ANALYTICAL METHODS OF OPTIMISATION

SOLUTIONS TO EXERCISES - 0486450341

Exercises 1

1.
$$C = 6x^2 + 8y^3 + 3(2x+y-1)^2$$

$$\frac{\partial C}{\partial x} = 12x + 12(2x+y-1) = 12(3x+y-1)$$

$$\frac{\partial C}{\partial y} = 24y^2 + 6(2x+y-1).$$

We find $\partial C/\partial x$, $\partial C/\partial y$ vanish at the points $\left(\frac{4}{9}, -\frac{1}{3}\right)$ and $\left(\frac{1}{4}, \frac{1}{4}\right)$.

$$\frac{\partial^2 C}{\partial x^2} = 36, \quad \frac{\partial^2 C}{\partial x \partial y} = 12, \quad \frac{\partial^2 C}{\partial y^2} = 48y+6.$$

Hence, at the stationary points, the matrix A has values

$$\begin{bmatrix} 36 & 12 \\ 12 & -10 \end{bmatrix}, \quad \begin{bmatrix} 36 & 12 \\ 12 & 18 \end{bmatrix}$$

respectively. The eigenvalues of the first matrix satisfy the equation $\alpha^2 - 26\alpha - 504 = 0$ and are accordingly of opposite signs; this stationary point is neither a maximum nor a minimum. The eigenvalues of the second matrix are 12 and 42; this stationary point is accordingly a local minimum.

The value of C at the second stationary point is 11/16 as stated.

2. Taking the mass of each particle to be unity, we find

$$I = \sum_i \left(x_i \cos\theta + y_i \sin\theta - p\right)^2$$

to be their moment of inertia about the line. Using the given notation, this leads to

$$C = I/n = \left(A+\bar{x}^2\right)\cos^2\theta + \left(B+\bar{y}^2\right)\sin^2\theta + 2\left(H+\bar{x}\bar{y}\right)\sin\theta\cos\theta - 2p\left(\bar{x}\cos\theta + \bar{y}\sin\theta\right) + p^2.$$

Thus

$$\frac{\partial C}{\partial p} = 2\left(p - \bar{x}\cos\theta - \bar{y}\sin\theta\right)$$

$$\frac{\partial C}{\partial \theta} = \left(B - A + \bar{y}^2 - \bar{x}^2\right)\sin 2\theta + 2\left(H + \bar{x}\bar{y}\right)\cos 2\theta + 2p\left(\bar{x}\sin\theta - \bar{y}\cos\theta\right)$$

and, hence, for C to be stationary

$$p = \bar{x}\cos\theta + \bar{y}\sin\theta$$

and $\partial C/\partial\theta$ then reduces to

$$\frac{\partial C}{\partial\theta} = (B-A)\sin 2\theta + 2H\cos 2\theta$$

requiring that $$\tan 2\theta = \frac{2H}{A-B}.$$

Further $$\frac{\partial^2 C}{\partial p^2} = 2, \quad \frac{\partial^2 C}{\partial p\partial\theta} = 2(\bar{x}\sin\theta - \bar{y}\cos\theta),$$

$$\frac{\partial^2 C}{\partial\theta^2} = 2\left(B - A + \bar{y}^2 - \bar{x}^2\right)\cos 2\theta - 4(H + \bar{x}\bar{y})\sin 2\theta + 2\mathrm{p}(\bar{x}\cos\theta + \bar{y}\sin\theta)$$

$$= 2\left(B - A + \bar{y}^2 - \bar{x}^2\right)\cos 2\theta - 4(H + \bar{x}\bar{y})\sin 2\theta + 2(\bar{x}\cos\theta + \bar{y}\sin\theta)^2$$

$$= 2(B-A)\cos 2\theta - 4H\sin 2\theta + 2(\bar{x}\sin\theta - \bar{y}\cos\theta)^2$$

for optimal values. The eigenvalues then satisfy the equation

$$\alpha^2 + P\alpha + Q = 0,$$

where $$P = -2 - 2(B-A)\cos 2\theta + 4H\sin 2\theta - 2(\bar{x}\sin\theta - \bar{y}\cos\theta)^2,$$

$$Q = 4(B-A)\cos 2\theta - 8H\sin 2\theta.$$

We note that $A = \frac{1}{n}\sum_i (x_i - \bar{x})^2$; hence $A > 0$. Similarly $B > 0$. H can be positive or negative.

There are 4 cases to consider:

(i) If $A > B$, $H > 0$, then $\tan 2\theta > 0$ and 2θ is an angle in the first or third quadrants. If 2θ is taken in the first quadrant, we find $Q < 0$ and the stationary value cannot be a minimum. Taking 2θ in the third quadrant, we find $Q > 0$, $P < 0$, the eigenvalues are both positive and the stationary value is a minimum..

(ii) If $A > B$, $H < 0$, then $\tan 2\theta < 0$ and 2θ lies in the second or fourth quadrants. Taking 2θ in the second quadrant, we find $Q > 0$, $P < 0$, as required for a minimum.

(iii) If $A < B$, $H > 0$, then $\tan 2\theta < 0$ and 2θ lies in the second or fourth quadrants. Taking 2θ in the fourth quadrant, we find $Q > 0$, $P < 0$, as required for a minimum.

(iv) If $A < B$, $H < 0$, then $\tan 2\theta > 0$ and 2θ lies in the first or third quadrants. Taking 2θ in the first quadrant, we find $Q > 0$, $P < 0$, as required for a minimum.

Having determined 2θ, θ will be arbitrary to the extent of the addition of π. However, if θ is increased by π, then the sign of the optimal value of p is reversed and the optimal straight line remains the same.

3. x_i $(i = 1, 2, \ldots, n)$ satisfy the constraint

$$x_1 x_2 \ldots x_n \quad = \quad G.$$

We need to minimise

$$C \quad = \quad x_1 + x_2 + \ldots + x_n.$$

Defining the Hamiltonian

$$H \quad = \quad x_1 + x_2 + \ldots + x_n + \lambda x_1 x_2 \ldots x_n,$$

it is necessary that

$$\frac{\partial H}{\partial x_i} \quad = \quad 1 + \lambda x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_n \quad = \quad 0.$$

i.e.

$$1 + \frac{\lambda G}{x_i} \quad = \quad 0, \qquad i = 1, 2, \ldots, n$$

using the constraint. Thus $x_i = -\lambda G$ **and all the** x_i are equal. Substituting in the constraint, we find that $\lambda \; = \; -G^{1/n-1}$, i.e. λ is a negative constant.

Treating $x_1, x_2, \ldots, x_{n-1}$ as control variables and x_n as a state variable, it follows from the constraint that

$$\frac{\partial x_n}{\partial x_r} \quad = \quad -\frac{G}{x_1 x_2 \ldots x_{n-1}} \cdot \frac{1}{x_r}, \qquad r = 1, 2, \ldots, (n-1),$$

$$= \quad -\frac{x_n}{x_r}.$$

Using (1.3.15), we now find that

$$\frac{\partial^2 D}{\partial x_r \partial x_s} \quad = \quad \lambda x_1 x_2 \ldots x_{r-1} x_{r+1} \ldots x_{n-1} \frac{\partial x_n}{\partial x_s} + \lambda x_1 x_2 \ldots x_{s-1} x_{s+1} \ldots x_{n-1} \frac{\partial x_n}{\partial x_r} + \frac{\partial^2 H}{\partial x_r \partial x_s}$$

where $r, s = 1, 2, \ldots, n-1$. In case $r \neq s$

$$\frac{\partial^2 H}{\partial x_r \partial x_s} = \lambda x_1 x_2 \ldots x_{r-1} x_{r+1} \ldots x_{s-1} x_{s+1} \ldots x_{n-1} = \frac{\lambda G}{x_r x_s}.$$

In case $r = s$

$$\frac{\partial^2 H}{\partial x_r^2} = 0.$$

Hence, in case $r = s$,

$$\frac{\partial^2 D}{\partial x_r \partial x_s} = -\frac{\lambda G}{x_r x_s} = -\frac{1}{\lambda G} = P,$$

and, in case $r = s$,

$$\frac{\partial^2 D}{\partial x_r^2} = -\frac{2\lambda G}{x_r^2} = -\frac{2}{\lambda G} = 2P.$$

after putting $x_r = x_s = -\lambda G$.. P is a positive constant.

Thus, the second sufficient condition for a minimum is that the quadratic form

$$P(x_1 + x_2 + \ldots + x_{n-1})^2 + P(x_1^2 + x_2^2 + \ldots + x_{n-1}^2)$$

be positive definite, which it obviously is.

4. The Hamiltonian for the problem is

$$H = x^3 + y^3 + z^3 + \lambda(x^2 + y^2 + z^2) + \mu(x + y + z).$$

For a stationary value of $x^3 + y^3 + z^3$, we must have

$$\frac{\partial H}{\partial x} = 3x^2 + 2\lambda x + \mu = 0$$

$$\frac{\partial H}{\partial y} = 3y^2 + 2\lambda y + \mu = 0$$

$$\frac{\partial H}{\partial z} = 3z^2 + 2\lambda z + \mu = 0$$

Eliminating λ and μ, we find

$$\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = 0$$

or $$(x - y)(y - z)(z - x) = 0.$$

Thus, $x = y$ or $y = z$ or $z = x$.

If $x = y = z$, the constraints are incompatible.

If $x = y$, solving the constraints we find

$x = y = 0,\ z = 1$ or $x = y = 2/3,\ z = -1/3.$

In the first case, $\lambda = -3/2,\ \mu = 0$, and in the second case $\lambda = -1/2,\ \mu = -2/3$.

Also, in the first case $x^3 + y^3 + z^3 = 1$ and in the second case $x^3 + y^3 + z^3 = 15/27$. We shall show the first is a maximum and the second is a minimum.

Treating x as a control variable and y, z as state variables, by differentiating the constraints we find

$$1 + \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} = 0, \qquad 2x + 2y\frac{\partial y}{\partial x} + 2z\frac{\partial z}{\partial x} = 0,$$

so that for both sets of stationary values $\frac{\partial y}{\partial x} = -1,\ \frac{\partial z}{\partial x} = 0.$

(1.3.15) now shows that

$$\frac{\partial^2 D}{\partial x^2} = \frac{\partial^2 H}{\partial y^2} - 2\frac{\partial^2 H}{\partial y \partial x} + \frac{\partial^2 H}{\partial x^2} = 2(3x + 3y + \lambda).$$

Thus, in the case $x = y = 0,\ z = 1,\ \lambda = -3/2$, we find $\frac{\partial^2 D}{\partial x^2} = -3$ corresponding to a maximum (value 1).

In the case $x = y = 2/3,\ z = -1/3,\ \lambda = -1/2$, we find $\frac{\partial^2 D}{\partial x^2} = 7$ corresponding to a minimum (value 15/27).

By permutation of the stationary values of x, y, z, we can identify two other maxima and two other minima.

5. We have $$AP^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$$

and AP is stationary with respect to variation of (x, y, z) subject to the condition $f(x, y, z) = 0$, if AP^2 is stationary.

We introduce the Hamiltonian

$$H = (x-a)^2 + (y-b)^2 + (z-c)^2 + \lambda f(x,y,z).$$

Then, AP^2 is stationary when

$$\frac{\partial H}{\partial x} = 2(x-a) + \lambda\frac{\partial f}{\partial x} = 0, \quad \frac{\partial H}{\partial y} = 2(y-b) + \lambda\frac{\partial f}{\partial y} = 0, \quad \frac{\partial H}{\partial z} = 2(z-c) + \lambda\frac{\partial f}{\partial z} = 0.$$

For such a point $P(x, y, z)$, we have

$$(x-a):(y-b):(z-c) = \frac{\partial f}{\partial x}:\frac{\partial f}{\partial y}:\frac{\partial f}{\partial z},$$

so that $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ are direction ratios for AP.

If $f(x,y,z) = x^2 + y^2 + z^2 - r^2$, we find that

$$(1+\lambda)x = a, \quad (1+\lambda)y = b, \quad (1+\lambda)z = c.$$

Substituting in $f = 0$, we calculate that

$$1+\lambda = \pm\sqrt{a^2+b^2+c^2}\,/r$$

leading to the result stated.

AP is, of course, a normal from A to the surface.

