LINEAR
ROBUST CONTROL

SOLUTIONS MANUAL

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MICHAEL GREEN
Australian National University
Canberra, Australia

DAVID J.N. LIMEBEER
Professor of Control Engineering
Imperial College of Science, Technology and Medicine
University of London
London, U.K.

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To Keith Brackenbury and Stanford Reynolds

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Preface

Every serious student of a technical scientific subject has spent late nights struggling with homework assignments at some time during their career. The frustrations which go along with this activity range from: "I don't have the foggiest idea how to do this exercise" to "this is probably right, but it would be nice to have my solution checked by an expert." It is our expectation that the student exercises in our book *Linear Robust Control*, published by Prentice-Hall, 1994, will generate both the above sentiments at some stage or other and many others besides!

Because we would like our book to be useful both as a teaching and as a research aid, we decided that a reasonably detailed solutions manual would have a role to play. We hope that most of the answers are informative and that some of them are interesting and even new. Some of the examples took their inspiration from research papers which we were unable to cover in detail in the main text. In some cases, and undoubtedly with the benefit of hindsight, we are able to supply different and possibly nicer solutions to the problems studied in this literature.

What about the answer to the question: "who should have access to the solutions manual?" We believe that in the first instance students should not have access to the solutions manual, because that would be like exploring the Grand Canyon from the window of a rental car—to really experience you have to actively partake.

In an attempt to steel the nerve for the task ahead, we thought it appropriate to repeat a quotation due to Brutus Hamilton (1957), from the book *Lore of Running* (Oxford University Press, 1992), by the South African sports scientist and ultramarathon runner Tim Noakes.

> "It is one of the strange ironies of this strange life that those who work the hardest, who subject themselves to the strictest discipline, who give up certain pleasurable things in order to achieve a goal are the happiest of people. When you see twenty or thirty men line up for a distance race, do not pity them, don't feel sorry for them. Better envy them."

After reading Noakes' book a little further, we couldn't help noticing a number of other analogies between doing student exercises and training for a marathon. Here are a few:

1. Nobody can do them for you.

2. At least in the beginning, there is no doubt that they are hard.

3. Like any acquired skill, the more effort that goes into the acquisition, and the more difficulties overcome, the more rewarding the result.

4. To achieve success there must always be the risk of failure no matter how hard you try.

5. Student exercises, like running, teach you real honesty. There is no luck. Results cannot be faked and there is no one but yourself to blame when things go wrong.

6. Don't make excuses like my feet are too big, I don't know enough mathematics, I am too old and so on. Overcoming such difficulties will only heighten the reward.

We have tried to tie the solutions manual to the main text as closely as possible and from time to time we refer to specific results there. Equation references of the form $(x.y.z)$ refer to equations in the main text. For example, equation (3.2.1) means equation (3.2.1) of *Linear Robust Control*, which will be the first equation in Section 3.2. Equations in the solutions manual have the form $(x.y)$. For example, equation (8.1) will be the first equation in the Solutions to Problems in Chapter 8. All cited works are as listed in the bibliography of *Linear Robust Control*.

We have made every effort within stringent time constraints to trap errors, but we cannot realistically expect to have found them all. If a solution is hard to follow or doesn't seem to make sense it could be wrong! Don't get mad, we have tried to help and we hope our solutions will assist teachers and students alike.

Michael Green
David Limebeer

London

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Solutions to Problems in Chapter 1

Solution 1.1. We will establish the three properties of a norm and then the submultiplicative property:

1. If $\boldsymbol{h} = 0$ $\sup_\omega |\boldsymbol{h}(j\omega)| = 0$. Conversely, if $\sup_\omega |\boldsymbol{h}(j\omega)| = 0$, $\boldsymbol{h}(j\omega) = 0$ for all ω, then $\boldsymbol{h} = 0$. If $\boldsymbol{h} \neq 0$, it is clear from the definition that $\|\boldsymbol{h}\|_\infty > 0$.

2.
$$\begin{aligned}
\|\alpha\boldsymbol{h}\|_\infty &= \sup_\omega |\alpha\boldsymbol{h}(j\omega)| \\
&= \sup_\omega |\alpha|.|\boldsymbol{h}(j\omega)| \\
&= |\alpha| \sup_\omega |\boldsymbol{h}(j\omega)| \\
&= |\alpha| \|\boldsymbol{h}\|_\infty.
\end{aligned}$$

3.
$$\begin{aligned}
\|\boldsymbol{h}+\boldsymbol{g}\|_\infty &= \sup_\omega |\boldsymbol{h}(j\omega)+\boldsymbol{g}(j\omega)| \\
&\leq \sup_\omega (|\boldsymbol{h}(j\omega)| + |\boldsymbol{g}(j\omega)|) \\
&\leq \sup_\omega |\boldsymbol{h}(j\omega)| + \sup_\omega |\boldsymbol{g}(j\omega)| \\
&= \|\boldsymbol{h}\|_\infty + \|\boldsymbol{g}\|_\infty.
\end{aligned}$$

This establishes that $\|\cdot\|_\infty$ is a norm. We now prove the submultiplicative property:

4.
$$\begin{aligned}
\|\boldsymbol{h}\boldsymbol{g}\|_\infty &= \sup_\omega |\boldsymbol{h}(j\omega)\boldsymbol{g}(j\omega)| \\
&= \sup_\omega (|\boldsymbol{h}(j\omega)|.|\boldsymbol{g}(j\omega)|) \\
&\leq \sup_\omega |\boldsymbol{h}(j\omega)| \sup_\omega |\boldsymbol{g}(j\omega)| \\
&= \|\boldsymbol{h}\|_\infty \|\boldsymbol{g}\|_\infty.
\end{aligned}$$

Solution 1.2. Set

$$\begin{aligned}\widehat{\boldsymbol{h}} &= \boldsymbol{w}(1-\boldsymbol{g}\boldsymbol{k})^{-1}\\ &= \boldsymbol{w}(1+\boldsymbol{g}\boldsymbol{q}), \qquad \boldsymbol{q}=\boldsymbol{k}(1-\boldsymbol{g}\boldsymbol{k})^{-1}.\end{aligned}$$

Therefore,

$$\boldsymbol{q}=\boldsymbol{g}^{-1}\boldsymbol{w}^{-1}(\widehat{\boldsymbol{h}}-\boldsymbol{w}).$$

In the case that $\alpha<0$, $\boldsymbol{g}^{-1}$ is stable and $\|\widehat{\boldsymbol{h}}\|_\infty$ may be made arbitrarily small by using the constant compensator $k\to\infty$. If $\alpha\geq 0$, $\widehat{\boldsymbol{h}}$ must satisfy the interpolation constraint:

$$\begin{aligned}\widehat{\boldsymbol{h}}(\alpha) &= \boldsymbol{w}(\alpha)\\ &= \frac{\alpha+4}{2(\alpha+1)}.\end{aligned}$$

Now

$$\begin{aligned}|\widehat{\boldsymbol{h}}(\alpha)| &< 1, \qquad \alpha\geq 0\\ \Leftrightarrow (\alpha+4) &< 2(\alpha+1)\\ \Leftrightarrow \alpha &> 2.\end{aligned}$$

Thus the problem has a solution if and only if $\alpha<0$ or if $\alpha>2$.

Solution 1.3. Since $\boldsymbol{e}=\boldsymbol{h}-\boldsymbol{g}\boldsymbol{f}$ with $\boldsymbol{f}\in\mathcal{RH}_\infty$,

$$\boldsymbol{f}=\boldsymbol{g}^{-1}(\boldsymbol{h}-\boldsymbol{e}).$$

In order for $\boldsymbol{f}$ to be stable, we need

$$\begin{aligned}\boldsymbol{e}(1) &= \boldsymbol{h}(1)\\ &= 1/5.\end{aligned}$$

Differentiating $\boldsymbol{e}=\boldsymbol{h}-\boldsymbol{g}\boldsymbol{f}$ gives

$$\frac{d\boldsymbol{e}}{ds}=\frac{d\boldsymbol{h}}{ds}-\boldsymbol{g}\frac{d\boldsymbol{f}}{ds}-\boldsymbol{f}\frac{d\boldsymbol{g}}{ds}.$$

Thus, $\boldsymbol{f}$ is also given by

$$\boldsymbol{f}=\left(\frac{d\boldsymbol{g}}{ds}\right)^{-1}\left(\frac{d\boldsymbol{e}}{ds}-\frac{d\boldsymbol{h}}{ds}+\boldsymbol{g}\frac{d\boldsymbol{f}}{ds}\right)$$

Since $\frac{d}{ds}\boldsymbol{g}|_{s=1} = 0$, the stability of $\boldsymbol{f}$ requires a second interpolation constraint:

$$\begin{aligned}\left.\frac{d\boldsymbol{e}}{ds}\right|_{s=1} &= \left.\frac{d\boldsymbol{h}}{ds}\right|_{s=1} \\ &= \left.\frac{-1}{(s+4)^2}\right|_{s=1} \\ &= -1/25.\end{aligned}$$

Solution 1.4.

1. It is sufficient for closed loop stability that $|\boldsymbol{gk}(1-\boldsymbol{gk})^{-1}(j\omega)|.|\delta(j\omega)| < 1$ for all real ω, including $\omega = \infty$. (This follows from the Nyquist criterion.) If $\|\delta\|_\infty < \alpha$, we need

$$\|\boldsymbol{gk}(1-\boldsymbol{gk})^{-1}\|_\infty \le \alpha^{-1},$$

with α maximized. This may be achieved via the following procedure:

Step 1: Factorize $\boldsymbol{gg}^\sim = \boldsymbol{mm}^\sim$, in which both $\boldsymbol{m}$ and $\boldsymbol{m}^{-1}$ are stable.

Step 2: Define the Blaschke product

$$\boldsymbol{a} = \prod_{i=1}^{m} \frac{\bar{p}_i + s}{p_i - s},$$

in which the p_i's are the right-half-plane poles of $\boldsymbol{g}$.

Step 3: If $\boldsymbol{q} = \boldsymbol{k}(1-\boldsymbol{gk})^{-1}$ and $\tilde{\boldsymbol{q}} = \boldsymbol{amq}$, we observe that

(i)

$$\begin{aligned}\|\boldsymbol{gq}\|_\infty &= \|\boldsymbol{gaq}\|_\infty, && \text{since } \boldsymbol{a} \text{ is allpass} \\ &= \|\boldsymbol{aqmm}^\sim(\boldsymbol{g}^\sim)^{-1}\|_\infty \\ &= \|\boldsymbol{aqm}\|_\infty, && \text{since } \boldsymbol{m}^\sim(\boldsymbol{g}^\sim)^{-1} \text{ is allpass} \\ &= \|\tilde{\boldsymbol{q}}\|_\infty.\end{aligned}$$

(ii) $\tilde{\boldsymbol{q}} \in \mathcal{RH}_\infty \Rightarrow \boldsymbol{q} \in \mathcal{RH}_\infty$.

(iii) $\tilde{\boldsymbol{q}} \in \mathcal{RH}_\infty \Rightarrow \boldsymbol{q}(p_i) = 0$.

(iv) $(1+\boldsymbol{gq})(p_i) = 0 \Leftrightarrow \tilde{\boldsymbol{q}}(p_i) = -(\boldsymbol{amg}^{-1})(p_i)$.

Step 4: Find a stable $\tilde{\boldsymbol{q}}$ of minimum infinity norm which satisfies

$$\tilde{\boldsymbol{q}}(p_i) = -(\boldsymbol{amg}^{-1})(p_i).$$

Step 5: Back substitute

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{q}(1+\boldsymbol{g}\boldsymbol{q})^{-1} \\ &= \tilde{\boldsymbol{q}}(\boldsymbol{a}\boldsymbol{m}+\boldsymbol{g}\tilde{\boldsymbol{q}})^{-1}. \end{aligned}$$

2. (i) If $\boldsymbol{g}$ is unstable and $\alpha \geq 1$, we can always destabilize the loop by setting $\delta = -1$ since in this case $\boldsymbol{g}(1+\delta)$ would be open loop.

