})(\xi^2 - 1)$$

$$N_3(\xi) = \frac{5\sqrt{5}}{8}(\xi + \frac{1}{\sqrt{5}})(1 - \xi^2)$$

$$N_4(\xi) = \frac{5}{8}(\xi + 1)(\xi^2 - \frac{1}{5})$$

7.3.2 Exercise 7, p. 445

I. Linear triangle:

For future reference, we calculate the consistent mass matrix, using area coordinates. Recall (3.I.30) and:

$$N_1 = r \qquad N_2 = s \qquad N_3 = t$$

$$\boldsymbol{m}^e = \left[m^e_{pq}\right] \qquad m^e_{pq} = \delta_{ij}\rho \int_{\Omega^e} N_a N_b \, d\Omega$$

$$p = n_{ed}(a-1)+i \qquad q = n_{ed}(b-1)+j$$

$$\int_{\Omega^e} N_1^2 \, d\Omega = \int_{\Omega^e} r^2 \, d\Omega = \frac{2A2!}{4!} = \frac{A}{6}$$

Thus,

$$m^e_{11} = \frac{\rho A}{6} = m^e_{ii}, \qquad i = 1, \ldots, n_{eq} \qquad \text{(no sum)}$$

$$\int_{\Omega^e} N_1 N_2 \, d\Omega = \int_{\Omega^e} rs \, d\Omega = \frac{2A1!}{4!} = \frac{A}{12}$$

and,

$$m^e_{13} = \frac{\rho A}{12} = m^e_{15} \quad \text{, etc.}$$

Consequently, the consistent mass matrix for the linear triangle is:

$$\boldsymbol{m}^e = \frac{\rho A}{12}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

a. Nodal quadrature:

Recall that numerical integration of triangles involves a scaling coefficient $c = \frac{1}{2}$ to reconcile the sum of the weights in the quadrature tables with the area of the parent triangle (see (3.I.34), p. 174).

$$m^e_{pq} = \begin{cases} \delta_{ij} \rho c j W_a & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

$j = 2A$ and $W_a = 1/3$, $a = 1, 2, 3$. Thus,

$$\boldsymbol{m}^e = \frac{\rho A}{3} \boldsymbol{I}_6$$

where $\boldsymbol{I}_6$ is the 6×6 identity matrix.

b. Row-sum technique:

$$m^e_{pq} = \begin{cases} \delta_{ij} \rho \int\limits_{\Omega^e} N_a \, d\Omega & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$\int\limits_{\Omega^e} N_a \, d\Omega = \frac{2\,A\,1!}{3!} = \frac{A}{3}$$

and therefore the resultant lumped mass matrix is identical to that obtained using the nodal quadrature method.

c. "Special-lumping technique":

$$m^e_{pq} = \begin{cases} \alpha \delta_{ij} \int\limits_{\Omega^e} \rho N_a^2 \, d\Omega & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

where

$$\alpha = \frac{\int\limits_{\Omega^e} \rho \, d\Omega}{\sum_{a=1}^{n_{en}} \int\limits_{\Omega^e} \rho N_a^2 \, d\Omega}$$

Since the denominator is just the the sum of the diagonal elements of the consistent mass matrix and ρ is constant,

$$\alpha = 2$$

and again the same lumped mass matrix is obtained as by the previous methods.

II. Bilinear quadrilateral:

Again for future reference we calculate the consistent mass matrix. We shall assume that the element is rectangular with edge lengths h_1 and h_2. In this case the Jacobian determinant $j = \text{constant} = h_1 h_2/4$.

$$\begin{aligned} m^e_{pq} &= \delta_{ij} \int_{\Omega^e} \rho N_a N_b \, d\Omega \\ &= \delta_{ij} \rho \frac{h_1 h_2}{64} \int_{-1}^{1} \int_{-1}^{1} (1 + \xi_a \xi)(1 + \xi_b \xi)(1 + \eta_a \eta)(1 + \eta_b \eta) \, d\xi \, d\eta \quad \text{(no sum on } a, b) \\ &= \delta_{ij} \frac{\rho h_1 h_2}{16} (1 + \frac{1}{3} \xi_a \xi_b)(1 + \frac{1}{3} \eta_a \eta_b) \end{aligned}$$

where ξ_a and η_a are given by Table 3.2.1, p. 113, of the text. Thus,

$$\boldsymbol{m}^e = \frac{\rho h_1 h_2}{36} \begin{bmatrix} 4 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 & 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 2 \\ 2 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 & 0 & 2 & 0 & 4 \end{bmatrix}$$

a. Nodal quadrature:

$$m^e_{pq} = \begin{cases} \delta_{ij} \rho j W_a & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

where

$$W_a = 1 \qquad a = 1, 2, 3, 4$$

Thus,

$$\boldsymbol{m}^e = \frac{\rho h_1 h_2}{4} \boldsymbol{I}_8$$

where $\boldsymbol{I}_8$ is the 8×8 identity matrix.

b. Row-sum technique:

$$m^e_{pq} = \begin{cases} \delta_{ij} \rho \int\limits_{\Omega^e} N_a \, d\Omega & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \int\limits_{\Omega^e} N_a \, d\Omega &= \int_{-1}^{1} \int_{-1}^{1} j N_a \, d\xi \, d\eta \\ &= \frac{h_1 h_2}{4} \int_{-1}^{1} \int_{-1}^{1} (1 + \xi_a \xi)(1 + \eta_a \eta) \, d\xi \, d\eta \\ &= \frac{h_1 h_2}{4} \end{aligned}$$

Consequently, we obtain an identical result to the matrix derived from nodal quadrature.

c. "Special-lumping technique":

$$m^e_{pq} = \begin{cases} \alpha \delta_{ij} \int\limits_{\Omega^e} \rho N_a^2 \, d\Omega & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

where

$$\alpha = \frac{\int\limits_{\Omega^e} \rho \, d\Omega}{\sum_{a=1}^{n_{en}} \int\limits_{\Omega^e} \rho N_a^2 \, d\Omega}$$

From the consistent mass matrix computed previously,

$$\alpha = \frac{9}{4}$$

Thus,

$$m^e_{pq} = \begin{cases} \delta_{ij}\frac{\rho h_1 h_2}{4} & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

which is the same result as for the previously computed lumped mass matrices.

7.3.2 Exercise 8, p. 445

Recall that for the three-node quadratic element,

$$N_1(\xi) = \frac{1}{2}\xi(\xi-1) \qquad N_2(\xi) = 1-\xi^2 \qquad N_3(\xi) = \frac{1}{2}\xi(\xi+1)$$

a. Row-sum technique:

$$m^e_{ab} = \begin{cases} \int\limits_{\Omega^e} \rho N_a \, d\Omega & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

Thus,

$$\boldsymbol{m}^e = \frac{\rho h}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b. "Special-lumping technique":

$$m^e_{ab} = \begin{cases} \alpha \int\limits_{\Omega^e} \rho N_a^2 \, d\Omega & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

where

$$\alpha = \frac{\int\limits_{\Omega^e} \rho \, d\Omega}{\sum_{a=1}^{n_{en}} \int\limits_{\Omega^e} \rho N_a^2 \, d\Omega}$$

$$\int\limits_{\Omega^e} \rho N_1^2 \, d\Omega = \frac{\rho h}{2}\frac{1}{4}\int_{-1}^{1} (\xi^4 - 2\xi^3 + \xi^2)\, d\xi = \frac{2\rho h}{15}$$

Similarly,

$$\int_{\Omega^e} \rho N_3^2 \, d\Omega \;=\; \int_{\Omega^e} \rho N_1^2 \, d\Omega \;=\; \frac{2\rho h}{15}$$

$$\int_{\Omega^e} \rho N_2^2 \, d\Omega \;=\; \frac{\rho h}{2} \int_{-1}^{1} (1 - 2\xi^2 + \xi^4) \, d\xi \;=\; \frac{8\rho h}{15}$$

Thus,

$$\alpha \;=\; \frac{\rho h}{\rho h(\frac{4}{15} + \frac{8}{15})} \;=\; \frac{5}{4}$$

$$\boldsymbol{m}^e \;=\; \frac{\rho h}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These matrices are identical to that obtained using Simpson's rule (7.3.39).

Appendix 7.I Exercise 1, p. 457

Recall the weak form of the heat equation:

$$(w \,,\, \rho c \, \dot{u}) + a(w \,,\, u) \;=\; (w \,,\, f) + (w, \mathrm{h})_\Gamma$$

Assume f, g, and h are zero. Since $g = 0$, $u \in \mathcal{V}$. Let $w = u \in \mathcal{V}$. Then

$$(u \,,\, \rho c \, \dot{u}) + a(u \,,\, u) \;=\; 0$$

which may be written as

$$\frac{1}{2} \frac{d}{dt} (u \,,\, \rho c \, u) \;=\; -a(u \,,\, u)$$

Notice that the inner product on the left-hand side induces a natural norm and that the right-hand side is non-positive, indicating the expected decay. Define $\|u\|_0^2 = (u \,,\, \rho c \, u)$. The first eigenvalue λ_1 minimizes the Rayleigh quotient, namely

$$\lambda_1 \;=\; \min_{u \in \mathcal{V}} \mathcal{R}(u)$$

where

$$\mathcal{R}(u) \;=\; \frac{a(u \,,\, u)}{\|u\|_0^2}$$

Therefore

$$\lambda_1 \|u\|_0^2 \;\leq\; a(u\,,\,u)$$

Thus

$$\begin{aligned}
\|u\|_0 \frac{d}{dt}\|u\|_0 &= \frac{1}{2}\frac{d}{dt}(u\,,\,\rho c\, u) \\
&= -a(u\,,\,u) \\
&\leq -\lambda_1 \|u\|_0^2
\end{aligned}$$

Consequently,

$$\frac{d}{dt}\|u\|_0 \;\leq\; -\lambda_1 \|u\|_0$$

Integrating this result yields

$$\|u(t)\|_0 \;\leq\; \exp(-\lambda_1 t)\|u(0)\|_0$$

Identical calculations may be used to obtain the decay inequality for the Galerkin formulation.

Appendix 7.I Exercise 2, p. 457

Recall the weak form of the equation of motion:

$$(\boldsymbol{w}\,,\,\rho\ddot{\boldsymbol{u}}) + a(\boldsymbol{w}\,,\,\boldsymbol{u}) \;=\; (\boldsymbol{w}\,,\,\boldsymbol{f}) + (\boldsymbol{w}\,,\,\mathbf{h})_\Gamma$$

Assume $\boldsymbol{f}$, $\boldsymbol{g}$, and $\mathbf{h}$ are zero. Since $\boldsymbol{g} = \mathbf{0}$, $\boldsymbol{u} \in \mathcal{V}$. Let $\boldsymbol{w} = \dot{\boldsymbol{u}} \in \mathcal{V}$. Then

$$(\dot{\boldsymbol{u}}\,,\,\rho\ddot{\boldsymbol{u}}) + a(\dot{\boldsymbol{u}}\,,\,\boldsymbol{u}) \;=\; \mathbf{0}$$

But

$$(\dot{\boldsymbol{u}}\,,\,\rho\ddot{\boldsymbol{u}}) \;=\; \frac{1}{2}\frac{d}{dt}(\dot{\boldsymbol{u}}\,,\,\rho\dot{\boldsymbol{u}}) \quad \text{and} \quad a(\dot{\boldsymbol{u}}\,,\,\boldsymbol{u}) \;=\; \frac{1}{2}\frac{d}{dt}a(\boldsymbol{u}\,,\,\boldsymbol{u})$$

Employing these expressions and integrating from 0 to t,

$$E(\boldsymbol{u}(t)\,,\,\dot{\boldsymbol{u}}(t)) - E(\boldsymbol{u}(0)\,,\,\dot{\boldsymbol{u}}(0)) \;=\; 0$$

where

$$E(\boldsymbol{u}(t)\,,\,\dot{\boldsymbol{u}}(t)) \;=\; \frac{1}{2}(\dot{\boldsymbol{u}}\,,\,\rho\dot{\boldsymbol{u}}) + \frac{1}{2}a(\boldsymbol{u}\,,\,\boldsymbol{u})$$

Thus,

$$E(\boldsymbol{u}(t)\,,\,\dot{\boldsymbol{u}}(t)) \quad = \quad E(\boldsymbol{u}(0)\,,\,\dot{\boldsymbol{u}}(0))$$

An identical proof of conservation of total energy may be performed for the Galerkin form of the equation of motion.

CHAPTER 8

8.1 Exercise 1, p. 462

$$M v_{n+1} + K d_{n+1} = F_{n+1} \quad (1)$$

$$d_{n+1} = d_n + \Delta t v_{n+\alpha} \quad (2)$$

$$v_{n+\alpha} = (1-\alpha) v_n + \alpha v_{n+\alpha} \quad (3)$$

$$\alpha(M v_{n+1} + K d_{n+1}) = \alpha F_{n+1} \quad (4)$$

$$(1-\alpha)(M v_n + K d_n) = (1-\alpha) F_n \quad (5)$$

Adding (4) and (5):

$$M v_{n+\alpha} + \alpha K d_{n+1} + (1-\alpha) K d_n = F_{n+\alpha} \quad (6)$$

where

$$F_{n+\alpha} = (1-\alpha) F_n + \alpha F_{n+1}$$

Using (2) and (6):

$$M(d_{n+1} - d_n) + \alpha \Delta t K d_{n+1} = \Delta t F_{n+\alpha} - (1-\alpha)\Delta t K d_n \quad (7)$$

Thus,

$$(M + \alpha \Delta t K) d_{n+1} = (M - (1-\alpha)\Delta t K) d_n + \Delta t F_{n+\alpha}$$

8.2.3 Exercise 1, p. 470

The lowest order error term that is added to the truncation expansion is $c\Delta t^q$ as given by the problem statement. Thus,

$$q \geq k$$

to retain kth order accuracy.

8.2.5 Exercise 2, p. 473

The solution to this problem is given in the text.

8.2.5 Exercise 3, p. 474

i. We need to manipulate equations (a) and (b) to arrive at the desired form:

$$d_{n+\frac{1}{2}} = (1 - \frac{1}{2}\lambda^h \Delta t)d_n \tag{a}$$

$$(1 + \frac{1}{2}\lambda^h \Delta t)d_{n+1} = d_{n+\frac{1}{2}} \tag{b}$$

Thus,

$$d_{n+1} = \left(\frac{1 - \frac{1}{2}\lambda^h \Delta t}{1 + \frac{1}{2}\lambda^h \Delta t}\right)d_n$$

and so

$$A = \frac{1 - \frac{1}{2}\lambda^h \Delta t}{1 + \frac{1}{2}\lambda^h \Delta t}$$

ii. Comparing the amplification factor derived above to the A for the generalized trapezoidal method, we see that they are equivalent for $\alpha = \frac{1}{2}$. Thus the fractional step method is equivalent to the midpoint rule for linear problems. Hence, it is unconditionally stable. Note also that it is implicit by equation (b).

iii. The local truncation error τ is defined by

$$\Delta t \tau(t_n) = d(t_{n+1}) - Ad(t_n)$$

We proceed as follows:

$$\begin{aligned}
\Delta t(1 + \frac{1}{2}\lambda^h \Delta t)\tau(t_n) &= (1 + \frac{1}{2}\lambda^h \Delta t)d(t_{n+1}) - (1 - \frac{1}{2}\lambda^h \Delta t)d(t_n) \\
&= (1 + \frac{1}{2}\lambda^h \Delta t)\Big(d(t_n) + \Delta t\dot{d}(t_n) + \frac{1}{2}\Delta t^2\ddot{d}(t_n) \\
&\qquad + \frac{1}{6}\Delta t^3 d^{(3)}(t_n)\Big) \\
&\quad - (1 - \frac{1}{2}\lambda^h \Delta t)d(t_n) + O(\Delta t^4) \\
&= (d(t_n) - d(t_n)) + \Delta t\Big((\frac{1}{2}\lambda^h + \frac{1}{2}\lambda^h)d(t_n) + \dot{d}(t_n)\Big) \\
&\quad + \Delta t^2(\frac{1}{2}\ddot{d}(t_n) + \frac{1}{2}\lambda^h \Delta t\dot{d}(t_n)) + O(\Delta t^3)
\end{aligned}$$

Thus,

$$\begin{aligned} \boldsymbol{K}\boldsymbol{v}_n &\overset{(6)}{=} -\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{d}_n \\ &= -(\boldsymbol{M}\boldsymbol{M}^{-1})\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{d}_n = -\boldsymbol{M}(\boldsymbol{M}^{-1}\boldsymbol{K})^2\boldsymbol{d}_n \end{aligned} \tag{7}$$

$$\boldsymbol{M}\boldsymbol{d}_{n+1} \overset{(5,7)}{=} \left(\boldsymbol{M} - \Delta t\boldsymbol{K} + \frac{1}{2}\Delta t^2 \boldsymbol{M}(\boldsymbol{M}^{-1}\boldsymbol{K})^2\right)\boldsymbol{d}_n \tag{8}$$

Now with

$$\boldsymbol{d}_n = \sum_{i=1}^{n_{eq}} d_{n(m)}\boldsymbol{\psi}_m \ , \qquad \boldsymbol{d}_{n+1} = \sum_{i=1}^{n_{eq}} d_{n+1(m)}\boldsymbol{\psi}_m$$

and

$$\boldsymbol{\psi}_i^T\boldsymbol{M}\boldsymbol{\psi}_j = \delta_{ij} \qquad \boldsymbol{\psi}_i^T\boldsymbol{K}\boldsymbol{\psi}_j = \delta_{ij}\lambda_j^h \qquad \text{(no sum)}$$

and using the given hint, the modal equation can be obtained from (8) as follows:

$$d_{n+1} \overset{(8)}{=} (1 - \lambda^h\Delta t + \frac{1}{2}(\lambda^h\Delta t)^2)d_n \tag{9}$$

and so

$$A = 1 - \lambda^h\Delta t + \frac{1}{2}(\lambda^h\Delta t)^2$$

(The plot is shown on the next page.)

ii. For stability $|A| \leq 1$:

$A \leq 1$:

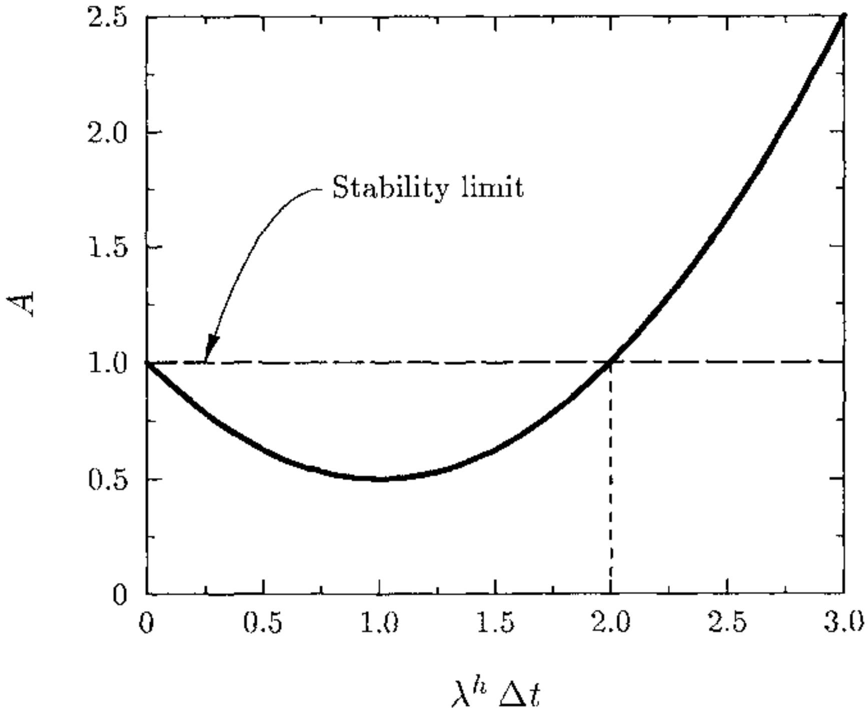
$$1 - \lambda^h\Delta t + \frac{1}{2}(\lambda^h\Delta t)^2 \leq 1 \implies \lambda^h\Delta t \leq 2 \tag{10}$$

$-1 \leq A$:

$$-1 \leq 1 - \lambda^h\Delta t + \frac{1}{2}(\lambda^h\Delta t)^2 \implies \lambda^h\Delta t(1 - \frac{1}{2}\lambda^h\Delta t) \leq 2$$

Since $\lambda^h\Delta t \geq 0$, this condition is less restrictive than (10) because

$$1 - \frac{1}{2}\lambda^h\Delta t \leq 1$$



So, for stability:

$$\lambda^h \Delta t \;\leq\; 2$$

iii. If $\boldsymbol{M}$ is diagonal, then, by (8), the method is explicit.

iv.