6. The Hamiltonian for the problem is

$$H = \frac{1}{2}p_{ij}x_i x_j + \frac{1}{2}q_{rs}u_r u_s + r_{ir}x_i u_r + \lambda_i(x_i - b_{ir}u_r)$$

and if C is to be minimised, we must have

$$\frac{\partial H}{\partial x_i} = p_{ij}x_j + r_{ir}u_r + \lambda_i = 0,$$

$$\frac{\partial H}{\partial u_r} = q_{rs}u_s + r_{ir}x_i - b_{ir}\lambda_i = 0.$$

Introducing matrices, these conditions can be written

$$Px + Ru + \lambda = 0, \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$Qu + R^T x - B^T\lambda = 0. \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

where P is $M \times M$ (symmetric), Q is $N \times N$ (symmetric), R and B are $M \times N$. The constraints can be written

$$x = Bu + c. \quad \ldots\ldots\ldots\ldots \quad \text{(iii)}$$

Eliminating x, λ by use of (i) and (iii), we find that the optimal control is

$$u = -\left(B^T PB + Q + B^T R + R^T B\right)^{-1}\left(B^T P + R^T\right)c$$

(N.B. Missing minus sign!)

Substitution in (iii) now yields

$$x = \left[I - B\left(B^T PB + Q + B^T R + R^T B\right)^{-1}\left(B^T P + R^T\right)\right]c$$

and then (i) gives

$$\lambda = \left[-P - (PB + R)\left(B^T PB + Q + B^T R + R^T B\right)^{-1}\left(B^T P + R^T\right)\right]c \qquad \text{....(iv)}$$

Since, from the constraints, $\partial x_i / \partial u_r = b_{ir}$, (1.3.15) leads to

$$\frac{\partial^2 D}{\partial u_r \partial u_s} = b_{ir} b_{js} p_{ij} + b_{ir} r_{is} + b_{is} r_{ir} + q_{rs},$$

we have in matrix form

$$\frac{\partial^2 D}{\partial u_r \partial u_s} \xi_r \xi_s = \xi^T\left(B^T PB + Q + B^T R + R^T B\right)\xi$$

which is a positive definite quadratic form, indicating that C is minimised.

Thus, using equations (i) - (iii),

$$C_{\min} = \frac{1}{2} x^T Px + \frac{1}{2} u^T Qu + x^T Ru$$

$$= -\frac{1}{2} x^T (Ru + \lambda) - \frac{1}{2} u^T\left(B^T \lambda + R^T x\right) + x^T Ru$$

$$= -\frac{1}{2}\left(x^T - u^T B^T\right)\lambda + \frac{1}{2} x^T Ru - \frac{1}{2} u^T R^T x = -\frac{1}{2}(x - Bu)^T \lambda$$

$$= -\frac{1}{2} c^T \lambda$$

since $x^T Ru = \left(x^T Ru\right)^T = u^T R^T x$. Finally, substituting for λ from (iv), we obtain

$$C_{\min} = \frac{1}{2} c^T \left[P + (PB + R)\left(B^T PB + Q + B^T R + R^T B\right)^{-1}\left(B^T P + R^T\right)\right]c.$$

(N.B. Omitted minus sign!)

At the point (2/3, 2/3, -1/3) where $x^3 + y^3 + z^3 = \dfrac{15}{27}$, the Lagrange multipliers have been shown to assume the values $\lambda = -1/2$, $\mu = -2/3$. Both these values are negative, so it follows that this point remains a minimum when the constraints are replaced by the inequalities

$$x + y + z \geq 1, \quad x^2 + y^2 + z^2 \geq 1.$$

Similary for the other two minima.

If we are to maximise $x^3 + y^3 + z^3$ subject to the constraint $x^2 + y^2 + z^2 \leq 1$ we first replace the constraint by $x^2 + y^2 + z^2 = 1$ and define the Hamiltonian

$$H \;=\; x^3 + y^3 + z^3 + \lambda\left(x^2 + y^2 + z^2\right).$$

Conditions for a stationary value are

$$\frac{\partial H}{\partial x} = 3x^2 + 2\lambda x = 0, \quad \frac{\partial H}{\partial y} = 3y^2 + 2\lambda y = 0, \quad \frac{\partial H}{\partial z} = 3z^2 + 2\lambda z = 0.$$

These are all satisfied together with the constraint at $x = y = 0, z = 1$, with $\lambda = -3/2$.

Treating x, y as control variables and z as a state variable, by differentiating the constraint we calculate that $\partial z/\partial x = -x/z$, $\partial z/\partial y = -y/z$ and both these derivatives vanish at $x = y = 0, z = 1$. Hence, from (1.3.15) we find that

$$\frac{\partial^2 D}{\partial x^2} \;=\; \frac{\partial^2 H}{\partial x^2} \;=\; 6x - 2\lambda \;=\; -3,$$

$$\frac{\partial^2 D}{\partial x \partial y} \;=\; \frac{\partial^2 H}{\partial x \partial y} \;=\; 0,$$

$$\frac{\partial^2 D}{\partial y^2} \;=\; \frac{\partial^2 H}{\partial y^2} \;=\; 6y - 2\lambda \;=\; -3,$$

at the point $x = y = 0, z = 1$. Since the quadratic form $-3\xi_1^2 - 3\xi_2^2$ is negative definite, the point $x = y = 0, z = 1$ is a minimum for $x^3 + y^3 + z^3$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Further, since λ is negative at this point, it remains a maximum when the constraint is replaced by $x^2 + y^2 + z^2 \leq 1$. (N.B. replacement of 'minimum' and $\geq$ in text by 'maximum' and $\leq$.)

8. The boundary planes of the admissible region of xyz - space are

$$x = 0,\ y = 0,\ z = 0,\ x + y + z = 3,\ 2x + 2y + z = 4,\ x + y = 0.$$

20 possible triads can be selected from these. The triad $z = 0$, $x + y + z = 3$, $2x + 2y + z = 4$ have no point in common. The remaining 19 all have common points and these are found to have coordinates (1/2, 1/2, 2), (0, 1, 2), (1, 0, 2), (0, 0, 3), (0, 0, 4), (0, 0, 2), (3/2, 3/2, 0), (0, 3, 0), (3, 0, 0), (1, 1, 0), (0, 2, 0), (2, 0, 0), (0, 0, 0), which are therefore candidates for the maximum point. However, not all the constraints are satisfied at the points (1, 0, 2), (0, 0, 4), (3/2, 3/2, 0), (0, 3, 0), (3, 0, 0), (2, 0, 0), thus leaving the points

(1/2, 1/2, 2), (0, 1, 2), (0, 0, 3), (0, 0, 2), (1, 1, 0), (0, 2, 0), (0, 0, 0).

At these points $4x - 2y - z$ takes the values -1, -4, -3, -2, 2, -4, 0 respectively. Clearly, the maximum value of 2 occurs at the point (1, 1, 0). The minimum value is -4 and occurs at both (0, 1, 2) and (0, 2, 0).

<u>**Exercises 2**</u>

1. Putting $\dot{x} = y$ we obtain equations in canonical form :

$$\dot{x} = y, \quad \dot{y} = -\frac{2}{t^2 - 1}x + \frac{2t}{t^2 - 1}y + \frac{1}{t^2 - 1}u.$$

Thus
$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{2}{t^2 - 1} & \dfrac{2t}{t^2 - 1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{t^2 - 1} \end{bmatrix}$$

Putting $u = 0$, we derive the state transition matrix by solving for x and y subject to the initial conditions $x = 1, y = 0$ and $x = 0, y = 1$ at $t = t_0$.

With $u = 0$, the second order equation clearly has a particular solution $x = t$. Thus, we change the dependent variable to v by substituting $x = tv$ and find that v satisfies

$$t(t^2 - 1)\ddot{v} - 2\dot{v} = 0.$$

It is now easy to solve for $\dot{v}$ and then v to yield

$$v = P\left(t + \frac{1}{t}\right) + Q,$$

so that
$$x = P(t^2 + 1) + Qt,$$

where P, Q are arbitrary constants.

Thus, the solution for initial conditions $x = 1, y = \dot{x} = 1$, is found to be

$$x = \frac{2tt_0 - t^2 - 1}{t_0^2 - 1}, \quad y = -\frac{2(t - t_0)}{t_0^2 - 1}$$

and that for the initial conditions $x = 0, y = \dot{x} = 1$, is

$$x = \frac{(t - t_0)(tt_0 - 1)}{t_0^2 - 1}, \quad y = \frac{2tt_0 - t_0^2 - 1}{t_0^2 - 1}.$$

We can now write down the state transition matrix, viz.:

$$\phi(t, t_0) = \frac{1}{t_0^2 - 1}\begin{bmatrix} 2tt_0 - t^2 - 1 & (t - t_0)(tt_0 - 1) \\ -2(t - t_0) & 2tt_0 - t_0^2 - 1 \end{bmatrix}$$

Then, since $x^0 = 0$, (2.2.20) yields the system's response to $u = (t^2 - 1)^2$ to be

$$\begin{bmatrix} x \\ y \end{bmatrix} = \int_2^t \frac{1}{\tau^2 - 1}\begin{bmatrix} 2t\tau - t^2 - 1 & (t - \tau)(t\tau - 1) \\ -2(t - \tau) & 2t\tau - \tau^2 - 1 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{\tau^2 - 1} \end{bmatrix}(\tau^2 - 1)^2 \, d\tau$$

$$= \int_2^t \begin{bmatrix} (t - \tau)(t\tau - 1) \\ 2t\tau - \tau^2 - 1 \end{bmatrix} d\tau$$

Hence

$$x = \int_2^t (t - \tau)(t\tau - 1)\,d\tau = \frac{1}{6}t^4 - \frac{5}{2}t^2 + \frac{14}{3}t - 2.$$

2. Since $\log z$ is not regular in any circle with its centre at $z = 0$, we proceed indirectly by studying $F(z) = \exp(\lambda + \mu z)$ and its associated matrix $\exp(\lambda I + \mu A)$.

The eigenvalues of A are 2 and 5 and we take $S(z) = s_0 + s_1 z$. Then we require

$$S(2) = s_0 + 2s_1 = \exp(\lambda + 2\mu), \quad S(5) = s_0 + 5s_1 = \exp(\lambda + 5\mu).$$

Solving for s_0, s_1, we find

$$s_0 = \frac{1}{3}\left[5\exp(\lambda + 2\mu) - 2\exp(\lambda + 5\mu)\right],$$

$$s_1 = \frac{1}{3}\left[\exp(\lambda + 5\mu) - \exp(\lambda + 2\mu)\right].$$

Thus $$\exp(\lambda I + \mu A) = S(A) = s_0 I + s_1 A.$$

We now choose λ, μ so that $s_0 = 0$, $s_1 = 1$, i.e. $\exp(\lambda I + \mu A) = A$. Thus

$$\exp(3\mu) = \frac{5}{2} \quad \text{or} \quad e^{\mu} = \left(\frac{5}{2}\right)^{1/3}$$

$$3 = e^{\lambda}\left(e^{5\mu} - e^{2\mu}\right) = e^{\lambda}\left[\left(\frac{5}{2}\right)^{5/3} - \left(\frac{5}{2}\right)^{2/3}\right] = \frac{3}{2}e^{\lambda}\left(\frac{5}{2}\right)^{2/3}.$$

Hence $$e^{\lambda} = 2\left(\frac{2}{5}\right)^{2/3}.$$

Thus $\lambda = \log\frac{2^{5/3}}{5^{2/3}} = \frac{1}{3}\log\frac{32}{25},\quad \mu = \frac{1}{3}\log\frac{5}{2}$, and

$$\log A = \lambda I + \mu A = \frac{1}{3}\left(I\log\frac{32}{25} + A\log\frac{5}{2}\right).$$

3. Putting $u_1 = u$, $u_2 = -e^{-t}u$, the system's equations can be written in the matrix form $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The eigenvalues of A are -1, -2 and the state transition matrix is

$$\phi(t,\tau) = e^{(t-\tau)A} = s_0 I + s_1 A,$$

where s_0, s_1 need to satisfy

$$s_0 - s_1 = e^{\tau - t}, \quad s_0 - 2s_1 = e^{2(\tau - t)}.$$

Thus $$\phi(t,\tau) = \begin{bmatrix} 3e^{\tau-t} - 2e^{2(\tau-t)} & -3e^{\tau-t} + 3e^{2(\tau-t)} \\ 2e^{\tau-t} - 2e^{2(\tau-t)} & -2e^{\tau-t} + 3e^{2(\tau-t)} \end{bmatrix}$$