   (ii) Suppose for a general plant that

$$\boldsymbol{g} = \frac{\boldsymbol{n}_+\boldsymbol{n}_-}{\boldsymbol{d}_+\boldsymbol{d}_-}$$

   in which $\boldsymbol{n}_-$ and $\boldsymbol{d}_-$ are polynomials that have all their roots in the closed-right-half plane, while $\boldsymbol{n}_+$ and $\boldsymbol{d}_+$ are polynomials that have all their roots in the open-left-half plane. Then

$$\boldsymbol{a} = \frac{\boldsymbol{d}_-^\sim}{\boldsymbol{d}_-}, \qquad \boldsymbol{m} = \frac{\boldsymbol{n}_+\boldsymbol{n}_-^\sim}{\boldsymbol{d}_+\boldsymbol{d}_-^\sim},$$

   and consequently

$$\begin{aligned} \boldsymbol{g}^{-1}\boldsymbol{m}\boldsymbol{a} &= \frac{\boldsymbol{d}_+\boldsymbol{d}_-}{\boldsymbol{n}_+\boldsymbol{n}_-}.\frac{\boldsymbol{n}_+\boldsymbol{n}_-^\sim}{\boldsymbol{d}_+\boldsymbol{d}_-^\sim}.\frac{\boldsymbol{d}_-^\sim}{\boldsymbol{d}_-} \\ &= \frac{\boldsymbol{n}_-^\sim}{\boldsymbol{n}_-}, \end{aligned}$$

   which is an unstable allpass function. The implication now follows from the fact that $|\boldsymbol{a}(p_i)| > 1$ for any unstable allpass function $\boldsymbol{a}$ and any $\mathrm{R}_e(p_i) > 0$.

3. In this case

$$\boldsymbol{g} = \left(\frac{s-2}{s-1}\right) \Rightarrow \boldsymbol{a} = \left(\frac{s+1}{1-s}\right) \text{ and } \boldsymbol{m} = \left(\frac{s+2}{s+1}\right),$$

   so that

$$\begin{aligned} \boldsymbol{g}^{-1}\boldsymbol{a}\boldsymbol{m} &= \left(\frac{s-1}{s-2}\right)\left(\frac{s+1}{1-s}\right)\left(\frac{s+2}{s+1}\right) \\ &= -\left(\frac{s+2}{s-2}\right). \end{aligned}$$

   Therefore

$$\begin{aligned} \boldsymbol{g}^{-1}\boldsymbol{a}\boldsymbol{m}(1) &= 3 \\ \Rightarrow \tilde{\boldsymbol{q}}_{opt} &= -3 \text{ and } \|\tilde{\boldsymbol{q}}_{opt}\|_\infty = 3. \end{aligned}$$

Since $\alpha_{\max} = 1/\|\tilde{\boldsymbol{q}}_{opt}\|_\infty$, we see that $\alpha_{\max} = 1/3$. Finally

$$\begin{aligned} \boldsymbol{k} &= \tilde{\boldsymbol{q}}_{opt}(\boldsymbol{am} + \boldsymbol{g}\tilde{\boldsymbol{q}}_{opt})^{-1} \\ &= -3\left(\left(\frac{s+2}{1-s}\right) - 3\left(\frac{s-2}{s-1}\right)\right)^{-1} \\ &= \frac{3(s-1)}{4(s-1)} \\ &= 3/4. \end{aligned}$$

Solution 1.5.

1. The closed loop will be stable provided

$$\begin{aligned} \|\boldsymbol{\delta k}(1-\boldsymbol{gk})^{-1}\|_\infty &< 1 \\ \Leftrightarrow \|\boldsymbol{\delta w}^{-1}\boldsymbol{wk}(1-\boldsymbol{gk})^{-1}\|_\infty &< 1 \\ \Leftarrow \|\boldsymbol{\delta w}^{-1}\|_\infty &< \frac{1}{\|\boldsymbol{wk}(1-\boldsymbol{gk})^{-1}\|_\infty}. \end{aligned}$$

   This last inequality will be satisfied if $|\boldsymbol{\delta}(j\omega)| < |\boldsymbol{w}(j\omega)|$ for all ω and $\|\boldsymbol{wk}(1-\boldsymbol{gk})^{-1}\|_\infty \leq 1$.

2. We will now describe the optimization procedure

   Step 1: Define

$$\tilde{\boldsymbol{q}} = \boldsymbol{waq},$$

   where

$$\boldsymbol{q} = \boldsymbol{k}(1-\boldsymbol{gk})^{-1} \quad \text{and} \quad \boldsymbol{a} = \prod_{i=1}^{m}\left(\frac{\bar{p}_i + s}{p_i - s}\right),$$

   in which the p_i's are the right-half-plane poles of $\boldsymbol{g}$.

   Step 2: Find a stable $\tilde{\boldsymbol{q}}$ of minimum infinity norm such that

$$\tilde{\boldsymbol{q}}(p_i) = -\boldsymbol{g}^{-1}\boldsymbol{aw}(p_i).$$

   Step 3: Back substitute using $\boldsymbol{k} = \tilde{\boldsymbol{q}}(\boldsymbol{aw} + \boldsymbol{g}\tilde{\boldsymbol{q}})^{-1}$.

3. Since

$$\boldsymbol{g} = \left(\frac{s+1}{s-2}\right),$$

we must have

$$\boldsymbol{a} = \left(\frac{s+2}{2-s}\right).$$

Therefore

$$\begin{aligned} \boldsymbol{g}^{-1}\boldsymbol{a}\boldsymbol{w} &= \left(\frac{s-2}{s+1}\right)\left(\frac{s+2}{2-s}\right)\left(\frac{s+1}{s+4}\right) \\ &= -\left(\frac{s+2}{s+4}\right). \end{aligned}$$

Consequently

$$\begin{aligned} \tilde{\boldsymbol{q}}_{opt} &= \left.\left(\frac{s+2}{s+4}\right)\right|_{s=2} \\ &= 2/3. \end{aligned}$$

Thus, a controller exists, since $\min_{\boldsymbol{k}} \|\boldsymbol{w}\boldsymbol{k}(1-\boldsymbol{g}\boldsymbol{k})^{-1}\|_\infty = \|\tilde{\boldsymbol{q}}_{opt}\|_\infty = 2/3 < 1$. The optimal controller is

$$\begin{aligned} \boldsymbol{k} &= \frac{2}{3}\left(\left(\frac{s+2}{2-s}\right)\left(\frac{s+1}{s+4}\right) + \frac{2}{3}\left(\frac{s+1}{s-2}\right)\right)^{-1} \\ &= -\left(\frac{2(s+4)}{s+1}\right). \end{aligned}$$

Solution 1.6.

1. If $\boldsymbol{E} = \boldsymbol{H} - \boldsymbol{G}\boldsymbol{F}$, it follows that

$$\boldsymbol{F} = \boldsymbol{G}^{-1}(\boldsymbol{H} - \boldsymbol{E}).$$

It is now immediate from the stability requirement on $\boldsymbol{F}$ that all the right half plane poles of $\boldsymbol{G}^{-1}$ must be cancelled by zeros of $(\boldsymbol{H} - \boldsymbol{E})$.

2. It follows from the standard theory of stable coprime matrix fractions that a cancellation between $\boldsymbol{G}^{-1}$ and $(\boldsymbol{H} - \boldsymbol{E})$ will occur if and only if

$$\left[\begin{array}{cc} \boldsymbol{H}(z_i) - \boldsymbol{E}(z_i) & \boldsymbol{G}(z_i) \end{array}\right]$$

looses rank at a zero z_i of $\boldsymbol{G}$. If such a loss of rank occurs, there exists a w_i^* such that

$$w_i^*\left[\begin{array}{cc} \boldsymbol{H}(z_i) - \boldsymbol{E}(z_i) & \boldsymbol{G}(z_i) \end{array}\right] = 0.$$

If

$$w_i^*\boldsymbol{H}(z_i) = v_i^*,$$

the vector valued interpolation constraints will be

$$w_i^* \boldsymbol{E}(z_i) = v_i^*.$$

Satisfaction of these constraints ensures the cancellation of the unstable poles of $\boldsymbol{G}^{-1}$.

Solutions to Problems in Chapter 2

Solution 2.1.

1.

$$\begin{aligned} \underline{\sigma}(I-Q) &\geq \underline{\sigma}(I)-\overline{\sigma}(Q) \\ &= 1-\overline{\sigma}(Q) \\ &> 0. \end{aligned}$$

2.

$$\begin{aligned} \overline{\sigma}(\sum_{k=0}^{\infty} Q^k) &\leq \sum_{k=0}^{\infty} \overline{\sigma}(Q^k) \\ &\leq \sum_{k=0}^{\infty} \{\overline{\sigma}(Q)\}^k \\ &= \frac{1}{1-\overline{\sigma}(Q)} \\ &< \infty. \end{aligned}$$

3. Consider

$$\begin{aligned} (I-Q)(\sum_{k=0}^{\infty} Q^k) &= \sum_{k=0}^{\infty} Q^k - \sum_{k=0}^{\infty} Q^{k+1} \\ &= I + \sum_{k=0}^{\infty} Q^{k+1} - \sum_{k=0}^{\infty} Q^{k+1} \\ &= I. \end{aligned}$$

Hence

$$(I-Q)^{-1} = \sum_{k=0}^{\infty} Q^k.$$

Solution 2.2.

1.

$$\begin{aligned}\boldsymbol{Q}(I-\boldsymbol{Q})^{-1} &= (I-\boldsymbol{Q})^{-1}(I-\boldsymbol{Q})\boldsymbol{Q}(I-\boldsymbol{Q})^{-1}\\ &= (I-\boldsymbol{Q})^{-1}(\boldsymbol{Q}-\boldsymbol{Q}^2)(I-\boldsymbol{Q})^{-1}\\ &= (I-\boldsymbol{Q})^{-1}\boldsymbol{Q}(I-\boldsymbol{Q})(I-\boldsymbol{Q})^{-1}\\ &= (I-\boldsymbol{Q})^{-1}\boldsymbol{Q}.\end{aligned}$$

2.

$$\begin{aligned}(I-\boldsymbol{Q})^{-1} &= (I-\boldsymbol{Q}+\boldsymbol{Q})(I-\boldsymbol{Q})^{-1}\\ &= I+\boldsymbol{Q}(I-\boldsymbol{Q})^{-1}.\end{aligned}$$

3.

$$\begin{aligned}\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K})^{-1} &= (I-\boldsymbol{K}\boldsymbol{G})^{-1}(I-\boldsymbol{K}\boldsymbol{G})\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K})^{-1}\\ &= (I-\boldsymbol{K}\boldsymbol{G})^{-1}(\boldsymbol{K}-\boldsymbol{K}\boldsymbol{G}\boldsymbol{K})(I-\boldsymbol{G}\boldsymbol{K})^{-1}\\ &= (I-\boldsymbol{K}\boldsymbol{G})^{-1}\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K})(I-\boldsymbol{G}\boldsymbol{K})^{-1}\\ &= (I-\boldsymbol{K}\boldsymbol{G})^{-1}\boldsymbol{K}.\end{aligned}$$

Solution 2.3. Suppose that $Q = Y\Sigma U^*$, where $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_p)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > 0$. Then

$$\begin{aligned}Q^{-1} &= U\Sigma^{-1}Y^*\\ &= U\text{diag}(\sigma_1^{-1}, \cdots, \sigma_p^{-1})Y^*,\end{aligned}$$

where $\sigma_p^{-1} \geq \cdots \geq \sigma_2^{-1} \geq \sigma_1^{-1}$. Hence

$$\begin{aligned}\overline{\sigma}(Q^{-1}) &= \sigma_p^{-1}\\ &= \frac{1}{\underline{\sigma}(Q)}.\end{aligned}$$

Solution 2.4.