$$\begin{aligned}
\Delta t \tau(t_n) &= d(t_{n+1}) - (1 - \lambda^h \Delta t + \frac{1}{2}(\lambda^h \Delta t)^2) d(t_n) \\
&= (d(t_n) + \Delta t \dot{d}(t_n) + \frac{1}{2}\Delta t^2 \ddot{d}(t_n) + \frac{1}{6}\Delta t^3 d^{(3)}(t_n)) \\
&\quad - (1 - \lambda^h \Delta t + \frac{1}{2}(\lambda^h \Delta t)^2) d(t_n) \; + \; O(\Delta t^4) \\
&= (d(t_n) - d(t_n)) \; + \; \Delta t(\dot{d}(t_n) + \lambda^h d(t_n)) \\
&\quad + \frac{1}{2}\Delta t^2(\ddot{d}(t_n) - (\lambda^h)^2 d(t_n)) \; + \; O(\Delta t^3)
\end{aligned}$$

Taking the time derivative of the heat equation,

$$\ddot{d}(t_n) \quad = \quad -\lambda^h \dot{d}(t_n) \quad = \quad (-\lambda^h)^2 d(t_n) \quad = \quad (\lambda^h)^2 d(t_n)$$

Consequently,

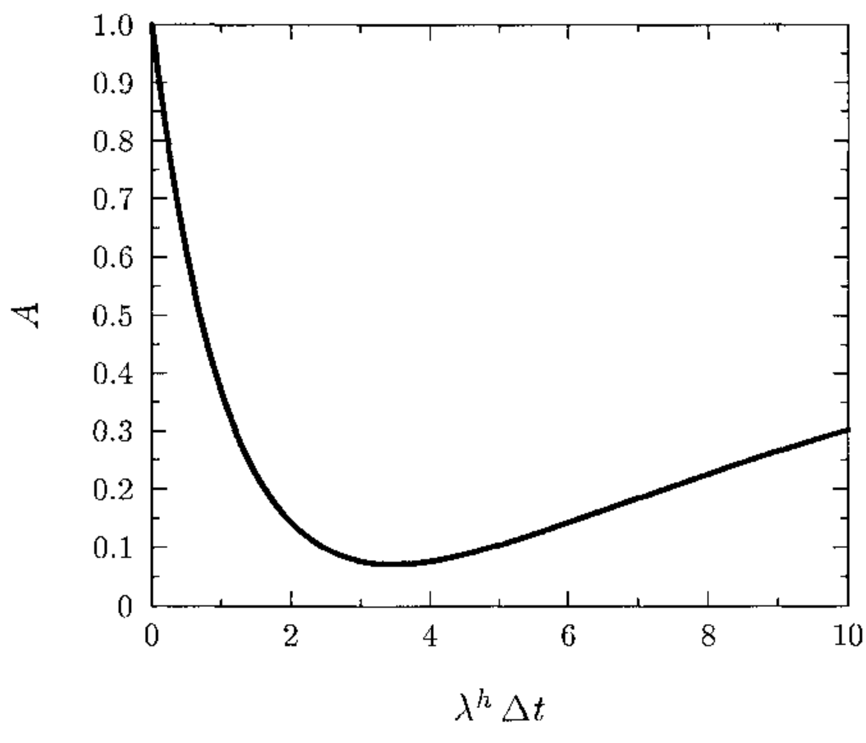
$$\Delta t \tau(t_n) \quad = \quad O(\Delta t^3)$$

v. Clearly from part (iv), $k \; = \; 2$.

8.2.5 Exercise 5, p. 475

i.

$$A \quad = \quad \frac{1 - \frac{1}{2}\lambda^h \Delta t + \frac{1}{12}(\lambda^h \Delta t)^2}{1 + \frac{1}{2}\lambda^h \Delta t + \frac{1}{12}(\lambda^h \Delta t)^2}$$



For stability, $-1 \leq A \leq 1$. Clearly, $A \leq 1$ since $\lambda^h \Delta t \geq 0$. With $A \geq -1$,

$$1 - \frac{1}{2}\lambda^h \Delta t + \frac{1}{12}(\lambda^h \Delta t)^2 \geq -1 - \frac{1}{2}\lambda^h \Delta t - \frac{1}{12}(\lambda^h \Delta t)^2$$

and therefore

$$2 + \frac{1}{6}(\lambda^h \Delta t)^2 \quad \geq \quad 0$$

which holds for all $\lambda^h \Delta t$ so the method is unconditionally stable.

ii.

$$\begin{aligned}
\Delta t \tau(t_n) &= \left(1 + \frac{1}{2}\lambda^h \Delta t + \frac{1}{12}(\lambda^h \Delta t)^2\right) d(t_{n+1}) \\
&\quad - \left(1 - \frac{1}{2}\lambda^h \Delta t + \frac{1}{12}(\lambda^h \Delta t)^2\right) d(t_n) \\
&\quad - \left(\frac{1}{2}\Delta t - \frac{1}{12}\Delta t^2 \lambda^h\right) F(t_n) - \left(\frac{1}{2}\Delta t + \frac{1}{12}\Delta t^2 \lambda^h\right) F(t_{n+1}) \\
&\quad - \frac{1}{12}\dot{F}(t_n) + \frac{1}{12}\Delta t^2 \dot{F}(t_{n+1}) \\
&= \left(1 + \frac{1}{2}\lambda^h \Delta t + \frac{1}{12}(\lambda^h \Delta t)^2\right) \Big(d(t_n) + \Delta t \dot{d}(t_n) + \frac{1}{2}\Delta t^2 \ddot{d}(t_n) \\
&\qquad\qquad + \frac{1}{6}\Delta t^3 d^{(3)}(t_n) + \frac{1}{24}\Delta t^4 d^{(4)}(t_n)\Big) \\
&\quad - \left(1 - \frac{1}{2}\lambda^h \Delta t + \frac{1}{12}(\lambda^h \Delta t)^2\right) d(t_n) - \left(\frac{1}{2}\Delta t - \frac{1}{12}\Delta t^2 \lambda^h\right) F(t_n) \\
&\quad - \left(\frac{1}{2}\Delta t + \frac{1}{12}\Delta t^2 \lambda^h\right) \Big(F(t_n) + \Delta t \dot{F}(t_n) \\
&\qquad\qquad + \frac{1}{2}\Delta t^2 \ddot{F}(t_n) + \frac{1}{6}\Delta t^3 F^{(3)}(t_n)\Big) \\
&\quad - \frac{1}{12}\Delta t^2 \dot{F}(t_n) + \frac{1}{12}\Delta t^2 \Big(\dot{F}(t_n) + \Delta t \ddot{F}(t_n) \\
&\qquad\qquad + \frac{1}{2}\Delta t^2 F^{(3)}(t_n)\Big) + O(\Delta t^5)
\end{aligned}$$

Evaluating the coefficient on each power of Δt separately, we have

Δt^0 term

$$d(t_n) - d(t_n) = 0$$

Δt^1 term

$$\frac{1}{2}\lambda^h d(t_n)(2) + \dot{d}(t_n) - \frac{1}{2}F(t_n)(2) = 0$$

Δt^2 term

$$\frac{1}{2}\ddot{d}(t_n) + \frac{1}{2}\lambda^h \dot{d}(t_n) + \frac{1}{12}(\lambda^h)^2 d(t_n) - \frac{1}{12}(\lambda^h)^2 d(t_n) + \frac{1}{12}\lambda^h F(t_n) - \frac{1}{12}\lambda^h F(t_n)$$

$$- \frac{1}{12}\dot{F}(t_n) + \frac{1}{12}\dot{F}(t_n) = \frac{1}{2}\Big(\ddot{d}(t_n) + \lambda^h \dot{d}(t_n) - \dot{F}(t_n)\Big) = 0$$

Δt^3 term

$$\frac{1}{6}d^{(3)}(t_n) + \frac{1}{4}\lambda^h \ddot{d}(t_n) + \frac{1}{12}(\lambda^h)^2 \dot{d}(t_n) - \frac{1}{4}\ddot{F}(t_n) - \frac{1}{12}\lambda^h \dot{F}(t_n) + \frac{1}{12}\ddot{F}(t_n)$$

$$= \frac{1}{6}\left(d^{(3)}(t_n) + \lambda^h \ddot{d}(t_n) - \ddot{F}(t_n)\right) + \frac{1}{12}\lambda^h \left(\ddot{d}(t_n) + \lambda^h \dot{d}(t_n) - \dot{F}(t_n)\right) = 0$$

Δt^4 term

$$\frac{1}{24}d^{(4)}(t_n) + \frac{1}{12}\lambda^h d^{(3)}(t_n) + \frac{1}{24}(\lambda^h)^2 \ddot{d}(t_n) - \frac{1}{12}F^{(3)}(t_n) - \frac{1}{24}\lambda^h \ddot{F}(t_n) + \frac{1}{24}F^{(3)}(t_n)$$

$$= \frac{1}{24}\Big(d^{(4)}(t_n) + \lambda^h d^{(3)}(t_n) - F^{(3)}(t_n)\Big) + \frac{1}{24}\lambda^h \Big(d^{(3)}(t_n) + \lambda^h \ddot{d}(t_n) - \ddot{F}(t_n)\Big) = 0$$

Thus, the truncation error is at least $O(\Delta t^4)$. To show that it is no greater that $O(\Delta t^4)$ we calculate the $O(\Delta t^5)$ terms assuming $\boldsymbol{F} \equiv \boldsymbol{0}$. If it does not vanish for this special case, it also will not vanish for the general case of $\boldsymbol{F} \neq \boldsymbol{0}$:

$\underline{\Delta t^5 \text{ term}}$ (assuming $\boldsymbol{F} \equiv \boldsymbol{0}$)

$$\begin{aligned}
\frac{1}{120}d^{(5)}(t_n) + \frac{1}{48}\lambda^h d^{(4)}(t_n) + \frac{1}{72}(\lambda^h)^2 d^{(3)}(t_n) &= \frac{1}{120}\Big(d^{(5)}(t_n) + \lambda^h d^{(4)}(t_n)\Big) \\
&\quad + \frac{3}{240}\lambda^h\Big(d^{(4)}(t_n) + \lambda^h d^{(3)}(t_n)\Big) \\
&\quad + \Big(\frac{1}{72} - \frac{1}{80}\Big)(\lambda^h)^2 d^{(3)}(t_n) \\
&= \frac{1}{720}(\lambda^h)^2 d^{(3)}(t_n) \\
&\neq 0
\end{aligned}$$

Therefore

$$\tau(t_n) = O(\Delta t^4)$$

8.2.5 Exercise 6, p. 475

$$v_{n+1} + \lambda^h d_{n+1} + \tilde{\lambda}^h \tilde{d}_{n+1} = F_{n+1} \tag{1}$$

$$\tilde{d}_{n+1} = d_n + \Delta t(1-\alpha)v_n \tag{2}$$

$$d_{n+1} = \tilde{d}_{n+1} + \Delta t \alpha v_{n+1} \tag{3}$$

where $\alpha \in [0\,,\,1]$

i. To obtain the amplification factor, we just need to evaluate the homogeneous case, but include the forcing term also so the resultant displacement difference form of the algorithm can be used to obtain the local truncation error equation for parts iii and iv.

$$\begin{aligned}
\alpha \Delta t F_{n+1} &\overset{(1)}{=} \alpha\Delta t v_{n+1} + \alpha\Delta t \lambda^h d_{n+1} + \alpha \Delta t \tilde{\lambda}^h \tilde{d}_{n+1} \\
&\overset{(3)}{=} (d_{n+1} - \tilde{d}_{n+1}) + \alpha\Delta t\lambda^h d_{n+1} + \alpha\Delta t \tilde{\lambda}^h \tilde{d}_{n+1} \\
&= (1 + \alpha\Delta t \lambda^h) d_{n+1} - (1 - \alpha \Delta t \tilde{\lambda}^h)\tilde{d}_{n+1} \\
&\overset{(2)}{=} (1 + \alpha\Delta t\lambda^h) d_{n+1} - (1 - \alpha\Delta t\tilde{\lambda}^h)(d_n + \Delta t(1-\alpha)v_n) \qquad (4)
\end{aligned}$$

$$
\begin{aligned}
F_n &\overset{(1)}{=} v_n + \lambda^h d_n + \tilde{\lambda}^h \tilde{d}_n \\
&\overset{(3)}{=} v_n + \lambda^h d_n + \tilde{\lambda}^h (d_n - \alpha \Delta t v_n) \\
&= (1 - \alpha \Delta t \tilde{\lambda}^h) v_n + (\lambda^h + \tilde{\lambda}^h) d_n \qquad (5)
\end{aligned}
$$

$$
v_n \overset{(5)}{=} (1 - \alpha \Delta t \tilde{\lambda}^h)^{-1} F_n - \frac{\lambda^h + \tilde{\lambda}^h}{1 - \alpha \Delta t \tilde{\lambda}^h} d_n \qquad (6)
$$

$$
\begin{aligned}
(1 + \alpha \Delta t \lambda^h) d_{n+1} &\overset{(4,6)}{=} \alpha \Delta t F_{n+1} + (1 - \alpha \Delta t \tilde{\lambda}^h) d_n \\
&\quad + (1 - \alpha) \Delta t F_n - (\lambda^h + \tilde{\lambda}^h)(1 - \alpha) \Delta t d_n \\
&= (1 - \alpha \Delta t \tilde{\lambda}^h - (1 - \alpha) \Delta t (\lambda^h + \tilde{\lambda}^h)) d_n \\
&\quad + \Delta t F_{n+\alpha} \qquad (7)
\end{aligned}
$$

where

$$
F_{n+\alpha} = (1 - \alpha) F_n + \alpha F_{n+1}
$$

Thus,

$$
A = \frac{1 - \alpha \Delta t \tilde{\lambda}^h - (1 - \alpha) \Delta t (\lambda^h + \tilde{\lambda}^h)}{1 + \alpha \Delta t \lambda^h}
$$

ii. For stability $-1 \le A \le 1$. Clearly $A \le 1$.

$-1 \le A$:

$$
1 - \alpha \Delta t \tilde{\lambda}^h - (1 - \alpha) \Delta t (\lambda^h + \tilde{\lambda}^h) \ge -1 - \alpha \Delta t \lambda^h
$$

So,

$$
(1 - \alpha) \Delta t \lambda^h - \alpha \Delta t \lambda^h + (1 - \alpha) \Delta t \tilde{\lambda}^h + \alpha \Delta t \tilde{\lambda}^h \le 2
$$

and

$$
\Delta t ((1 - 2\alpha) \lambda^h + \tilde{\lambda}^h) \le 2
$$

iii.

$$
\begin{aligned}
\Delta t\tau(t_n) &= (1+\alpha\Delta t\lambda^h)d(t_{n+1}) - (1-(1-\alpha)\Delta t\lambda^h - \Delta t\tilde{\lambda}^h)d(t_n) \\
&- \Delta t(1-\alpha)F(t_n) - \Delta t\alpha F(t_{n+1}) \\
&= (1+\alpha\Delta t\lambda^h)\Big(d(t_{n+\alpha}) + (1-\alpha)\Delta t\dot{d}(t_{n+\alpha}) + (1-\alpha)^2\frac{1}{2}\Delta t^2\ddot{d}(t_{n+\alpha})\Big) \\
&- \Big(1-(1-\alpha)\Delta t\lambda^h - \Delta t\tilde{\lambda}^h\Big)\Big(d(t_{n+\alpha}) - \alpha\Delta t\dot{d}(t_{n+\alpha}) \\
&\qquad + \frac{1}{2}\alpha^2\Delta t^2\ddot{d}(t_{n+\alpha})\Big) \\
&- \Delta t(1-\alpha)\Big(F(t_{n+\alpha}) - \alpha\Delta t\dot{F}(t_{n+\alpha})\Big) \\
&- \Delta t\alpha\Big(F(t_{n+\alpha}) + (1-\alpha)\Delta t\dot{F}(t_{n+\alpha})\Big) + O(\Delta t^3)
\end{aligned}
$$

Evaluating the coefficient on each power of Δt separately, we have

Δt^0 term

$$
d(t_{n+\alpha}) - d(t_{n+\alpha}) = 0
$$

Δt^1 term

$$
\begin{aligned}
&\alpha\lambda^h d(t_{n+\alpha}) + (1-\alpha)\dot{d}(t_{n+\alpha}) + (1-\alpha)\lambda^h d(t_{n+\alpha}) + \alpha\dot{d}(t_{n+\alpha}) + \tilde{\lambda}^h d(t_{n+\alpha}) \\
&\quad - (1-\alpha)F(t_{n+\alpha}) - \alpha F(t_{n+\alpha}) \\
&\quad = \dot{d}(t_{n+\alpha}) + (\lambda^h + \tilde{\lambda}^h)d(t_{n+\alpha}) - F(t_{n+\alpha}) \\
&\quad = 0
\end{aligned}
$$

Δt^2 term

$$
\begin{aligned}
&\frac{1}{2}(1-\alpha)^2\ddot{d}(t_{n+\alpha}) + \alpha(1-\alpha)\lambda^h\dot{d}(t_{n+\alpha}) - \frac{1}{2}\alpha^2\ddot{d}(t_{n+\alpha}) - (1-\alpha)\alpha\lambda^h\dot{d}(t_{n+\alpha}) \\
&\quad - \alpha\tilde{\lambda}^h\dot{d}(t_{n+\alpha}) + \alpha(1-\alpha)\dot{F}(t_{n+\alpha}) - \alpha(1-\alpha)\dot{F}(t_{n+\alpha}) \\
&\quad = (\frac{1}{2}-\alpha)\ddot{d}(t_{n+\alpha}) - \alpha\tilde{\lambda}^h\dot{d}(t_{n+\alpha})
\end{aligned}
$$

Thus,

$$r(t_n) \quad = \quad \Delta t\Big((\frac{1}{2}-\alpha)\ddot{d}(t_{n+\alpha}) - \alpha\tilde{\lambda}^h\dot{d}(t_{n+\alpha})\Big) \;+\; O(\Delta t^2)$$

iv. Clearly, from part (iii), $k \;=\; 1$.

8.2.5 Exercise 7, p. 476

The solution to this problem is given in the text.

8.3 Exercise 1, p. 480

Introduce the perturbation $\delta_n(m)$ which satisfies the algorithmic equation. Assume that,

$$\delta_n(m) \quad = \quad \zeta^n e^{im\xi}$$

a. BTCS algorithm:

Given the above perturbation,

$$\zeta^{n+1}e^{im\xi} \quad = \quad \zeta^n e^{im\xi} + r\Big(\zeta^{n+1}e^{i(m+1)\xi} - 2\zeta^{n+1}e^{im\xi} + \zeta^{n+1}e^{i(m-1)\xi}\Big)$$

Dividing both sides by $\zeta^n e^{im\xi}$ results in:

$$\begin{aligned}\zeta \quad &= \quad 1+\zeta r\Big(e^{i\xi}+e^{-i\xi}-2\Big) \quad = \quad 1+\zeta r\Big(2\,\cos\xi-2\Big) \\ &= \quad 1+2\zeta r(\,\cos\xi-1) \qquad = \quad 1-4r\zeta\,\sin^2(\frac{1}{2}\xi)\end{aligned}$$

Thus,

$$\zeta \quad = \quad \frac{1}{1+4r\,\sin^2(\frac{1}{2}\xi)}$$

For stability, it is necessary that $|\zeta| \;\leq\; 1 \;\;\forall\; \xi$. Clearly, $|\zeta| \;\leq\; 1$ since $r \;\geq\; 0$. Thus BTCS is unconditionally stable.

b. Crank-Nicolson:

Given the above perturbation,

$$\zeta^{n+1}e^{im\xi} = \zeta^n e^{im\xi} + \frac{1}{2}r\zeta^{n+1}\Big(e^{i\xi(m+1)} - 2e^{im\xi} + e^{i\xi(m-1)}\Big)$$
$$+ \frac{1}{2}r\zeta^n\Big(e^{i\xi(m+1)} - 2e^{im\xi} + e^{i\xi(m-1)}\Big)$$

Dividing both sides by $\zeta^n e^{im\xi}$ results in:

$$\Big(1 - \frac{1}{2}r(e^{i\xi} + e^{-i\xi} - 2)\Big)\zeta = 1 + \frac{1}{2}r(e^{i\xi} + e^{-i\xi} - 2)$$

As in part (a),

$$e^{i\xi} - e^{-i\xi} - 2 = 2\cos\xi - 2 = -4\sin^2(\frac{1}{2}\xi)$$

So,

$$\zeta = \frac{1 - 2r\ \sin^2(\frac{1}{2}\xi)}{1 + 2r\ \sin^2(\frac{1}{2}\xi)}$$

It is clear that $\zeta \leq 1$ so we must also prove that $\zeta \geq -1$. This follows immediately from

$$1 - 2r\ \sin^2(\frac{1}{2}\xi) \geq -1 - 2r\ \sin^2(\frac{1}{2}\xi)$$

so the Crank-Nicolson algorithm is also unconditionally stable.

c. Leap frog:

Given the above perturbation,

$$\zeta^{n+1}e^{im\xi} = \zeta^{n-1}e^{im\xi} + 2r\zeta^n e^{im\xi}\Big(e^{i\xi} + e^{-i\xi} - 2\Big)$$

Dividing both sides by $\zeta^{n-1}e^{im\xi}$ results in:

$$\zeta^2 = 1 - 8r\,\zeta\,\sin^2(\frac{1}{2}\xi)$$

The roots of this quadratic equation are

$$
\begin{aligned}
\zeta_{1,2} &= -4r\ \sin^2(\frac{1}{2}\xi) \pm \left((4r\ \sin^2(\frac{1}{2}\xi))^2 + 1\right)^{\frac{1}{2}} \\
\zeta_2 &= -4r\ \sin^2(\frac{1}{2}\xi) - \left(1 + (4r\ \sin^2(\frac{1}{2}\xi))^2\right)^{\frac{1}{2}} \\
&\leq -1
\end{aligned}
$$

Consequently $|\zeta_2| \geq 1$, and therefore the leap frog method is unconditionally unstable.