The response to $u = \left[e^{-t}, -e^{-2t}\right]^T$ from $x = [1,1]^T$ at $t = 0$, is given by (2.2.20) to be

$$x = \begin{bmatrix} 3e^{-t} - 2e^{-2t} & -3e^{-t} - 3e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -2e^{-t} + 3e^{-2t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} +$$

$$-\int_0^t \begin{bmatrix} 3e^{\tau-t} - 2e^{2(\tau-t)} & -3e^{\tau-t} - 3e^{2(\tau-t)} \\ 2e^{\tau-t} - 2e^{2(\tau-t)} & -2e^{\tau-t} - 3e^{2(\tau-t)} \end{bmatrix} \begin{bmatrix} e^{-\tau} \\ -e^{-2\tau} \end{bmatrix} d\tau$$

Thus

$$\begin{aligned} x_1(t) &= e^{-2t} + \int_0^t \left[3e^{-t} - 2e^{\tau-2t} - 3e^{-\tau-t} - 3e^{-2t}\right] d\tau \\ &= (3t+1)e^{-t} - 3te^{-2t}, \end{aligned}$$

$$\begin{aligned} x_2(t) &= e^{-2t} + \int_0^t \left[2e^{-t} - 2e^{\tau-2t} + 2e^{-\tau-t} - 3e^{-2t}\right] d\tau \\ &= 2te^{-t} - (3t-1)e^{-2t}. \end{aligned}$$

4. The equations of motion for the vehicle are

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{u} = \cos\theta, \quad \dot{v} = \sin\theta.$$

Here, (x, y, u, v) are state variables and θ is the control variable. Introducing the associated Lagrange multipliers $(\lambda_x, \lambda_y, \lambda_u, \lambda_v)$, the Hamiltonian is

$$H = \lambda_x u + \lambda_y v + \lambda_u \cos\theta + \lambda_v \sin\theta$$

and for optimality, we require

$$\frac{\partial H}{\partial \theta} = -\lambda_u \sin\theta - \lambda_v \cos\theta = 0,$$

$$\text{i.e. } \tan\theta = \lambda_v / \lambda_u.$$

Hamilton's equations for the multipliers are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0, \quad \dot{\lambda}_u = -\frac{\partial H}{\partial u} = -\lambda_x, \quad \dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_y.$$

The multipliers also have to satisfy the conditions

$$\lambda_x = \frac{\partial C}{\partial x_1} = 0, \quad \lambda_y = \frac{\partial C}{\partial y_1} = 1, \quad \lambda_u = \frac{\partial C}{\partial u_1} = 1, \quad \lambda_v = \frac{\partial C}{\partial v_1} = 0.$$

at $t = 1$. It follows that

$$\lambda_x = 0, \quad \lambda_y = 1, \quad \lambda_u = 1, \quad \lambda_v = 1 - t.$$

Thus, for optimality,

$$\tan\theta = 1 - t, \quad \cos\theta = \frac{1}{\sqrt{1+(1-t)^2}}, \quad \sin\theta = \frac{1-t}{\sqrt{1+(1-t)^2}}.$$

We can now solve the state equations under the initial conditions $x = y = u = v = 0$ at $t = 0$, using the standard integrals

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}x, \quad \int \sinh^{-1}x\,dx = x\sinh^{-1}x - \sqrt{1+x^2},$$

$$\int \frac{x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2}, \quad \int \sqrt{1+x^2}\,dx = \frac{1}{2}\sinh^{-1}x + \frac{1}{2}x\sqrt{1+x^2},$$

to obtain the results stated.

5. The Hamiltonian for the problem is

$$H = x^2 + \frac{1}{3}u^2 + \lambda(-x+u).$$

For optimal control

$$\frac{\partial H}{\partial u} = \frac{2}{3}u + \lambda = 0.$$

The adjoint equation is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2x + \lambda$$

and the state equation is $\qquad \dot{x} = -x + u.$

The boundary conditions are $x = 1$ at $t = 0$ and $\lambda = 0$ at $t = 1$.

Eliminating u and λ between the last 3 equations, we find

$$\ddot{x} - 4x = 0$$

so that $\qquad x = Ae^{2t} + Be^{-2t}.$

A and B being constants. Thus

$$u = \dot{x} + x = 3Ae^{2t} - Be^{-2t},$$
$$\lambda = -\frac{2}{3}u = -2Ae^{2t} + \frac{2}{3}Be^{-2t}.$$

The boundary conditions require

$$A = \frac{1}{3e^4 + 1}, \quad B = \frac{3e^4}{3e^4 + 1},$$

so that the optimal control is

$$u = \frac{3e^{2t} - 3e^{4-2t}}{3e^4 + 1} = \frac{3e^{2t-2} - 3e^{2-2t}}{3e^2 + e^{-2}} = \frac{3\sinh 2(t-1)}{2\cosh 2 + \sinh 2}.$$

We now calculate that

$$x^2 + \frac{1}{3}u^2 = 4A^2e^{4t} + \frac{4}{3}B^2e^{-4t}$$

so that $C = A^2\left(e^4 - 1\right) + \frac{1}{3}B^2\left(1 - e^{-4}\right) = \frac{e^4 - 1}{3e^4 + 1},$

which is equivalent to the result stated.

6. In canonical form, the state equations are

$$\dot{x} = y, \quad \dot{y} = -x + u.$$

Introducing Lagrange multipliers λ_x, λ_y, the Hamiltonian is

$$H = u^2 + \lambda_x y + \lambda_y(-x + u)$$

For optimal control, we require

$$\frac{\partial H}{\partial u} = 2u + \lambda_y = 0.$$

The adjoint equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = \lambda_y, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x.$$

Solving for λ_x, λ_y, we obtain

$$\lambda_x = A\cos t + B\sin t, \quad \lambda_y = -A\sin t + B\cos t.$$

At $t = \pi/2$, the multipliers must satisfy the equations $\lambda_x = 2x_1$, $\lambda_y = 0$. Thus, $A = 0$ and $B = 2x_1$.

$$u = -x_1 \cos t.$$

We now have

$$\ddot{x} + x = -x_1 \cos t,$$

whose general solution is

$$x = C\cos t + D\sin t - \frac{1}{2}x_1 t \sin t.$$

The initial conditions $x = 0$, $\dot{x} = 1$ at $t = 0$, require that $C = 0$, $D = 1$, so that

$$x = \left(1 - \frac{1}{2}x_1 t\right)\sin t.$$

Putting $t = \pi/2$, we find that

$$x_1 = \frac{4}{\pi + 4},$$

and thus that

$$u = -\frac{4\cos t}{\pi + 4}.$$

It now follows that

$$C_{\min} = \frac{16}{(\pi + 4)^2}\left[1 + \int_0^{\pi/2} \cos^2 t \, dt\right] = \frac{4}{\pi + 4}.$$

7. Introducing Lagrange multipliers λ_x, λ_y, the Hamiltonian is

$$H = x^2 + u^2 + \lambda_x y + \lambda_y u.$$

For optimal control, we require

$$\frac{\partial H}{\partial u} = 2u + \lambda_y = 0.$$

The adjoint equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -2x, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x.$$

Introducing the operator $D = d/dt$, we now find

$$D^4\lambda_y = -D^3\lambda_x = 2D^2 x = 2Dy = 2u = -\lambda_y.$$

Thus $$(D^4 + 1)\lambda_y = 0.$$

The equation giving the characteristic roots, viz. $m^4 + 1 = 0$, can be factorised thus:

$$(m^2 + \sqrt{2}m + 1)(m^2 - \sqrt{2}m + 1) = 0,$$

so that these roots are $\frac{1}{\sqrt{2}}(-1 \pm i)$, $\frac{1}{\sqrt{2}}(1 \pm i)$, and the general solution for λ_y is

$$\lambda_y = e^{-t/\sqrt{2}}\left(A\cos\frac{t}{\sqrt{2}} + B\sin\frac{t}{\sqrt{2}}\right) + e^{t/\sqrt{2}}\left(C\cos\frac{t}{\sqrt{2}} + D\sin\frac{t}{\sqrt{2}}\right).$$

As $t \to \infty$, we require $\lambda_y \to 0$; hence $C = D = 0$. The values of A and B will be determined by the initial values of x and y at $t = 0$.

Since $$\lambda_y = e^{-t/\sqrt{2}}\left(A\cos\frac{t}{\sqrt{2}} + B\sin\frac{t}{\sqrt{2}}\right)$$

is the general solution of the equation

$$(D^2 + \sqrt{2}D + 1)\lambda_y = 0,$$

λ_y satisfies this equation. Operating with $(D - 1/\sqrt{2})$ on this equation, we deduce that

$$\left(D^3 + \frac{1}{\sqrt{2}}D^2 - \frac{1}{\sqrt{2}}\right)\lambda_y = 0$$

which leads to the equation

$$2y + \sqrt{2}x + \sqrt{2}u = 0$$

and this is equivalent to the stated control equation.

8. Introducing a factor 1/2 into the specification of the cost in Ex.5 (clearly of no significance), we can take $P = 1, Q = 0, R = 1/3, A = -1, B = 1$ in (2.7.22) to yield the Riccati equation

$$\frac{dK}{dt} = 3K^2 + 2K - 1 = (3K - 1)(K + 1).$$

Thus $$\frac{dt}{dK} = \frac{1}{4}\left[\frac{3}{3K-1} - \frac{1}{K+1}\right]$$

which integrates immediately and since $K = 0$ at $t = 1$, leads to the stated result.

(2.7.19) now shows that

$$\dot{x} = -(1+3K)x = \frac{2e^{-4(1-t)} - 6}{e^{-4(1-t)} - 3}x$$

Integrating we deduce that

$$\begin{aligned}\ln x &= \int \frac{2e^{-4(1-t)} - 6}{e^{-4(1-t)} - 3}dt \\ &= \frac{1}{2}\ln\left[e^{-4(1-t)} - 3\right] + \frac{1}{2}\ln\left[1 - 3e^{4(1-t)}\right] + \text{constant}\end{aligned}$$

Using the initial condition $x = 1$ at $t = 0$, this leads to the result

$$x = \frac{3e^{2(1-t)} + e^{-2(1-t)}}{3e^{2} + e^{-2}}.$$

Using (2.7.26), we now calculate the optimal control to be

$$u = -3Kx = -\frac{3\sinh 2(1-t)}{2\cosh 2 + \sinh 2},$$

as before.

9. With $\dot{x} = y$, we take

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},\quad Q = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\quad R = 1/4,\quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad K = \begin{bmatrix} k & m \\ m & n \end{bmatrix}.$$

(2.7.29) takes the form

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} k & m \\ m & n \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k & m \\ m & n \end{bmatrix} - \begin{bmatrix} k & m \\ m & n \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 4 \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} k & m \\ m & n \end{bmatrix} = 0,$$

leading to the equations

$$2 - 2m - 4m^2 = 0,\quad k - n - 4mn = 0,\quad 2m - 4n^2 = 0.$$

Since n is real, we deduce that $m = 1/2$, $n = \pm 1/2$, $k = \pm 3/2$. We thus calculate the matrix (2.7.30) to be

$$\begin{bmatrix} 0 & 1 \\ -3 & \pm 2 \end{bmatrix}$$

For this matrix to have negative eigenvalues, it is necessary to take the negative sign and, hence,

$$K = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

Then, using (2.7.19), we have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and, hence, $\ddot{x} + 2\dot{x} + 3x = 0$. The solution of this equation satisfying the initial conditions $x = 0,\ \dot{x} = 1$ is

$$x = \frac{1}{\sqrt{2}} e^{-t} \sin(\sqrt{2}t).$$

The optimal control now follows from (2.7.26) to be

$$u = -4\begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = -2e^{-t}\cos(\sqrt{2}t).$$

The minimal value of C is

$$\int_0^\infty \left[\frac{1}{2}e^{-2t}\sin^2(\sqrt{2}t) + \frac{1}{2}e^{-2t}\cos^2(\sqrt{2}t)\right]dt = \frac{1}{2}\int_0^\infty e^{-2t}\,dt = \frac{1}{4}.$$

10. The state equations are

$$\dot{x} = y, \quad \dot{y} = -x - \varepsilon x^3 + u.$$

Introducing Lagrange multipliers λ, μ, the Hamiltonian takes the form

$$H = x^2 + \frac{1}{8}u^2 + \lambda y + \mu(-x - \varepsilon x^3 + u)$$

and the adjoint equations are accordingly

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2x + \mu(1 + 3\varepsilon x^2), \quad \dot{\mu} = -\frac{\partial H}{\partial y} = -\lambda.$$

For optimal control, we require

$$\frac{\partial H}{\partial u} = \frac{1}{4}u - \mu = 0,$$

The end conditions are $x = 0$, $y = 1$ at $t = 0$ and λ, μ must tend to zero as $t \to \infty$.