1. Let $Q = WJW^{-1}$, in which J is the Jordan form of Q. Then

$$\begin{aligned}\det Q &= \det(W)\det(J)\det(W^{-1})\\ &= \det(W)\det(J)\frac{1}{\det(W)}\\ &= \det(J)\\ &= \prod_{i=1}^{p}\lambda_i(Q).\end{aligned}$$

Now let $Q = Y\Sigma U^*$, with $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_p)$ be a singular value decomposition of Q. Then

$$\begin{aligned} \det(Q) &= \det(Y)\det(\Sigma)\det(U^*) \\ &= e^{i\theta_Y} \prod_{i=1}^{p} \sigma_i(Q) e^{-i\theta_U}, \qquad \text{since } Y \text{ and } U \text{ are unitary} \\ &= e^{i\theta} \prod_{i=1}^{p} \sigma_i(Q). \end{aligned}$$

2. It is well known that

$$\underline{\sigma}(Q) \leq \frac{\|Qu\|}{\|u\|} \leq \overline{\sigma}(Q)$$

for any non-zero vector u. Now if $Qw_i = \lambda_i w_i$, we see that

$$\underline{\sigma}(Q) \leq \frac{\|Qw_i\|}{\|w_i\|} = |\lambda_i| \leq \overline{\sigma}(Q).$$

Solution 2.5. Nothing can be concluded in general. To see this consider the system

$$\boldsymbol{G} = \begin{bmatrix} 1 + \alpha\frac{s+1}{s-1} & \frac{1}{\epsilon} \\ 0 & 1 + \alpha\frac{s+1}{s-1} \end{bmatrix}.$$

Each eigenvalue $\lambda_i(j\omega)$ makes one encirclement of $+1$, and $|1 - \lambda_i(j\omega)| = \alpha$ for all ω and any value of ϵ. It is easy to check that the (constant) additive perturbation

$$A = \begin{bmatrix} 0 & 0 \\ \epsilon\alpha^2 & 0 \end{bmatrix}$$

will destabilize the loop, since $\det\big(I - (\boldsymbol{G}(j\omega) + A)\big)$ passes through the origin (at $\omega = 0$). Since $\lim_{\epsilon \to 0} \|A\| = 0$, we see that the loop may be made arbitrarily close to instability for any value of α.

Solution 2.6.

1. These are just the Nyquist plots of $\frac{1}{s+1}$ and $\frac{2}{s+2}$, which are circles cutting the real axis at 0 and 1. (*i.e.*, both are circles with center 1/2 and radius 1/2.)

2. This can be checked by evaluating $C(sI - A)^{-1}B$ with

$$\begin{aligned} A &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\ B &= \begin{bmatrix} 7 & -8 \\ -12 & 14 \end{bmatrix} \\ C &= \begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix}. \end{aligned}$$

3. We begin by setting $k_1 = k + \delta$ and $k_2 = k - \delta$. This gives

$$\begin{aligned}\det(sI - A - BKC) &= \det\left(\begin{bmatrix} s+1-k-97\delta & -112\delta \\ 168\delta & s+2-2k+194\delta \end{bmatrix}\right) \\ &= s^2 + s(3-3k+97\delta) + 2\big((1-k)^2 - \delta^2\big).\end{aligned}$$

We therefore require positivity of the linear coefficient $3 - 3k + 97\delta$. Now

$$\begin{aligned}3 - 3k + 97\delta &= 3 - 50(k-\delta) + 47(k+\delta) \\ &= 3 + 47k_1 - 50k_2.\end{aligned}$$

Solution 2.7. The plots given in Figures 2.1, 2.2 and 2.3 can be verified using MATLAB procedures.

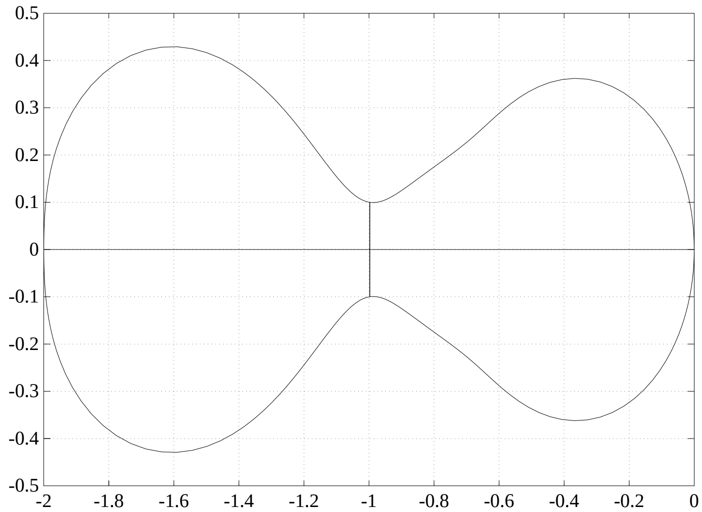


Figure 2.1: Generalized Nyquist diagram when $\epsilon = 0.005$.

Solution 2.8. The proof of Theorem 2.4.3 has to be modified to make use of

$$\begin{aligned}\Gamma_\epsilon &= \det(I - (I + \epsilon\Delta_1(s))\boldsymbol{GK}(s)) \\ &= \det(I - \epsilon\Delta_1\boldsymbol{GK}(I - \boldsymbol{GK}(s))^{-1})\det(I - \boldsymbol{GK}(s)).\end{aligned}$$

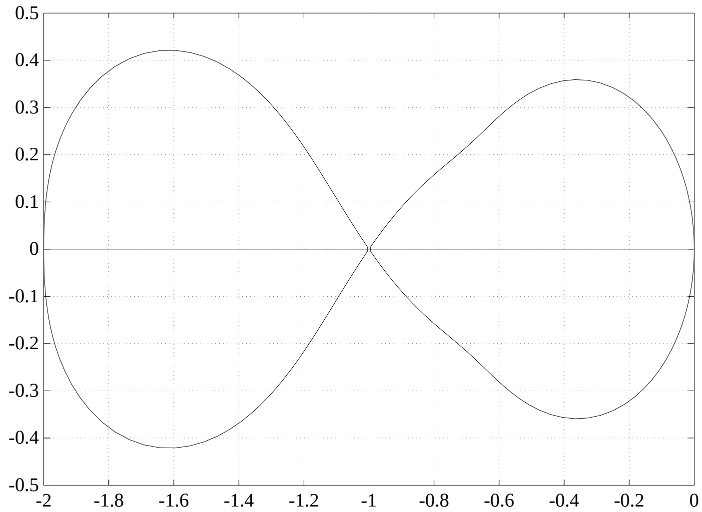


Figure 2.2: Generalized Nyquist diagram when $\epsilon = 0$.

It is now clear that Γ_ϵ will not cross the origin for any $\epsilon \in [0, 1]$ if

$$\begin{aligned} \overline{\sigma}\big(\Delta_1(s)\big)\overline{\sigma}\big(\boldsymbol{GK}(I - \boldsymbol{GK}(s))^{-1}\big) &< 1 \\ \Leftrightarrow \overline{\sigma}\big(\Delta_1(s)\big) &< \frac{1}{\overline{\sigma}\big(\boldsymbol{GK}(I - \boldsymbol{GK}(s))^{-1}\big)} \end{aligned}$$

for all s on the Nyquist contour D_R.

Solution 2.9. In this case we use

$$\begin{aligned} \Gamma_\epsilon &= \det\big(I - (I - \epsilon\Delta_2(s))^{-1}\boldsymbol{GK}(s)\big) \\ &= \det\big(I - \epsilon\Delta_2(s) - \boldsymbol{GK}(s)\big)\det\big((I - \epsilon\Delta_2(s))^{-1}\big). \end{aligned}$$

We need $\overline{\sigma}\big(\Delta_2(s)\big) < 1$ to ensure the existence of $\big(I - \epsilon\Delta_2(s)\big)^{-1}$, and we need

$$\overline{\sigma}\big(\Delta_2(s)\big) < \underline{\sigma}\big(I - \boldsymbol{GK}(s)\big)$$

to ensure that Γ_ϵ will not cross the origin for any $\epsilon \in [0, 1]$. Thus a sufficient condition for closed loop stability is

$$\overline{\sigma}(\Delta_2(s)) < \min\{1, \underline{\sigma}(I - \boldsymbol{GK}(s))\}$$

for all s on the Nyquist contour D_R.

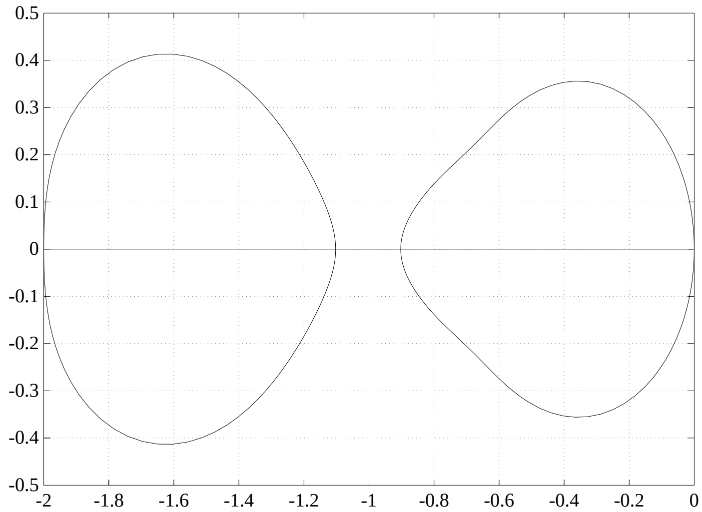


Figure 2.3: Generalized Nyquist diagram when $\epsilon = -0.005$.

Solution 2.10. To see that the implication in (2.4.8) is true we argue that

$$\begin{aligned}
\overline{\sigma}(\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K})^{-1}) &\leq \gamma \\
\Rightarrow \overline{\sigma}(\boldsymbol{K})\underline{\sigma}((I-\boldsymbol{G}\boldsymbol{K})^{-1}) &\leq \gamma \\
\Leftrightarrow \frac{\overline{\sigma}(\boldsymbol{K})}{\overline{\sigma}(I-\boldsymbol{G}\boldsymbol{K})} &\leq \gamma \\
\Rightarrow \frac{\overline{\sigma}(\boldsymbol{K})}{1+\overline{\sigma}(\boldsymbol{G})\overline{\sigma}(\boldsymbol{K})} &\leq \gamma \\
\Leftrightarrow \overline{\sigma}(\boldsymbol{K}) &\leq \frac{\gamma}{1-\gamma\overline{\sigma}(\boldsymbol{G})}, \quad \text{for } 1-\gamma\overline{\sigma}(\boldsymbol{G}) > 0.
\end{aligned}$$

For the implication in (2.4.9),

$$\begin{aligned}
\overline{\sigma}(\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K})^{-1}) &\leq \gamma \\
\Leftarrow \frac{\overline{\sigma}(\boldsymbol{K})}{1-\overline{\sigma}(\boldsymbol{G})\overline{\sigma}(\boldsymbol{K})} &\leq \gamma \\
\Leftrightarrow \overline{\sigma}(\boldsymbol{K}) &\leq \frac{\gamma}{1+\gamma\overline{\sigma}(\boldsymbol{G})}.
\end{aligned}$$

To establish the inequalities given in (2.4.13), we argue that

$$\overline{\sigma}(\boldsymbol{Q}(I-\boldsymbol{Q})^{-1}) \leq \gamma$$

$$\begin{aligned}
\Rightarrow \frac{\overline{\sigma}(\boldsymbol{Q})}{\overline{\sigma}(I-\boldsymbol{Q})} &\leq \gamma \\
\Rightarrow \frac{\overline{\sigma}(\boldsymbol{Q})}{1+\overline{\sigma}(\boldsymbol{Q})} &\leq \gamma \\
\Leftrightarrow \overline{\sigma}(\boldsymbol{Q}) &\leq \frac{\gamma}{1-\gamma}, \quad \text{for } \gamma < 1.
\end{aligned}$$

Also,

$$\begin{aligned}
\overline{\sigma}(\boldsymbol{Q}(I-\boldsymbol{Q})^{-1}) &\leq \gamma \\
\Leftarrow \frac{\overline{\sigma}(\boldsymbol{Q})}{\underline{\sigma}(I-\boldsymbol{Q})} &\leq \gamma \\
\Leftarrow \frac{\overline{\sigma}(\boldsymbol{Q})}{1-\overline{\sigma}(\boldsymbol{Q})} &\leq \gamma \\
\Leftrightarrow \overline{\sigma}(\boldsymbol{Q}) &\leq \frac{\gamma}{1+\gamma}.
\end{aligned}$$