8.3 Exercise 2, p. 481

Recall that

$$\Delta t\tau(t_n) = \Delta t O(\Delta t^k , h^l)$$

a. BTCS:

$$u_{n+1}^h(m) = u_n^h + r\Big(u_{n+1}^h(m+1) - 2u_{n+1}^h(m) + u_{n+1}^h(m-1)\Big)$$

$$
\begin{aligned}
\Delta t\tau(t_n) &= u(x_m, t_{n+1}) - u(x_m, t_n) - r\Big(u(x_{m+1}, t_{n+1}) - 2u(x_m, t_{n+1}) \\
&\qquad + u(x_{m-1}, t_{n+1})\Big) \\
&= u(x_m, t_{n+1}) \\
&\quad - \Big(u(x_m, t_{n+1}) - \Delta t u_{,t}(x_m, t_{n+1}) + \frac{1}{2}\Delta t^2 u_{,tt}(x_m, t_{n+1})\Big) \\
&\quad - r\Big(u(x_m, t_{n+1}) + h u_{,x}(x_m, t_{n+1}) + \frac{1}{2}h^2 u_{,xx}(x_m, t_{n+1}) \\
&\qquad + \frac{1}{6}h^3 u_{,xxx}(x_m, t_{n+1}) - 2u(x_m, t_{n+1}) \\
&\qquad + u(x_m, t_{n+1}) - h u_{,x}(x_m, t_{n+1}) + \frac{1}{2}h^2 u_{,xx}(x_m, t_{n+1}) \\
&\qquad - \frac{1}{6}h^3 u_{,xxx}(x_m, t_{n+1})\Big) \\
&\quad + O(\Delta t^3, h^4)
\end{aligned}
$$

Recall that $r = k\Delta t/h^2$. So,

$$\Delta t\tau(t_n) = \Big(u(x_m, t_{n+1}) - u(x_m, t_{n+1})\Big)$$

$$+ \Delta t\Big(h^{-2}(-k(1-2+1)u(x_m, t_{n+1}))$$

$$+ h^{-1}(k(1-1)u_{,x}(x_m, t_{n+1}))$$

$$+ (u_{,t}(x_m, t_{n+1}) - ku_{,xx}(x_m, t_{n+1}))$$

$$+ h(\frac{1}{6}(1-1)ku_{,xxx}(x_m, t_{n+1}))\Big)$$

$$+ O(\Delta t^2, \Delta th^2)$$

Thus,

$$\tau(t_n) = O(\Delta t^1, h^2) \implies k = 1 \quad l = 2$$

b. Crank-Nicolson:

$$u^h_{n+1}(m) = u^h_n(m) + \frac{1}{2}r\Big(u^h_{n+1}(m+1) - 2u^h_{n+1}(m) + u^h_{n+1}(m-1)\Big)$$

$$+ \frac{1}{2}r\Big(u^h_n(m+1) - 2u^h_n(m) + u^h_n(m-1)\Big)$$

$$\Delta t\tau(t_n) = u(x_m, t_{n+1}) - u(x_m, t_n)$$

$$-\frac{1}{2}\frac{k\Delta t}{h^2}\Big(u(x_{m+1}, t_{n+1}) - 2u(x_m, t_{n+1}) + u(x_{m-1}, t_{n+1})\Big)$$

$$-\frac{1}{2}\frac{k\Delta t}{h^2}\Big(u(x_{m+1}, t_n) - 2u(x_m, t_n) + u(x_{m-1}, t_n)\Big)$$

From part (a) it can be seen that,

$$-\frac{1}{2}\frac{k\Delta t}{h^2}\Big(u(x_{m+1}, t_{n+1}) - 2u(x_m, t_{n+1}) + u(x_{m-1}, t_{n+1})\Big) =$$

$$-\frac{1}{2}k\Delta tu_{,xx}(x_m, t_{n+1}) + O(\Delta th^2)$$

Also from part (a),

$$u(x_m, t_{n+1}) - u(x_m, t_n) \;=\; \Delta t u_{,t}(x_m, t_{n+1}) - \frac{1}{2}\Delta t^2 u_{,tt}(x_m, t_{n+1}) + O(\Delta t^3)$$

Now evaluating the last term in the truncation error:

$$\begin{aligned}
-\frac{1}{2}\frac{k\Delta t}{h^2}\Big(u(x_{m+1}, t_n) - 2u(x_m, t_n) + u(x_{m-1}, t_n)\Big) \\
&= -\frac{1}{2}k\Delta t u_{,xx}(x_m, t_n) + O(\Delta t h^2) \\
&= -\frac{1}{2}k\Delta t\Big(u_{,xx}(x_m, t_{n+1}) - \Delta t u_{,xxt}(x_m, t_{n+1})\Big) \\
&\quad + O(\Delta t^3, \Delta t h^2)
\end{aligned}$$

So,

$$\begin{aligned}
\Delta t \tau(t_n) &= \Delta t\Big(u_{,t}(x_m, t_{n+1}) - k u_{,xx}(x_m, t_{n+1})\Big) \\
&\quad + \Delta t^2\Big(-\frac{1}{2}u_{,tt}(x_m, t_{n+1}) + \frac{1}{2}k u_{,xxt}(x_m, t_{n+1})\Big) \\
&\quad + O(\Delta t^3, \Delta t h^2)
\end{aligned}$$

Using,

$$u_{,t} \;=\; k u_{,xx}$$

and differentiating both sides by t,

$$u_{,tt} \;=\; k u_{,xxt}$$

and substituting into the truncation error equation yields,

$$\tau(t_n) \;=\; O(\Delta t^2, h^2) \quad \Longrightarrow \quad k \;=\; l \;=\; 2$$

c. Leap frog:

$$u^h_{n+1}(m) \;=\; u^h_{n-1}(m) + 2r\Big(u^h_n(m+1) - 2u^h_n(m) + u^h_n(m-1)\Big)$$

$$\Delta t\tau(t_n) = u(x_m, t_{n+1}) - u(x_m, t_{n-1}) - 2r\Big(u(x_{m+1}, t_n) - 2u(x_m, t_n) + u(x_{m-1}, t_n)\Big)$$

Using the results from the first two parts of this exercise,

$$-2r\Big(u(x_m, t_{n+1}) - 2u(x_m, t_n) + u(x_{m-1}, t_n)\Big) = -2k\Delta t u_{,xx}(x_m, t_n) + O\left(\Delta t h^2\right)$$

$$\begin{aligned} u(x_m, t_{n+1}) - u(x_m, t_{n-1}) &= \Big(u(x_m, t_n) + \Delta t u_{,t}(x_m, t_n) + \frac{1}{2}\Delta t^2 u_{,tt}(x_m, t_n)\Big) \\ &\quad - \Big(u(x_m, t_n) - \Delta t u_{,t}(x_m, t_n) + \frac{1}{2}\Delta t^2 u_{,tt}(x_m, t_n)\Big) \\ &\quad + O\left(\Delta t^3\right) \\ &= 2\Delta t u_{,t}(x_m, t_n) + O\left(\Delta t^3\right) \end{aligned}$$

Thus,

$$\Delta t\tau(t_n) = \Delta t\Big(2u_{,t}(x_m, t_n) - 2ku_{,xx}(x_m, t_n)\Big) + O\left(\Delta t^3, \Delta t h^2\right)$$

and therefore $k = l = 2$.

8.3 Exercise 3, p. 481

DuFort-Frankel:

$$u_{n+1}^h(m) = u_{n-1}^h(m) + 2r\Big(u_n^h(m+1) - u_{n-1}^h(m) - u_{n+1}^h(m) + u_n^h(m-1)\Big)$$

$$(1+2r)u_{n+1}^h(m) = (1-2r)u_{n-1}^h(m) + 2r\Big(u_n^h(m+1) + u_n^h(m-1)\Big)$$

Replace $u_n^h(m)$ by $\delta_n(m) = \zeta^n e^{im\xi}$. Then after dividing both sides of the previous equation by $\zeta^{n-1}e^{im\xi}$, we get

$$\begin{aligned} (1+2r)\zeta^2 &= (1-2r) + 2r\zeta(e^{i\xi} + e^{-i\xi}) \\ &= (1-2r) + 4r\,\zeta\,\cos\xi \end{aligned}$$

Thus,

$$(1+2r)\zeta^2 - (4r\ \cos\xi)\zeta - (1-2r) \quad = \quad 0$$

and

$$\begin{aligned}
\zeta_{1,2} &= \frac{1}{1+2r}\Big(2r\ \cos\xi \pm \sqrt{(2r\ \cos\xi)^2 + 1 - 4r^2}\Big) \\
&= \frac{1}{1+2r}\Big(2r\ \cos\xi \pm \sqrt{1-4r^2\ \sin^2\xi}\Big)
\end{aligned}$$

If ζ is real

$$\begin{aligned}
|\zeta| &\le \frac{1}{1+2r}\left(|2r\ \cos\xi| + \sqrt{1-4r^2\ \sin^2\xi}\right) \\
&\le \frac{1}{1+2r}(2r+1) = 1
\end{aligned}$$

Otherwise

$$\begin{aligned}
|\zeta| &= \frac{1}{1+2r}\sqrt{4r^2\ \cos^2\xi + 1 - 4r^2\ \sin^2\xi} \\
&= \frac{1}{1+2r}\sqrt{4r^2(\ \cos^2\xi - \ \sin^2\xi) + 1} \\
&= \frac{1}{1+2r}\sqrt{1+4r^2\ \cos(2\xi)} \\
&\le \frac{1}{1+2r}\sqrt{1+4r^2} \\
&\le 1
\end{aligned}$$

So the method is unconditionally stable.

$$\begin{aligned}
\Delta t\tau(t_n) \;=\; & (1+2r)u(x_m,t_{n+1}) - (1-2r)u(x_m,t_{n-1}) \\
& - 2r\Big(u(x_{m+1},t_n) + u(x_{m-1},t_n)\Big)
\end{aligned}$$

$$
\begin{aligned}
\Delta t\tau(t_n) \;=\;& (1+\frac{2k\Delta t}{h^2})\Big(u(x_m,t_n)+\Delta t u_{,t}(x_m,t_n)+\frac{1}{2}\Delta t^2 u_{,tt}(x_m,t_n) \\
&\qquad + \frac{1}{6}\Delta t^3 u_{,ttt}(x_m,t_n)\Big) \\
&- (1-\frac{2k\Delta t}{h^2})\Big(u(x_m,t_n)-\Delta t u_{,t}(x_m,t_n)+\frac{1}{2}\Delta t^2 u_{,tt}(x_m,t_n) \\
&\qquad - \frac{1}{6}\Delta t^3 u_{,ttt}(x_m,t_n)\Big) \\
&- \frac{2k\Delta t}{h^2}\Big(u(x_m,t_n)+h u_{,x}(x_m,t_n)+\frac{1}{2}h^2 u_{,xx}(x_m,t_n) \\
&\qquad + \frac{1}{6}h^3 u_{,xxx}(x_m,t_n)\Big) \\
&- \frac{2k\Delta t}{h^2}\Big(u(x_m,t_n)-h u_{,x}(x_m,t_n)+\frac{1}{2}h^2 u_{,xx}(x_m,t_n) \\
&\qquad - \frac{1}{6}h^3 u_{,xxx}(x_m,t_n)\Big) + O\left(\Delta t^4, \Delta t h^2\right) \\
=\;& \Delta t^0\Big((1-1)u(x_m,t_n)\Big) \\
&+ \Delta t\Bigg(h^{-2}(2k+2k-2k-2k)u(x_m,t_n) \\
&\qquad + h^{-1}(-2k+2k)u_{,x}(x_m,t_n) \\
&\qquad + h^0(2u_{,t}(x_m,t_n)-2k u_{,xx}(x_m,t_n)) \\
&\qquad + h^1(-\frac{1}{3}k+\frac{1}{3}k)u_{,xxx}(x_m,t_n)\Bigg) \\
&+ \Delta t^2\Bigg(h^{-2}(2k-2k)u_{,t}(x_m,t_n)+(\frac{1}{2}-\frac{1}{2})u_{,tt}(x_m,t_n)\Bigg) \\
&+ \Delta t^3\Bigg(h^{-2}(k+k)u_{,tt}(x_m,t_n)+(\frac{1}{6}+\frac{1}{6})u_{,ttt}(x_m,t_n)\Bigg) \\
&+ O\left(\Delta t^4, \Delta t h^2\right)
\end{aligned}
$$

Thus,

$$\tau(t_n) = O\left(h^2, \Delta t^2, \left(\frac{\Delta t}{h}\right)^2\right)$$

8.3 Exercise 4, p. 481

Saul'yev's method:

$$u^h_{n+1}(m) = u^h_n(m) + r(u^h_n(m+1) - u^h_n(m) - u^h_{n+1}(m) + u^h_{n+1}(m-1))$$

a. Stability:

Replacing $u^h_n(m)$ by $\delta_n(m) = \zeta^n e^{im\xi}$; and dividing both sides by $\zeta^n e^{im\xi}$ yields,

$$\zeta\Big(1 + r(1 - e^{-i\xi})\Big) = (1 - r) + re^{i\xi}$$

So,

$$\zeta = \frac{1 - r(1 - e^{i\xi})}{1 + r(1 - e^{-i\xi})}$$

$$|\zeta|^2 = \frac{(1-r)^2 + r^2}{(1+r)^2 + r^2} \leq 1$$

So the method is unconditionally stable.

b. Consistency:

Let

$$u \equiv u(x_m, t_n)$$

Then,

$$\begin{aligned}
\Delta t\tau(t_n) &= (1+r)u(x_m,t_{n+1}) - ru(x_{m-1},t_{n+1}) - (1-r)u(x_m,t_n) \\
&\quad - ru(x_{m+1},t_n) \\
&= (1+\frac{k\Delta t}{h^2})\Big(u + \Delta t u_{,t} + \frac{1}{2}\Delta t^2 u_{,tt} + O\left(\Delta t^3\right)\Big) \\
&\quad - \frac{k\Delta t}{h^2}\Bigg((u + \Delta t u_{,t} + \frac{1}{2}\Delta t^2 u_{,tt} + O\left(\Delta t^3\right)) \\
&\quad - h(u_{,x} + \Delta t u_{,xt} + \frac{1}{2}\Delta t^2 u_{,xtt} + O\left(\Delta t^3\right)) \\
&\quad + \frac{1}{2}h^2(u_{,xx} + \Delta t u_{,xxt} + O\left(\Delta t^2\right)) \\
&\quad - \frac{1}{6}h^3(u_{,xxx} + \Delta t u_{,xxxt} + O\left(\Delta t^2\right)) + O\left(h^4\right)\Bigg) \\
&\quad - (1-r)u - \frac{k\Delta t}{h^2}\Big(u + hu_{,x} + \frac{1}{2}h^2 u_{,xx} + \frac{1}{6}h^3 u_{,xxx} + O\left(h^4\right)\Big) \\
&= \Delta t(u_{,t} - ku_{,xx}) + \Delta t^2\frac{1}{2}\left(u_{,tt} - ku_{,xxt}\right) + \frac{\Delta t^2}{h}ku_{,xt} + \frac{1}{6}k\Delta t^2 h u_{,xxxt} \\
&\quad + O\left(\Delta t h^2, \Delta t^3\right) \\
\tau(t_n) &= O\left(h^2, \Delta t^2, h\Delta t, \frac{\Delta t}{h}\right)
\end{aligned}$$

8.3 Exercise 5, p. 481

With $f = 0$,

$$\boldsymbol{M}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{0} \tag{1}$$

where

$$\boldsymbol{M} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \boldsymbol{m}^e \qquad \boldsymbol{K} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \boldsymbol{k}^e \tag{2}$$

For heat conduction, it follows from (1.15.3) that

$$\boldsymbol{k}^e = \frac{\kappa}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3}$$

and from Sec. 7.2, Exercise 3, it follows that

$$\boldsymbol{m}^e = \frac{\rho c h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{4}$$

Isolating the assembly of elements about the mth degree of freedom results in

$$\frac{\rho c h}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{d}_{m-1} \\ \dot{d}_m \\ \dot{d}_{m+1} \end{Bmatrix} + \frac{\rho c h}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{d}_{m-1} \\ \dot{d}_m \\ \dot{d}_{m+1} \end{Bmatrix}$$

$$+ \frac{\kappa}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} d_{m-1} \\ d_m \\ d_{m+1} \end{Bmatrix} + \frac{\kappa}{h}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} d_{m-1} \\ d_m \\ d_{m+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus, from the completed mth row:

$$\frac{\rho c h}{6}\left(\dot{d}_{m-1}(t) + 4\dot{d}_m(t) + \dot{d}_{m+1}(t)\right) - \frac{\kappa}{h}\left(d_{m-1}(t) - 2d_m(t) + d_{m+1}(t)\right) = 0$$

8.3 Exercise 6, p. 482

The generalized trapezoidal method, applied to the given stencil is:

$$\begin{aligned} &\frac{\rho c h}{6}\Big(v_{m-1}(t_{n+1}) + 4v_m(t_{n+1}) + v_{m+1}(t_{n+1})\Big) \\ &- \frac{\kappa}{h}\Big(d_{m-1}(t_{n+1}) - 2d_m(t_{n+1}) + d_{m+1}(t_{n+1})\Big) = 0 \end{aligned} \tag{1}$$

$$d_m(t_{n+1}) = d_m(t_n) + \Delta t v_m(t_{n+\alpha}) \tag{2}$$

$$v_m(t_{n+\alpha}) = (1-\alpha)v_m(t_n) + \alpha v_m(t_{n+1}) \tag{3}$$

Equation (1) holds with $n+1$ replaced by n . Equations (2) and (3) hold with m replaced by $m+1$. Multiply (1) by $\alpha\Delta t$ and multiply (1) written at time step n by $(1-\alpha)\Delta t$ Adding these two equations yields:

$$\begin{aligned}
&\Delta t\frac{\rho ch}{6}\Big(v_{m-1}(t_{n+\alpha})+4v_m(t_{n+\alpha})+v_{m+1}(t_{n+\alpha})\Big)\\
&-\frac{\Delta t\kappa}{h}\Big(d_{m-1}(t_{n+\alpha})-2d_m(t_{n+\alpha})+d_{m+1}(t_{n+\alpha})\Big) = 0 \qquad (4)
\end{aligned}$$

where

$$d_m(t_{n+\alpha}) = \alpha d_m(t_{n+1})+(1-\alpha)d_m(t_n) \qquad \text{, etc.} \qquad (5)$$

Using (2)-(5)

$$\begin{aligned}
&\frac{\rho ch}{6}\Bigg((d_{m-1}(t_{n+1})-d_{m-1}(t_n))+4(d_m(t_{n+1})-d_m(t_n))\\
&\quad+(d_{m+1}(t_{n+1})-d_{m+1}(t_n))\Bigg)\\
&\quad-\frac{\Delta t\kappa}{h}\Bigg(d_{m-1}(t_{n+\alpha})-2d_m(t_{n+\alpha})+d_{m+1}(t_{n+\alpha})\Bigg) = 0 \qquad (6)
\end{aligned}$$

Let

$$r = \frac{6\kappa\Delta t}{\rho ch^2}$$

Then,

$$\begin{aligned}
d_{m-1}(t_{n+1})+4d_m(t_{n+1})+d_{m+1}(t_{n+1}) &=\\
&(d_{m-1}(t_n)+4d_m(t_n)+d_{m+1}(t_n))\\
&+r\Bigg((1-\alpha)\Big(d_{m-1}(t_n)-2d_m(t_n)+d_{m+1}(t_n)\Big)\\
&+\alpha\Big(d_{m-1}(t_{n+1})-2d_m(t_{n+1})+d_{m+1}(t_{n+1})\Big)\Bigg) \qquad (7)
\end{aligned}$$

a. Stability:

In (7) replace $d_m(t_n)$ by $\delta_n(m) = \zeta^n e^{im\xi}$. Dividing both sides by $\zeta^n e^{im\xi}$ yields:

$$
\begin{aligned}
(4 + e^{i\xi} + e^{-i\xi})\zeta \stackrel{(7)}{=} & (4 + e^{i\xi} + e^{-i\xi}) + (1-\alpha)r(e^{i\xi} + e^{-i\xi} - 2) \\
& + \alpha r(e^{i\xi} + e^{-i\xi} - 2)\zeta
\end{aligned}
\tag{8}
$$

Recall

$$
e^{i\xi} + e^{-i\xi} = 2\cos\xi \qquad \text{and} \qquad 1 - \cos\xi = 2\sin^2(\frac{1}{2}\xi) \tag{9}
$$

So, by (8),

$$
\left((2 + \cos\xi) + 2\alpha r \sin^2(\frac{1}{2}\xi)\right)\zeta = (2 + \cos\xi) - (1-\alpha)r\, 2\sin^2(\frac{1}{2}\xi) \tag{10}
$$

Thus,

$$
\zeta = \frac{(2 + \cos\xi) - (1-\alpha)r\, 2\sin^2(\frac{1}{2}\xi)}{(2 + \cos\xi) + \alpha r\, 2\sin^2(\frac{1}{2}\xi)} \tag{11}
$$

For stability, we require that $|\zeta| \leq 1$. Since $\alpha \in [0, 1]$ and $r \geq 0$

$$
\zeta < 1 \qquad \forall\ \xi
$$

The critical case is $\zeta \geq -1$. This imposes the following condition on (11):

$$
(2 + \cos\xi) - (1-\alpha)r\, 2\sin^2(\frac{1}{2}\xi) \geq -(2 + \cos\xi) - \alpha r\, 2\sin^2(\frac{1}{2}\xi)
$$

Thus,

$$
- \qquad r \leq \frac{2 + \cos\xi}{(1-2\alpha)\sin^2(\frac{1}{2}\xi)} \equiv r_c
$$

r_c has a minimum when $\xi = n\pi$, $n = (2j-1)$, $j = 1, 2, \ldots$ So for the critical limit, i.e., $\xi = n\pi$,

$$
r \leq \frac{2}{1 - 2\alpha}
$$

which is the same stability condition previously derived.

b. Local truncation error:

We shall adopt the "u" notation rather than the "d" notation in calculating the local truncation error.

$$\begin{aligned}\Delta t\tau(x_m, t_n) \quad &= \quad (u_{m-1}^{n+1} + 4u_m^{n+1} + u_{m+1}^{n+1}) \\ &\quad - (u_{m-1}^{n} + 4u_m^{n} + u_{m+1}^{n}) \\ &\quad + \alpha r(-u_{m-1}^{n+1} + 2u_m^{n+1} - u_{m+1}^{n+1}) \\ &\quad + (1-\alpha)r(-u_{m-1}^{n} + 2u_m^{n} - u_{m+1}^{n})\end{aligned}$$

where

$$u_m^n \quad = \quad u(x_m, t_n)$$

Henceforth, let

$$u \quad \equiv \quad u(x_m, t_n)$$

Expanding in a Taylor series in x and t about (x_m, t_n) :

$$\begin{aligned}\Delta t\tau(x_m, t_n) \; &= \; \Big(6u + h^2 u_{,xx}\Big) \\ &\quad - \Big(6u + h^2 u_{,xx}\Big) \\ &\quad + \alpha r\Big(- h^2 u_{,xx} - \frac{1}{12} h^4 u_{,xxxx}\Big) \\ &\quad + (1-\alpha) r\Big(- h^2 u_{,xx} - \frac{1}{12} h^4 u_{,xxxx}\Big) \\ &\quad + \Delta t\Big(6u_{,t} + h^2 u_{,xxt}\Big) \\ &\quad + \Delta t\alpha r(-h^2 u_{,xxt}) \\ &\quad + \frac{1}{2}\Delta t^2\Big(6u_{,tt} + h^2 u_{,xxtt}\Big) \\ &\quad + O\left(\Delta t^3, \Delta t h^4\right)\end{aligned}$$

Collecting terms:

$$\Delta t\tau(x_m,t_n) = \Delta t\left(6(u_{,t} - \frac{\kappa}{\rho c}u_{,xx}) + \frac{1}{2}h^2(2u_{,xxt} - \frac{\kappa}{\rho c}u_{,xxxx})\right)$$

$$+ \Delta t^2\left(6(\frac{1}{2}u_{,tt} - \alpha\frac{\kappa}{\rho c}u_{,xxt}) + \frac{1}{2}h^2u_{,xxtt}\right)$$

$$+ O(\Delta t^3, \Delta t h^4)$$

$$= \Delta t(\frac{1}{2}h^2 u_{,xxt})$$

$$+ \Delta t^2\left(6(\frac{1}{2}(1-2\alpha)\frac{\kappa}{\rho c}u_{,xxt}) + \frac{1}{2}h^2u_{,xxtt}\right)$$

$$+ O(\Delta t^3, \Delta t h^4)$$

Thus,

$$\tau = O(\Delta t^k, h^l)$$

where $l = 2$ and

$$k = \begin{cases} 1 & \text{if } \alpha \neq \frac{1}{2} \\ 2 & \text{if } \alpha = \frac{1}{2} \end{cases}$$

8.3 Exercise 7, p. 482

a.

$$\boldsymbol{m}^e = \frac{\rho c h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \rho c h(\frac{1}{6} - \nu)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The additional mass term is proportional to the stiffness matrix. Thus, using the results from Sec. 8.3, Exercise 5, the added terms to the equation are:

$$\rho c h(\frac{1}{6} - \nu)\Big(- \dot{d}_{m-1}(t) + 2\dot{d}_m(t) - d_{m+1}(t)\Big)$$

which, when added to (8.3.16) results in:

$$\rho c h\Big(\nu \dot{d}_{m-1}(t) + (1-2\nu)\dot{d}_m(t) + \nu \dot{d}_{m+1}(t)\Big)$$
$$-\frac{\kappa}{h}\Big(d_{m-1}(t) - 2d_m(t) + d_{m+1}(t)\Big) \qquad = \quad 0$$

b. Local truncation error:

Again, we shall adopt the "u" notation in computing the local truncation error. Using the trapezoidal algorithm $(\alpha = \frac{1}{2})$ and setting $\nu = \frac{1}{12}$, the local truncation error is

$$\begin{aligned}\Delta t\tau(x_m, t_n) \quad = \quad & \Big(u_{m-1}^{n+1} + 10u_m^{n+1} + u_{m+1}^{n+1}\Big) \\ & - \Big(u_{m-1}^{n} + 10u + u_{m+1}^{n}\Big) \\ & + \frac{1}{2}\tilde{r}\Big(- u_{m-1}^{n+1} + 2u_m^{n+1} - u_{m+1}^{n+1}\Big) \\ & + \frac{1}{2}\tilde{r}\Big(- u_{m-1}^{n} + 2u - u_{m+1}^{n}\Big)\end{aligned}$$

where

$$\tilde{r} \quad = \quad 12\frac{\kappa \Delta t}{\rho c h^2} \qquad\qquad u \quad = \quad u(x_m, t_n)$$

Expanding in a Taylor series in x and t about (x_m, t_n) :

$$\begin{aligned}\Delta t\tau(x_m, t_n) \; = \; & \Big(12u + h^2 u_{,xx}\Big) - \Big(12u + h^2 u_{,xx}\Big) \\ & + \frac{1}{2}\tilde{r}\Big(- h^2 u_{,xx} - \frac{1}{12}h^4 u_{,xxxx}\Big) + \frac{1}{2}\tilde{r}\Big(- h^2 u_{,xx} - \frac{1}{12}h^4 u_{,xxxx}\Big) \\ & + \Delta t\Big(12u_{,t} + h^2 u_{,xxt}\Big) \\ & + \frac{1}{2}\tilde{r}\Delta t\Big(- h^2 u_{,xxt} - \frac{1}{12}h^4 u_{,xxxxt}\Big) + \frac{1}{2}\Delta t^2\Big(12u_{,tt} + h^2 u_{,xxtt}\Big) \\ & + O\left(\Delta t^3 , \Delta t h^4\right)\end{aligned}$$

Collecting terms:

$$\begin{aligned}\Delta t\tau(x_m, t_n) &= \Delta t\left(\left(12u_{,t} - 12\frac{\kappa}{\rho c}u_{,xx}\right) + h^2\left(u_{,xxt} - \frac{\kappa}{\rho c}u_{,xxxx}\right)\right)\\ &+ \Delta t^2\left(6\left(u_{,tt} - \frac{\kappa}{\rho c}u_{,xxt}\right) + \frac{1}{2}h^2\left(u_{,xxtt} - \frac{\kappa}{\rho c}u_{,xxxxt}\right)\right)\\ &+ O\left(\Delta t^3, \Delta t h^4\right)\end{aligned}$$

By virtue of the fact that

$$u_{,t} - \frac{\kappa}{\rho c}u_{,xx} = 0$$

the following hold:

$$\left(u_{,t} - \frac{\kappa}{\rho c}u_{,xx}\right)_{,xx} = u_{,txx} - \frac{\kappa}{\rho c}u_{,xxxx} = 0$$

$$\left(u_{,t} - \frac{\kappa}{\rho c}u_{,xx}\right)_{,t} = u_{,tt} - \frac{\kappa}{\rho c}u_{,xxt} = 0$$

$$\left(u_{,t} - \frac{\kappa}{\rho c}u_{,xx}\right)_{,xxt} = u_{,xxtt} - \frac{\kappa}{\rho c}u_{,xxxxt} = 0$$

So,

$$\tau = O\left(\Delta t^2, h^4\right)$$

8.3 Exercise 8, p. 482

The model equation now becomes:

$$\rho c u_{,t} - \kappa u_{,xx} = f$$

where

$$u = u(x_m, t_n) \qquad f = f(x_m, t_n)$$

To obtain the same rate of convergence for the inhomogeneous problem as for the homogeneous problem, the "f-terms" in the Taylor series expansion of the truncation error must be

$$-\frac{1}{\rho c}\left(\Delta t\Big(12f + h^2 f_{,xx}\Big) + \Delta t^2\Big(6f_{,t} + \frac{1}{2}h^2 f_{,xxt}\Big)\right) + O\left(\Delta t^3\,,\,\Delta t h^4\right)$$

to match up with the homogeneous terms as obtained in the previous exercise.

Applying the trapezoidal rule to the above semidiscrete equation, and arranging the equation into displacement difference form, the inhomogeneous term becomes:

$$\begin{aligned} F &= \frac{12\Delta t}{\rho c}\frac{1}{2}\Big(\nu f_{m-1}^{n+1} + (1-2\nu)f_m^{n+1} + \nu f_{m+1}^{n+1}\Big) \\ &+ \frac{12\Delta t}{\rho c}\frac{1}{2}\Big(\nu f_{m-1}^{n} + (1-2\nu)f_m^{n} + \nu f_{m+1}^{n}\Big) \end{aligned}$$

Setting $\nu = \frac{1}{12}$ expanding F in a Taylors series expansion in x and t about f :

$$\begin{aligned} F &= \frac{6\Delta t}{\rho c}f + \frac{6\Delta t}{\rho c}f + \frac{6\Delta t}{\rho c}(\frac{h^2}{12}f_{,xx}) \\ &+ \frac{6\Delta t}{\rho c}(\frac{h^2}{12}f_{,xx}) + \frac{6\Delta t}{\rho c}\Delta t f_{,t} + \frac{6\Delta t}{\rho c}(\frac{h^2\Delta t}{12}f_{,xxt}) \\ &+ O\left(\Delta t h^4, \Delta t^3\right) \\ &= \frac{\Delta t}{\rho c}(12f + h^2 f_{,xx}) + \frac{\Delta t^2}{\rho c}(6f_{,t} + \frac{1}{2}h^2 f_{,xxt}) \\ &+ O\left(\Delta t h^4\,,\,\Delta t^3\right) \end{aligned}$$

This result, combined with the results of the previous exercise, results in

$$\tau = O\left(\Delta t^2\,,\,h^4\right)$$

CHAPTER 9

9.1 Exercise 1, p. 492

$$\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{C}\boldsymbol{v}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1} = \boldsymbol{F}_{n+1} \quad (1)$$

$$\boldsymbol{d}_{n+1} = \tilde{\boldsymbol{d}}_{n+1} + \beta\Delta t^2 \boldsymbol{a}_{n+1} \quad (2)$$

$$\boldsymbol{v}_{n+1} = \tilde{\boldsymbol{v}}_{n+1} + \gamma\Delta t \boldsymbol{a}_{n+1} \quad (3)$$

$$\beta\Delta t^2 \boldsymbol{a}_{n+1} \overset{(2)}{=} \boldsymbol{d}_{n+1} - \tilde{\boldsymbol{d}}_{n+1} \quad (4)$$

$$\begin{aligned}
\beta\Delta t^2 \boldsymbol{F}_{n+1} &\overset{(1)}{=} \beta\Delta t^2 \{\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{C}\boldsymbol{v}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1}\} \\
&\overset{(4)}{=} \boldsymbol{M}(\boldsymbol{d}_{n+1} - \tilde{\boldsymbol{d}}_{n+1}) + \boldsymbol{C}(\beta\Delta t^2 \boldsymbol{v}_{n+1}) + \beta\Delta t^2 \boldsymbol{K}\boldsymbol{d}_{n+1} \\
&\overset{(3)}{=} \{\boldsymbol{M} + \gamma\Delta t\boldsymbol{C} + \beta\Delta t^2 \boldsymbol{K}\}\boldsymbol{d}_{n+1} - \boldsymbol{M}\tilde{\boldsymbol{d}}_{n+1} \\
&\quad + \beta\Delta t^2 \boldsymbol{C}\tilde{\boldsymbol{v}}_{n+1} - \gamma\Delta t\boldsymbol{C}\tilde{\boldsymbol{d}}_{n+1} \quad (5)
\end{aligned}$$

$$\begin{aligned}
\{\boldsymbol{M} + \gamma\Delta t\boldsymbol{C} + \beta\Delta t^2 \boldsymbol{K}\}\boldsymbol{d}_{n+1} &= \beta\Delta t^2 \boldsymbol{F}_{n+1} + \boldsymbol{M}\tilde{\boldsymbol{d}}_{n+1} \\
&\quad + \boldsymbol{C}(\gamma\Delta t\tilde{\boldsymbol{d}}_{n+1} - \beta\Delta t^2 \tilde{\boldsymbol{v}}_{n+1}) \quad (6)
\end{aligned}$$

9.1 Exercise 2, p. 495

The central difference algorithm is:

$$\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1} = \boldsymbol{F}_{n+1} \quad (1)$$

$$\boldsymbol{d}_{n+1} = \boldsymbol{d}_n + \Delta t\boldsymbol{v}_n + \frac{1}{2}\Delta t^2 \boldsymbol{a}_n \quad (2)$$

$$\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \frac{1}{2}\Delta t(\boldsymbol{a}_n + \boldsymbol{a}_{n+1}) \quad (3)$$

$$\boldsymbol{d}_{n+1} - \boldsymbol{d}_n \overset{(2)}{=} \Delta t\boldsymbol{v}_n + \frac{1}{2}\Delta t^2 \boldsymbol{a}_n \quad (4)$$

$$\boldsymbol{d}_n - \boldsymbol{d}_{n-1} \overset{(2)}{=} \Delta t\boldsymbol{v}_{n-1} + \frac{1}{2}\Delta t^2 \boldsymbol{a}_{n-1} \quad (5)$$

Adding (4) and (5)

$$\boldsymbol{d}_{n+1} - \boldsymbol{d}_{n-1} \quad = \quad \Delta t \boldsymbol{v}_n + \Delta t \boldsymbol{v}_{n-1} + \frac{1}{2}\Delta t^2(\boldsymbol{a}_n + \boldsymbol{a}_{n-1}) \tag{6}$$

$$\begin{aligned}\boldsymbol{v}_n \quad &= \quad \frac{(\boldsymbol{d}_{n+1} - \boldsymbol{d}_{n-1})}{\Delta t} - \left(\boldsymbol{v}_{n-1} + \frac{1}{2}\Delta t(\boldsymbol{a}_n + \boldsymbol{a}_{n-1})\right) \\ &\overset{(3)}{=} \quad \frac{(\boldsymbol{d}_{n+1} - \boldsymbol{d}_{n-1})}{\Delta t} - \boldsymbol{v}_n \end{aligned} \tag{7}$$

Thus,

$$\boldsymbol{v}_n \quad \overset{(7)}{=} \quad \frac{(\boldsymbol{d}_{n+1} - \boldsymbol{d}_{n-1})}{2\Delta t} \tag{8}$$

$$\begin{aligned}\boldsymbol{a}_n \quad &\overset{(2)}{=} \quad \frac{2}{\Delta t^2}(\boldsymbol{d}_{n+1} - \boldsymbol{d}_n) - \frac{2}{\Delta t}\boldsymbol{v}_n \\ &\overset{(8)}{=} \quad \frac{2}{\Delta t^2}(\boldsymbol{d}_{n+1} - \boldsymbol{d}_n) - \frac{1}{\Delta t^2}(\boldsymbol{d}_{n+1} - \boldsymbol{d}_{n-1}) \\ &= \quad \frac{(\boldsymbol{d}_{n+1} - 2\boldsymbol{d}_n + \boldsymbol{d}_{n-1})}{\Delta t^2}\end{aligned} \tag{9}$$

9.1 Exercise 3, p. 495

With $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, the Newmark method is:

$$\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{C}\boldsymbol{v}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1} \quad = \quad \boldsymbol{F}_{n+1} \tag{1}$$

$$\boldsymbol{d}_{n+1} \quad = \quad \boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{1}{4}\Delta t^2(\boldsymbol{a}_n + \boldsymbol{a}_{n+1}) \tag{2}$$

$$\boldsymbol{v}_{n+1} \quad = \quad \boldsymbol{v}_n + \frac{1}{2}\Delta t(\boldsymbol{a}_n + \boldsymbol{a}_{n+1}) \tag{3}$$

$$\boldsymbol{a}_n \quad \overset{(1)}{=} \quad \boldsymbol{M}^{-1}(\boldsymbol{F}_n - \boldsymbol{C}\boldsymbol{v}_n - \boldsymbol{K}\boldsymbol{d}_n) \tag{4}$$

$$\boldsymbol{a}_{n+1} \quad \overset{(1)}{=} \quad \boldsymbol{M}^{-1}(\boldsymbol{F}_{n+1} - \boldsymbol{C}\boldsymbol{v}_{n+1} - \boldsymbol{K}\boldsymbol{d}_{n+1}) \tag{5}$$

$$\boldsymbol{v}_{n+1} - \boldsymbol{v}_n \overset{(3)}{=} \frac{1}{2}\Delta t(\boldsymbol{a}_n + \boldsymbol{a}_{n+1}) \tag{6}$$

$$\begin{aligned}\boldsymbol{d}_{n+1} &\overset{(2,6)}{=} \boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{1}{2}\Delta t(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n) \\ &= \boldsymbol{d}_n + \frac{1}{2}\Delta t(\boldsymbol{v}_{n+1} + \boldsymbol{v}_n)\end{aligned} \tag{7}$$

$$\begin{aligned}\boldsymbol{v}_{n+1} \overset{(3-5)}{=} \boldsymbol{v}_n + \frac{1}{2}\Delta t\Big(&\boldsymbol{M}^{-1}(\boldsymbol{F}_n - \boldsymbol{C}\boldsymbol{v}_n - \boldsymbol{K}\boldsymbol{d}_n) \\ &+ \boldsymbol{M}^{-1}(\boldsymbol{F}_{n+1} - \boldsymbol{C}\boldsymbol{v}_{n+1} - \boldsymbol{K}\boldsymbol{d}_{n+1})\Big)\end{aligned} \tag{8}$$

$$\begin{aligned}\begin{Bmatrix} \boldsymbol{d}_{n+1} \\ \boldsymbol{v}_{n+1} \end{Bmatrix} \overset{(7-8)}{=} & \begin{Bmatrix} \boldsymbol{d}_n \\ \boldsymbol{v}_n \end{Bmatrix} \\ & + \frac{1}{2}\Delta t \begin{Bmatrix} \boldsymbol{v}_{n+1} \\ \boldsymbol{M}^{-1}(\boldsymbol{F}_{n+1} - \boldsymbol{C}\boldsymbol{v}_{n+1} - \boldsymbol{K}\boldsymbol{d}_{n+1}) \end{Bmatrix} \\ & + \frac{1}{2}\Delta t \begin{Bmatrix} \boldsymbol{v}_n \\ \boldsymbol{M}^{-1}(\boldsymbol{F}_n - \boldsymbol{C}\boldsymbol{v}_n - \boldsymbol{K}\boldsymbol{d}_n) \end{Bmatrix}\end{aligned} \tag{9}$$

which is equivalent to:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}\Delta t(\boldsymbol{z}_{n+1} + \boldsymbol{z}_n) \tag{10}$$