Expanding all dependent variables as power series in ε, we write

$$x = x_0 + \varepsilon x_1 + \dots \quad \text{etc.}$$

Substituting in the above equations and equating the coefficients of like powers of ε on the two sides of each, we obtain

$$\dot{x}_0 = y_0, \quad \dot{y}_0 = -x_0 + u_0, \quad \dot{\lambda}_0 = -2x_0 + \mu_0, \quad \dot{\mu}_0 = -\lambda_0, \quad u_0 = -4\mu_0,$$

by consideration of terms in ε^0, and

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -x_1 + u_1 - x_0^3, \quad \dot{\lambda}_1 = -2x_1 + \mu_1 + 3\mu_0 x_0^2,$$

$$\dot{\mu}_1 = -\lambda_1, \quad u_1 = -4\mu_1,$$

by consideration of terms in ε^1. It follows that

$$(D^2 + 1)x_0 = -4\mu_0, \quad (D^2 + 1)\mu_0 = 2x_0,$$
......................................(i)

$$(D^2 + 1)x_1 = -4\mu_1 - x_0^3, \quad (D^2 + 1)\mu_1 = 2x_1 - 3\mu_0 x_0^2,$$
......................................(ii)

where $D = d/dt$.

Substitution in the end conditions leads to the equations

$$x_0 = 0, \quad Dx_0 = 1, \quad x_1 = 0, \quad Dx_1 = 0, \quad \text{at } t = 0,$$
$$\mu_0 = 0, \quad D\mu_0 = 0, \quad \mu_1 = 0, \quad D\mu_1 = 0, \quad \text{at } t = \infty.$$

We can now solve the equations (i) for x_0 and μ_0. Then, substituting for these variables in the equations (ii), we can solve for x_1 and μ_1.

Notes: The characteristic equation for the differential equations satisfied by x_0, x_1, μ_0, μ_1 is $m^4 + 2m^2 + 9 = 0$ and thus has roots $1 \pm \sqrt{2}i$, $-1 \pm \sqrt{2}i$. The identities

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta), \quad \sin^2\theta \cos\theta = \frac{1}{4}(\cos\theta - \cos 3\theta),$$

will be found useful.

Exercises 3

1. Defining the Hamiltonian by

$$H = u^2 - \lambda_x y - \lambda_y(-x + u)$$

the adjoint equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = \lambda_y, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x$$

and for optimal control

$$\frac{\partial H}{\partial u} = 2u + \lambda_y = 0.$$

The state variables need to satisfy the end conditions $x = y = 0$ at $t = 0$ and $x = 1, y = 0$ at $t = \pi/2$. There are no end conditions on λ_x, λ_y.

Since $\ddot{\lambda}_y + \lambda_y = 0$, we can assume

$$u = A\cos t + B\sin t$$

and then have to solve the state equation

$$\ddot{x} + x = A\cos t + B\sin t.$$

Its general solution is

$$x = C\cos t + D\sin t + \frac{1}{2}t(A\sin t - B\cos t).$$

and the end conditions require that

$$C = 0, \quad \frac{1}{4}\pi A + D = 1, \quad -\frac{1}{2}B + D = 0, \quad \frac{1}{2}A + \frac{1}{4}\pi B - C = 0.$$

Thus

$$x = \frac{(2\pi t - 4)\sin t - 4t\cos t}{\pi^2 - 4},$$

$$u = \frac{4\pi\cos t - 8\sin t}{\pi^2 - 4},$$

leading to $$C_{\min} = \frac{4\pi}{\pi^2 - 4}.$$

2. For both problems the Hamiltonian is

$$H = x^2 - \frac{1}{3}u^2 - \lambda(x - u)$$

and the adjoint and optimal control equations are

$$\dot{\lambda} = -2x - \lambda, \quad \frac{2}{3}u + \lambda = 0.$$

Solving these equations with the state equation, we find

$$x = Ae^{2t} + Be^{-2t}, \quad \lambda = -\frac{2}{3}Ae^{2t} + 2Be^{-2t}, \quad u = Ae^{2t} - 3Be^{-2t} \ldots\ldots(\text{i})$$

In the first problem, we have end conditions $x = 1, t = 0$ and $\lambda = 0, t = 1$, so that

$$x = \frac{e^{2(1-t)} + 3e^{-2(1-t)}}{e^{2} + 3e^{-2}}, \quad u = \frac{3e^{-2(1-t)} - 3e^{2(1-t)}}{e^{2} + 3e^{-2}}.$$

Since $v = u + 3/u$, $1 \le u \le e^{2}$, has a minimum of $2\sqrt{3}$ at $u = \sqrt{3}$, x has a minimum of

$$\frac{2\sqrt{3}e^{2}}{e^{4} + 3} \text{ at } t = 1 - \frac{1}{4}\ln 3.$$

Clearly $$x(1) = \frac{4e^{2}}{e^{4} + 3}.$$

Thus, x first decreases to its minimum value and, after $t = 1 - \frac{1}{4}\ln 3$, increases to its final value.

In the second problem, the end conditions are $x = 1$ at $t = 0$, $x = e^{-1}$ at $t = t_1$; also, the second condition (3.1.33), requires that $H = 0$ at $t = t_1$.

Referring to equations (i) above, we conclude that

$$A + B = 1, \quad Ae^{2t_1} + Be^{-2t_1} = e^{-1}, \quad H = \frac{16}{3}AB = 0.$$

If $B = 0$, $A = 1$ and, then, $e^{2t_1} = e^{-1}$. Since t_1 cannot be negative, this possibility must be rejected. Hence $A = 0$, $B = 1$ and $t_1 = \frac{1}{2}$.

Hence $x = e^{-2t}$, $u = -3e^{-2t}$ and

$$C_{opt} = \int_0^{1/2} 4e^{-4t}\,dt = 1 - e^{-2}.$$

3. Taking rectangular axes Oxy in the plane of flight, with the y-axis parallel to the wind, the state equations for the aeroplane are

$$\dot{x} = V\cos\theta, \quad \dot{y} = V\sin\theta - w.$$

where θ is the variable under the control of the pilot. At $t = 0$, if $x = x_0$, $y = y_0$, we shall suppose $\theta = \theta(t)$ such that at $t = T$, $x = x_0$, $y = y_0$ again; clearly, unless $V > w$, this assumption is invalid. The area enclosed by the circuit is given by

$$A = -\int_0^T y\dot{x}\,dt = -V\int_0^T y\cos\theta\,dt$$

Introducing multipliers λ, μ, we define the Hamiltonian

$$H = -Vy\cos\theta + \lambda V\cos\theta + \mu(V\sin\theta - w),$$

leading to the adjoint equations

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0, \quad \dot{\mu} = -\frac{\partial H}{\partial y} = V\cos\theta.$$

Optimal control requires that

$$\frac{\partial H}{\partial \theta} = V(y - \lambda)\sin\theta + \mu V\cos\theta = 0$$

i.e. $$\tan\theta = \frac{\mu}{\lambda - y}.$$

Since $\dot{\mu} = \dot{x}$, we have $\mu = x + \alpha$ (α constant). Taking $\lambda = -\beta$ (constant), the optimal control equation becomes

$$\tan\theta = -\frac{x + \alpha}{y + \beta}.$$

Dividing the state equations, we get

$$\frac{dy}{dx} = \tan\theta + \frac{w}{V}\sec\theta = -\frac{x+\alpha}{y+\beta} + \varepsilon\sqrt{1 + \left(\frac{x+\alpha}{y+\beta}\right)^2},$$

where $\varepsilon = w/V < 1$. Shifting the origin to the point $(-\alpha, -\beta)$ in the xy - plane, this equation reduces to

$$\frac{dy}{dx} = -\frac{x}{y} + \varepsilon\sqrt{1 + \frac{x^2}{y^2}},$$

which can be integrated by changing the dependent variable to v by the substitution $y = vx$ to yield

$$x\frac{dv}{dx} = -v - \frac{1}{v} + \varepsilon\sqrt{1 + \frac{1}{v^2}}.$$

The variables are now separable, thus:

$$\frac{dx}{x} = \frac{v\,dv}{-v^2 - 1 + \varepsilon\sqrt{v^2 + 1}}.$$

Putting $z^2 = v^2 + 1$, this reduces to

$$\frac{dx}{x} = -\frac{dz}{z - \varepsilon},$$

which integrates to yield

$$\ln x = -\ln(z - \varepsilon) + \ln C,$$

where C is some constant. Hence $x = C/(z - \varepsilon)$, which is equivalent to

$$\sqrt{x^2 + y^2} - \varepsilon x = C \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

After some manipulation, this is seen to be equivalent to the equation

$$\left[\left(1 - \varepsilon^2\right)x - \varepsilon C\right]^2 + \left(1 - \varepsilon^2\right)y^2 = C^2,$$

which is the equation of an ellipse with centre

$$x = \frac{\varepsilon C}{1 - \varepsilon^2}, \quad y = 0,$$

and semi-axes

$$\frac{C}{1 - \varepsilon^2}, \quad \frac{C}{\sqrt{1 - \varepsilon^2}}.$$

Its area is accordingly

$$A = \frac{\pi C^2}{\left(1 - \varepsilon^2\right)^{3/2}} \ldots\ldots\text{(ii)}$$

Since $\tan\theta = -x/y$, equation (i) requires that

$$\operatorname{cosec}\theta = C/x + \varepsilon$$

or

$$x = \frac{C\sin\theta}{1 - \varepsilon\sin\theta}.$$

Diifferentiating with respect to t, we find

$$V\cos\theta = \dot{x} = \frac{C\cos\theta}{(1 - \varepsilon\sin\theta)^2}\dot{\theta}.$$

i.e. $$C\dot{\theta} = V(1-\varepsilon\sin\theta)^2.$$

Integrating from θ = - π to θ = π (i.e. over one circuit of the ellipse), we deduce that

$$T = \frac{C}{V}\int_{-\pi}^{\pi}\frac{d\theta}{(1-\varepsilon\sin\theta)^2} = \frac{2\pi C}{V(1-\varepsilon^2)^{3/2}}.$$

(Note: The integral can be evaluated by conversion to a contour integral around the unit circle $|z| = 1$ in the complex plane.) C can now be evaluated in terms of T and, then, substitution in equation (ii) yields the result stated.

4. We take

$$H = u^2 + \lambda(y-x) + \mu(u-y)$$

yielding the adjoint equations

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = \lambda, \quad \dot{\mu} = -\frac{\partial H}{\partial y} = -\lambda - \mu.$$

For optimal control

$$\frac{\partial H}{\partial u} = 2u - \mu = 0.$$

At $t = 1$, μ has to satisfy the end condition μ = 0. Since $(D-1)^2\mu = 0 (D = d/dt)$,

$$\mu = A(1-t)e^t.$$

Thus, the state equations are

$$(D+1)x = y, \quad (D+1)y = A(1-t)e^t.$$

Solving, we find

$$x = (B + Ct)e^{-t} + \frac{1}{4}Ae^t(2-t).$$

The end conditions $x = \dot{x} = 0$ at $t = 0$, $x = 1$ at $t = 1$ require that

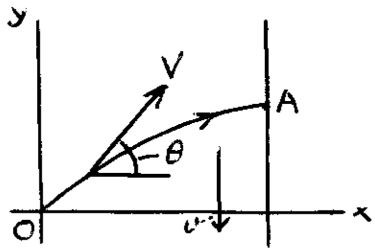
$$B + \frac{1}{2}A = 0, \quad C - B + \frac{1}{4}A = 0, \quad (B + C)e^{-1} + \frac{1}{4}Ae = 1.$$

Solving, we find $A = \dfrac{4e}{e^2 - 5}$ and the optimal control is accordingly

$$u = -\frac{2e^{1+t}}{e^2 - 5}(1-t)$$

Thus $\quad C_{opt} \quad = \quad \int_0^1 \frac{4e^{2t+2}(1-t)^2}{(e^2-5)^2}\,dt \quad = \quad \frac{e^2}{e^2-5}$

5.