Solution 2.11. The aim of this question is to construct a rational additive perturbation $\boldsymbol{A}$ of minimum norm such that

$$I - \boldsymbol{AK}(I-\boldsymbol{GK})^{-1}$$

is singular at ω_0. The frequency point ω_0 is selected to be $\omega_0 = \arg\max_{s=j\omega} \overline{\sigma}(\boldsymbol{K}(I-\boldsymbol{GK})^{-1}(j\omega))$. If $\boldsymbol{K}(I-\boldsymbol{GK})^{-1}(j\omega_0)$ has singular value decomposition

$$\boldsymbol{K}(I-\boldsymbol{GK})^{-1}(j\omega_0) = \sum_{i=1}^{2} \sigma_i v_i u_i^*,$$

then a constant complex perturbation with the correct properties is given by

$$A = \sigma_1^{-1} u_1 v_1^*,$$

since $\|A\| = \sigma_1^{-1}$. To realize this as a physical system, we set

$$v_1^* = \begin{bmatrix} r_1 e^{-i\phi_1} & r_2 e^{-i\phi_2} \end{bmatrix} \text{ and } u_1 \begin{bmatrix} r_3 e^{-i\theta_3} \\ r_4 e^{-i\theta_4} \end{bmatrix},$$

in which the signs of the r_i's are chosen to make all the angles positive. We then select the α_i's and β_i's, both nonnegative, so that the phase of

$$\left(\frac{j\omega_0 - \alpha_i}{j\omega_0 + \alpha_i}\right)$$

is given by ϕ_i and the phase of

$$\left(\frac{j\omega_0 - \beta_i}{j\omega_0 + \beta_i}\right)$$

is given by θ_i. The perturbation is then given by

$$\boldsymbol{A} = \sigma_1^{-1} \begin{bmatrix} r_3 \left(\frac{s-\alpha_1}{s+\alpha_1}\right) \\ r_4 \left(\frac{s-\alpha_2}{s+\alpha_2}\right) \end{bmatrix} \begin{bmatrix} r_1 \left(\frac{s-\beta_1}{s+\beta_1}\right) & r_2 \left(\frac{s-\beta_2}{s+\beta_2}\right) \end{bmatrix}.$$

To find the pole locations for the allpass functions we argue as follows:

$$re^{-i\phi} = x + iy$$

and so

$$\frac{x+iy}{\pm\sqrt{x^2+y^2}} = \left(\frac{j\omega_0 - \alpha}{j\omega_0 + \alpha}\right).$$

This gives

$$(x+iy)(j\omega_0 + \alpha) = \pm\sqrt{x^2+y^2}(j\omega_0 - \alpha)$$

and equating real parts yields

$$\alpha = \frac{y\omega_0}{x \pm \sqrt{x^2+y^2}}$$

in which the sign is selected to ensure $\alpha \geq 0$. If $\alpha_i \geq 0$ and $\beta_i \geq 0$, $\boldsymbol{A}$ will be stable.

These ideas are implemented in the follow MATLAB[1] code.

```
%
% Enter the transfer function and find a state-space model for it
%
d=[1 3 2];
num=[0 -47 2;0 -42 0;0 56 0; 0 50 2];
[a,b,c,d]=tfm2ss(num,d,2,2);
%
% Find the frequency response singular values of (I - GK)
%
w=logspace(-2,3,100);
[sv]=sigma(a,b,c,eye(2)+d,1,w);
svp=log10(sv(2,:));
%
% Find K(I - GK)^-1
%
[af,bf,cf,df]=feedbk(a,b,c,d,2);
```

[1]MATLAB is a registered trademark of The MathWorks, Inc.

```
%
% Find the frequency response singular values of K(I − GK)^−1
%
w=logspace(-2,3,100);
[sv]=sigma(af,bf,cf,df,1,w);
svp2=-log10(sv(1,:));
semilogx(w,svp,w,svp2)
grid
pause
%
% Find the singular values and singular vectors of K(I − GK)^−1(j3)
% with K = −I
%
wp=3
ac=a-b*inv(eye(2)+d)*c;
bc=b*inv(eye(2)+d);
cc=inv(eye(2)+d)*c;
dd=-inv(eye(2)+d);
g=dd+cc*inv(j*wp*eye(4)-ac)*bc;
[u,s,v]=svd(g);
zz=u';
u1=zz(1,:);
v1=v(:,1);
s2=1/s(1,1);
%
% Find the constant for the first allpass function
%
x=real(u1(1,1));
y=imag(u1(1,1));
%
% Select the sign of r1 so that β1 is positive
%
r1=-abs(u1(1,1))
alp1=y*wp/(x+r1)
pause
%
% Find the constant for the second allpass function
%
x=real(u1(1,2));
y=imag(u1(1,2));
%
% Select the sign of r2 so that β2 is positive
%
r2=-abs(u1(1,2))
```

```
alp2=y*wp/(x+r2)
pause
%
% Assemble the first part of the perturbation
%
aper=[-alp1 0;0 -alp2];
bper=[-2*alp1*r1*s2 0 ;0 -2*alp2*r2*s2];
cper=[1 1];
dper=[s2*r1 s2*r2];
%
% Find the constant for the third allpass function
%
x=real(v1(1,1));
y=imag(v1(1,1));
%
% Select the sign of r3 so that α3 is positive
%
r3=abs(v1(1,1))
alp3=y*wp/(x+r3)
pause
%
% Find the constant for the forth allpass function
%
x=real(v1(2,1));
y=imag(v1(2,1));
%
% Select the sign of r3 so that α3 is positive
%
r4=-abs(v1(2,1))
alp4=y*wp/(x+r4)
pause
%
% Assemble the second part of the perturbation
%
aper1=[-alp3 0;0 -alp4];
bper1=[1;1];
cper1=[-2*alp3*r3 0; 0 -2*alp4*r4];
dper1=[r3; r4];
%
% Assemble the full perturbation
%
adel=[aper1 bper1*cper;zeros(2,2) aper];
bdel=[bper1*dper;bper];
cdel=[cper1 dper1*cper];
```

```
ddel=dper1*dper;
%
% Plot the perturbation's frequency response to check that it is allpass
%
[sv]=sigma(adel,bdel,cdel,ddel,1,w);
loglog(w,sv(1,:))
grid
pause
%
% Check results by assembling and plotting I - AK(I - GK)^-1
%
at=a-b*inv(eye(2)+d)*c;
bt=b*inv(eye(2)+d);
ct=inv(eye(2)+d)*c;
dt=-inv(eye(2)+d);
w=logspace(0,1,400);
[ae,be,ce,de]=series(at,bt,ct,dt,adel,bdel,cdel,ddel);
[sv]=sigma(ae,be,-ce,eye(2)-de,1,w);
loglog(w,sv)
grid
%
% As a last check check, find the poles of the perturbed closed
% loop system
%
[A,B,C,D]=addss(a,b,c,d,adel,bdel,cdel,ddel);
eig(A-B*inv(eye(2)+D)*C)
```

These are:

```
  0.0000
  0.0000 + 3.0000i
  0.0000 - 3.0000i
 -2.0000
 -1.0000
 -0.0587
 -0.0616
 -0.0382
```

You will note, that in this case, the perturbation has an unobservable mode at the origin which is finding its way into the closed loop pole set.

Solution 2.12. It is immediate from Figure 2.13 that

$$y = \boldsymbol{G}(I - \boldsymbol{K}\boldsymbol{G})^{-1}r,$$

so

$$r - y = (I - \boldsymbol{GK})^{-1}(I - \boldsymbol{G}(\boldsymbol{K} + \boldsymbol{R}))r.$$

We can thus argue

$$\begin{aligned}
\overline{\sigma}\big((I - \boldsymbol{GK})^{-1}(I - \boldsymbol{G}(\boldsymbol{K} + \boldsymbol{R}))\big) &\leq \gamma \\
\Leftarrow \frac{\overline{\sigma}\big(I - \boldsymbol{G}(\boldsymbol{K} + \boldsymbol{R})\big)}{\underline{\sigma}(I - \boldsymbol{GK})} &\leq \gamma \\
\Leftarrow \frac{\overline{\sigma}\big(I - \boldsymbol{G}(\boldsymbol{K} + \boldsymbol{R})\big)}{\underline{\sigma}(\boldsymbol{GK} - 1} &\leq \gamma \\
\Leftrightarrow 1 + \frac{\overline{\sigma}(I - \boldsymbol{G}(\boldsymbol{K} + \boldsymbol{R}))}{\gamma} &\leq \underline{\sigma}(\boldsymbol{GK}).
\end{aligned}$$

Solution 2.13. Suppose that

$$\begin{aligned}
\rho\overline{\sigma}(\boldsymbol{S}) &< 1 - \delta\overline{\sigma}(\boldsymbol{GKS}) \\
\Rightarrow \rho\overline{\sigma}(\boldsymbol{S}) &< 1 - \overline{\sigma}(\boldsymbol{\Delta GKS}) \\
\Rightarrow \rho\overline{\sigma}(\boldsymbol{S}) &< \underline{\sigma}(I - \boldsymbol{\Delta GKS}) \\
\Leftrightarrow \rho\overline{\sigma}(\boldsymbol{S})\overline{\sigma}\big((I - \boldsymbol{\Delta GKS})^{-1}\big) &< 1 \\
\Rightarrow \rho\overline{\sigma}\big(\boldsymbol{S}(I - \boldsymbol{\Delta GKS})^{-1}\big) &< 1.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\rho\overline{\sigma}\big(\boldsymbol{S}(I - \boldsymbol{\Delta GKS})^{-1}\big) &< 1 \\
\Rightarrow \rho\overline{\sigma}(\boldsymbol{S})\underline{\sigma}\big((I - \boldsymbol{\Delta GKS})^{-1}\big) &< 1 \\
\Rightarrow \rho\overline{\sigma}(\boldsymbol{S}) &< \overline{\sigma}(I - \boldsymbol{\Delta GKS}) \\
\Rightarrow \rho\overline{\sigma}(\boldsymbol{S}) &< 1 + \delta\overline{\sigma}(\boldsymbol{GKS}).
\end{aligned}$$

Solution 2.14. It follows from Figure 2.4 that

$$y_c = \boldsymbol{G}_t\boldsymbol{K}(I - \boldsymbol{G}_t\boldsymbol{K})^{-1}r.$$

Therefore

$$\begin{aligned}
r + y_c &= (I + \boldsymbol{G}_t\boldsymbol{K}(I - \boldsymbol{G}_t\boldsymbol{K})^{-1})r \\
&= (I - \boldsymbol{G}_t\boldsymbol{K})^{-1}r \\
&= (I - (I + \Delta_2)^{-1}\boldsymbol{GK})^{-1}r \\
&= (I + \Delta_2 - \boldsymbol{GK})^{-1}(I + \Delta_2)r.
\end{aligned}$$

This means that

$$
\begin{aligned}
\overline{\sigma}\big((I-\boldsymbol{G}_t\boldsymbol{K})^{-1}\big) &\le \rho(j\omega)\\
\Leftrightarrow \overline{\sigma}\big((I+\Delta_2-\boldsymbol{GK})^{-1}(I+\Delta_2)\big) &\le \rho(j\omega)\\
\Leftarrow \overline{\sigma}\big((I+\Delta_2-\boldsymbol{GK})^{-1}\big)\overline{\sigma}(I+\Delta_2) &\le \rho(j\omega)\\
\Leftrightarrow \frac{\overline{\sigma}(I+\Delta_2)}{\underline{\sigma}(I+\Delta_2-\boldsymbol{GK})} &\le \rho(j\omega)\\
\Leftarrow \frac{1+\delta(j\omega)}{\underline{\sigma}(I-\boldsymbol{GK})-\delta(j\omega)} &\le \rho(j\omega)\\
\Leftrightarrow \overline{\sigma}(\boldsymbol{S}) &\le \frac{\rho(j\omega)}{1+\delta(j\omega)(1+\rho(j\omega))}
\end{aligned}
$$