9.1 Exercise 4, p. 498

$\boldsymbol{y}_{n+1} = \boldsymbol{A}\boldsymbol{y}_n$ where $\boldsymbol{A} = \boldsymbol{A}_1^{-1}\boldsymbol{A}_2$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 + \beta\Omega^2 & \beta\Delta t 2\xi\Omega \\ \Delta t^{-1}\gamma\Omega^2 & 1 + \gamma 2\xi\Omega \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 1 - \frac{1}{2}(1-2\beta)\Omega^2 & \frac{1}{2}\Delta t(2 - (1-2\beta)2\xi\Omega) \\ -\Delta t^{-1}(1-\gamma)\Omega^2 & 1 - (1-\gamma)2\xi\Omega \end{bmatrix}$$

So,

$$A_1^{-1} \quad = \quad \frac{1}{D}\begin{bmatrix} 1+\gamma 2\xi\Omega & -\beta\Delta t 2\xi\Omega \\ -\gamma\Delta t^{-1}\Omega^2 & 1+\beta\Omega^2 \end{bmatrix}$$

where $D \quad = \quad 1+\gamma 2\xi\Omega+\beta\Omega^2$.

$$A \;=\; \begin{bmatrix} 1-\frac{\Omega^2}{D}[\frac{1}{2}+2\xi\Omega(\frac{1}{2}\gamma-\beta)] & \Delta t(1-\frac{1}{D}[\xi\Omega+(\beta+(\frac{1}{2}\gamma-\beta)(2\xi)^2)\Omega^2]) \\ -\frac{\Omega^2}{\Delta t}(1-\frac{1}{D}[\gamma 2\xi\Omega+\frac{1}{2}\gamma\Omega^2]) & 1-\frac{1}{D}[2\xi\Omega+\gamma\Omega^2-(\frac{1}{2}\gamma-\beta)2\xi\Omega^3] \end{bmatrix}$$

$$\begin{aligned} A_1 \quad &= \quad \frac{1}{2}\mathrm{trace}A \quad = \quad 1-\frac{1}{4}\frac{\Omega^2}{D}-\frac{\xi\Omega}{D}-\frac{1}{2}\frac{\gamma\Omega^2}{D} \\ &= \quad 1-\frac{1}{D}[\xi\Omega+\frac{1}{2}(\gamma+\frac{1}{2})\Omega^2] \end{aligned}$$

$$\begin{aligned} A_2 \quad &= \quad \det A \\ &= \quad \left(1-\frac{\Omega^2}{D}[\frac{1}{2}+2\xi\Omega(\frac{1}{2}\gamma-\beta)]\right)\left(1-\frac{1}{D}[2\xi\Omega+\gamma\Omega^2-(\frac{1}{2}\gamma-\beta)2\xi\Omega^3]\right) \\ &\quad + \left(1-\frac{1}{D}[\xi\Omega+(\beta+(\frac{1}{2}\gamma-\beta)(2\xi)^2)\Omega^2]\right)\Omega^2\left(1-\frac{1}{D}[\gamma 2\xi\Omega+\frac{1}{2}\gamma\Omega^2]\right) \\ &= \quad 1-\frac{1}{D}[2\xi\Omega+\Omega^2(\gamma-\frac{1}{2})] \end{aligned}$$

9.1 Exercise 5, p. 501

$$\ddot{d}+(\omega^h)^2 d \quad = \quad 0 \tag{1}$$

$$a_{n+1}+(\omega^h)^2 d_{n+1} \quad = \quad 0 \tag{2}$$

$$d_{n+1} \quad = \quad d_n+\Delta t((1-\alpha)v_n+\alpha v_{n+1}) \tag{3}$$

$$v_{n+1} \quad = \quad v_n+\Delta t((1-\alpha)a_n+\alpha a_{n+1}) \tag{4}$$

$$d_{n+1} - \alpha \Delta t v_{n+1} \overset{(3)}{=} d_n + \Delta t(1-\alpha) v_n$$
$$v_{n+1} - \alpha \Delta t a_{n+1} \overset{(4)}{=} v_n + \Delta t(1-\alpha) a_n \tag{5}$$

By using (2) we obtain

$$v_{n+1} + \alpha \Delta t (\omega^h)^2 d_{n+1} = v_n - \Delta t (1-\alpha)(\omega^h)^2 d_n \tag{6}$$

Using (5) and (6) we can set up the following system:

$$\boldsymbol{y}_n = \begin{Bmatrix} d_n \\ v_n \end{Bmatrix} \tag{7}$$

$$\boldsymbol{y}_{n+1} = \boldsymbol{A}\boldsymbol{y}_n \tag{8}$$

where $\boldsymbol{A} = \boldsymbol{A}_1^{-1}\boldsymbol{A}_2$,

$$\boldsymbol{A}_1^{-1} = \frac{1}{D}\begin{bmatrix} 1 & \alpha\Delta t \\ -\alpha\Delta t(\omega^h)^2 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 1 & \Delta t(1-\alpha) \\ -\Delta t(1-\alpha)(\omega^h)^2 & 1 \end{bmatrix}$$

$$D = 1 + \alpha^2\Omega^2$$

and

$$\boldsymbol{A} = \frac{1}{D}\begin{bmatrix} 1-\alpha(1-\alpha)\Omega^2 & \Delta t \\ -\Delta t(\omega^h)^2 & 1-\alpha(1-\alpha)\Omega^2 \end{bmatrix}$$

$$A_1 = \frac{1}{2}\text{trace}\boldsymbol{A} = \frac{1}{D}(1-\alpha(1-\alpha)\Omega^2)$$

$$A_2 \quad = \quad \det \boldsymbol{A} \quad = \quad \frac{1}{D^2}(1 + (1 - 2\alpha + 2\alpha^2)\Omega^2 + \alpha^2(1-\alpha)^2\Omega^4)$$

or,

$$A_1 \quad = \quad 1 - \frac{\alpha\Omega^2}{D} \qquad A_2 \quad = \quad 1 - \frac{(2\alpha - 1)\Omega^2}{D}$$

Stability:

The stability regions are given by (9.1.52) and (9.1.53); see also Fig. 9.1.1. We begin with (9.1.52):

$$-1 \ \le \ A_2 \ < \ 1\,, \qquad -\frac{1}{2}(A_2+1) \ \le \ A_1 \ \le \ \frac{1}{2}(A_2+1)$$

i. First check $A_2 \ < \ 1$.

$$\Longrightarrow \frac{1}{D}(2\alpha - 1)\Omega^2 \ > \ 0 \Longrightarrow \alpha \ > \ \frac{1}{2}$$

ii. $A_2 \ \ge \ -1$.

$$\frac{1}{D}(2\alpha - 1)\Omega^2 \ \le \ 2 \Longrightarrow 2[(\alpha(1-\alpha) - \frac{1}{2})\Omega^4 - \Omega^2] \ \le \ 0$$

which is satisfied $\forall \alpha \in [0,1]$.

iii. $A_1 \ \le \ \frac{1}{2}(A_2 + 1)$.

$$1 - \frac{1}{D}\alpha\Omega^2 \ \le \ \frac{1}{2}(2 - \frac{1}{D}(2\alpha - 1)\Omega^2)$$

$$= \quad 1 - \frac{1}{D}\alpha\Omega^2 + \frac{1}{2D}\Omega^2$$

$$\Longrightarrow \quad \frac{1}{2D}\Omega^2 \ \ge \ 0 \qquad \text{which is always satisfied.}$$

iv. $A_1 \ge -\frac{1}{2}(A_2 + 1)$.

$$1 - \frac{1}{D}\alpha\Omega^2 \ \ge \ -\frac{1}{2}(2 - \frac{1}{D}(2\alpha - 1)\Omega^2) \quad = \quad -1 + \frac{1}{D}\alpha\Omega^2 - \frac{1}{2D}\Omega^2$$

$$\Longrightarrow \quad \frac{1}{D}(2\alpha - \frac{1}{2})\Omega^2 - 2 \ \le \ 0$$

$$\Longrightarrow \quad (-2\alpha^2 + 2\alpha - \frac{1}{2})\Omega^2 - 2 \ \le \ 0$$

which is satisfied $\forall \alpha \in [0,1]$.

We may also attain stability by satisfying (9.1.53), namely $-1 < A_1 < 1$, $A_2 = 1$. $A_1 < 1$ is obvious and we can use simple algebra to show that $A_1 > -1$. The condition $A_2 = 1$ implies that $\alpha = \frac{1}{2}$. Thus, for stability,

$$\alpha \geq \frac{1}{2}$$

Order of accuracy:

Using the results from Exercise 2, Sec. 9.1, p. 495, it is clear that $\alpha = \frac{1}{2}$ corresponds to the Newmark method with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ and therefore is second-order accurate. Otherwise, the method is first-order accurate.

9.1 Exercise 6, p. 501

$$\begin{Bmatrix} d_{n+1} \\ v_{n+1} \end{Bmatrix} = \boldsymbol{A} \begin{Bmatrix} d_n \\ v_n \end{Bmatrix}$$

where $\boldsymbol{A}$ was derived in Exercise 3, Sec. 9.1, p. 495. Thus we must evaluate $\lim_{\Omega\to\infty} \boldsymbol{A}$. We assume $\gamma = 2\beta = \frac{1}{2}$ and $\xi = 0$.

$$\begin{aligned}
\lim_{\Omega\to\infty} \boldsymbol{A} &= \lim_{\Omega\to\infty} \begin{bmatrix} 1 - \frac{1}{2}\frac{\Omega^2}{D} & \Delta t(1 - \frac{1}{D}\beta\Omega^2) \\ -\Delta t^{-1}\Omega^2(1 - \frac{1}{2}\gamma\frac{\Omega^2}{D}) & 1 - \frac{\gamma\Omega^2}{D} \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}$$

So, for $\Omega \to \infty$,

$$d_{n+1} = -d_n\,, \qquad v_{n+1} = -v_n$$

and

$$\begin{aligned}
Ma_{n+1} &= -Cv_{n+1} - Kd_{n+1} = Cv_n + Kd_n \\
&= -Ma_n \\
&\implies a_{n+1} = -a_n
\end{aligned}$$

The reader is invited to generalize the argument to the case $\xi > 0$. (This amounts to very simple observations.)

9.1 Exercise 7, p. 502

$$\begin{aligned}
\rho_\infty &= \lim_{\Omega\to\infty} \max\{|\lambda_1|, |\lambda_2|\} \\
\lambda_1, \lambda_2(\boldsymbol{A}) &= A_1 \pm (A_1^2 - A_2)^{\frac{1}{2}}
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= 1 - \frac{1}{D}[\xi\Omega + \frac{1}{2}\Omega^2(\gamma + \frac{1}{2})] \\
A_2 &= 1 - \frac{1}{D}[2\xi\Omega + \Omega^2(\gamma - \frac{1}{2})] \\
D &= 1 + \gamma 2\xi\Omega + \beta\Omega^2 \\
\lim_{\Omega\to\infty} A_1 &= 1 - \frac{(\gamma + \frac{1}{2})}{2\beta} \quad = \quad 1 - \frac{\gamma}{2\beta} - \frac{1}{4\beta} \\
\lim_{\Omega\to\infty} A_2 &= 1 - \frac{(\gamma - \frac{1}{2})}{\beta} \quad = \quad 1 - \frac{\gamma}{\beta} + \frac{1}{2\beta}
\end{aligned}$$

i. $2\beta = \gamma$

$$\lim_{\Omega\to\infty} A_1 = -\frac{1}{4\beta} \qquad \lim_{\Omega\to\infty} A_2 = -1 + \frac{1}{2\beta}$$

$$\begin{aligned}
\lim_{\Omega\to\infty} \{\lambda_1, \lambda_2\} &= -\frac{1}{4\beta} \pm [(\frac{1}{4\beta})^2 + 1 - \frac{1}{2\beta}]^{\frac{1}{2}} \\
&= -\frac{1}{4\beta} \pm (1 - \frac{1}{4\beta}) \\
&= \{-1\,,\, 1 - \frac{1}{2\beta}\}
\end{aligned}$$

Since $\beta \;\geq\; \frac{1}{4}, \quad \max|\lambda| \;=\; 1$. Thus,

$$\rho_\infty = 1$$

ii. $\beta = \frac{1}{4}(\gamma + \frac{1}{2})^2$

$$\lim_{\Omega\to\infty} A_1 = 1 - \frac{2}{(\gamma+\frac{1}{2})} \qquad \lim_{\Omega\to\infty} A_2 = 1 - \frac{4(\gamma-\frac{1}{2})}{(\gamma+\frac{1}{2})^2}$$

$$\lim_{\Omega\to\infty} A_1^2 = 1 - \frac{4}{(\gamma+\frac{1}{2})} + \frac{4}{(\gamma+\frac{1}{2})^2} = 1 - \frac{4(\gamma-\frac{1}{2})}{(\gamma+\frac{1}{2})^2}$$

So,

$$\lim_{\Omega\to\infty} A_1^2 = \lim_{\Omega\to\infty} A_2$$

$$\Longrightarrow \quad \lim_{\Omega\to\infty} \lambda_1 = \lim_{\Omega\to\infty} \lambda_2$$

and therefore

$$\bar{\rho}_\infty = \left|1 - \frac{2}{(\gamma+\frac{1}{2})}\right|$$

iii. $\frac{1}{2}\gamma \leq \beta < \frac{1}{4}(\gamma+\frac{1}{2})^2$

From the upper bound we can derive

$$\frac{2}{\gamma+\frac{1}{2}} < \frac{\gamma+\frac{1}{2}}{2\beta} \qquad \text{and} \qquad \left(\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2 - \frac{1}{\beta} > 0$$

In general,

$$\rho_\infty = \lim_{\Omega\to\infty} \{|\lambda_1|, |\lambda_2|\} = 1 - \frac{\gamma+\frac{1}{2}}{2\beta} \pm \frac{1}{\beta}\left[\frac{1}{4}(\gamma+\frac{1}{2})^2 - \beta\right]^{\frac{1}{2}}$$

If $\beta < \frac{1}{4}(\gamma+\frac{1}{2})^2$, then $[\frac{1}{4}(\gamma+\frac{1}{2})^2 - \beta]^{\frac{1}{2}}$ is a real number, so

$$\rho_\infty = \left|1 - \frac{\gamma+\frac{1}{2}}{2\beta}\right| + \left[\left(\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2 - \frac{1}{\beta}\right]^{\frac{1}{2}}$$

Note that

$$\frac{\gamma+\frac{1}{2}}{2\beta} > \left[\left(\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2 - \frac{1}{\beta}\right]^{\frac{1}{2}}$$

implies

$$\left[\left(\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2-\frac{1}{\beta}\right]^{\frac{1}{2}} > \left[\left(\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2-\frac{1}{\beta}\right]\frac{2\beta}{\gamma+\frac{1}{2}}$$

$$= \frac{\gamma+\frac{1}{2}}{2\beta}-\frac{2}{\gamma+\frac{1}{2}}$$

Consider two cases: If $\gamma \leq \frac{3}{2}$ then

$$\tilde{\rho}_\infty = \frac{2}{\gamma+\frac{1}{2}}-1$$

and

$$\begin{aligned}
\rho_\infty &= \left|1-\frac{\gamma+\frac{1}{2}}{2\beta}\right|+\left[\left(\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2-\frac{1}{\beta}\right]^{\frac{1}{2}} \\
&> \left|1-\frac{\gamma+\frac{1}{2}}{2\beta}\right| \\
&= \max\left\{1-\frac{\gamma+\frac{1}{2}}{2\beta},\frac{\gamma+\frac{1}{2}}{2\beta}-1\right\} \\
&> \max\left\{1-\frac{\gamma+\frac{1}{2}}{2\beta},\frac{2}{\gamma+\frac{1}{2}}-1\right\} \\
&\geq \tilde{\rho}_\infty
\end{aligned}$$

On the other hand, if $\gamma \geq \frac{3}{2}$ then

$$\tilde{\rho}_\infty = 1-\frac{2}{\gamma+\frac{1}{2}}$$

and

$$
\begin{aligned}
\rho_\infty &= \left|1-\frac{\gamma+\frac{1}{2}}{2\beta}\right| + \left[(\frac{\gamma+\frac{1}{2}}{2\beta})^2 - \frac{1}{\beta}\right]^{\frac{1}{2}} \\
&> \max\left\{1-\frac{\gamma+\frac{1}{2}}{2\beta}, \frac{\gamma+\frac{1}{2}}{2\beta} - 1\right\} + \frac{\gamma+\frac{1}{2}}{2\beta} - \frac{2}{\gamma+\frac{1}{2}} \\
&= \max\left\{1-\frac{2}{\gamma+\frac{1}{2}}, \frac{\gamma+\frac{1}{2}}{\beta} - 1 - \frac{2}{\gamma+\frac{1}{2}}\right\} \\
&\geq \bar{\rho}_\infty
\end{aligned}
$$

iv. $\beta > \frac{1}{4}(\gamma+\frac{1}{2})^2$

For this case, $[\frac{1}{4}(\gamma+\frac{1}{2})^2-\beta]^{\frac{1}{2}}$ is imaginary,

$$
\begin{aligned}
\rho_\infty &= \left[\left(1-\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2 + \frac{1}{\beta} - \left(\frac{\gamma+\frac{1}{2}}{2\beta}\right)^2\right]^{\frac{1}{2}} \\
&= \left(1-\frac{\gamma-\frac{1}{2}}{\beta}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\rho_\infty^2 &= 1-\frac{\gamma-\frac{1}{2}}{\beta} + \bar{\rho}_\infty^2 - \underbrace{\left[1-4\frac{\gamma-\frac{1}{2}}{\left(\gamma+\frac{1}{2}\right)^2}\right]}_{=\rho_\infty^2} \\
&= \bar{\rho}_\infty^2 + 4\underbrace{\frac{\gamma-\frac{1}{2}}{\left(\gamma+\frac{1}{2}\right)^2}}_{\geq 0}\underbrace{\left[1-\frac{\left(\gamma+\frac{1}{2}\right)^2}{4\beta}\right]}_{>0} \\
&\geq \rho_\infty^2
\end{aligned}
$$

Equality holds only for $\gamma = \frac{1}{2}$.

v. From part (ii),

$$
0 = \bar{\rho}_\infty = \left|1-\frac{2}{\gamma+\frac{1}{2}}\right|
$$

$$\Longrightarrow \qquad \gamma = \frac{3}{2} \qquad \beta = \frac{(\gamma+\frac{1}{2})^2}{4} = 1$$

9.1 Exercise 8, p. 502

The solution to this problem is given in the text.

9.1 Exercise 9, p. 512

The solution to this problem is given in the text.

9.1 Exercise 10, p. 513

The trapezoidal rule is:

$$a_{n+1} + (\omega^h)^2 d_{n+1} = 0 \tag{1}$$

$$d_{n+1} = d_n + \Delta t v_n + \frac{1}{4}\Delta t^2 (a_n + a_{n+1}) \tag{2}$$

$$v_{n+1} = v_n + \frac{1}{2}\Delta t (a_n + a_{n+1}) \tag{3}$$

$$\Delta t v_n \overset{(3)}{=} \Delta t^2 v_{n-1} + \frac{1}{2}\Delta t^2 (a_{n-1} + a_n) \tag{4}$$

$$\Delta t v_{n-1} \overset{(2)}{=} d_n - d_{n-1} - \frac{1}{4}\Delta t^2 (a_{n-1} + a_n) \tag{5}$$

$$\Delta t v_n \overset{(4,5)}{=} d_n - d_{n-1} + \frac{1}{4}\Delta t^2 (a_{n-1} + a_n) \tag{6}$$

$$d_{n+1} \overset{(2,6)}{=} 2d_n - d_{n-1} + \frac{1}{4}\Delta t^2 (a_{n+1} + 2a_n + a_{n-1}) \tag{7}$$

$$d_{n+1} - 2d_n + d_{n-1} \overset{(1,7)}{=} -\frac{1}{4}\Omega^2 (d_{n+1} + 2d_n + d_{n-1}) \tag{8}$$

$$(d_{n+1} + 2d_n + d_{n-1})[1 + \frac{1}{4}\Omega^2] \overset{(8)}{=} 4d_n \tag{9}$$

Thus,

$$
\begin{aligned}
d_n &\stackrel{(9)}{=} [1+\frac{1}{4}\Omega^2]\frac{d_{n+1}+2d_n+d_{n-1}}{4} \\
&= [1+\frac{1}{4}\Omega^2]x_n
\end{aligned}
$$

or

$$
x_n = \frac{d_n}{[1+\frac{1}{4}\Omega^2]}
$$

9.2 Exercise 1, p. 518

$$
\boldsymbol{k}^e = \frac{E}{h}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \qquad \boldsymbol{m}^e = \frac{\rho h}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
\{\boldsymbol{k}^e - \lambda \boldsymbol{m}^e\}\boldsymbol{\psi}^e = \boldsymbol{0}
$$

There is one rigid body mode: $\lambda_1 = 0$, $\boldsymbol{\psi} = \left\{\begin{matrix} 1 \\ 1 \end{matrix}\right\}$. The non-zero eigenvalue can be calculated knowing that $\boldsymbol{\psi} = \left\{\begin{matrix} 1 \\ -1 \end{matrix}\right\}$. Therefore

$$
\frac{E}{h}\left\langle 1 \quad -1 \right\rangle \left\{\begin{matrix} 1 \\ -1 \end{matrix}\right\} - \frac{1}{2}\lambda_2 \rho h \left\langle 1 \quad 0 \right\rangle \left\{\begin{matrix} 1 \\ -1 \end{matrix}\right\} = 0
$$

which implies

$$
\frac{2E}{h} - \frac{\rho h}{2}\lambda_2 = 0 \implies \lambda_2 = \frac{4E}{\rho h^2}
$$

and

$$
\omega^h_{\max} = \sqrt{\lambda_2} = \frac{2}{h}\sqrt{\frac{E}{\rho}} = \frac{2c}{h}
$$

The consistent mass case can be worked out in similar fashion.

$$\Longrightarrow \qquad \gamma = \frac{3}{2} \qquad \beta = \frac{(\gamma+\frac{1}{2})^2}{4} = 1$$

9.1 Exercise 8, p. 502

The solution to this problem is given in the text.