If (x, y) are the coordinates of the ferry at time t, the state equations are

$$\dot{x} \quad = \quad V\cos\theta, \quad \dot{y} \quad = \quad V\sin\theta - v.$$

The Hamiltonian is

$$H \quad = \quad \lambda V\cos\theta - \mu(V\sin\theta - v)$$

and the adjoint equations are

$$\dot{\lambda} \quad = \quad -\frac{\partial H}{\partial x} \quad = \quad \mu\frac{dv}{dx}, \quad \dot{\mu} \quad = \quad -\frac{\partial H}{\partial y} \quad = \quad 0.$$

O, A are the given terminals of the ferry's path and the times at these points are $t = 0$, $t = t_1$. Since t_1 is to be minimised, the end condition (3.1.33) is operable and requires $H = -1$ at $t = t_1$. However, since H does not depend explicitly on t, we can assume $H = -1$ for all t.

For optimal control

$$\frac{\partial H}{\partial \theta} \quad = \quad -\lambda V\sin\theta - \mu V\cos\theta \quad = \quad 0.$$

Eliminating λ between this equation and $H = -1$, we find

$$\mu(V\operatorname{cosec}\theta - v) \quad = \quad -1,$$

where μ is constant. Thus

$$\sin\theta \quad = \quad \frac{V}{v-\alpha},$$

where $\alpha = 1/\mu$. It follows that

$$\cos\theta \quad = \quad \frac{\sqrt{(v-\alpha)^2 - V^2}}{v-\alpha}$$

Dividing the state equations, we obtain

$$\frac{dy}{dx} = \frac{V^2 - v(v-\alpha)}{V\sqrt{(v-\alpha)^2 - V^2}}$$

determining the optimal path. Integration from $x = 0$ to x now yields the stated result.

If v is constant, the optimal path is a straight line

$$y = \frac{V^2 - v(v-\alpha)}{V\sqrt{(v-\alpha)^2 - V^2}}$$

and if A lies on the x-axis, it is necessary that

$$V^2 = v(v-\alpha),$$

determining α. Thus, $\alpha = (v^2 - V^2)/v$ and, thus, $\cos\theta = \sqrt{V^2 - v^2}/V$. Since $\dot{x} = V\cos\theta = \sqrt{V^2 - v^2}$, the time to cross the river is $d/\sqrt{V^2 - v^2}$.

6. The state equations are

$$\dot{x} = u \qquad \dot{y} = v$$
$$\dot{u} = a\cos\theta \qquad \dot{v} = a\sin\theta - g.$$

We assume that at $t = 0, x = y = u = v = 0$ and that at $t = T$, $u = u_1$ is the prescribed orbital velocity, $v = 0$, i.e. the rocket is moving parallel to the earth's surface. At $t = T$, $y = y_1$ is to be maximised.

The Hamiltonian is

$$H = \lambda_x u + \lambda_y v + \lambda_u a\cos\theta + \lambda_v(a\sin\theta - g),$$

so that the adjoint equations are

$$\dot{\lambda}_x = 0, \quad \dot{\lambda}_y = 0, \quad \dot{\lambda}_u = -\lambda_x, \quad \dot{\lambda}_v = -\lambda_y.$$

For optimal control

$$\frac{\partial H}{\partial \theta} = -\lambda_u a\sin\theta + \lambda_v a\cos\theta = 0$$

$$\text{i.e.} \quad \tan\theta = \lambda_v/\lambda_u.$$

Since x_1, y_1 are not prescribed, λ_x and λ_y must satisfy end conditions (3.1.33) at $t = T$, viz.

$$\lambda_x = \frac{\partial G}{\partial x} = 0, \quad \lambda_y = \frac{\partial G}{\partial y} = 1.$$

Since λ_x, λ_y are constants, these equations are valid for all t. It follows that

$$\lambda_u = \text{constant}, \quad \lambda_v = -t + \text{constant}$$

so that

$$\tan\theta = \alpha - \beta t.$$

α, β have to be chosen so that $u = u_1$, $v = 0$ at $t = T$. Since we have

$$\tan\theta_0 = \alpha, \quad \tan\theta_1 = \alpha - \beta T,$$

α, β will be determined if θ_0, θ_1 are determined.

Integrating the state equation $\dot{u} = a\cos\theta$ from $t = 0$ to $t = T$, we find

$$u_1 = a\int_0^T \cos\theta dt = \frac{a}{\beta}\int_{\theta_1}^{\theta_0} \sec\theta d\theta = \frac{a}{\beta}\ln\frac{\sec\theta_0 + \tan\theta_0}{\sec\theta_1 + \tan\theta_1}$$

Since $\beta T = \tan\theta_0 - \tan\theta_1$, this leads to the second of the stated equations.

Integrating the state equation $\dot{v} = a\sin\theta - g$ from $t = 0$ to $t = T$, we find

$$0 = \int_0^T (a\sin\theta - g)dt = \frac{a}{\beta}\int_{\theta_1}^{\theta_0} \sin\theta\sec^2\theta d\theta - gT$$

$$= \frac{a}{\beta}(\sec\theta_0 - \sec\theta_1) - gT.$$

Since $\beta T = \tan\theta_0 - \tan\theta_1$, this is equivalent to

$$a(\sec\theta_0 - \sec\theta_1) = g(\tan\theta_0 - \tan\theta_1).$$

(Note: A term has been omitted in the first stated equation.)

7. The Hamiltonian for the problem is

$$H = u^2 + \lambda(-x + u) + \mu x^2,$$

where μ is a constant multiplier. The adjoint equation is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = \lambda - 2\mu x.$$

Eliminating x between this last equation and the state equation, we arrive at the equation

$$(D^2 - 1 - \mu)\lambda = 0,$$

where $D = d/dt$. Putting $\mu = \alpha^2 - 1$, since λ must vanish as $t \to \infty$, we have

$$\lambda = Ae^{-\alpha t}.$$

The optimal control equation is

$$\partial H/\partial u = 2u + \lambda = 0$$

and, hence, $u = -\frac{1}{2}Ae^{-\alpha t}$.

Substituting for λ in the adjoint equation, we find

$$x = \frac{A}{2(\alpha - 1)}e^{-\alpha t}$$

and, since $x = 1$ at $t = 0$, $A = 2(\alpha - 1)$. Thus, the integral constraint requires that

$$\int_0^{\infty} e^{-2\alpha t}\,dt = \frac{1}{4}$$

and, hence, $\alpha = 2$. It then follows that $A = 2$ and the optimal control is

$$u = -\frac{1}{2}Ae^{-\alpha t} = -e^{-2t}.$$

8. The Hamiltonian is

$$H = \lambda_x y + \lambda_y u e^{(1-u^2)/2}$$

so that the adjoint equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x.$$

Hence $\lambda_x = a, \quad \lambda_y = -at + b,$

where a, b are constants. For optimal control

$$\frac{\partial H}{\partial u} = \lambda_y(1 - u^2)e^{(1-u^2)/2} = 0$$

If λ_y were identically zero, H would also vanish and the end condition $H = -1$, which must be satisfied if t_1 is to be minimised, would fail. We conclude that either $u = +1$ or $u = -1$. Neither of these controls is itself compatible with the end conditions $y = 0$ at $t = 0$ and $t = t_1$. Hence, both forms of control must be utilised and there is a discontinuity at $t = \tau$ say.

Suppose $u = +1$ for $0 \le t < \tau$, $u = -1$ for $\tau < t \le t_1$. Then, integrating the state equations under the initial conditions $x = y = 0$ at $t = 0$ and assuming x, y are continuous at $x = \tau$, we calculate that

$$x = \frac{1}{2}t^2, \quad y = t, \quad 0 \le t \le \tau,$$

$$x = -\frac{1}{2}t^2 + 2\tau t - \tau^2, \quad y = -t + 2\tau, \quad \tau \le t \le t_1.$$

At $t = t_1$, we require $x = 1, y = 0$. Hence $\tau = 1$, $t_1 = 2$.

At the discontinuity, λ_x and λ_y are continuous and, hence, $\lambda_x = a$, $\lambda_y = -at - b$ for all values of t. Thus, $H_- = a\tau - (b - a\tau)$, $H_+ = a\tau - (b - a\tau)$ and H is continuous provided $b = a\tau$. The condition $H(t_1) = -1$ requires that $at_1 - b = -1$. Since $\tau = 1, t_1 = 2$, we find $a = b = -1$ and $\lambda_x = -1, \lambda_y = t - 1$.

If we assume instead that $u = -1$ for $0 \le t < \tau$, $u = +1$ for $\tau < t \le t_1$, it will be found that real values of t_1 and τ cannot be found to satisfy the necessary conditions.

Exercises 4

The state equations are

$$\dot{x} = y, \quad \dot{y} = u$$

and the Hamiltonian is

$$H = \lambda_x y + \lambda_y u.$$

The cost function is $C = t_1$ and, hence, $H = -1$ at $t = t_1$.

By Pontryagin's principle, $u = +1$ if $\lambda_y > 0$ and $u = -1$ if $\lambda_y < 0$.

The adjoint equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x,$$

so that $\lambda_x = -a$, $\lambda_y = at + b$, where a, b are constants. Clearly λ_y can change sign at most once during the motion and it follows that u can switch from $+1$ to -1 or from -1 to $+1$ at most once.

If $u = +1$, then from the state equations $dy/dx = 1/y$. Integration leads to

$$y^2 = 2x + \text{constant},$$

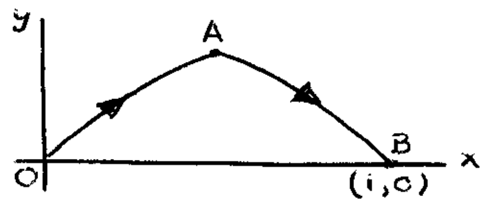
determining a family of parabolas in state space all with their axes along the x - axis.

If $u = -1$, we calculate similarly that

$$y^2 = -2x + \text{constant}$$

again determining a family of parabolas but with their axes in the opposite sense of the x-axis.

The space trajectory must connect the end points (0, 0) and (1, 0) in state space and can only comprise two arcs, one from each of the families above. These arcs are indicated in the figure as OA and AB. Along OA,

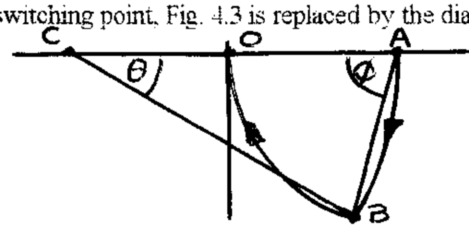


$u = +1$ and $y^2 = 2x$. Along AB, $u = -1$ and $y^2 = 2(1-x)$. Clearly, A is the point $x = 1/2, y = 1$.

Since $\dot{y} = +1$ along OA, the time lapse from O to A is 1. Since $\dot{y} = -1$ along AB, the time lapse from A to B is also 1. Thus, the minimum time for the transfer from O to B is 2.

λ_y must change sign as t increases through $t = 1$. Hence $a + b = 0$. Further, at $t = 2$, $\lambda_y = 2a + b$, so that $H = -(2a + b)$ at this instant. The end condition $H = -1$ requires that $a = 1, b = -1$ therefore. Thus $\lambda_y = t - 1$ and $\lambda_y < 0$ along OA and $\lambda_y > 0$ along AB, in agreement with the values of u along these arcs.

2. With the notation of Problem 28, the initial point A of the state space trajectory is now the point (1, 0) on the x - axis. Assuming one switching point, Fig. 4.3 is replaced by the diagram below:



C is the centre of the circular arc AB and A is the centre of the circular arc BO. Thus $AB = 1, BC = 2, AC = 2$ and ΔABC is isosceles. Thus,

$$\theta + \phi = \pi - \phi = \pi - \cos^{-1}(1/4)$$

Since $\theta + \phi$ is the time taken for the point (x, y) to describe the arcs AB and BO, the problem has been solved.