Solution 2.15. The solution to this problem is similar to the previous one and we will therefore simply present an annotated working MATLAB code.

```
%
% Enter the batch reactor model...
%
a=[ 1.3800 -0.2077 6.7150 -5.6760;
-0.5814 -4.2900 0 0.6750;
1.0670 4.2730 -6.6540 5.8930;
0.0480 4.2730 1.3430 -2.1040]
b=[ 0.0 0.0 ;
5.6790 0;
1.1360 -3.1460;
1.1360 0]
c=[1 0 1 -1;
0 1 0 0]
d=[ 0.0 0.0;
0.0 0.0]
%
% and now the controller
%
ac=[0 0;0 0];
bc=[1 0;0 1];
cc=[0 2;-8 0];
dc=[0 2;-5 0];
%
% Evaluate the frequency response of 1/σ̄(GK(I − GK)^{-1}(jω))
%
w=logspace(-2,3,100);
```

```
[A,B,C,D]=series(ac,bc,cc,dc,a,b,c,d);
[af,bf,cf,df]=feedbk(A,B,C,D,2);
[sv]=sigma(af,bf,cf,df,1,w);
svp1=-log10(sv(1,:));
semilogx(w,svp1)
grid
pause
%
% Find GK(I - GK)^-1(j2.5)
%
wp=2.5
g=df+cf*inv(j*wp*eye(6)-af)*bf;
[u,s,v]=svd(g);
u1=u(:,1)
v1=v(:,1)
s2=1/s(1,1);
%
% Evaluate the first pair of allpass function constants
%
x=real(v1(1,1));
y=imag(v1(1,1));
r1=abs(v1(1,1))
alp1=wp*y/(r1+x)
%
x=real(v1(2,1));
y=imag(v1(2,1));
r2=abs(v1(2,1))
alp2=wp*y/(r2+x)
%
% Assemble the first part of the perturbation
%
aper=[-alp1 0;0 -alp2];
bper=[1;1];
cper=[-2*alp1*r1*s2 0;0 -2*alp2*r2*s2];
dper=[s2*r1;s2*r2];
%
% Evaluate the second pair of allpass function constants
%
x=real(u1(1,1));
y=-imag(u1(1,1));
r3=abs(u1(1,1))
alp3=wp*y/(r3+x)
%
x=real(u1(2,1));
```

```
y=-imag(u1(2,1));
r4=-abs(u1(2,1))
alp4=wp*y/(r4+x)
%
% Assemble the second part of the perturbation
%
aper1=[-alp3 0;0 -alp4];
bper1=[-2*alp3*r3 0;0 -2*alp4*r4];
cper1=[1 1];
dper1=[r3 r4];
%
% Put the whole perturbation together
%
adel=[aper bper*cper1;zeros(2,2) aper1];
bdel=[bper*dper1;bper1];
cdel=[cper dper*cper1];
ddel=dper*dper1;
%
% Plot the frequency response of the perturbation to check that
% it is allpass
%
w=logspace(0,1,400);
[sv]=sigma(adel,bdel,cdel,ddel,1,w);
loglog(w,sv(1,:))
grid
pause
%
% Assemble and plot I − ΔGK(I − GK)^{-1} and check that it is
% singular at ω = 2.5
%
[ae,be,ce,de]=series(af,bf,cf,df,adel,bdel,cdel,ddel);
[sv]=sigma(ae,be,-ce,eye(2)-de,1,w);
loglog(w,sv)
grid
%
% As a last check, find the closed loop poles of the perturbed system
%
ach11=af+bf*inv(eye(2)-ddel*df)*ddel*cf;
ach12=bf*inv(eye(2)-ddel*df)*cdel;
ach21=bdel*inv(eye(2)-df*ddel)*cf;
ach22= adel+bdel*inv(eye(2)-df*ddel)*df*cdel;
ach=[ach11 ach12; ach21 ach22];
eig(ach)
```

These are:

```
  0.0
-14.6051
-10.8740
 -0.0000 + 2.5000i
 -0.0000 - 2.5000i
 -3.2794
 -2.3139
 -0.6345 + 0.3534i
 -0.6345 - 0.3534i
 -0.7910
```

Solutions to Problems in Chapter 3

Solution 3.1.

1. For any finite $T > 0$ and any $0 < \epsilon < T$,

$$\begin{aligned}\int_\epsilon^T |f(t)|^2\,dt &= \int_\epsilon^T t^{2\alpha}\,dt \\ &= \begin{cases} \frac{1}{2\alpha+1}(T^{2\alpha+1} - \epsilon^{2\alpha+1}) & \text{for} \quad \alpha \neq -\frac{1}{2} \\ \log(T/\epsilon) & \text{for} \quad \alpha = -\frac{1}{2}. \end{cases}\end{aligned}$$

   If $\alpha > -\frac{1}{2}$, then $\|f\|_{2,[0,T]} = \frac{T^{2\alpha+1}}{2\alpha+1} < \infty$ for any finite T. If $\alpha \leq -\frac{1}{2}$, then f is not in $\mathcal{L}_2[0,T]$ for any T.

2. For any finite $T > 0$

$$\begin{aligned}\int_0^T |g(t)|^2\,dt &= \int_0^T (t+1)^{2\alpha}\,dt \\ &= \begin{cases} \frac{1}{2\alpha+1}((T+1)^{2\alpha+1} - 1) & \text{for} \quad \alpha \neq -\frac{1}{2} \\ \log(T+1) & \text{for} \quad \alpha = -\frac{1}{2}. \end{cases}\end{aligned}$$

   Hence $g \in \mathcal{L}_2[0,T]$ for any finite T, which is to say $g \in \mathcal{L}_{2e}$. The integral $\int_0^T |g(t)|^2\,dt$ remains finite as $T \to \infty$ if and only if $\alpha < -\frac{1}{2}$, so this is a necessary and sufficient condition for $g \in \mathcal{L}_2[0,\infty)$.

Solution 3.2. $XX^{-1} = I$. Therefore

$$\begin{aligned}0 &= \frac{d}{dt}(X(t)X^{-1}(t)) \\ &= \big(\frac{d}{dt}X(t)\big)X^{-1}(t) + X(t)\frac{d}{dt}X^{-1}(t).\end{aligned}$$

The result follows upon multiplying on the left by $X^{-1}(t)$.

Solution 3.3.

1. Let $v \in \mathbb{R}^n$ and consider the differential equation

$$\dot{x}(t) = A(t)x(t) \qquad x(t_1) = v.$$

The unique solution is $x(t) = \Phi(t, t_1)v$, for all t. Choose any real τ and consider the differential equation

$$\dot{y}(t) = A(t)y(t) \qquad y(\tau) = \Phi(\tau, t_1)v,$$

which has unique solution $y(t) = \Phi(t, \tau)\Phi(\tau, t_1)v$ for all t. Since $y(\tau) = x(\tau)$, it follows from the uniqueness of solutions to linear differential equations that $y(t) = x(t)$ for all t. Therefore $\Phi(t, t_1)v = \Phi(t, \tau)\Phi(\tau, t_1)v$ for all t, τ, t_1 and all v. Consequently, $\Phi(t_2, t_1) = \Phi(t_2, \tau)\Phi(\tau, t_1)$ for all t_2, τ, t_1.

2. From Item 1, $\Phi(\tau, t)\Phi(t, \tau) = \Phi(\tau, \tau) = I$. Hence $\Phi^{-1}(t, \tau) = \Phi(\tau, t)$.

3.

$$\begin{aligned}
\frac{d}{d\tau}\Phi(t, \tau) &= \frac{d}{d\tau}\Phi^{-1}(\tau, t) \\
&= -\Phi^{-1}(\tau, t)[\frac{d}{d\tau}\Phi(\tau, t)]\Phi^{-1}(\tau, t) \\
&= -\Phi^{-1}(\tau, t)A(\tau)\Phi(\tau, t)\Phi^{-1}(\tau, t) \\
&= -\Phi^{-1}(\tau, t)A(\tau) \\
&= -\Phi(t, \tau)A(\tau).
\end{aligned}$$

Solution 3.4.

1.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\alpha + j\omega)^* f(\alpha + j\omega)\, d\omega &= \int_{-\infty}^{\infty} \frac{1}{(\alpha - a)^2 + \omega^2}\, d\omega \\
&= \frac{1}{\alpha - a}\left[\tan^{-1}(\frac{\omega}{\alpha - a})\right]_{-\infty}^{\infty} \\
&= \frac{\pi}{\alpha - a}.
\end{aligned}$$

Alternatively,

$$\int_{-\infty}^{\infty} f(\alpha + j\omega)^* f(\alpha + j\omega)\, d\omega$$

$$\begin{aligned} &= \frac{1}{2(\alpha-a)}\int_{-\infty}^{\infty}\frac{1}{j\omega+(\alpha-a)}-\frac{1}{j\omega-(\alpha-a)}d\omega \\ &= \frac{1}{2(\alpha-a)j}\int_{D_R}\frac{1}{s-(\alpha-a)}-\frac{1}{s+(\alpha-a)}ds \\ &= \frac{\pi}{\alpha-a} \text{ by Cauchy's integral formula.} \end{aligned}$$

(The contour D_R is a standard semicircular contour in the right-half plane of radius $R > \alpha - a$ and is traversed in an anticlockwise direction.) The result follows, since $\frac{1}{\alpha-a}$ is maximized by setting $\alpha = 0$.

2. f satisfies

$$\begin{aligned} \dot{x}(t) &= ax(t) \qquad x(0) = 1 \\ f(t) &= x(t). \end{aligned}$$

Since $a < 0$, $f \in \mathcal{L}_2[0,\infty)$. Furthermore, the observability gramian q, which is the solution to

$$2aq + 1 = 0,$$

is given by $q = \frac{1}{-2a}$.

Solution 3.5.

1. Choose an arbitrary $x_0 \in \mathbb{R}^n$ and let $x(t)$ be the solution to

$$\begin{aligned} \dot{x}(t) &= Ax \qquad x(0) = x_0 \\ z(t) &= Cx. \end{aligned}$$

Noting that $\lim_{t\to\infty} x(t) = 0$ we obtain

$$\begin{aligned} x_0 Q x_0 &= -\int_0^{\infty}\frac{d}{dt}(x'Qx)\,dt \\ &= -\int_0^{\infty} x'A'Qx + x'QAx\,dt \\ &= \int_0^{\infty} z'z\,dt \\ &\geq 0. \end{aligned}$$

2. Let $Ax = \lambda x$. Then

$$\begin{aligned} 0 &= x^*(QA + A'Q + C'C)x \\ &= (\lambda+\bar{\lambda})x'Qx + \|Cx\|^2. \end{aligned}$$

Since $x'Qx \geq 0$, it follows that either (a) $\|Cx\| = 0$ or (b) $\lambda + \bar{\lambda} < 0$. That is, x is either asymptotically stable or is unobservable.

Solution 3.6.

1. Let $z_i = \boldsymbol{G}_i w$. Then $z = z_1 + z_2$ is the output of $\boldsymbol{G}_1 + \boldsymbol{G}_2$ and

$$z = C_1x_1 + C_2x_2 + (D_1 + D_2)w.$$

2. The input to $\boldsymbol{G}_2$ is $z_1 = C_1x_1 + D_1w$. Therefore

$$\begin{aligned}\dot{x}_2 &= A_2x_2 + B_2(C_1x_1 + D_1w)\\ &= B_2C_1x_1 + A_2x_2 + B_2D_1w\end{aligned}$$

and the output of G_2 is

$$\begin{aligned}z &= C_2x_2 + D_2(C_1x_1 + D_1w)\\ &= D_2C_1x_1 + C_2x_2 + D_2D_1w.\end{aligned}$$

3.
$$\begin{bmatrix}\dot{x}_1\\ w_1\end{bmatrix} = \begin{bmatrix}A_1 & B_1\\ 0 & I\end{bmatrix}\begin{bmatrix}x_1\\ w_1\end{bmatrix}; \quad \begin{bmatrix}x_1\\ y_1\end{bmatrix} = \begin{bmatrix}I & 0\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}x_1\\ w_1\end{bmatrix}.$$

Hence
$$\begin{bmatrix}\dot{x}_1\\ w_1\end{bmatrix} = \begin{bmatrix}A_1 & B_1\\ 0 & I\end{bmatrix}\begin{bmatrix}I & 0\\ C_1 & D_1\end{bmatrix}^{-1}\begin{bmatrix}x_1\\ y_1\end{bmatrix}.$$

4.
$$\begin{bmatrix}\dot{x}\\ z_1\\ z_2\end{bmatrix} = \begin{bmatrix}A & B\\ C_1 & D_1\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}$$

Hence
$$\begin{bmatrix}x\\ z_1\end{bmatrix} = \begin{bmatrix}I & 0\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}$$

and
$$\begin{bmatrix}\dot{x}\\ z_2\end{bmatrix} = \begin{bmatrix}A & B\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}I & 0\\ C_1 & D_1\end{bmatrix}^{-1}\begin{bmatrix}x\\ z_1\end{bmatrix}.$$

A more pedestrian approach:

$$\boldsymbol{G}_1^{-1} \stackrel{s}{=} \left[\begin{array}{c|c}A - BD_1^{-1}C_1 & BD_1^{-1}\\ \hline -D_1^{-1}C_1 & D_1^{-1}\end{array}\right].$$

Hence, by the series formula,

$$\boldsymbol{G}_2\boldsymbol{G}_1^{-1} \stackrel{s}{=} \left[\begin{array}{cc|c}A - BD_1^{-1}C_1 & 0 & BD_1^{-1}\\ -BD_1^{-1}C_1 & A & BD_1^{-1}\\ \hline -D_2D_1^{-1}C_1 & C_2 & D_2D_1^{-1}\end{array}\right].$$

Now apply the state transformation $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$ to obtain

$$\begin{aligned} \boldsymbol{G}_2\boldsymbol{G}_1^{-1} &\stackrel{s}{=} \left[\begin{array}{cc|c} A - BD_1^{-1}C_1 & 0 & BD_1^{-1} \\ 0 & A & 0 \\ \hline C_2 - D_2D_1^{-1}C_1 & C_2 & D_2D_1^{-1} \end{array}\right] \\ &\stackrel{s}{=} \left[\begin{array}{c|c} A - BD_1^{-1}C_1 & BD_1^{-1} \\ \hline C_2 - D_2D_1^{-1}C_1 & D_2D_1^{-1} \end{array}\right]. \end{aligned}$$

The final step follows since the states associated with A are uncontrollable.