9.1 Exercise 9, p. 512

The solution to this problem is given in the text.

9.1 Exercise 10, p. 513

The trapezoidal rule is:

$$a_{n+1} + \left(\omega^h\right)^2 d_{n+1} = 0 \tag{1}$$

$$d_{n+1} = d_n + \Delta t v_n + \frac{1}{4}\Delta t^2 (a_n + a_{n+1}) \tag{2}$$

$$v_{n+1} = v_n + \frac{1}{2}\Delta t (a_n + a_{n+1}) \tag{3}$$

$$\Delta t v_n \overset{(3)}{=} \Delta t^2 v_{n-1} + \frac{1}{2}\Delta t^2 (a_{n-1} + a_n) \tag{4}$$

$$\Delta t v_{n-1} \overset{(2)}{=} d_n - d_{n-1} - \frac{1}{4}\Delta t^2 (a_{n-1} + a_n) \tag{5}$$

$$\Delta t v_n \overset{(4,5)}{=} d_n - d_{n-1} + \frac{1}{4}\Delta t^2 (a_{n-1} + a_n) \tag{6}$$

$$d_{n+1} \overset{(2,6)}{=} 2d_n - d_{n-1} + \frac{1}{4}\Delta t^2 (a_{n+1} + 2a_n + a_{n-1}) \tag{7}$$

$$d_{n+1} - 2d_n + d_{n-1} \overset{(1,7)}{=} -\frac{1}{4}\Omega^2 (d_{n+1} + 2d_n + d_{n-1}) \tag{8}$$

$$(d_{n+1} + 2d_n + d_{n-1})[1 + \frac{1}{4}\Omega^2] \overset{(8)}{=} 4d_n \tag{9}$$

9.2 Exercise 2, p. 518

The solution to this problem is given in the text.

9.2 Exercise 3, p. 519

Comparing (9.2.15) and the lumped mass matrix (9.2.11), their only difference is the value of the rotatory inertias. It is obvious that this difference will have no effect on the two rigid body modes.

i. Assuming the bending mode to be $\left\langle 0 \quad 1 \quad 0 \quad -1 \right\rangle^T$, we verify this assumption and obtain the corresponding eigenvalue by:

$$\begin{aligned}
&(\boldsymbol{k}_b + \boldsymbol{k}_s - \lambda^h \boldsymbol{m}) \left\langle 0 \quad 1 \quad 0 \quad -1 \right\rangle^T \\
&= \left(\frac{2EI}{h} - \frac{1}{2}\alpha\rho\hat{A}h\lambda^h\right) \left\langle 0 \quad 1 \quad 0 \quad -1 \right\rangle^T \\
&= 0 \qquad \Longleftrightarrow \qquad \lambda^h = 4\frac{E}{\rho h^2}(I\hat{A}^{-1}\alpha^{-1})
\end{aligned}$$

Thus,

$$\omega^h = \frac{2c}{h}\left(\frac{I}{\alpha\hat{A}}\right)^{\frac{1}{2}}$$

ii. Shear Mode

Designating the eigenvector as $\left\langle a_1 \quad a_2 \quad a_3 \quad a_4 \right\rangle^T$, orthogonality with respect to the three other eigenvectors (in the "$\boldsymbol{m}$-norm") requires:

$$\begin{aligned}
\text{rigid translation :} \quad a_1 + a_3 &= 0 \\
\text{rigid rotation :} \quad -a_1 + \frac{2}{h}\alpha a_2 + a_3 + \frac{2}{h}\alpha a_4 &= 0 \\
a_2 - a_4 &= 0
\end{aligned}$$

These equations are identical to those obtained for the the trapezoidal lumped mass case with

$$\frac{\hat{A}h}{2I} \leftarrow \frac{1}{2}h\alpha^{-1}$$

so the eigenvector is $\left\langle 1 \quad \frac{1}{2}h\alpha^{-1} \quad -1 \quad \frac{1}{2}h\alpha^{-1} \right\rangle^T$. The eigenvalue is obtained by using the following substitution in the trapezoidal lumped mass case, $\hat{A}I^{-1} \leftarrow \alpha^{-1}$, which results in

$$\begin{aligned}\lambda^h &= \frac{4c_s^2}{h^2}(1+\alpha^{-1}(\frac{1}{2}h)^2) \\ \omega^h &= \frac{2c_s}{h}(1+\alpha^{-1}(\frac{1}{2}h)^2)^{\frac{1}{2}}\end{aligned}$$

9.2 Exercise 4, p. 520

The rigid body modes of the thin beam element (cf. p. 518) are the same for both the consistent and the lumped-mass cases. The eigenvectors corresponding to rigid translation and rotation are $\left\langle 1 \quad 0 \quad 1 \quad 0 \right\rangle^T$ and $\left\langle -1 \quad \frac{2}{h} \quad 1 \quad \frac{2}{h} \right\rangle^T$, respectively, with eigenvalues $\lambda^h_{1,2} = 0$.

i. Consistent mass:

Recall that the consistent mass matrix for a thin beam was evaluated in Exercise 2, Sec. 7.3, p. 433. Guessing the third mode to be of the form $\left\langle \alpha \quad 1 \quad \alpha \quad -1 \right\rangle^T$ we solve

$$[\boldsymbol{k}^e - \lambda^h \boldsymbol{m}_1^e] \left\langle \alpha \quad 1 \quad \alpha \quad -1 \right\rangle^T = \boldsymbol{0}$$

and obtain $\alpha = -h/6$. Since

$$[\boldsymbol{k}^e - \lambda^h \boldsymbol{m}_1^e] \left\langle -\frac{h}{6} \quad 1 \quad -\frac{h}{6} \quad -1 \right\rangle^T = \boldsymbol{0} \quad \Longleftrightarrow \quad \left(\frac{2EI}{h} - \lambda^h \frac{\rho \hat{A} h^3}{360}\right) = 0$$

Thus the corresponding eigenvalue is $\lambda_3^h = 720c^2 I/(h^4 \hat{A})$. For the fourth mode we take the eigenvector to be $\left\langle 1 \quad a_2 \quad a_3 \quad a_4 \right\rangle^T$ and orthogonalize it to the three known ones (w.r.t. $\boldsymbol{m}_1^e$).

$$\begin{aligned}6 + ha_2 + 6a_3 - ha_4 &= 0 \\ 12 + ha_2 - 12a_3 + ha_4 &= 0 \\ a_2 - a_4 &= 0\end{aligned}$$

The resulting eigenvector is $\left\langle 1 \quad -\frac{12}{h} \quad -1 \quad -\frac{12}{h} \right\rangle^T$ with an eigenvalue of $\lambda_4^h = 8400c^2 I/(h^4 \hat{A})$. The time step for the central difference operator is bounded by

$$\Delta t \;\leq\; \frac{2}{\omega_{\max}^h}$$

In our case $\omega_{\max}^h = \sqrt{\lambda_4^h}$ so

$$\Delta t \;\leq\; \frac{h^2}{c}\sqrt{\frac{\hat{A}}{2100I}}$$

ii. Lumped mass:

In this case the third mode is pure bending with an eigenvector of $\left\langle 0 \quad 1 \quad 0 \quad -1 \right\rangle^T$ and an eigenvalue of $\lambda_3^h = 4c^2 I/(a\hat{A}h^4)$. The fourth eigenvector is found by orthogonality (w.r.t. $\boldsymbol{m}_2^e$) to be $\left\langle 1 \quad \frac{1}{2ah} \quad -1 \quad \frac{1}{2ah} \right\rangle^T$ and its corresponding eigenvalue is $\lambda_4^h = 4c^2(12a+3)I/(a\hat{A}h^4)$. Hence

$$\Delta t \;\leq\; \frac{h^2}{c}\sqrt{\frac{a\hat{A}}{(12a+3)I}}$$

iii. For any value of $a > \frac{1}{696}$ the critical time step for the case of consistent mass is smaller than that of the lumped mass case. For example, take $a = \frac{1}{12}$. The critical time step for lumped mass, namely

$$\Delta t \;\leq\; \sqrt{\frac{\hat{A}}{48I}}\frac{h^2}{c}$$

is almost one order of magnitude larger than for the consistent case.

9.2 Exercise 5, p. 521

Consider the axial and torsional contributions to the C^0 beam element with only active degrees of freedom of each shown. By comparing the equations in Sec. 5.4 pertaining to axial (5.4.87), (5.4.91), (5.4.96) and torsional (5.4.88), (5.4.92), (5.4.97) behavior of the beam is is clear that the stiffness and lumped mass matrices for the torsional mode can be obtained from those of the axial mode by replacing E with μ and A with J. For the linear element

$$\boldsymbol{k}_t^e = \frac{\mu J}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \boldsymbol{m}_t^e = \frac{\rho h J}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The answer in the text can now be obtained by analogy to the linear rod problem, equation (9.2.2), since J cancels in the eigenvalue analysis.

9.2 Exercise 6, p. 521

The solution to this problem is given in the text.

9.2 Exercise 7, p. 523

$$u_{,tt} = c^2 u_{,xx} \tag{1}$$

$$\begin{aligned} u^h_{n+1}(m) - 2u^h_n(m) + u^h_{n-1}(m) &= r^2(u^h_n(m+1) - 2u^h_n(m) \\ &\quad + u^h_n(m-1)) \end{aligned} \tag{2}$$

where $r = c\Delta t/h$. Using the following replacement,

$$u^h_n(m) \leftarrow \delta_n(m) = \zeta^n e^{im\xi}$$

substituting into (2) and dividing by $\zeta^{n-1}e^{im\xi}$, yields:

$$\begin{aligned} \zeta^2 - 2\zeta + 1 &= r^2(e^{i\xi} + e^{-i\xi} - 2)\zeta \\ &= 2r^2(\cos\xi - 1)\zeta = -4r^2\sin^2(\frac{1}{2}\xi)\zeta \end{aligned}$$

So,

$$\zeta^2 + 2(2r^2\sin^2(\frac{1}{2}\xi) - 1)\zeta + 1 \quad = \quad 0$$

For stability, $|\zeta| \;\leq\; 1$.

$$\zeta_{1,2} \quad = \quad 1 - 2r^2\sin^2(\frac{1}{2}\xi) \pm [(1 - 2r^2\sin^2(\frac{1}{2}\xi))^2 - 1]^{\frac{1}{2}}$$

i. $\zeta_1 \;=\; \zeta_2$ implies

$$1 - 2r^2\sin^2(\frac{1}{2}\xi) \;=\; 1 \quad \Longrightarrow \quad r^2\sin^2(\frac{1}{2}\xi) \;=\; 1$$

ii. ζ_1, ζ_2 real implies

$$(1 - 2r^2\sin^2(\frac{1}{2}\xi))^2 \;\geq\; 1 \quad \Longrightarrow \quad r^2\sin^2(\frac{1}{2}\xi) \;\geq\; 1$$

But if this holds,

$$\begin{aligned}\zeta_2 \quad &= \quad 1 - 2r^2\sin^2(\frac{1}{2}\xi) - [(1 - 2r^2\sin^2(\frac{1}{2}\xi))^2 - 1]^{\frac{1}{2}} \\ &\leq \; -1\end{aligned}$$

So, for stability, $r^2\sin^2(\frac{1}{2}\xi) \;\not\geq\; 1$. This leads to the only possible stable case:

iii. ζ_1, ζ_2 complex, and

$$\begin{aligned}|\zeta_1| \quad &= \quad |\zeta_2| \\ &= \quad [2(1 - 2r^2\sin^2(\frac{1}{2}\xi))^2 - 1]^{\frac{1}{2}}\end{aligned}$$

Stability requires that

$$-1 \;\leq\; 1 - 2r^2\sin^2(\frac{1}{2}\xi) \;\leq\; 1 \quad \Longrightarrow \quad 2r^2\sin^2(\frac{1}{2}\xi) \;\leq\; 2$$

Thus, $r \;\leq\; 1 \Longrightarrow \Delta t \;\leq\; h/c$.

The local truncation error is determined as follows:

$$\begin{aligned} \Delta t^2 \tau &= u(x_m, t_{n+1}) - 2u(x_m, t_n) + u(x_m, t_{n-1}) \\ &\quad - r^2[u(x_{m+1}, t_n) - 2u(x_m, t_n) + u(x_{m-1}, t_n)] \end{aligned} \tag{3}$$

$$\begin{aligned} u(x_m, t_{n\pm1}) &= u(x_m, t_n) \pm \Delta t u_{,t}(x_m, t_n) + \frac{1}{2}\Delta t^2 u_{,tt}(x_m, t_n) \\ &\quad \pm \frac{1}{6}\Delta t^3 u_{,ttt}(x_m, t_n) + O(\Delta t^4) \end{aligned} \tag{4}$$

$$\begin{aligned} u(x_{m\pm1}, t_n) &= u(x_m, t_n) \pm h u_{,x}(x_m, t_n) + \frac{1}{2}h^2 u_{,xx}(x_m, t_n) \\ &\quad \pm \frac{1}{6}h^3 u_{,xxx}(x_m, t_n) + O(h^4) \end{aligned} \tag{5}$$

Substituting (4) and (5) into (3) ,

$$\begin{aligned} \Delta t^2 \tau &= \Delta t^2[u_{,tt}(x_m, t_n) - c^2 u_{,xx}(x_m, t_n)] + O(\Delta t^4) + O(\Delta t^2 h^2) \\ &= \Delta t^2[O(\Delta t^2, h^2)] \end{aligned}$$

Hence, $\tau \;=\; O(\Delta t^2, h^2)$.

9.3 Exercise 1, p. 524

We want to write: $\boldsymbol{M}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}(t)$ in the form: $\dot{\boldsymbol{y}} = \boldsymbol{G}\boldsymbol{y} + \boldsymbol{H}(t)$. Clearly, with $\boldsymbol{y} = \boldsymbol{d}$, $\boldsymbol{G} = -\boldsymbol{M}^{-1}\boldsymbol{K}$, and $\boldsymbol{H}(t) = \boldsymbol{M}^{-1}\boldsymbol{F}(t)$, the associated eigenproblem is

$$(\boldsymbol{G} - \lambda(\boldsymbol{G})\boldsymbol{I})\boldsymbol{\psi} = \boldsymbol{0}$$

Using the definition of $\boldsymbol{G}$ and pre-multiplying by $\boldsymbol{M}$, yields,

$$(\boldsymbol{K} + \lambda(\boldsymbol{G})\boldsymbol{M})\boldsymbol{\psi} = \boldsymbol{0}$$

This is the eigenproblem of Chapter 8, Sec. 2.1 with $\lambda(\boldsymbol{G}) = -\lambda$. Since λ is real and positive, $\lambda(\boldsymbol{G})$ is real and negative.

9.3 Exercise 2, p. 524

$$\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{C}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}(t) \tag{1}$$

We want to write (1) in the form:

$$\dot{\boldsymbol{y}} = \boldsymbol{G}\boldsymbol{y} + \boldsymbol{H}(t) \tag{2}$$

where $\boldsymbol{y} = \begin{Bmatrix} \boldsymbol{d} \\ \dot{\boldsymbol{d}} \end{Bmatrix}$. Therefore

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix}$$

$$\boldsymbol{H}(t) = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{F}(t) \end{Bmatrix}$$

Let $\boldsymbol{\psi}(\boldsymbol{G}) = \begin{Bmatrix} \boldsymbol{\psi}_1(\boldsymbol{G}) \\ \boldsymbol{\psi}_2(\boldsymbol{G}) \end{Bmatrix}$. Then, the eigenproblem has the form:

$$\left(\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} - \begin{bmatrix} \lambda(\boldsymbol{G})\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \lambda(\boldsymbol{G})\boldsymbol{I} \end{bmatrix} \right) \boldsymbol{\psi}(\boldsymbol{G}) = \boldsymbol{0}$$

Thus,

$$\boldsymbol{\psi}_2(\boldsymbol{G}) = \lambda(\boldsymbol{G})\boldsymbol{\psi}_1(\boldsymbol{G})$$

$$\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{\psi}_1(\boldsymbol{G}) + (\boldsymbol{M}^{-1}\boldsymbol{C} + \lambda(\boldsymbol{G})\boldsymbol{I})\boldsymbol{\psi}_2(\boldsymbol{G}) = \boldsymbol{0}$$

and

$$(\lambda^2(\boldsymbol{G})\boldsymbol{M} + \lambda(\boldsymbol{G})\boldsymbol{C} + \boldsymbol{K})\boldsymbol{\psi}_1(\boldsymbol{G}) = \boldsymbol{0}$$

Using the modal reduction procedure of Chapter 7 yields,

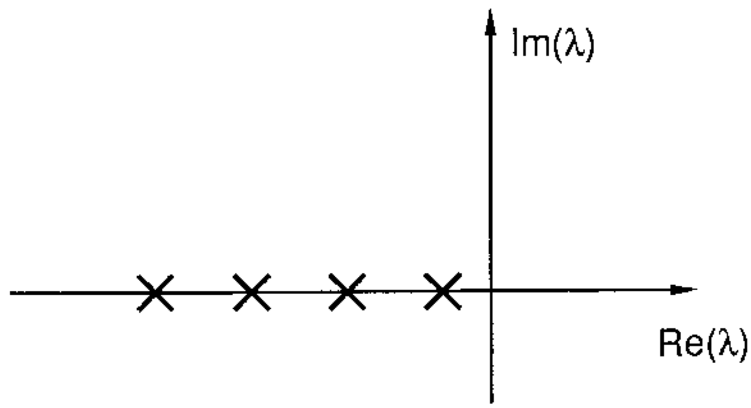
$$\lambda^2(\boldsymbol{G}) + 2\xi\omega^h\lambda(\boldsymbol{G}) + \left(\omega^h\right)^2 = 0$$

Consequently,

$$\lambda_{1,2}(\boldsymbol{G}) = -\xi\omega^h \pm \omega^h\sqrt{\xi^2 - 1}$$

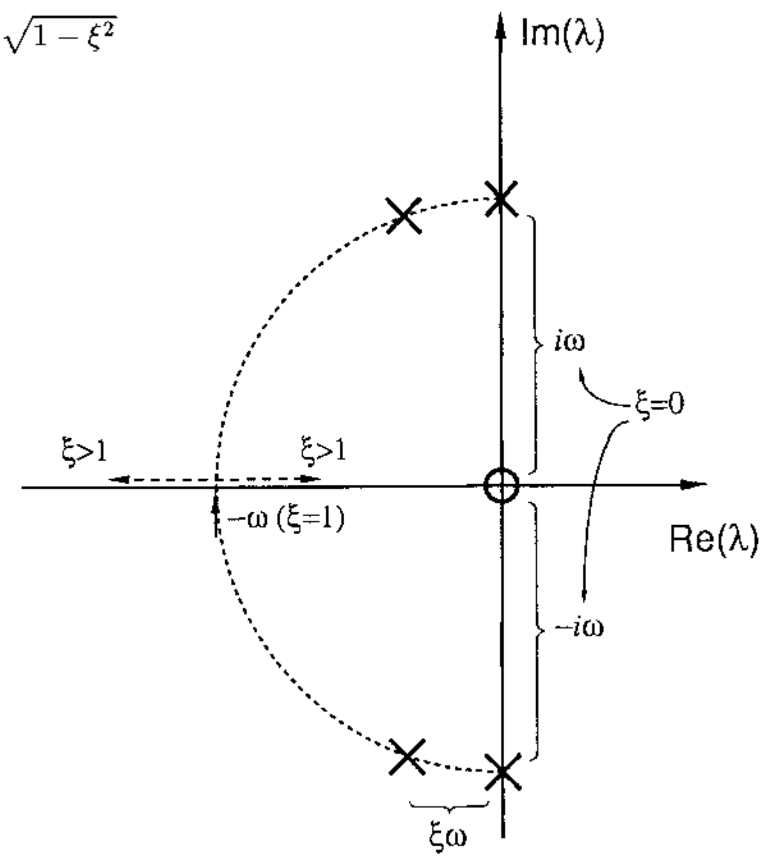
9.3 Exercise 3, p. 524

(i) Heat equation, $\lambda(\boldsymbol{G}) = -\lambda$



(ii) Equation of motion,

$\lambda(\boldsymbol{G}) = -\xi\omega \pm i\omega\sqrt{1-\xi^2}$



9.3 Exercise 4, p. 527

$$M a_{n+1} + C v_{n+1} + K d_{n+1} \quad = \quad F_{n+1} \tag{1}$$

$$d_{n+1} \quad = \quad d_n + \Delta t v_n + \frac{1}{2}\Delta t^2((1-2\beta)a_n + 2\beta a_{n+1}) \tag{2}$$

$$v_{n+1} \quad = \quad v_n + \Delta t((1-\gamma)a_n + \gamma a_{n+1}) \tag{3}$$

$$\beta\Delta t^2 a_{n+1} \quad \overset{(2)}{=} \quad d_{n+1} - d_n - \Delta t v_n - \frac{1}{2}\Delta t^2(1-2\beta)a_n \tag{4}$$

$$\begin{aligned} \beta\Delta t^2 v_{n+1} \quad &\overset{(3,4)}{=} \quad \beta\Delta t^2(v_n + \Delta t(1-\gamma)a_n) \\ &\qquad + \gamma\Delta t(d_{n+1} - d_n - \Delta t v_n - \frac{1}{2}\Delta t^2(1-2\beta)a_n) \\ &= \quad \gamma\Delta t(d_{n+1} - d_n) + (\beta-\gamma)\Delta t^2 v_n + (\beta - \frac{1}{2}\gamma)\Delta t^3 a_n \end{aligned} \tag{5}$$