3. Taking $$H = \lambda_x y + \lambda_y (x + u)$$

since t_1 is to be minimised, $H = -1$ at $t = t_1$.

The adjoint equations are

$$\dot{\lambda}_x = -\lambda_y, \quad \dot{\lambda}_y = -\lambda_x,$$

and so $$\lambda_y = Ae^t + Be^{-t},$$

where A, B are constants.

By Pontryagin's principle, if $\lambda_y > 0$ then $u = -1$, and if $\lambda_y < 0$ then $u = +1$. But λ_y can only change sign by passing through the value zero and then

$$e^{2t} = -B/A.$$

This equation has but one real solution for t. It follows that u can change its value at most once.

From the state equations we find

$$y\frac{dy}{dx} = x - u.$$

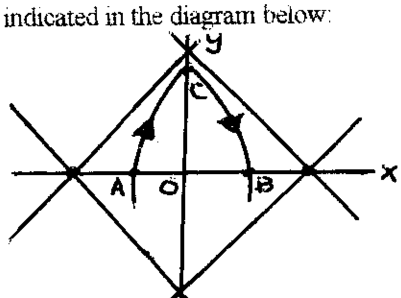
On an optimal trajectory $u = \pm 1$. If $u = +1$, by integration we find

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + \text{constant},$$

which is equivalent to the first of the stated equations. If $u = -1$, integration leads to the second equation.

The asymptotes of the first set of rectangular hyperbolae both pass through the point $(-1, 0)$ and make an angle of $\pi/4$ with the x-axis. The asymptotes of the second set pass through $(1, 0)$ and are parallel to the first set. Since, $\dot{x} = y$, x must be increasing with t in the region $y > 0$ and be decreasing with t in the region $y < 0$.

If the initial point of the optimal trajectory is A $(-1/2, 0)$ and the terminal point is B $(1/2, 0)$, the only possible optimal trajectory is ACB as indicated in the diagram below:



Along AC, $u = +1$ and $(x+1)^2 - y^2 = \frac{1}{4}$. Along CB, $u = -1$ and $(x-1)^2 - y^2 = \frac{1}{4}$. Thus, C is the point $(0, \sqrt{3}/2)$.

Along AC, we have

$$\dot{x} = y, \quad \dot{y} = x + 1.$$

Solving for x under the initial conditions $x = -1/2$, $\dot{x} = 0$ at $t = 0$, we find

$$x = \frac{1}{2}\cosh t - 1.$$

At C $x = 0$ and, hence, $t = \cosh^{-1} 2$. Clearly, the time from C to B is the same and so $t_1 = 2\cosh^{-1} 2$ is the minimal time of transfer.

Putting $\tau = \cosh^{-1} 2$, it may be verified that the condition $\lambda_y = 0$ at $t = \tau$ and $H = -1$ at $t = 2\tau$ lead to the result

$$\lambda_y = \frac{2}{\sqrt{3}}\sinh(t - \tau)$$

showing that $\lambda_y < 0$ on AC and $\lambda_y > 0$ on CB in conformity with Pontryagin's principle.

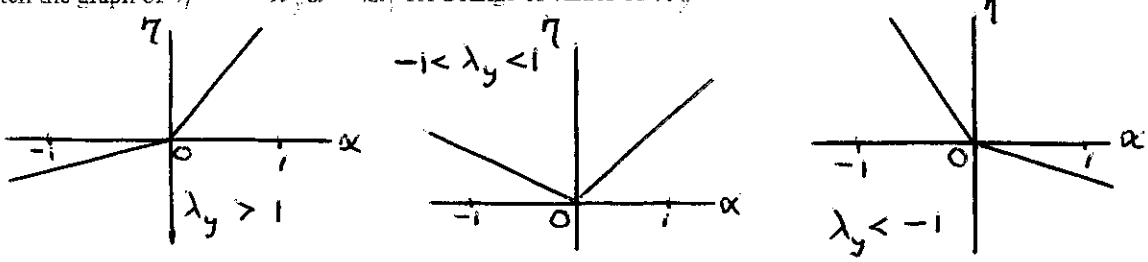
4. The state equations are $\qquad \dot{x} = y, \quad \dot{y} = \alpha.$

The Hamiltonian is $\qquad H = \lambda_x y + \lambda_y \alpha + |\alpha|.$

The adjoint equations are $\qquad \dot{\lambda}_x = 0, \quad \dot{\lambda}_y = -\lambda_x,$

Hence $\lambda_y = at + b$, where a, b are constants.

For optimal control, we need to minimise H with respect to α, $-1 \le \alpha \le 1$. In the diagrams below, we sketch the graph of $\eta = \lambda_y \alpha + |\alpha|$ for a range of values of λ_y:



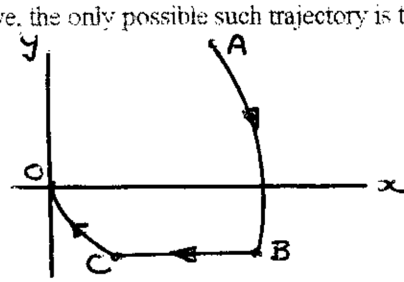
Clearly, if $\lambda_y > 1$ we take $\alpha = -1$, if $-1 < \lambda_y < 1$ we take $\alpha = 0$, if $\lambda_y < -1$ we take $\alpha = +1$, to minimise H. Hence, since λ_y either increases or decreases uniformly, α ranges through the values $-1, 0, +1$ in that order or in the reverse order.

Dividing the state equations, we find $y\,dy/dx = \alpha$ which integrates to give

$$y^2 = 2\alpha x + \text{constant}.$$

Thus, if $\alpha = -1$, the state space trajectory is a parabola with axis $x = 0$ and vertex directed towards $x = +\infty$. If $\alpha = 0$, the state space trajectory is a straight line parallel to the x-axis. If $\alpha = +1$, the state space trajectory is a parabola with axis $x = 0$ and vertex directed towards $x = -\infty$. If $y > 0$, x increases, whereas if $y < 0$, x decreases; this determines the sense of description of these trajectories.

The initial point of the optimal space trajectory is $A\,(1, 1)$ and the final point is the origin. In conformity with what has been said above, the only possible such trajectory is that indicated below:



Along AB $\alpha = -1$, along BC $\alpha = 0$, along CO $\alpha = +1$. Thus B and C are switching points.

From the state equations, along AB we have

$$x = 1 + t - \frac{1}{2}t^2, \quad y = 1 - t.$$

Thus, if $t = \tau$ at B,

$$x_B = 1 + \tau - \frac{1}{2}\tau^2, \quad y_B = 1 - \tau. \qquad \text{(i)}$$

Along CO we have

$$x = \frac{1}{2}t^2 - 4t + 8, \quad y = t - 4.$$

Thus, if $t = \theta$ at C,

$$x_C = \frac{1}{2}\theta^2 - 4\theta + 8, \quad y_C = \theta - 4. \qquad \text{(ii)}$$

But x decreases uniformly along BC at rate $y_B = y_C$ and hence

$$\frac{x_C - x_B}{\theta - \tau} = y_B = y_C. \qquad \text{(iii)}$$

Equations (i) - (iii) now determine

$$x_B = \frac{3}{4}\sqrt{3}, \quad x_C = \frac{3}{4}\left(2 - \sqrt{3}\right), \quad y_B = y_C = \frac{1}{2}\left(\sqrt{3} - 3\right),$$
$$\tau = \frac{1}{2}\left(5 - \sqrt{3}\right), \quad \theta = \frac{1}{2}\left(5 + \sqrt{3}\right).$$

Hence $\quad C_{\min} = \int_0^\tau dt + \int_\theta^4 dt = \tau + 4 - \theta = 4 - \sqrt{3}.$

5. The state equations are

$$\dot{x} = y, \quad \dot{y} = u - y$$

and the Hamiltonian is

$$H = \lambda_x y + \lambda_y (u - y)$$

leading to adjoint equations

$$\dot{\lambda}_x = 0, \quad \dot{\lambda}_y = -\lambda_x + \lambda_y.$$

Solving for the multipliers, we find

$$\lambda_x = a, \quad \lambda_y = a + be^t,$$

where a, b are constants.

Pontryagin's principle shows that $u = +1$ if $\lambda_y > 0$ and $u = -1$ if $\lambda_y < 0$. Since λ_y increases steadily or decreases steadily as t increases, we conclude that the optimal motion comprises at most two phases during which the control u takes its extreme values.

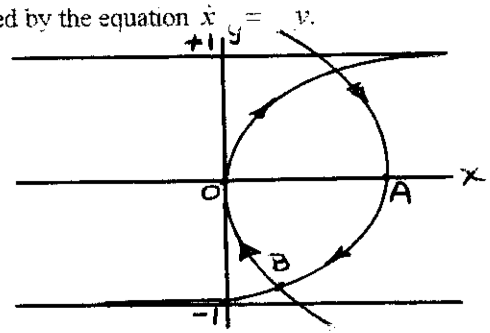
Dividing the state equations, we find that

$$\frac{dx}{dy} = \frac{u}{u - y} - 1,$$

which integrates to yield

$$x = -u \ln|u - y| - y + \text{constant}.$$

In state space, the optimal trajectory has initial point (1, 0) and terminal point the origin. It comprises at most two arcs, each determined by an equation of the last type with $u = -1$ or $u = +1$. Thus the trajectory must be ABO as sketched in the diagram below, with $u = +1$ along AB and $u = -1$ along BO. The sense of description of the arcs is determined by the equation $\dot{x} = y$.



Supposing $t = 0$ at A, $t = \tau$ at B and $t = t_1$ at O, we have to solve

$$\ddot{x} + \dot{x} = -1 \quad \text{with } x = 1, \dot{x} = 0 \text{ at } t = 0,$$

$$\ddot{x} + \dot{x} = -1 \quad \text{with } x = 0, \dot{x} = 0 \text{ at } t = t_1.$$

We calculate that

$$x = 2 - t - e^{-t}, \quad 0 \le t \le \tau,$$

$$x = t - t_1 - 1 - e^{t_1 - t}, \quad \tau \le t \le t_1.$$

Since $x, \dot{x}$ are continuous at $t = \tau$, it is necessary that

$$2 - \tau - e^{-\tau} = \tau - t_1 - 1 - e^{t_1 - \tau}, \quad -1 - e^{-\tau} = 1 - e^{t_1 - \tau}.$$

Thus $\quad e^{-\tau} + e^{t_1 - \tau} = 3 - 2\tau + t_1 = 2,$

showing that $t_1 = 2\tau - 1$. Eliminating τ, we find that

$$e^{(t_1 - 1)/2} + e^{-(t_1 + 1)/2} = 2.$$

Putting $e^{t_1/2} = \mu$, this reduces to the quadratic equation

$$\mu^2 - 2\sqrt{e}\,\mu + 1 = 0.$$

Solving for μ, we deduce that

$$t_1 = 2\ln\mu = 2\ln\left(\sqrt{e} + \sqrt{e-1}\right).$$

(Note: $\ln\left(\sqrt{e} - \sqrt{e-1}\right) < 0$.) Thus

$$\tau = \frac{1}{2} + \ln\left(\sqrt{e} + \sqrt{e-1}\right)$$

is the switching time. Approximately, $t_1 \approx 2.17$, $\tau \approx 1.58$.

Using the conditions $\lambda_y = 0$ at $t = \tau$, $H = -1$ at $t = t_1$, we calculate that

$$a = \sqrt{\frac{e}{e-1}}, \quad b = -\frac{1}{e-1+\sqrt{e^2-e}},$$

showing that λ_y decreases steadily as t increases, being positive for $t < \tau$ and negative thereafter.

6. The state equations are

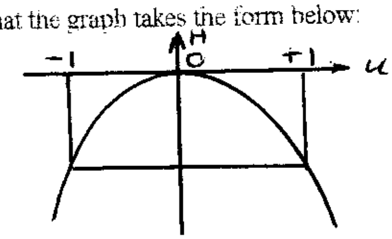
$$\dot{x} = u, \quad \dot{y} = u^2,$$

and the Hamiltonian is $\quad H = \lambda_x u + \lambda_y u^2,$

with adjoint equations $\quad \dot{\lambda}_x = 0, \quad \dot{\lambda}_y = 0.$

Thus, λ_x and λ_y are constant.