Solution 3.7.

1. Since any state-space system has finite $\mathcal{L}_2[0,T]$ induced norm, we may set $\epsilon_2 = \|\boldsymbol{G}\|_{[0,T]} < \infty$. Since $\boldsymbol{G}^{-1}$ has realization $(A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$, it too has finite $\mathcal{L}_2[0,T]$ induced norm as we may take $\epsilon_1 = 1/\|\boldsymbol{G}^{-1}\|_{[0,T]}$.

2. Take $\epsilon_2 = \|\boldsymbol{G}\|_\infty$ and $\epsilon_1 = 1/\|\boldsymbol{G}^{-1}\|_\infty$.

Solution 3.8.

1.

$$\begin{aligned} &\boldsymbol{G}^*(s)\boldsymbol{G}(s) \\ &= \big(D' + B'(\bar{s}I - A')^{-1}C'\big)\big(D + C(sI - A)^{-1}B\big) \\ &= D'D + D'C(sI - A)^{-1}B + B'(\bar{s}I - A')^{-1}C'D \\ &\quad +B'(\bar{s}I - A')^{-1}C'C(sI - A)^{-1}B \\ &= I - B'Q(sI - A)^{-1}B - B'(\bar{s}I - A')^{-1}QB \\ &\quad +B'(\bar{s}I - A')^{-1}C'C(sI - A)^{-1}B \\ &= I + B'(\bar{s}I - A')^{-1}\left(C'C - (\bar{s}I - A')Q - Q(sI - A)\right)(sI - A)^{-1}B \\ &= I - (s + \bar{s})B'(\bar{s}I - A')^{-1}Q(sI - A)^{-1}B. \end{aligned}$$

The conclusion that $\boldsymbol{G}^*(s)\boldsymbol{G}(s) \leq I$ if $Q \geq 0$ and $s + \bar{s} \geq 0$ is immediate.

2. Since $D'D = I$, there exists a matrix D_e such that $\begin{bmatrix} D & D_e \end{bmatrix}$ is a square orthogonal matrix—the columns of D_e are an orthonormal basis for the orthogonal complement of the range of D. To show that $B_e = -Q^{\#}C'D_e$, in which $Q^{\#}$ denotes the Moore-Penrose pseudo inverse, satisfies $D_e'C + B_e'Q = 0$, we need to show that $\ker C \subset \ker Q$. Let $Qx = 0$. Then $0 = x'(QA + A'Q + C'C)x = \|Cx\|^2$, giving $Cx = 0$.

Solution 3.9. Suppose, without loss of generality, that

$$\begin{aligned} A &= \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \\ C &= \begin{bmatrix} C_1 & C_2 & 0 \end{bmatrix} \end{aligned}$$

in which A_1 and $-A_2$ are asymptotically stable and A_3 has only imaginary axis eigenvalues. Let

$$\begin{aligned} 0 &= Q_1 A_1 + A_1 Q_1 + C_1' C_1 \\ 0 &= Q_2(-A_2) + (-A_1) Q_2 + C_2' C_2 \end{aligned}$$

which exist by Theorem 3.1.1. Set

$$Q = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & -Q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution 3.10.

1. Consider $f(x) = |x|$, which is not differentiable at the origin. Then $|(|x_1| - |x_2|)| \leq |x_1 - x_2|$. It follows that $\gamma(\boldsymbol{f}) = 1$.

2. The inequality
$$\left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right| \leq \gamma(\boldsymbol{f})$$
shows that $|\frac{df}{dx}| \leq \gamma(\boldsymbol{f})$. On the other hand, $f(x_2) = f(x_1) + (x_2 - x_1)\frac{df}{dx}|_{x=x_1} + 0((x_2 - x_1)^2)$ shows that $\sup_x |\frac{df}{dx}| = \gamma(\boldsymbol{f})$.

   If f is differentiable except at isolated points x_i then $\sup_{x \neq x_i} |\frac{df}{dx}| = \gamma(\boldsymbol{f})$.

3. See Figure 3.1

Solution 3.11. Notice that

$$(XY')_{ii} = \sum_j x_{ij} y_{ij}.$$

Therefore,

$$\text{trace}(XY') = \sum_{i,j} x_{ij} y_{ij}.$$

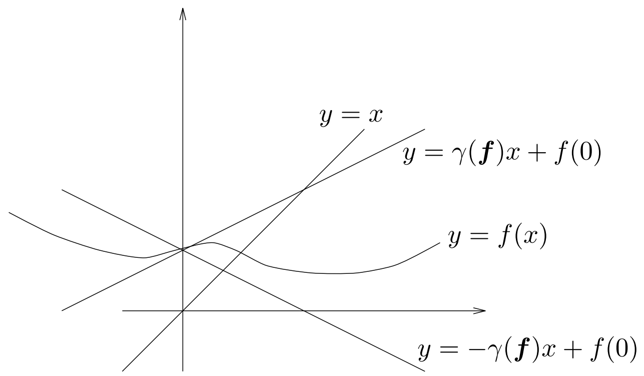


Figure 3.1: Illustration showing that $\gamma(\boldsymbol{f}) < 1$ implies $x = \boldsymbol{f}x$ has a solution.

1.
$$\begin{aligned}\text{trace}(XY') &= \sum_{i,j} x_{ij}y_{ij}\\ &= \sum_{i,j} y_{ij}x_{ij}\\ &= \text{trace}(YX').\end{aligned}$$

2. Follows from the above.

3. Let $\|X\| = \sqrt{\text{trace}(XX')}$.

   (a) $\sqrt{\text{trace}(XX')} \geq 0$ is obvious, and $\sqrt{\text{trace}(XX')} = 0 \Leftrightarrow x_{i,j} = 0$ for all i, j. That is $\sqrt{\text{trace}(XX')} = 0 \Rightarrow X = 0$.

   (b) $\sqrt{\text{trace}(\alpha X \alpha X')} = \sqrt{\sum_{i,j} \alpha^2 x_{ij}} = |\alpha|\sqrt{\text{trace}(XX')}$.

   (c)
$$\begin{aligned}&\text{trace}\big((X+Y)(X+Y)'\big)\\ &= \sum_{i,j}(x_{ij}+y_{ij})^2\\ &= \sum_{i,j} x_{ij}^2 + 2x_{ij}y_{ij} + y_{ij}^2\\ &= \text{trace}(XX') + \text{trace}(YY') + 2\sum_{i,j} x_{ij}y_{ij}\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \operatorname{trace}(XX') + \operatorname{trace}(YY') + 2\sqrt{\sum_{i,j} x_{ij}^2}\sqrt{\sum_{i,j} y_{ij}^2} \\
&\qquad \text{by Cauchy Schwartz} \\
&= \quad (\sqrt{\operatorname{trace}(XX')} + \sqrt{\operatorname{trace}(YY')})^2.
\end{aligned}
$$

Solution 3.12. Consider $\boldsymbol{g} = \frac{1}{s-a_1}$ and $\boldsymbol{h} = \frac{1}{s-a_2}$, with $a_i < 0$. Then

$$
\|\boldsymbol{g}\|_2 \|\boldsymbol{h}\|_2 = \frac{1}{2\sqrt{a_1 a_2}}.
$$

Also,

$$
\boldsymbol{hg} \stackrel{s}{=} \left[\begin{array}{cc|c} a_1 & 0 & 1 \\ 1 & a_2 & 0 \\ \hline 0 & 1 & 0 \end{array}\right].
$$

The observability gramian of this realization is given by

$$
Q = \begin{bmatrix} \frac{1}{-2a_1 a_2 (a_1+a_2)} & \frac{1}{2a_2(a_1+a_2)} \\ \frac{1}{2a_2(a_1+a_2)} & \frac{1}{-2a_2} \end{bmatrix}.
$$

Hence

$$
\|\boldsymbol{hg}\|_2 = \frac{1}{\sqrt{-2a_1 a_2 (a_1 + a_2)}}.
$$

It follows that $\|\boldsymbol{hg}\|_2 > \|\boldsymbol{g}\|_2 \|\boldsymbol{h}\|_2$ for any a_1, a_2 such that $-2 < a_1 + a_2$. For example, choose $a_1 = -1/2$. Then $\|\boldsymbol{g}\|_2^2 = 1$ and $\|\boldsymbol{g}^2\|_2 = \sqrt{2}$.

Solution 3.13.

1.

$$
\begin{aligned}
\|\boldsymbol{GB}\| &= \sup_{w \neq 0} \frac{\|\boldsymbol{GB}w\|_{\mathcal{S}_2}}{\|w\|_{\mathcal{S}_0}} \\
&= \sup_{z \neq 0} \frac{\|\boldsymbol{G}z\|_{\mathcal{S}_2}}{\|z\|_{\mathcal{S}_1}} \\
&= \|\boldsymbol{G}\|.
\end{aligned}
$$

2. Take the infinite horizon 2-norm

$$
\begin{aligned}
\|\boldsymbol{AG}\|_2 &= \int_{-\infty}^{\infty} \operatorname{trace}\big(\boldsymbol{G}(j\omega)^* \boldsymbol{A}(j\omega)^* \boldsymbol{A}(j\omega) \boldsymbol{G}(j\omega)\big)\, d\omega \\
&= \int_{-\infty}^{\infty} \operatorname{trace}\big(\boldsymbol{G}(j\omega)^* \boldsymbol{G}(j\omega)\big)\, d\omega \\
&= \|\boldsymbol{G}\|_2
\end{aligned}
$$

since $\boldsymbol{A}(j\omega)^* \boldsymbol{A}(j\omega) = I$.

Solution 3.14.

Sufficiency Suppose $\boldsymbol{Z} \in \mathcal{H}_\infty$ is strictly positive real. The condition $\boldsymbol{Z} \in \mathcal{H}_\infty$ implies that $\boldsymbol{Z}$ has finite incremental gain. Equation (3.9.1) gives

$$\inf_{\sigma_0>0} \underline{\sigma}\big(\boldsymbol{Z}(\sigma_0 + j\omega_0) + \boldsymbol{Z}^*(\sigma_o + j\omega_0)\big) \geq 2\epsilon,$$

which implies that $\boldsymbol{Z}(j\omega) + \boldsymbol{Z}^*(j\omega) \geq 2\epsilon I$ for all real ω.

Suppose the system is relaxed at time t_0, let w be *any* signal in $\mathcal{L}_2[t_0, T]$ and let $z = \boldsymbol{Z}w$. Define the $\mathcal{L}_2(-\infty, \infty)$ signals w_e and z_e by

$$w_e(t) = \begin{cases} w(t) & \text{for } t \in [t_0, T] \\ 0 & \text{otherwise} \end{cases}$$

and

$$z_e(t) = \begin{cases} z(t) & \text{for } t \geq t_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\langle z, w\rangle_{[t_0,T]} &= \int_{t_0}^{T} w'(t)z(t)\,dt \\
&= \int_{-\infty}^{\infty} w_e'(t)z_e(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{w}_e^*(j\omega)\widehat{z}_e(j\omega)\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{w}_e^*(j\omega)\boldsymbol{Z}(j\omega)\widehat{w}_e(j\omega)\,d\omega \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} \widehat{w}_e^*(j\omega)[\boldsymbol{Z}(j\omega) + \boldsymbol{Z}^*(j\omega)]\widehat{w}_e(j\omega)\,d\omega \\
&\geq \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} \widehat{w}_e^*(j\omega)\widehat{w}_e(j\omega)\,d\omega \\
&= \epsilon\int_{-\infty}^{\infty} w_e'(t)w_e(t)\,dt \\
&= \epsilon\int_{t_0}^{T} w'(t)w(t)\,dt.
\end{aligned}$$

In the above, $\widehat{w}_e$ and $\widehat{z}_e$ denote the Fourier transforms of w_e and z_e. Consequently, $\boldsymbol{Z}$ defines an incrementally strictly passive system.