Multiplying (1) by $\beta\Delta t^2$ results in,

$$\begin{aligned} &(M + \gamma\Delta t C + \beta\Delta t^2 K)d_{n+1} + (-M - \gamma\Delta t C)d_n \\ &+ \boxed{-\Delta t M v_n + (\beta - \tfrac{1}{2})\Delta t^2 M a_n + (\beta-\gamma)\Delta t^2 C v_n + (\beta - \tfrac{1}{2}\gamma)\Delta t^3 C a_n} \\ &\overset{(1,4,5)}{=} \quad \beta\Delta t^2 F_{n+1} \end{aligned} \tag{6}$$

$$\begin{aligned} \Delta t v_n \quad &\overset{(3)}{=} \quad \Delta t v_{n-1} + \Delta t^2((1-\gamma)a_{n-1} + \gamma a_n) \\ &\overset{(2)}{=} \quad d_n - d_{n-1} - \Delta t^2((\frac{1}{2} - \beta)a_{n-1} + \beta a_n) \\ &\qquad + \Delta t^2((1-\gamma)a_{n-1} + \gamma a_n) \\ &= \quad d_n - d_{n-1} + \Delta t^2((\beta + \frac{1}{2} - \gamma)a_{n-1} + (\gamma - \beta)a_n) \end{aligned} \tag{7}$$

$$(1-\gamma)\Delta t(\boldsymbol{d}_n - \boldsymbol{d}_{n-1}) \overset{(2)}{=} (1-\gamma)\Delta t^2 \boldsymbol{v}_{n-1} + (\frac{1}{2}-\beta)(1-\gamma)\Delta t^3 \boldsymbol{a}_{n-1} + \beta(1-\gamma)\Delta t^3 \boldsymbol{a}_n \tag{8}$$

$$(\frac{1}{2}-\beta)(1-\gamma)\Delta t^3 \boldsymbol{a}_{n-1} \overset{(3)}{=} (\frac{1}{2}-\beta)\Delta t^2(\boldsymbol{v}_n - \boldsymbol{v}_{n-1}) - \gamma(\frac{1}{2}-\beta)\Delta t^3 \boldsymbol{a}_n \tag{9}$$

$$(\beta - \frac{1}{2}\gamma)\Delta t^3 \boldsymbol{a}_n \overset{(8,9)}{=} (1-\gamma)\Delta t(\boldsymbol{d}_n - \boldsymbol{d}_{n-1}) + (\beta - \frac{1}{2})\Delta t^2 \boldsymbol{v}_n + (\gamma - \beta - \frac{1}{2})\Delta t^2 \boldsymbol{v}_{n-1} \tag{10}$$

$$\boxed{-\Delta t \boldsymbol{M}\boldsymbol{v}_n + (\beta - \tfrac{1}{2})\Delta t^2 \boldsymbol{M}\boldsymbol{a}_n + (\beta-\gamma)\Delta t^2 \boldsymbol{C}\boldsymbol{v}_n + (\beta - \tfrac{1}{2}\gamma)\Delta t^3 \boldsymbol{C}\boldsymbol{a}_n}$$

$$\overset{(7,10)}{=} (-\boldsymbol{M} + (1-\gamma)\Delta t\boldsymbol{C})\boldsymbol{d}_n + (\boldsymbol{M} - (1-\gamma)\Delta t\boldsymbol{C})\boldsymbol{d}_{n-1} + \Delta t^2(2\beta - \gamma - \frac{1}{2})(\boldsymbol{M}\boldsymbol{a}_n + \boldsymbol{C}\boldsymbol{v}_n) + \Delta t^2(\gamma - \frac{1}{2} - \beta)(\boldsymbol{M}\boldsymbol{a}_{n-1} + \boldsymbol{C}\boldsymbol{v}_{n-1})$$

$$\overset{(1)}{=} (-\boldsymbol{M} + (1-\gamma)\Delta t\boldsymbol{C} + (\frac{1}{2}+\gamma-2\beta)\Delta t^2\boldsymbol{K})\boldsymbol{d}_n + (\boldsymbol{M} - (1-\gamma)\Delta t\boldsymbol{C} + (\frac{1}{2}+\beta-\gamma)\Delta t^2\boldsymbol{K})\boldsymbol{d}_{n-1} + \Delta t^2((2\beta-\gamma-\frac{1}{2})\boldsymbol{F}_n + (\gamma - \frac{1}{2} - \beta)\boldsymbol{F}_{n-1}) \tag{11}$$

The proof is completed using (11) and (6) and defining

$$\bar{\boldsymbol{F}}_n = -\beta\boldsymbol{F}_{n+1} + (2\beta - \gamma - \frac{1}{2})\boldsymbol{F}_n + (\gamma - \frac{1}{2} - \beta)\boldsymbol{F}_{n-1}$$

9.3 Exercise 5, p. 527

With $\bar{F}_n = 0$, and using standard modal decomposition arguments as presented in Sec. 8.2.1, the following is obtained:

$$\begin{aligned}(1+\gamma 2\xi\Omega+\beta\Omega^2)d_{n+1}+(-2+(1-2\gamma)2\xi\Omega+(\frac{1}{2}-2\beta+\gamma)\Omega^2)d_n & \\ +(1-(1-\gamma)2\xi\Omega+(\frac{1}{2}+\beta-\gamma)\Omega^2)d_{n-1} &= 0\end{aligned}$$

Taking $d_n = \zeta^n$ and comparing the above to (9.3.7), the following is obtained:

$$\alpha_0 = \alpha_2 = 1\,, \quad \alpha_1 = -2$$

$$\beta_0 = \gamma\,, \quad \beta_1 = 1-2\gamma\,, \quad \beta_2 = -(1-\gamma)$$

$$\gamma_0 = \beta\,, \quad \gamma_1 = \frac{1}{2}-2\beta+\gamma\,, \quad \gamma_2 = \frac{1}{2}+\beta-\gamma$$

So, comparing (9.1.44)–(9.1.51) to (9.3.7) and the above equation, it can be seen that:

$$\begin{aligned} D &= 1+\gamma 2\xi\Omega+\beta\Omega^2 = \alpha_0+\beta_0 2\xi\Omega+\gamma_0\Omega^2 \\ A_1 &= -(\alpha_1+\beta_1 2\xi\Omega+\gamma_1\Omega^2)/(2D) \\ A_2 &= (\alpha_2+\beta_2 2\xi\Omega+\gamma_2\Omega^2)/D \end{aligned}$$

Thus the equations are identical, i.e., (9.1.44) = (9.3.7), and $\zeta = \lambda(\boldsymbol{A})$.

9.3 Exercise 6, p. 527

We want to express

$$\sum_{i=0}^{k}(\alpha_i+\Delta t\lambda\beta_i)z_{n+1-i} = 0 \tag{1}$$

in the form

$$\boldsymbol{Z}_{n+1} = \boldsymbol{A}\boldsymbol{Z}_n$$

where $\boldsymbol{Z}_n = \left\langle z_n \quad z_{n-1} \quad \cdots \quad z_{n+1-k} \right\rangle^T$ and $\boldsymbol{A}$ is a $k\times k$ matrix.

$$z_{n+1} = -\sum_{i=1}^{k}\left(\frac{\alpha_i+\Delta t\lambda\beta_i}{\alpha_0+\Delta t\lambda\beta_0}\right)z_{n+1-i}$$

Let

$$\gamma_i = \left(\frac{\alpha_i + \Delta t \lambda \beta_i}{\alpha_0 + \Delta t \lambda \beta_0} \right)$$

Then,

$$\boldsymbol{A} = \begin{bmatrix} -\gamma_1 & -\gamma_2 & \dots & \cdot & -\gamma_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The stability polynomial of (1) is:

$$\zeta^k + \sum_{i=1}^{k} \gamma_i \zeta^{k-i} = 0 \tag{2}$$

We want an expression for $\det(\boldsymbol{A} - \lambda(\boldsymbol{A})\boldsymbol{I})$. Expanding in the cofactors of the first row:

$$\begin{aligned} \det\,(\boldsymbol{A} - \lambda(\boldsymbol{A})\boldsymbol{I}) &= (-\gamma_1 - \lambda(\boldsymbol{A}))(-\lambda(\boldsymbol{A}))^{k-1} - (-\gamma_2)(-\lambda(\boldsymbol{A}))^{k-2} + (-\gamma_3)(-\lambda(\boldsymbol{A}))^{k-3} \\ &\quad + \cdots + (-1)^{k-2}(-\gamma_{k-1})(-\lambda(\boldsymbol{A})) + (-1)^{k-1}(-\gamma_k) \\ &= (-1)^k[(\lambda(\boldsymbol{A}))^k + \sum_{i=1}^{k} \gamma_i (\lambda(\boldsymbol{A}))^{k-i}] = 0 \end{aligned} \tag{3}$$

So,

$$(\lambda(\boldsymbol{A}))^k + \sum_{i=1}^{k} \gamma_i (\lambda(\boldsymbol{A}))^{k-i} = 0$$

which is identical to (2).

9.3 Exercise 7, p. 528

$$\zeta^3 - 2A_1\zeta^2 + A_2\zeta - A_3 = 0$$

Using the results of the previous exercise,

$$A = \begin{bmatrix} \frac{-(\alpha_1+\Delta t\lambda\beta_1)}{\alpha_0+\Delta t\lambda\beta_0} & \frac{-(\alpha_2+\Delta t\lambda\beta_2)}{\alpha_0+\Delta t\lambda\beta_0} & \frac{-(\alpha_3+\Delta t\lambda\beta_3)}{\alpha_0+\Delta t\lambda\beta_0} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{trace } A = \frac{-(\alpha_1 + \Delta t\lambda\beta_1)}{\alpha_0 + \Delta t\lambda\beta_0}$$

$$\text{sum of the principal minors of } A = \frac{\alpha_2 + \Delta t\lambda\beta_2}{\alpha_0 + \Delta t\lambda\beta_0}$$

$$\det A = \frac{-(\alpha_3 + \Delta t\lambda\beta_3)}{\alpha_0 + \Delta t\lambda\beta_0}$$

From the previous exercise

$$\zeta^3 + \sum_{k=1}^{3}\left(\frac{\alpha_k + \Delta t\lambda\beta_k}{\alpha_0 + \Delta t\lambda\beta_0}\right)\zeta^{3-k} = 0$$

or

$$\zeta^3 - 2A_1\zeta^2 + A_2\zeta - A_3 = 0$$

where

$$A_1 = \frac{1}{2}\text{ trace } A$$
$$A_2 = \text{sum of the principal minors of } A$$
$$A_3 = \det A$$

9.3 Exercise 8, p. 528

i. For the parabolic case, recall from Exercise 1 that $\mathrm{Re}(\lambda^h\Delta t) \leq 0$, $\mathrm{Im}(\lambda^h\Delta t) = 0$. From the figure, the method is stable for $-2.7 \leq \lambda^h\Delta t \leq 0$.

ii. For the undamped hyperbolic case, from Sec. 9.3, Exercise 2,

$$\lambda^h\Delta t = \pm i\omega\Delta t$$

that is, the roots lie on the imaginary axis. So, stability requires $\omega\Delta t \leq 2.5$.

9.3 Exercise 11, p. 528

The Euler forward-difference algorithm is given by:

$$\alpha_0 = -\alpha_1 = 1 , \quad \beta_0 = 0 , \quad \beta_1 = -1$$

or

$$y_{n+1} = (1 + \lambda^h \Delta t) y_n$$

So,

$$A = 1 + \lambda^h \Delta t$$

$$\mathrm{Re}(A) = 1 + \mathrm{Re}(\lambda^h \Delta t) \qquad \mathrm{Im}(A) = \mathrm{Im}(\lambda^h \Delta t)$$

The equation $|A| = 1$ defines a unit circle in the complex plane $\lambda^h \Delta t$ with center at $\{-1, 0i\}$. The region of absolute stability lies within this circle.

9.3 Exercise 12, p. 529

The local truncation error for the first-order system (9.3.2) is

$$\Delta t \tau_n = \sum_{i=0}^{k} \left(\alpha_i \boldsymbol{y}(t_{n+1-i}) + \Delta t \beta_i \boldsymbol{f}(\boldsymbol{y}(t_{n+1-i}) , t_{n+1-i}) \right)$$

The local truncation error for (9.3.6) is defined analogously, i.e.,

$$\Delta t^2 \tau_n = \sum_{i=0}^{k} \Bigg(\alpha_i \boldsymbol{y}(t_{n+1-i}) + \Delta t \beta_i \boldsymbol{G}_1 \boldsymbol{y}(t_{n+1-i}) + \Delta t^2 \gamma_i [\boldsymbol{G}_0 \boldsymbol{y}(t_{n+1-i}) + \boldsymbol{H}(t_{n+1-i})] \Bigg)$$

The Δt^2 coefficient on the left-hand side is needed since displacement difference forms are used to approximate $\ddot{\boldsymbol{y}}$.

9.3 Exercise 13, p. 530

$$\sum_{i=0}^{3}\Big(\alpha_i \boldsymbol{y}_{n+1-i} + \Delta t \beta_i \boldsymbol{f}(\boldsymbol{y}_{n+1-i}\,,\, t_{n+1-i})\Big) \quad = \quad 0$$

where

$$\boldsymbol{y}_n \quad = \quad \begin{Bmatrix} \boldsymbol{d}_n \\ \boldsymbol{v}_n \end{Bmatrix}$$

$$\boldsymbol{f}(\boldsymbol{y}_n\,,\, t_n) \quad = \quad \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \boldsymbol{y}_n + \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{F}_n \end{Bmatrix}$$

For Park's method,

$$\alpha_0 \;=\; -1\,,\quad \alpha_1 \;=\; 1.5\,,\quad \alpha_2 \;=\; -0.6\,,\quad \alpha_3 \;=\; 0.1$$

$$\beta_0 \;=\; 0.6\,,\quad \beta_1 \;=\; \beta_2 \;=\; \beta_3 \;=\; 0$$

Thus,

$$-\boldsymbol{d}_{n+1} + \frac{3}{2}\boldsymbol{d}_n - \frac{3}{5}\boldsymbol{d}_{n-1} + \frac{1}{10}\boldsymbol{d}_{n-2} + \frac{3}{5}\Delta t \boldsymbol{v}_{n+1} \quad = \quad 0$$

$$\boldsymbol{v}_{n+1} \quad = \quad \Delta t^{-1}\Big(\frac{5}{3}\boldsymbol{d}_{n+1} - \frac{5}{2}\boldsymbol{d}_n + \boldsymbol{d}_{n-1} - \frac{1}{6}\boldsymbol{d}_{n-2}\Big)$$

$$(\boldsymbol{I} + \frac{3}{5}\Delta t \boldsymbol{M}^{-1}\boldsymbol{C})\boldsymbol{v}_{n+1} \quad = \quad \frac{3}{2}\boldsymbol{v}_n - \frac{3}{5}\boldsymbol{v}_{n-1} + \frac{1}{10}\boldsymbol{v}_{n-2} - \frac{3}{5}\Delta t \boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{d}_{n+1}$$

$$(\boldsymbol{I} + \frac{3}{5}\Delta t \boldsymbol{M}^{-1}\boldsymbol{C})\Bigg(\frac{5}{3}\boldsymbol{d}_{n+1} - \frac{5}{2}\boldsymbol{d}_n + \boldsymbol{d}_{n+1} - \frac{1}{6}\boldsymbol{d}_{n-2}\Bigg) \quad = \quad -\frac{3}{5}\Delta t^2 \boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{d}_{n+1} + \frac{3}{2}\Delta t \boldsymbol{v}_n - \frac{3}{5}\Delta t \boldsymbol{v}_{n-1} + \frac{1}{10}\Delta t \boldsymbol{v}_{n-2}$$

$$
\begin{aligned}
&(\frac{5}{3}\boldsymbol{I}+\Delta t\boldsymbol{M}^{-1}\boldsymbol{C}+\frac{3}{5}\Delta t^2\boldsymbol{M}^{-1}\boldsymbol{K})\boldsymbol{d}_{n+1}\\
&\quad+(\boldsymbol{I}+\frac{3}{5}\Delta t\boldsymbol{M}^{-1}\boldsymbol{C})(-\frac{5}{2}\boldsymbol{d}_n+\boldsymbol{d}_{n-1}-\frac{1}{6}\boldsymbol{d}_{n-2}) \quad = \\
&\quad\frac{3}{2}\Delta t\boldsymbol{v}_n-\frac{3}{5}\Delta t\boldsymbol{v}_{n-1}+\frac{1}{10}\Delta t\boldsymbol{v}_{n-2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{3}{2}\Delta t\boldsymbol{v}_n-\frac{3}{5}\Delta t\boldsymbol{v}_{n-1}+\frac{1}{10}\Delta t\boldsymbol{v}_{n-2} &= \frac{3}{2}\Big(\frac{5}{3}\boldsymbol{d}_n-\frac{5}{2}\boldsymbol{d}_{n-1}+\boldsymbol{d}_{n-2}-\frac{1}{6}\boldsymbol{d}_{n-3}\Big)\\
&\quad-\frac{3}{5}\Big(\frac{5}{3}\boldsymbol{d}_{n-1}-\frac{5}{2}\boldsymbol{d}_{n-2}+\boldsymbol{d}_{n-3}-\frac{1}{6}\boldsymbol{d}_{n-4}\Big)\\
&\quad+\frac{1}{10}\Big(\frac{5}{3}\boldsymbol{d}_{n-2}-\frac{5}{2}\boldsymbol{d}_{n-3}+\boldsymbol{d}_{n-4}-\frac{1}{6}\boldsymbol{d}_{n-5}\Big)\\
&= \frac{5}{2}\boldsymbol{d}_n-\frac{19}{4}\boldsymbol{d}_{n-1}+\frac{19}{6}\boldsymbol{d}_{n-2}-\frac{11}{10}\boldsymbol{d}_{n-3}\\
&\quad+\frac{1}{5}\boldsymbol{d}_{n-4}-\frac{1}{60}\boldsymbol{d}_{n-5}
\end{aligned}
$$

So,

$$
\begin{aligned}
&(\frac{5}{3}\boldsymbol{I}+\Delta t\boldsymbol{M}^{-1}\boldsymbol{C}+\frac{3}{5}\Delta t^2\boldsymbol{M}^{-1}\boldsymbol{K})\boldsymbol{d}_{n+1}-(5\boldsymbol{I}+\frac{3}{2}\Delta t\boldsymbol{M}^{-1}\boldsymbol{C})\boldsymbol{d}_n\\
&+(\frac{23}{4}\boldsymbol{I}+\frac{3}{5}\Delta t\boldsymbol{M}^{-1}\boldsymbol{C})\boldsymbol{d}_{n-1}-(\frac{10}{3}\boldsymbol{I}+\frac{1}{10}\Delta t\boldsymbol{M}^{-1}\boldsymbol{C})\boldsymbol{d}_{n-2}\\
&+\frac{11}{10}\boldsymbol{d}_{n-3}-\frac{1}{5}\boldsymbol{d}_{n-4}+\frac{1}{60}\boldsymbol{d}_{n-5} \quad = \quad 0
\end{aligned}
$$

This is a six-step method.