The graph of H against u is a parabola through the origin and, in all cases except one, H will be minimised for values of u in the range $-1 \le u \le 1$ by only one value of u. The exception is when $\lambda_x = 0$ and λ_y is negative, so that the graph takes the form below:



In this case, Pontryagin's principle requires that $u = -1$ or $u = +1$.

If u assumes one value only during the optimal motion, suppose $t = t_0$ initially and $t = t_1$ finally. Then, from the state equations we find that

$$x_1 - x_0 = u(t_1 - t_0), \quad y_1 - y_0 = u^2(t_1 - t_0).$$

Hence

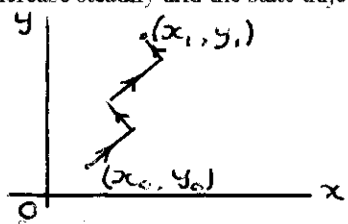
$$u = \frac{y_1 - y_0}{x_1 - x_0}, \quad t_1 - t_0 = \frac{(x_1 - x_0)^2}{y_1 - y_0}.$$

This motion is only possible if $y_1 - y_0 \le |x_1 - x_0|$.

If u alternates between the values - 1 and + 1, the optimal state trajectory is constructed from straight lines from the families

$$y = x + \text{constant}, \quad y = -x + \text{constant}.$$

Since $\dot{y} = +1$, y must increase steadily and the state trajectory takes the form shown below:



Since $\dot{y} = +1$, the time of transit is $y_1 - y_0$.

Clearly, $|x_1 - x_0| < y_1 - y_0$, so that the alternative constant thrust trajectory is not available.

<u>Exercises 5</u>

1. The Hamiltonian is

$$H = x^{1/2}\sec\theta + \lambda\tan\theta$$

and Hamilton's equations are accordingly

$$\dot{x} = \tan\theta, \quad \dot{\lambda} = -\frac{1}{2}x^{-1/2}\sec\theta, \quad x^{1/2}\sec\theta\tan\theta + \lambda\sec^2\theta = 0.$$

Thus, $\lambda = -x^{1/2}\sin\theta$ and substituting in the second of these equations, we find

$$-\frac{1}{2}x^{-1/2}\sec\theta = -\frac{1}{2}x^{-1/2}\dot{x}\sin\theta - x^{1/2}\cos\theta\,\dot{\theta}.$$

Putting $\dot{x} = \tan\theta$, this reduces to $x\dot{\theta} = \frac{1}{2}$. Hence

$$\frac{1}{x}\frac{dx}{d\theta} = \frac{\dot{x}}{x\dot{\theta}} = 2\tan\theta,$$

which integrates to give $x = \alpha\sec^2\theta$, where α is constant.

It now follows that

$$\dot{\theta} = \frac{1}{2\alpha}\cos^2\theta.$$

which integrates to yield

$$\tan\theta = \frac{1}{2\alpha}(t + \beta),$$

β being a second constant of integration. Hence

$$x = \alpha\sec^2\theta = \alpha + \frac{1}{4\alpha}(t + \beta)^2 \qquad \text{............(i)}$$

The end conditions $x = 2$, $t = 0$ and $x = 5$, $t = 2$ now determine α and β. There are two possibilities, viz. $\alpha = 1$, $\beta = 2$ and $\alpha = 1/13$, $\beta = -10/13$, leading to solutions

$$x = \frac{1}{4}t^2 + t + 2, \quad x = \frac{13}{4}t^2 - 5t + 2,$$

respectively.

Differentiating the general solution (i) of the state equation with respect to α and β, we obtain

$$\frac{\partial x}{\partial \alpha} = 1 - \frac{1}{4\alpha^2}(t + \beta)^2, \quad \frac{\partial x}{\partial \beta} = \frac{1}{2\alpha}(t + \beta).$$

If $\alpha = 1$, $\beta = 2$, then

$$\frac{\partial x}{\partial \alpha} = -\frac{1}{4}t^2 - t, \quad \frac{\partial x}{\partial \beta} = \frac{1}{2}t + 1.$$

Thus, the general solution of the accessory equations for the first variation η of x can be taken to be

$$\eta = A\left(t^2 + 4t\right) + B(t + 2).$$

If $\alpha = 1/13$, $\beta = -10/13$, then

$$\frac{\partial x}{\partial \alpha} = -\frac{169}{4}t^2 + 65t - 24, \quad \frac{\partial x}{\partial \beta} = \frac{13}{2}t - 5,$$

and the general solution for η is

$$\eta = A\left(169t^2 - 260t + 96\right) + B(13t - 10).$$

In the first case, if $\eta = 0$ at $t = 0$ and $t = \tau$, we must have

$$2B = 0, \quad A\left(\tau^2 + 4\tau\right) + B(\tau + 2) = 0$$

and if A, B are not both to vanish, $\tau = -4$. Thus, there is no point conjugate to the initial point such that $0 < \tau < 5$ on this state space trajectory and Jacobi's condition is satisfied.

In the second case, if $\eta = 0$ at $t = 0$ and $t = \tau$, we have

$$96A - 10B = 0, \quad A\left(169\tau^2 - 260\tau + 96\right) + B(13\tau - 10) = 0.$$

The condition that A and B do not both vanish is then

$$96(13\tau - 10) + 10\left(169\tau^2 - 260\tau + 96\right) = 0,$$

i.e. $\tau = 4/5$ and Jacobi's condition is not satisfied.

If $\alpha = 1$, $\beta = 2$, we find

$$x = \frac{1}{4}t^2 + t + 2, \quad \tan\theta = \frac{1}{2}t + 1$$

and so

$$C = \int_0^2 \left(\frac{1}{4}t^2 + t + 2\right)dt = \frac{20}{3}.$$

2. The Hamiltonian is

$$H = \frac{1}{x}\sec\theta + \lambda\tan\theta$$

and Hamilton's equations are

$$\dot x = \tan\theta, \quad \dot\lambda = \frac{1}{x^2}\sec\theta, \quad \frac{1}{x}\sec\theta\tan\theta - \lambda\sec^2\theta = 0$$

Thus, $\lambda = \frac{1}{x}\sin\theta$ and, substituting in the second equation, we deduce that

$$\frac{1}{x^2}\sec\theta = \dot\lambda = \frac{1}{x^2}\dot x\sin\theta - \frac{1}{x}\cos\theta\,\dot\theta.$$

Putting $\dot x = \tan\theta$, we find that $x\dot\theta = -1$. It then follows that

$$\frac{1}{x}\frac{dx}{d\theta} = \frac{\dot x}{x\dot\theta} = -\dot x = -\tan\theta,$$

which integrates to give

$$x = \alpha\cos\theta,$$

where α is constant. Since $x\dot\theta = -1$, we have

$$\cos\theta\,\dot\theta = -\frac{1}{\alpha}.$$

which integrates to yield

$$\sin\theta \;=\; -\frac{1}{\alpha}(t+\beta),$$

β being a second integration constant. It now follows that

$$x \;=\; \alpha\cos\theta \;=\; \sqrt{\alpha^2-(t+\beta)^2}\,.$$

The end conditions $x = 1$, $t = 0$ and $x = \sqrt{2}$, $t = 1$, now determine $\alpha = \sqrt{2}$, $\beta = -1$ and thus

$$x \;=\; \sqrt{1+2t-t^2}\,.$$

Differentiating the general solution for x with respect to α and β, we get

$$\frac{\partial x}{\partial\alpha} \;=\; \frac{\alpha}{\sqrt{\alpha^2-(t+\beta)^2}}, \quad \frac{\partial x}{\partial\beta} \;=\; -\frac{t+\beta}{\sqrt{\alpha^2-(t+\beta)^2}}.$$

Along the optimal trajectory, $\alpha = \sqrt{2}$ and $\beta = -1$, so that

$$\frac{\partial x}{\partial\alpha} \;=\; \frac{\sqrt{2}}{\sqrt{1+2t-t^2}}, \quad \frac{\partial x}{\partial\beta} \;=\; -\frac{t-1}{\sqrt{1+2t-t^2}},$$

and the general solution of the accessory equations for η is

$$\eta \;=\; \frac{A+B(t-1)}{\sqrt{1+2t-t^2}}.$$

Suppose $\eta = 0$ at $t = 0$ and $t = \tau$. Then

$$A+B \;=\; 0, \quad \frac{A+B(\tau-1)}{\sqrt{1+2\tau-\tau^2}} \;=\; 0,$$

implying that $\tau = 0$. We conclude that there is no point on the optimal trajectory conjugate to the point $t = 0$ and, hence, that Jacobi's condition is satisfied.

On the optimal trajectory, $\sec\theta = \alpha/x$, so that the integrand of C is $\alpha/x^2 = \sqrt{2}/(1+2t-t^2)$ and

$$C_{min} \;=\; \int_0^1 \frac{\sqrt{2}\,dt}{2-(t-1)^2} \;=\; \sqrt{2}\left[\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right]_0^1$$

$$= \ln(\sqrt{2}+1).$$

3. The Hamiltonian is

$$H = x + \frac{1}{4}u^2 + \lambda u$$

and Hamilton's equations are

$$\dot{x} = u, \quad \dot{\lambda} = -1, \quad \frac{1}{2}u + \lambda = 0.$$

Integration leads to the results

$$x = t^2 + \alpha t + \beta, \quad \lambda = -t - \frac{1}{2}\alpha, \quad u = 2t + \alpha.$$

where α, β are constants of integration.

The end conditions $x = 0, t = 0$ and $x = 2, t = 1$ require that $\alpha = 1, \beta = 0$, so that

$$x = t^2 + t.$$

Referring to the general solution of the state equation, we note that

$$\frac{\partial x}{\partial \alpha} = t, \quad \frac{\partial x}{\partial \beta} = 1,$$

so that the general solution of the accessory equations for the first variation η of x is

$$\eta = At + B.$$

Clearly, η can only vanish at one point on an optimal trajectory so that Jacobi's condition is satisfied.

4. The Hamiltonian is

$$H = \sqrt{x^2 + u^2} + \lambda u$$

and Hamilton's equations are

$$\dot{x} = u, \quad \dot{\lambda} = -\frac{x}{\sqrt{x^2 + u^2}}, \quad \frac{u}{\sqrt{x^2 + u^2}} + \lambda = 0.$$

Clearly, $\dot{\lambda}^2 + \lambda^2 = 1$ or

$$\frac{1}{\sqrt{1 - \lambda^2}}\frac{d\lambda}{dt} = 1.$$

Integration yields $\sin^{-1}\lambda \;=\; t - \alpha$ or

$$\lambda = \sin(t - \alpha),$$

where α is constant. It now follows that

$$\frac{\dot{x}}{x} \;=\; \frac{u}{x} \;=\; \frac{\dot{\lambda}}{\lambda} \;=\; \tan(t - \alpha),$$

which integrates to yield

$$x = \beta \sec(t - \alpha), \qquad \text{..................................(i)}$$

where β is another integration constant.

The end conditions $x = 1$, $t = 0$ and $x = \sqrt{2}$, $t = \pi/4$ require that $\alpha = 0$, $\beta = 1$, so that $x = \sec t$ along the optimal state trajectory. Then, $u \;=\; \dot{x} \;=\; \sec t \tan t$.

Differentiating (i), we find

$$\frac{\partial x}{\partial \alpha} \;=\; \beta \sec(t - \alpha)\tan(t - \alpha), \quad \frac{\partial x}{\partial \beta} \;=\; \sec(t - \alpha),$$

so that the general solution of the accessory equations for η along the optimal state trajectory $\alpha = 0$, $\beta = 1$ is

$$\eta \;=\; A \sec t \tan t + B \sec t.$$

If $\eta = 0$ at $t = 0$ and $t = \tau$, then

$$B = 0, \quad A \sec\tau \tan\tau + B \sec\tau = 0.$$

If A, B do not both vanish, then $\sec\tau\tan\tau = 0$; this last equation has no root such that $0 < \tau < \pi/4$, so that Jacobi's condition is satisfied.