Necessity Suppose the system defined by $\boldsymbol{Z}$ has finite incremental gain and is incrementally strictly passive. The finite incremental gain assumption implies that $\boldsymbol{Z} \in \mathcal{H}_\infty$.

Notice that for any complex numbers z and w, $\mathrm{R}_e(z)\mathrm{R}_e(w) = \frac{1}{2}\mathrm{R}_e(\bar{w}z + wz)$. Choose $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 = \mathrm{R}_e(s_0) > 0$ and choose $x \in \mathbb{C}^n$. Consider the input

$$w(t) = \mathrm{R}_e(xe^{s_0 t}1(t - t_0))$$

in which $1(\cdot)$ denotes the unit step function. For $t_0 \to -\infty$, the response to this input is

$$z(t) = \mathrm{R}_e(\boldsymbol{Z}(s_0)xe^{s_0 t}1(t - t_0)).$$

Therefore,

$$w'(t)z(t) = \frac{1}{2}\mathrm{R}_e\left(e^{2\sigma_0 t}x^*\boldsymbol{Z}(s_0)x + e^{2s_0 t}x'\boldsymbol{Z}(s_0)x\right).$$

Integrating from $-\infty$ to some finite time T we have

$$\int_{-\infty}^{T} w'(t)z(t)\,dt = \frac{1}{2}\mathrm{R}_e\left(\frac{1}{2\sigma_0}e^{2\sigma_0 T}x^*\boldsymbol{Z}(s_0)x + \frac{1}{2s_0}e^{2s_0 T}x'\boldsymbol{Z}(s_0)x\right).$$

Also,

$$\epsilon\int_{-\infty}^{T} w'(t)w(t)\,dt = \frac{\epsilon}{2}\mathrm{R}_e\left(\frac{1}{2\sigma_0}e^{2\sigma_0 T}x^*x + \frac{1}{2s_0}e^{2s_0 T}x'x\right).$$

Suppose $\epsilon > 0$ is such that

$$\langle z, w\rangle_{[-\infty,T]} \geq \epsilon\|w\|_{2,[-\infty,T]}$$

for all finite T (such an ϵ exists by the assumption that $\boldsymbol{Z}$ is incrementally strictly passive). Then

$$\mathrm{R}_e\left(x^*\boldsymbol{Z}(s_0)x - \epsilon x^*x\right) \geq \frac{2\sigma_0}{e^{2\sigma_0 T}}\mathrm{R}_e\left(\frac{e^{2s_0 T}}{2s_0}(x'\boldsymbol{Z}(s_0)x - x'x)\right). \tag{3.1}$$

If $w_0 = 0$ (i.e., s_0 is real), we choose x real and obtain $x'\boldsymbol{Z}(s_0)x - \epsilon x'x \geq 0$. Since x may be any real vector and since $\boldsymbol{Z}^*(s_0) = \boldsymbol{Z}'(s_0)$ for $s_0 \in \mathbb{R}$, we conclude that $\boldsymbol{Z}(s_0) + \boldsymbol{Z}^*(s_0) \geq 2\epsilon I$ for all real $s_0 > 0$.

If $\omega \neq 0$, we notice that $\arg e^{2s_0 T} = 2\omega_0 T$ takes all values between 0 and 2π as T varies. This implies that the right-hand side of (3.1) is nonnegative for some values of T (which will depend on the choice of s_0 and x). Because the left-hand side of (3.1) is independent of T, we conclude that $\mathrm{R}_e\left(x^*\boldsymbol{Z}(s_0)x - \epsilon x^*x\right) \geq 0$. Consequently,

$$\boldsymbol{Z}(s_0) + \boldsymbol{Z}^*(s_0) \geq 2\epsilon I$$

for all s_0 such that $\mathrm{R}_e(s_0) > 0$.

Solution 3.15. Since $\boldsymbol{Z} \in \mathcal{H}_\infty$, the complex function $f(s) = v^*\boldsymbol{Z}(s)v$ is also in $\mathcal{H}_\infty$ for any (complex) vector v. Also,

$$g(s) = e^{-f(s)}$$

is analytic and nonzero in the closed-right-half plane. It follows from the maximum modulus principle that $\max_{\mathrm{R}_e(s)\geq 0} |g(s)| = \max_\omega |g(j\omega)|$. Now note that

$$|g(s)| = e^{-\mathrm{R}_e\left(f(s)\right)}.$$

Therefore, $\min_{\mathrm{R}_e(s)\geq 0} \mathrm{R}_e\big(f(s)\big) = \min_\omega \mathrm{R}_e\big(g(j\omega)\big)$. The result follows.

Solution 3.16.

1. The nonsingularity of $\boldsymbol{Z}(s)$ follows from the definition:

$$\boldsymbol{Z}(s_0)v = 0 \Rightarrow v^*\big(\boldsymbol{Z}^*(s_0) + \boldsymbol{Z}(s_0)\big)v = 0$$

   and it follows that $v = 0$.

2. Since $\boldsymbol{Z}$ is strictly positive real, D is nonsingular. The eigenvalues of $A - BD^{-1}C$ are either (a) zeros of $\boldsymbol{Z}$ or (b) unobservable modes of (A, C) or uncontrollable modes of (A, B). Since A is asymptotically stable, the realization (A, B, C, D) has no uncontrollable or unobservable modes in the closed-right-half plane and any eigenvalue of $A - BD^{-1}C$ which is in the closed-right-half plane is a zero of $\boldsymbol{Z}$. Since $\boldsymbol{Z}$ has no zeros in the closed-right-half plane, $A - BD^{-1}C$ is asymptotically stable.

Solution 3.17. Notice that

$$\begin{bmatrix} I + \boldsymbol{G} \\ I - \boldsymbol{G} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & I + D \\ -C & I - D \end{array}\right].$$

Hence

$$\begin{aligned} (I - \boldsymbol{G})(I + \boldsymbol{G})^{-1} &\stackrel{s}{=} \begin{bmatrix} A & B \\ -C & I - D \end{bmatrix} \begin{bmatrix} I & 0 \\ C & I + D \end{bmatrix}^{-1} \\ &\stackrel{s}{=} \begin{bmatrix} A - B(I + D)^{-1}C & B(I + D)^{-1} \\ -C - (I - D)(I + D)^{-1}C & (I - D)(I + D)^{-1} \end{bmatrix} \\ &\stackrel{s}{=} \begin{bmatrix} A - B(I + D)^{-1}C & B(I + D)^{-1} \\ -2(I + D)^{-1}C & (I - D)(I + D)^{-1} \end{bmatrix}. \end{aligned}$$

Solution 3.18. Consider Figure 3.2 and define

$$z_2 = \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} \quad ; \quad w_2 = \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix}.$$

Then

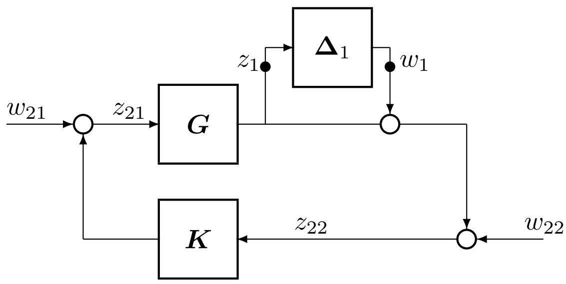


Figure 3.2:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \boldsymbol{P} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

in which

$$\boldsymbol{P} = \left[\begin{array}{c|cc} \boldsymbol{SGK} & \boldsymbol{SG} & \boldsymbol{SGK} \\ \hline \boldsymbol{KS} & (I - \boldsymbol{KG})^{-1} & \boldsymbol{KS} \\ \boldsymbol{S} & \boldsymbol{SG} & \boldsymbol{S} \end{array}\right],$$

with $\boldsymbol{S} = (I - \boldsymbol{GK})^{-1}$. The nominal closed loop is stable if and only if $\boldsymbol{P}$ is stable.

1. The result follows by a direct application of Theorem 3.6.1.

2. By Theorem 3.5.7, the closed loop will be stable provided that $-\boldsymbol{GKS}$ is incrementally strictly passive. By Lemma 3.5.6, this is equivalent to $\gamma\big((I + \boldsymbol{GKS})(I - \boldsymbol{GKS})^{-1}\big) < 1$, which can be simplified to

$$\gamma\big((I - 2\boldsymbol{GK})^{-1}\big) < 1.$$

Solution 3.19.

1. The condition $\min \gamma(\boldsymbol{\Delta})\gamma(\boldsymbol{DGD}^{-1}) < 1$ implies that there exists a $\boldsymbol{D} \in \mathcal{D}$ such that $\gamma(\boldsymbol{\Delta})\gamma(\boldsymbol{DGD}^{-1}) < 1$. Therefore, there exists a $\boldsymbol{D} \in \mathcal{D}$ such that $\gamma(\boldsymbol{D\Delta D}^{-1})\gamma(\boldsymbol{DGD}^{-1}) < 1$, by virtue of the commutative property of $\boldsymbol{D}$. The stability of the closed loop is now immediate from Corollary 3.5.2.

2. For any matrix valued $\boldsymbol{\Delta}_i$, the corresponding block-diagonal entry $\boldsymbol{D}_i$ in $\boldsymbol{D}$ must have the form $\boldsymbol{\alpha} I$, for some scalar transfer function $\boldsymbol{\alpha} \in \mathcal{H}_\infty$ such that $\boldsymbol{\alpha}^{-1} \in \mathcal{H}_\infty$. For any $\boldsymbol{\Delta}_i$ that is of the form $\boldsymbol{\beta} I$, the corresponding block-diagonal entry $\boldsymbol{D}_i$ in $\boldsymbol{D}$ must satisfy $\boldsymbol{D}^{\pm 1} \in \mathcal{H}_\infty$. The other block-entries of $\boldsymbol{D}$ are zero.

Solution 3.20. Firstly note that at least one solution always exists (see Solution 3.9). Also, if Q_1 and Q_2 are any two solutions, then $X = Q_2 - Q_1$ satisfies

$$XA + A'X = 0,$$

which we may write as

$$\begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} A.$$

But

$$\begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} A,$$

so if $\lambda_i(A) \neq \lambda_j(-A')$, which is to say $\lambda_i(A) + \lambda_j(A) \neq 0$, for all i, j, then $X = 0$ by the uniqueness properties of eigenvalue decompositions.

Conversely, if $\lambda_i(A) + \lambda_j(A) = 0$ for some i, j, we have

$$\begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} A$$

and

$$\begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} A, \text{ for some } X \neq 0.$$

Therefore, Q and $Q + X \neq Q$ are two solutions.

To illustrate these nonuniqueness properties, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $XA + A'X = 0$ has the solution set $X = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. As another example, if $A = \begin{bmatrix} 0 & w \\ -\omega & 0 \end{bmatrix}$, then $X = \alpha I$ is a solution.

Solution 3.21.

1. Let $Hx = \lambda x$, $x \neq 0$.

$$\begin{aligned} x^*SH &= x^*(SH)' \quad \text{by the Hamiltonian property} \\ &= x^*H'S' \\ &= \bar{\lambda} x^* S' \\ &= -\bar{\lambda} x^* S \quad \text{since } S' = -S. \end{aligned}$$

Noting that $x^*S \neq 0$, we conclude that $-\bar{\lambda}$ is an eigenvalue of H.

2.

$$\begin{aligned}(X'SX)\Lambda + \Lambda'(X'SX) &= X'SHX + X'H'SX \\ &= X'\big(SH - (SH)'\big)X \quad \text{since } S' = -S \\ &= 0 \quad \text{by the Hamiltonian property.}\end{aligned}$$

We conclude that $X'SX = 0$, because linear matrix equations of the form $Y\Lambda + \Lambda'Y = 0$ in which $\mathrm{R}_e(\lambda_i(\Lambda)) < 0$ have the unique solution $Y = 0$ (see Problem 3.20).

3. From Item 2, $X_1'X_2 = X_2'X_1$. Hence $X_2X_1^{-1} = (X_1')^{-1}X_2'$ and P is symmetric. Also from Item 2, $X'SHX = X'SX\Lambda = 0$. Hence

$$\begin{aligned}0 &= (X_1')^{-1}X'SHXX_1^{-1} \\ &= \begin{bmatrix} P & -I \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} \\ &= PH_{11} - H_{21} + PH_{12}P - H_{22}P \\ &= PH_{11} + H_{11}'P + PH_{12}P - H_{21}.\end{aligned}$$

The final equality is valid because the Hamiltonian property implies that $H_{22} = -H_{11}'$.