9.3 Exercise 14, p. 530

Houbolt's method:

$$
\boldsymbol{M}\boldsymbol{a}_{n+1}+\boldsymbol{C}\boldsymbol{v}_{n+1}+\boldsymbol{K}\boldsymbol{d}_{n+1} = \boldsymbol{F}_{n+1}
$$

$$
\boldsymbol{a}_{n+1} = \Delta t^{-2}(2\boldsymbol{d}_{n+1}-5\boldsymbol{d}_n+4\boldsymbol{d}_{n-1}-\boldsymbol{d}_{n-2})
$$

$$
\boldsymbol{v}_{n+1} = \frac{1}{6\Delta t}(11\boldsymbol{d}_{n+1}-18\boldsymbol{d}_n+9\boldsymbol{d}_{n-1}-2\boldsymbol{d}_{n-2})
$$

i. Applying the standard modal reduction technique yields:

$$\begin{aligned} 2d_{n+1} - 5d_n + 4d_{n-1} - d_{n-2} & \\ + \frac{2\xi\Omega}{6}(11d_{n+1} - 18d_n + 9d_{n-1} - 2d_{n-2}) + \Omega^2 d_{n+1} &= F_{n+1} \end{aligned}$$

Let

$$D = 2 + \frac{11}{6}(2\xi\Omega) + \Omega^2$$

Then for $F_{n+1} = 0$

$$\begin{aligned} \begin{Bmatrix} d_{n+1} \\ d_n \\ d_{n-1} \end{Bmatrix} &= \boldsymbol{A} \begin{Bmatrix} d_n \\ d_{n-1} \\ d_{n-2} \end{Bmatrix} \\ &= \begin{bmatrix} (5+6\xi\Omega)/D & -(4+3\xi\Omega)/D & (1+\frac{2}{3}\xi\Omega)/D \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} d_n \\ d_{n-1} \\ d_{n-2} \end{Bmatrix} \end{aligned}$$

ii. The spectral radius for Houbolt's method is plotted in Fig. 9.3.1.

iii. Note that the displacement difference form has been obtained in part i, where

$$\begin{aligned} G_1 &= -2\xi\omega^h & G_0 &= -\left(\omega^h\right)^2 \\ H(t_{n+1}) &= F_{n+1} & H(t_{n+1-i}) &= 0, \quad i = 1,2,3 \end{aligned}$$

and

i	α_i	β_i	γ_i
0	2	$-\frac{11}{6}$	-1
1	-5	3	0
2	4	$-\frac{3}{2}$	0
3	-1	$\frac{1}{3}$	0

The local truncation error is obtained by substituting these values into the expression in the answer to Exercise 12 for LMS methods of the type (9.3.6). It can be verified that the local truncation error is $O(\Delta t^2)$.

iv.

$$
\begin{aligned}
A_1 &= \frac{1}{2}\text{ trace } \boldsymbol{A} = (5+6\xi\Omega)/(2D) \\
A_2 &= \text{sum of the principal minors of } \boldsymbol{A} = (4+3\xi\Omega)/D \\
A_3 &= \det \boldsymbol{A} = (1+\frac{2}{3}\xi\Omega)/D
\end{aligned}
$$

(a)

$$
\begin{aligned}
1-2A_1+A_2-A_3 &= 1-(5+6\xi\Omega)/D+(4+3\xi\Omega)/D-(1+\frac{2}{3}\xi\Omega)/D \\
&= (D-2-\frac{11}{3}\xi\Omega)/D \\
&= \Omega^2/D \geq 0
\end{aligned}
$$

(b)

$$
\begin{aligned}
3-2A_1-A_2+3A_3 &= \frac{1}{D}\Big(3(2+\frac{11}{3}\xi\Omega+\Omega^2)-(5+6\xi\Omega)-(4+3\xi\Omega)+3(1+\frac{2}{3}\xi\Omega)\Big) \\
&= (4\xi\Omega+3\Omega^2)/D \geq 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
3+2A_1-A_2-3A_3 &= \frac{1}{D}\Big(3(2+\frac{11}{3}\xi\Omega+\Omega^2)+(5+6\xi\Omega)-(4+3\xi\Omega)-3(1+\frac{2}{3}\xi\Omega)\Big) \\
&= (4+12\xi\Omega+3\Omega^2)/D \geq 0
\end{aligned}
$$

(d)

$$
\begin{aligned}
1+2A_1+A_2+A_3 &= \frac{1}{D}\Big((2+\frac{11}{3}\xi\Omega+\Omega^2)+(5+6\xi\Omega)+(4+3\xi\Omega)+(1+\frac{2}{3}\xi\Omega)\Big) \\
&\geq 0
\end{aligned}
$$

(e)

$$
\begin{aligned}
1 - A_2 + A_3(2A_1 - A_3) &= \frac{1}{D}\Big((2 + \frac{11}{3}\xi\Omega + \Omega^2) - (4 + 3\xi\Omega)\Big) \\
&\quad + \frac{1}{D^2}\Big((1 + \frac{2}{3}\xi\Omega)((5 + 6\xi\Omega) - (1 + \frac{2}{3}\xi\Omega))\Big) \\
&= \frac{1}{D^2}\Big((2 + \frac{1}{3}\xi\Omega + \Omega^2)(-2 + \frac{2}{3}\xi\Omega + \Omega^2) \\
&\quad + (1 + \frac{2}{3}\xi\Omega)(4 + \frac{16}{3}\xi\Omega)\Big) \\
&= \frac{1}{D^2}\Big(\frac{26}{3}\xi\Omega + \frac{34}{9}(\xi\Omega)^2 + \xi\Omega^3 + \Omega^4\Big) \quad \geq \quad 0
\end{aligned}
$$

Thus, the Houbolt method is unconditionally stable.

9.4 Exercise 15, p. 540

The solution to this problem is given in the text.

9.4 Exercise 16, p. 542

The solution to this problem is given in the text.

9.4 Exercise 1, p. 553

Explicit predictor-corrector method:

$$\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{C}\tilde{\boldsymbol{v}}_{n+1} + \boldsymbol{K}\tilde{\boldsymbol{d}}_{n+1} \quad = \quad \boldsymbol{F}_{n+1}$$

$$\boldsymbol{d}_{n+1} \quad = \quad \tilde{\boldsymbol{d}}_{n+1} + \beta\Delta t^2\boldsymbol{a}_{n+1}$$

$$\boldsymbol{v}_{n+1} \quad = \quad \tilde{\boldsymbol{v}}_{n+1} + \gamma\Delta t\boldsymbol{a}_{n+1}$$

Let $\gamma \; = \; 1/2$ and $\forall n \;\; \boldsymbol{F}_n \; = \; \boldsymbol{0}$. Then,

$$(\boldsymbol{M} - \frac{1}{2}\Delta t\boldsymbol{C} - \beta\Delta t^2\boldsymbol{K})\boldsymbol{a}_{n+1} + \boldsymbol{C}\boldsymbol{v}_{n+1} + \boldsymbol{K}\boldsymbol{d}_{n+1} \quad = \quad \boldsymbol{0}$$

Denote

$$\boldsymbol{M}^* \quad = \quad \boldsymbol{M} - \frac{1}{2}\Delta t\boldsymbol{C} - \beta\Delta t^2\boldsymbol{K}$$

Then, using the displacement difference equation derived for the Newmark method, (9.3.8), replacing $\boldsymbol{M}$ by $\boldsymbol{M}^*$, and setting $\gamma = \frac{1}{2}$, results in

$$\boldsymbol{M}\boldsymbol{d}_{n+1} + (-2\boldsymbol{M} + \Delta t\boldsymbol{C} + \Delta t^2\boldsymbol{K})\boldsymbol{d}_n + (\boldsymbol{M} - \Delta t\boldsymbol{C})\boldsymbol{d}_{n-1} \quad = \quad \boldsymbol{0}$$

This is equivalent to the Newmark displacement difference equation with $\beta = 0$ and $\gamma = \frac{1}{2}$ *only* if $\boldsymbol{C} = \boldsymbol{0}$. It can be seen that the predictor-corrector method above is first-order accurate if $\boldsymbol{C} \neq \boldsymbol{0}$ (as noted in Remark 7, p. 558) whereas the Newmark method is second-order accurate for $\gamma = \frac{1}{2}$.

9.4 Exercise 2, p. 555

Recall that the α-method is given by:

$$\boldsymbol{M}\boldsymbol{a}_{n+1} + (1+\alpha)\boldsymbol{C}\boldsymbol{v}_{n+1} + (1+\alpha)\boldsymbol{K}\boldsymbol{d}_{n+1} - \alpha\boldsymbol{C}\boldsymbol{v}_n - \alpha\boldsymbol{K}\boldsymbol{d}_n \;=\; \boldsymbol{F}(t_{n+\alpha})$$

$$\boldsymbol{d}_{n+1} \quad = \quad \boldsymbol{d}_n + \Delta t\boldsymbol{v}_n + \frac{1}{2}\Delta t^2((1-2\beta)\boldsymbol{a}_n + 2\beta\boldsymbol{a}_{n+1})$$

$$\boldsymbol{v}_{n+1} \quad = \quad \boldsymbol{v}_n + \Delta t((1-\gamma)\boldsymbol{a}_n + \gamma\boldsymbol{a}_{n+1})$$

Define $\tilde{\boldsymbol{d}}_{n+1}$ and $\tilde{\boldsymbol{v}}_{n+1}$ by (9.4.3) and (9.4.4). Then the implicit-explicit algorithm has the form:

$$\begin{aligned}\boldsymbol{M}\boldsymbol{a}_{n+1} + (1+\alpha)\boldsymbol{C}^I\boldsymbol{v}_{n+1} + (1+\alpha)\boldsymbol{C}^E\tilde{\boldsymbol{v}}_{n+1} + (1+\alpha)\boldsymbol{K}^I\boldsymbol{d}_{n+1}& \\ + (1+\alpha)\boldsymbol{K}^E\tilde{\boldsymbol{d}}_{n+1} \quad &= \quad \boldsymbol{F}(t_{n+\alpha}) + \alpha\boldsymbol{C}\boldsymbol{v}_n + \alpha\boldsymbol{K}\boldsymbol{d}_n\end{aligned}$$

9.4 Exercise 3, p. 564

The solution to this problem is given in the text.

CHAPTER 10

10.1 Exercise 1, p. 572

$$
\begin{aligned}
0 &= (\tilde{\boldsymbol{K}}^* - \lambda^* \boldsymbol{I})\boldsymbol{\psi}^* = (\bar{\boldsymbol{U}}^{-T}\boldsymbol{K}^*\bar{\boldsymbol{U}}^{-1} - \lambda^* \boldsymbol{I})\bar{\boldsymbol{U}}\boldsymbol{\psi}^* \\
&= (\bar{\boldsymbol{U}}^{-T}\boldsymbol{K}^* - \lambda^* \bar{\boldsymbol{U}})\boldsymbol{\psi}^*
\end{aligned}
$$

Premultiplying by $\bar{\boldsymbol{U}}^T$ and using (10.1.11) yields,

$$
\begin{aligned}
0 &= (\bar{\boldsymbol{U}}^T\bar{\boldsymbol{U}}^{-T}\boldsymbol{K}^* - \lambda^* \bar{\boldsymbol{U}}^T\bar{\boldsymbol{U}})\boldsymbol{\psi}^* \\
&= (\boldsymbol{K}^* - \lambda^* \boldsymbol{M}^*)\boldsymbol{\psi}^*
\end{aligned}
$$

10.1 Exercise 2, p. 572

$$
\boldsymbol{d}(t) = \sum_{l=1}^{n_{eq}} d_{(l)}(t)\boldsymbol{\psi}_l \qquad \ddot{\boldsymbol{d}}(t) = \sum_{l=1}^{n_{eq}} \ddot{d}_{(l)}(t)\boldsymbol{\psi}_l \qquad \boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}
$$

Substituting the first two equations in the third and performing the usual steps of modal reduction (e.g., see Sec. 8.2.1) results in

$$
\ddot{d}_{(l)} + \lambda_l d_{(l)} = F_{(l)}
$$

where $F_{(l)} = \boldsymbol{\psi}_l^T \boldsymbol{F}$. Using Duhamel's integral,

$$
d_{(l)}(t) = \frac{1}{\omega_l}\int_0^t F_{(l)}(\tau)\sin\omega_l(t-\tau)\,d\tau
$$

where $\omega_l = \sqrt{\lambda_l}$ and the total solution is given by (10.1.13), namely

$$
\boldsymbol{d}(t) = \sum_{l=1}^{n_{eq}} d_{(l)}(t)\boldsymbol{\psi}_l
$$

10.1 Exercise 3, p. 572

Again, see Sec. 8.2.1 for background. The associated eigenproblem for $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$ is

$$(\boldsymbol{K} - \lambda \boldsymbol{I})\boldsymbol{\psi} = \boldsymbol{0}$$

where

$$\boldsymbol{\psi}_l^T \boldsymbol{I} \boldsymbol{\psi}_m = \delta_{lm} \quad \text{and} \quad \boldsymbol{\psi}_l^T \boldsymbol{K} \boldsymbol{\psi}_m = \delta_{lm}\lambda_m \qquad \text{(no sum)}$$

Let

$$\boldsymbol{d}(t) = \sum_{m=1}^{n_{eq}} d_{(m)}(t)\boldsymbol{\psi}_m$$

Then premultiplying the first equation by $\boldsymbol{\psi}_l^T$ yields

$$\lambda_l d_{(l)} = F_{(l)}$$

where $F_{(l)} = \boldsymbol{\psi}_l^T \boldsymbol{F}$. Solving for $d_{(l)}$,

$$d_{(l)} = \frac{F_{(l)}}{\lambda_l}$$

Therefore,

$$\boldsymbol{d} = \sum_{l=1}^{n_{eq}} \frac{1}{\lambda_l} F_{(l)} \boldsymbol{\psi}_l$$

10.1 Exercise 4, p. 572

$$\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{C}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$$

where

$$\boldsymbol{C} = a\boldsymbol{M} + b\boldsymbol{K}$$

Defining $\xi_l = \frac{1}{2}(a/\omega_l + b\,\omega_l)$ and using the same approach as in Exercise 3,

$$\ddot{d}_{(l)} + 2\xi_l\,\omega_l \dot{d}_{(l)} + \omega_l^2 d_{(l)} = F_{(l)}$$

Using Duhamel's integral and considering non-zero initial conditions,

$$
\begin{aligned}
d_{(l)}(t) &= e^{-\xi_l\omega_l t}\Big\{d_{0(l)}\cos(\hat{\omega}_l t) + \frac{\dot{d}_{0(l)} + \xi_l\,\omega_l d_{0(l)}}{\hat{\omega}_l}\sin(\hat{\omega}_l t)\Big\} \\
&\quad + \frac{1}{\hat{\omega}_l}\int_0^t F_{(l)}(\tau)e^{\xi_l\omega_l(t-\tau)}\sin(\hat{\omega}_l(t-\tau))\,d\tau
\end{aligned}
$$

where

$$
\hat{\omega}_l = \sqrt{1-\xi_l^2}\,\omega_l \qquad d_{0(l)} = \boldsymbol{\psi}_l^T \boldsymbol{M}\boldsymbol{d}(0) \qquad \dot{d}_{0(l)} = \boldsymbol{\psi}_l^T \boldsymbol{M}\dot{\boldsymbol{d}}(0)
$$

The solution is

$$
\boldsymbol{d}(t) = \sum_{l=1}^{n_{eq}} d_{(l)}(t)\boldsymbol{\psi}_l
$$

10.3 Exercise 1, p. 575

$$
\begin{aligned}
\boldsymbol{K}^* &= \boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{K}_{11} - \boldsymbol{K}_{12}\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21} \\ \boldsymbol{K}_{21} - \boldsymbol{K}_{21} \end{bmatrix} \\
&= \boldsymbol{K}_{11} - \boldsymbol{K}_{12}\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21} = \boldsymbol{K}_{11}^* \\
\boldsymbol{M}^* &= \boldsymbol{R}^T\boldsymbol{M}\boldsymbol{R} = \boldsymbol{R}^T \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{R} = \boldsymbol{M}_{11}
\end{aligned}
$$

10.5 Exercise 1, p. 579

The solution to this problem is given in the text.

10.6 Exercise 1, p. 584

The solution to this problem is given in the text.

CHAPTER 11

11.2 Exercise 1, p. 643

i. Crout factors:

$$D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ & 1 & -1 & 0 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}$$

ii. Forward reduction:

$$U^T z = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \Longrightarrow \quad z = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Diagonal scaling:

$$Dy = z \quad \Longrightarrow \quad y = z$$

Back substitution:

$$Ud = y \quad \Longrightarrow \quad d = \begin{Bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{Bmatrix}$$

iii. The verification is straightforward.

11.2 Exercise 2, p. 644

In components $A_{ij} = \sum_{k=1}^{j} G_{ki} G_{kj}$, where $G_{ij} = 0$ for $i > j$. Expanding:

$$A_{11} = G_{11}^2$$

$$\begin{aligned} A_{12} &= G_{11}G_{12} + G_{21}G_{22} \\ &= G_{11}G_{12} \\ A_{22} &= G_{12}^2 + G_{22}^2 \end{aligned}$$

$$\begin{aligned} A_{13} &= G_{11}G_{13} + G_{21}G_{23} + G_{31}G_{33} \\ &= G_{11}G_{13} \\ A_{23} &= G_{12}G_{13} + G_{22}G_{23} + G_{32}G_{33} \\ &= G_{12}G_{13} + G_{22}G_{23} \\ A_{33} &= G_{13}^2 + G_{23}^2 + G_{33}^2 \end{aligned}$$

$$\begin{aligned} A_{14} &= G_{11}G_{14} + G_{21}G_{24} + G_{31}G_{34} + G_{41}G_{44} \\ &= G_{11}G_{14} \\ A_{24} &= G_{12}G_{14} + G_{22}G_{24} + G_{32}G_{34} + G_{42}G_{44} \\ &= G_{12}G_{14} + G_{22}G_{24} \\ A_{34} &= G_{13}G_{14} + G_{23}G_{24} + G_{33}G_{34} + G_{43}G_{44} \\ &= G_{13}G_{14} + G_{23}G_{24} + G_{33}G_{34} \\ A_{44} &= G_{14}^2 + G_{24}^2 + G_{33}^2 + G_{44}^2 \end{aligned}$$

And so on. Solving for the components of $\boldsymbol{G}$:

$$G_{11} = \sqrt{A_{11}}$$

$$\begin{aligned} G_{12} &= A_{12}/G_{11} \\ G_{22} &= (A_{22} - G_{12}^2)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} G_{13} &= A_{13}/G_{11} \\ G_{23} &= (A_{23} - G_{12}G_{13})/G_{22} \\ G_{33} &= (A_{33} - G_{13}^2 - G_{23}^2)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} G_{14} &= A_{14}/G_{11} \\ G_{24} &= (A_{24} - G_{12}G_{14})/G_{22} \\ G_{34} &= (A_{34} - G_{13}G_{14} - G_{23}G_{24})/G_{33} \end{aligned}$$

$$G_{44} = (A_{44} - G_{14}^2 - G_{24}^2 - G_{34}^2)^{\frac{1}{2}}$$

And so on. Summarizing,

For $j = 1, 2, \ldots, n_{eq}$

For $i = 1, 2, \ldots, j-1$

$$G_{ij} = \left(A_{ij} - \sum_{k=1}^{i-1} G_{ki} G_{kj}\right) / G_{ii}$$

$$G_{jj} = \left(A_{jj} - \sum_{k=1}^{j-1} G_{kj}^2\right)^{\frac{1}{2}}$$

11.5 Exercise 1, p. 716

The fundamental frequency of the rod may be estimated as

$$\omega_1 = \frac{\pi c}{2L} = \frac{100\pi}{2(10.0)} = 15.71$$

The required stiffness-proportional damping ratio can be calculated from

$$\xi_1 = 0.10 = \frac{1}{2}\left(\frac{\text{RDAMPM}}{\omega_1} + \text{RDAMPK} \times \omega_1\right)$$

thus $\text{RDAMPK} = 2\xi_1/\omega_1 = 1.273 \times 10^{-2}$. The maximum frequency

$$\omega_{\max}^h \leq \frac{2c}{h} = 400 \Longrightarrow \xi_{\max} = 2.546$$

and so the reduction factor in critical time step is

$$\frac{([\xi_{\max}^2 + 2\gamma]^{\frac{1}{2}} - \xi_{\max})/\gamma}{\sqrt{2/\gamma}} = 0.1893$$

For a rod having 210 elements of length 0.05

$$\omega_{\max}^h \leq \frac{2c}{h} = 4000 \Longrightarrow \xi_{\max} = 25.46$$

and the reduction factor in critical time step is

$$\frac{([\xi_{\max}^2 + 2\gamma]^{\frac{1}{2}} - \xi_{\max})/\gamma}{\sqrt{2/\gamma}} = 1.963 \times 10^{-2}$$

This example illustrates the significant restriction stiffness-proportional damping can place upon critical time-step size.