Since $x^2 + u^2 \;=\; \sec^4 t$ on the optimal trajectory,

$$C_{\min} \;=\; \int_0^{\pi/4} \sec^2 t \, dt \;=\; [\tan t]_0^{\pi/4} \;=\; 1.$$

8. The Hamiltonian is

$$H \;=\; u^2 + \lambda_x y + \lambda_y u,$$

yielding Hamilton's equations

$$\dot{x} \;=\; y, \quad \dot{y} \;=\; u, \quad \dot{\lambda}_x \;=\; 0, \quad \dot{\lambda}_y \;=\; -\lambda_x, \quad 2u + \lambda_y \;=\; 0$$

Clearly, $u = at + b$, where a, b are constants, and then

$$x = \frac{1}{6}at^3 + \frac{1}{2}bt^2 + ct + d, \quad y = \frac{1}{2}at^2 + bt + c.$$

where c, d are further constants of integration. These 4 constants are determined by the end conditions.

Since

$$\frac{\partial x}{\partial a} = \frac{1}{6}t^3, \quad \frac{\partial y}{\partial a} = \frac{1}{2}t^2,$$

a solution of the accessory equations for η_x, η_y is

$$\eta_x = \frac{1}{6}t^3, \quad \eta_y = \frac{1}{2}t^2.$$

Three other solutions are obtained by differentiations with respect to b, c and d, viz.

$$\begin{aligned} \eta_x &= \frac{1}{2}t^2, & \eta_y &= t, \\ \eta_x &= t, & \eta_y &= 1, \\ \eta_x &= 1, & \eta_y &= 0, \end{aligned}$$

and the general solution is accordingly

$$\eta_x = \frac{1}{6}At^3 + \frac{1}{2}Bt^2 + Ct + D, \quad \eta_y = \frac{1}{2}At^2 + Bt + C.$$

To calculate points conjugate to the initial point $t = t_0$ on the optimal trajectory, we require η_x **and** η_y to vanish at $t = t_0$ **and** $t = \tau$. This leads to 4 linear equations for A, B, C, D and the condition these should not all vanish is

$$\begin{vmatrix} \frac{1}{6}t_0^3 & \frac{1}{2}t_0^2 & t_0 & 1 \\ \frac{1}{6}\tau^3 & \frac{1}{2}\tau^2 & \tau & 1 \\ \frac{1}{2}t_0^2 & t_0 & 1 & 0 \\ \frac{1}{2}\tau^2 & \tau & 1 & 0 \end{vmatrix} = 0.$$

Expanding the determinant, we arrive at the condition

$$-\frac{1}{12}(t_0 - \tau)^4 = 0.$$

i.e. $t_c = \tau$. Hence there is no point on the optimal trajectory which is conjugate to the initial point and Jacobi's condition is satisfied.

b. With $$H = x^{1/2}\sec\theta + \lambda\tan\theta$$

all matrices are 1×1 and in particular

$$f_x = 0, \quad f_u = \sec^2\theta, \quad H_{xx} = -\frac{1}{4x^{3/2}}\sec\theta, \quad H_{xu} = \frac{1}{2x^{1/2}}\sec\theta\ \tan\theta,$$

$$H_{uu} = x^{1/2}\left(\sec\theta\tan^2\theta + \sec^3\theta\right) + 2\lambda\sec^2\theta\tan\theta.$$

On the extremal to be considered, we have

$$x = \sec^2\theta, \quad \lambda = -x^{1/2}\sin\theta = -\tan\theta$$

and hence

$$H_{xx} = -\frac{1}{4}\cos^2\theta, \quad H_{xu} = \frac{1}{2}\tan\theta, \quad H_{uu} = \sec^2\theta.$$

Thus, the accessory Riccati equation (5.4.11) can be written down as

$$\frac{dK}{dt} = \left(K\sec^2\theta + \frac{1}{2}\tan\theta\right)^2\cos^2\theta + \frac{1}{4}\cos^2\theta.$$

But $\dot{\theta} = 1/2x = \frac{1}{2}\cos^2\theta$, so that

$$\frac{dK}{d\theta} = 2\left(K\sec^2\theta + \frac{1}{2}\tan\theta\right)^2 + \frac{1}{2},$$

which is equivalent to the equation stated.

It is now straightforward to verify that $K = -\cot\theta/2$ is a special solution of the Riccati equation. Then, putting $K = -\frac{1}{2}\cot\theta + \frac{1}{v}$, we find that v satisfies the linear equation

$$\frac{dv}{d\theta} - 2v\sec^2\theta\cot\theta = -2\sec^4\theta.$$

The integrating factor for this equation is $\cot^2\theta$, bringing it to the form

$$\frac{d}{d\theta}\left(y\cot^2\theta\right) = -2\sec^2\theta\,\mathrm{cosec}^2\theta.$$

Integrating, we find

$$y = 2\left(\tan\theta + A\tan^2\theta - \tan^3\theta\right),$$

where A is a constant of integration.

On the extremal under consideration, $\tan\theta = (t+2)/2$ and so

$$K = -\frac{1}{t+2} + \frac{4}{4(t+2) + 2A(t+2)^2 - (t+2)^3}.$$

By proper choice of A, we can now find a solution for K that is finite from $t = 0$ to $t = 2$.

7. Since $H = x + \dfrac{1}{4}u^2 + \lambda u$, we find

$$f_x = 0, \quad f_u = 1, \quad H_{xx} = 0, \quad H_{xu} = 0, \qquad H_{uu} = \frac{1}{2}$$

and the accessory Riccati equation is

$$\dot{K} = 2K^2.$$

This integrates immediately to give

$$K = \frac{1}{A - 2t}.$$

Clearly, by proper choice of the constant A, K can be rendered finite on any extremal.

8. Since $H = \left(x^2 + u^2\right)^{1/2} + \lambda u$, we have

$$f_x = 0, \quad f_u = 1, \quad H_{xx} = \frac{u^2}{\left(x^2 + u^2\right)^{3/2}}, \quad H_{xu} = -\frac{xu}{\left(x^2 + u^2\right)^{3/2}},$$

$$H_{uu} = \frac{x^2}{\left(x^2 + u^2\right)^{3/2}},$$

and the accessory Riccati equation takes the form

$$\frac{dK}{dt} - \left[K - \frac{xu}{\left(x^2 + u^2\right)^{3/2}}\right]^2 \frac{\left(x^2 + u^2\right)^{3/2}}{x^2} + \frac{u^2}{\left(x^2 + u^2\right)^{3/2}} = 0.$$

On the extremal to be considered, $x = \sec t$ and $u = \sec t \tan t$. Substituting in the last equation, we arrive at the result stated.

Then, putting $K = 1/J$, we are led to the equation

$$\frac{dJ}{dt} - 2J \tan t = -\sec^4 t.$$

This is a linear equation with integrating factor $\cos^2 t$. Hence

$$\frac{d}{dt}\left(J \cos^2 t\right) = -\sec^2 t.$$

which integrates to yield

$$J \cos^2 t = -\tan t + A.$$

Thus

$$K = \frac{\cos^2 t}{A - \tan t}.$$

If $0 \leq t \leq \pi/4$, by choosing $A > 1$ we arrive at a solution for K which is finite over the whole extremal arc.

If, however, $0 \leq t \leq \pi$, then $\tan t$ will take values from $-\infty$ to $+\infty$ and any choice of A will permit the denominator to vanish for some value of t and K to become infinite.

9. The state equations and Hamiltonian are

$$\dot{x} = y, \quad \dot{y} = u, \quad H = u^2 + \lambda_x y + \lambda_y u.$$

Thus, $M = 2$ and $N = 1$ and

$$f_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H_{xx} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_{xu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad H_{uu} = 2.$$

The result stated is in error, having assumed f_x to be the null 2×2 matrix. Apologies!

To solve the correct Riccati equation, we need to generalise a method which has been successful in the case where $M = 1$.

Suppose $K = K_0$ is a particular solution of the general $(M \times M)$ Riccati equation. Substituting

$$K = K_0 + F^{-1},$$

where Y is a symmetric $M \times M$ matrix, we need to be able to calculate $\dfrac{dY^{-1}}{dt}$. Differentiating the equation $YY^{-1} = I$, we find

$$\frac{d}{dt}\left(YY^{-1}\right) = \frac{dY}{dt}Y^{-1} + Y\frac{d}{dt}\left(Y^{-1}\right) = O,$$

where O is the null $M \times M$ matrix. Multiplying on the left by Y^{-1}, this gives

$$\frac{d}{dt}\left(Y^{-1}\right) = -Y^{-1}\frac{dY}{dt}Y^{-1}.$$

Now, substituting for K in (5.4.11), we obtain

$$\begin{aligned} &\frac{dK_0}{dt} - Y^{-1}\frac{dY}{dt}Y^{-1} - \left(K_0 f_u + H_{xu} + Y^{-1} f_u\right) H_{uu}^{-1}\left(f_u^{\,T} K_0 + H_{xu}^{\,T} + f_u^{\,T} Y^{-1}\right) \\ &+ K_0 f_x + Y^{-1} f_x + f_x^{\,T} K_0 + f_x^{\,T} Y^{-1} + H_{xx} = O. \end{aligned}$$

Since K_0 is a particular solution of the Riccati equation, this reduces to

$$\begin{aligned} &-Y^{-1}\frac{dY}{dt}Y^{-1} - \left(K_0 f_u + H_{xu}\right) H_{uu}^{-1} f_u^{\,T} Y^{-1} - Y^{-1} f_u H_{uu}^{-1}\left(f_u^{\,T} K_0 + H_{xu}^{\,T}\right) \\ &- Y^{-1} f_u H_{uu}^{-1} f_u^{\,T} Y^{-1} + Y^{-1} f_x + f_x^{\,T} Y^{-1} = O. \end{aligned}$$

Multiplying on the left and right by Y, we have finally

$$\begin{aligned} &\frac{dY}{dt} + Y\left(K_0 f_u + H_{xu}\right) H_{uu}^{-1} f_u^{\,T} + f_u H_{uu}^{-1}\left(f_u^{\,T} K_0 + H_{xu}^{\,T}\right) Y \\ &+ f_u H_{uu}^{-1} f_u^{\,T} - f_x Y - Y f_x^{\,T} = O. \end{aligned}$$

This equation is linear in Y and can be solved in the usual way.

In the case under consideration, since H_{xu} **and** H_{xx} are null matrices, $K_0 = O$ is a particular solution of the Riccati equation and the equation for Y reduces to

$$\frac{dY}{dt} + f_u H_{uu}^{-1} f_u^{\,T} - f_x Y - Y f_x^{\,T} = O.$$

Taking
$$Y = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$$

we find that the last matrix equation reduces to the equations

$$\frac{dp}{dt} - 2q = 0, \quad \frac{dq}{dt} - r = 0, \quad \frac{dr}{dt} - \frac{1}{2} = 0.$$

Integration gives

$$\begin{aligned} p &= -\frac{1}{6}t^3 - \frac{1}{2}\alpha t^2 - 2\beta t - \gamma, \\ q &= -\frac{1}{4}t^2 - \frac{1}{2}\alpha t - \beta, \\ r &= -\frac{1}{2}t - \frac{1}{2}\alpha, \end{aligned}$$

where α, β, γ are constants of integration.

We now calculate that

$$K = Y^{-1} = \frac{1}{D}\begin{bmatrix} -\frac{1}{2}t - \frac{1}{2}\alpha & \frac{1}{4}t^2 + \frac{1}{2}\alpha t + \beta \\ \frac{1}{4}t^2 + \frac{1}{2}\alpha t + \beta & -\frac{1}{6}t^3 - \frac{1}{2}\alpha t^2 - 2\beta t - \gamma \end{bmatrix},$$

where $D = \frac{1}{48}t^4 + \frac{1}{12}\alpha t^3 + \frac{1}{2}\beta t^2 + \frac{1}{2}\gamma t + \frac{1}{2}\alpha\gamma - \beta^2$.

If we take $\beta = \frac{1}{4}\alpha^2$, $\gamma = \frac{1}{6}\alpha^3$, it will be found that D reduces to $\frac{1}{48}(t + \alpha)^4$. In this case, K is finite for all values of t except $t = -\alpha$. It follows that by choosing α appropriately, we can ensure that K is finite over every extremal.