Finally,

$$\begin{aligned}H_{11} + H_{12}P &= \begin{bmatrix} I & 0 \end{bmatrix} HXX_1^{-1} \\ &= \begin{bmatrix} I & 0 \end{bmatrix} X\Lambda X_1^{-1} \\ &= X_1\Lambda X_1^{-1}.\end{aligned}$$

Solution 3.22.

1. Since $\Phi(T,T) = I$, it is immediate that $P(T) = \Sigma$.

$$\begin{aligned}-\dot{P} &= -\dot{X}_2X_1^{-1} + P\dot{X}_1X_1^{-1} \\ &= -(\dot{\Phi}_{21} + \dot{\Phi}_{22}\Sigma)X_1^{-1} + P(\dot{\Phi}_{11} + \dot{\Phi}_{12}\Sigma)X_1^{-1} \\ &= \begin{bmatrix} P & -I \end{bmatrix} \dot{\Phi} \begin{bmatrix} I \\ \Sigma \end{bmatrix} X_1^{-1} \\ &= \begin{bmatrix} P & -I \end{bmatrix} H\Phi \begin{bmatrix} I \\ \Sigma \end{bmatrix} X_1^{-1} \\ &= \begin{bmatrix} P & -I \end{bmatrix} H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} X_1^{-1} \\ &= \begin{bmatrix} P & -I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix}.\end{aligned}$$

The result follows upon expansion of the right-hand side and noting that $H_{22} = -H'_{11}$.

2. Matrix addition, multiplication and inversion are continuous operations. $(A+B)_{ij}$ and $(AB)_{ij}$ are continuous functions of the entries of A and B, and $(A^{-1})_{ij}$ is a continuous function of the entries of A, provided A is nonsingular. The result follows from these facts and Item 1.

Solution 3.23.

1. Write the Riccati equation as

$$\Pi A + A'\Pi + \begin{bmatrix} C' & \gamma^{-1}\Pi B \end{bmatrix} \begin{bmatrix} C \\ \gamma^{-1}B'\Pi \end{bmatrix} = 0$$

   Therefore Π is the observability gramian of $(A, \begin{bmatrix} C \\ \gamma^{-1}B'\Pi \end{bmatrix})$. Hence (A, C) observable implies that Π is nonsingular.

2. Write the Riccati equation as

$$\Pi(A + \gamma^{-2}BB'\Pi) = -A'\Pi - C'C.$$

   Suppose that $Ax = \lambda x$ and $Cx = 0$, $x \neq 0$. Note that $\mathrm{R}_e(\lambda) < 0$ since A is asymptotically stable. It follows that

$$x^*\Pi(A + \gamma^{-2}BB'\Pi) = -\bar{\lambda}x^*\Pi.$$

   Since $A + \gamma^{-2}BB'\Pi$ is asymptotically stable and $\mathrm{R}_e(-\bar{\lambda}) > 0$, we conclude that $\Pi x = 0$. Hence the unobservable subspace is contained in $\ker \Pi$.

   Suppose now that M_2 is a basis for $\ker \Pi$. Then $M_2'(3.7.17)M_2$ yields $CM_2 = 0$ and $(3.7.17)M_2$ results in $\Pi A M_2 = 0$. That is, $\ker \Pi$ is an A invariant subspace contained in $\ker C$, which shows that $\ker \Pi$ is a subset of the unobservable subspace. We therefore conclude that $\ker \Pi$ is the unobservable subspace.

3. Follows immediately from Item 2.

Solution 3.24.

1. Let $X(t) = \Pi - P(t)$. Differentiating $X(t)$ and using the two Riccati equations, we obtain

$$\begin{aligned} \dot{X} &= XA + A'X + \gamma^{-2}\Pi BB'\Pi - \gamma^{-2}PBB'P \\ &= X(A + \gamma^{-2}BB'\Pi) + (A + \gamma^{-2}BB'\Pi)'X - \gamma^{-2}XBB'X. \qquad (3.2) \end{aligned}$$

Choose a $t^* \leq T$ and an x such that $X(t^*)x = 0$. Then $x'(3.2)x$ gives $x'\dot{X}(t)x|_{t=t^*} = 0$, which is equivalent to $\dot{X}(t)x|_{t=t^*} = 0$ (since $P(t)$ and hence $X(t)$ are monotonic, which implies $\dot{X}$ is semidefinte). Consequently, $(3.2)x$ gives $X(t^*)(A + \gamma^{-2}BB'\Pi)x = 0$. That is, the kernel of $X(t^*)$ is invariant under multiplication by $A + \gamma^{-2}BB'\Pi$.

Consider the differential equation

$$\dot{x}(t) = \big((A+\gamma^{-2}BB'\Pi)' - \gamma^{-2}XBB'\big)x(t) + X(A+\gamma^{-2}BB'\Pi)x, \quad x(t^*) = 0.$$

One solution is $x_1(t) = X(t)x$. Another solution is $x_2(t) = 0$ for all t. Hence $X(t)x = 0$ for all t by the uniqueness of solutions to differential equations.

2. Let $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ be any nonsingular matrix such that M_2 is a basis for the kernel of Π. Since $P(T) = 0$, M_2 is also a basis for the kernel of $\Pi - P(t)$ for all $t \leq T$ by Item 1. It follows that $\big(\Pi - P(t)\big)M_2 = 0$ and $M_2'[\Pi - P(t)] = 0$ for all $t \leq T$ and that $\Pi_1 - P_1(t) = M_1'\big(\Pi - P(t)\big)M_1$ is nonsingular for all $t \leq T$.

3. The matrix $\Pi - P(t)$ is nonsingular for all $t \leq T$ if and only if Π is nonsingular (this follows from $P(T) = 0$ and Item 1).

   If Π is nonsingular, the assumptions used in the text hold and $A + \gamma^{-2}BB'\Pi$ is asymptotically stable.

   In the case that Π is singular, let M be as in Item 2. By the solution to Problem 3.23 Item 1,

$$\begin{aligned} M^{-1}AM &= \begin{bmatrix} \widehat{A}_{11} & 0 \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix}, \\ CM &= \begin{bmatrix} \widehat{C}_1 & 0 \end{bmatrix}. \end{aligned}$$

   (Note that we cannot, and do not, assume that $(\widehat{A}_{11}, \widehat{C}_1)$ is observable). Furthermore, $M_1'\big(\Pi - P(t)\big)M_1$ is nonsingular and $P_1(t) = M_1'P(t)M_1$ satisfies

$$\dot{P}_1 = \widehat{A}_{11}P_1 + P_1\widehat{A}_{11}' + \gamma^{-1}P_1\widehat{B}_1\widehat{B}_1'P_1 + \widehat{C}_1'\widehat{C}_1 \qquad P_1(T) = 0$$

   with $\lim_{t\to-\infty} P_1(t) = \Pi_1$ in which $\Pi_1 = M_1\Pi M_1$. Applying the argument of the text to this subspace shows that $\widehat{A}_{11} + \gamma^{-2}\widehat{B}_1\widehat{B}_1\Pi_1$ is asymptotically stable. We conclude that $A + \gamma^{-2}BB'\Pi$ is asymptotically stable, since

$$A + \gamma^{-2}BB'\Pi = M \begin{bmatrix} \widehat{A}_{11} + \gamma^{-2}\widehat{B}_1\widehat{B}_1\Pi_1 & 0 \\ \widehat{A}_{21} + \gamma^{-2}\widehat{B}_2\widehat{B}_1\Pi_1 & \widehat{A}_{22} \end{bmatrix} M^{-1}.$$

Solution 3.25. Define

$$\begin{aligned}
\boldsymbol{G} &= (I-\boldsymbol{Z})(I+\boldsymbol{Z})^{-1}\\
&\overset{s}{=} \left[\begin{array}{c|c} A-B(I+D)^{-1}C & B(I+D)^{-1} \\ \hline -2(I+D)^{-1}C & (I-D)(I+D)^{-1}\end{array}\right]\\
&\overset{s}{=} \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & \widehat{D}\end{array}\right].
\end{aligned}$$

Since $\boldsymbol{Z}$ defines an incrementally strictly passive system with finite incremental gain, $\gamma(\boldsymbol{G}) < 1$ by Lemma 3.5.6. This is equivalent to $\|\boldsymbol{G}\|_\infty < 1$ since $\boldsymbol{G}$ is linear, time-invariant and rational. Now verify

$$\begin{aligned}
\widehat{R} &= I-\widehat{D}'\widehat{D}\\
&= I-(I+D')^{-1}(I-D')(I-D)(I+D)^{-1}\\
&= (I+D')^{-1}\big((I+D')(I+D)-(I-D')(I-D)\big)(I+D)^{-1}\\
&= (I+D')^{-1}\big(I+D'+D+D'D-I+D'+D-D'D\big)(I+D)^{-1}\\
&= 2(I+D')^{-1}(D+D')(I+D)^{-1}.
\end{aligned}$$

Condition 1 of the bounded real lemma says that $\widehat{R} > 0$. Therefore $R = D+D' > 0$. Using this identity, we easily obtain

$$\widehat{B}\widehat{R}^{-1}\widehat{B}' = \frac{1}{2}BR^{-1}B'$$

and

$$\begin{aligned}
\widehat{A}+\widehat{B}\widehat{R}^{-1}\widehat{D}'\widehat{C} &= A-B(I+D)^{-1}C-BR^{-1}(I-D')(I+D)^{-1}C\\
&= A-B\big(I+R^{-1}(I-D')\big)(I+D)^{-1}C\\
&= A-BR^{-1}\big(D+D'+I-D'\big)(I+D)^{-1}C\\
&= A-BR^{-1}C.
\end{aligned}$$

Finally, notice that for any matrix X with $I-X'X$ nonsingular,

$$\begin{aligned}
I+X(I-X'X)^{-1}X' &= I+(I-XX')^{-1}XX'\\
&= (I-XX')^{-1}(I-XX'+XX')\\
&= (I-XX')^{-1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
I+\widehat{D}\widehat{R}^{-1}\widehat{D}' &= (I-\widehat{D}\widehat{D}')^{-1}\\
&= \Big((I+D)^{-1}\big((I+D)(I+D')-(I-D)(I-D')\big)(I+D')^{-1}\Big)^{-1}\\
&= \frac{1}{2}(I+D')(D+D')^{-1}(I+D)
\end{aligned}$$

which results in

$$\widehat{C}'(I + \widehat{D}\widehat{R}^{-1}\widehat{D}')\widehat{C} = 2C'R^{-1}C.$$

Condition 2 of the Bounded Real Lemma ensures the existence of a $\widehat{P}$ such that

$$\widehat{P}(A - BR^{-1}C) + (A - BR^{-1}C)'\widehat{P} + \frac{1}{2}\widehat{P}BR^{-1}B'\widehat{P} + 2C'R^{-1}C = 0$$

with $A - BR^{-1}C + \frac{1}{2}BR^{-1}B'\widehat{P}$ asymptotically stable (and $\widehat{P} \geq 0$). We therefore define $P = \frac{1}{2}\widehat{P}$.

Solution 3.26.

1. Suppose $\boldsymbol{\Phi} = \boldsymbol{W}^{\sim}\boldsymbol{W}$. Then $\boldsymbol{\Phi}^{\sim} = \boldsymbol{W}^{\sim}\boldsymbol{W} = \boldsymbol{\Phi}$. Furthermore, $\boldsymbol{W}^{-1} \in \mathcal{RH}_\infty$ implies that $\boldsymbol{\Phi}(j\omega) > 0$.

   Now suppose that $\boldsymbol{\Phi} = \boldsymbol{V}^{\sim}\boldsymbol{V}$. Then

   $$\boldsymbol{W}\boldsymbol{V}^{-1} = (\boldsymbol{W}^{\sim})^{-1}\boldsymbol{V}^{\sim}.$$

   The elements of the left-hand side have no poles in $\mathrm{R}_e s \geq 0$ and the elements of the right-hand side have no poles in $\mathrm{R}_e s \leq 0$. Hence $\boldsymbol{W}\boldsymbol{V}^{-1} = M$, a constant matrix, which satisfies $M = M^*$. We conclude that M is real (hence orthogonal) by noting that $\boldsymbol{W}$ and $\boldsymbol{V}$ are implicitly assumed to be real systems.

2. Since $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\sim}$, the poles of $\boldsymbol{\Phi}$ are symmetric about the imaginary axis and

   $$\boldsymbol{\Phi} = \sum_i \sum_{j=1} \frac{M_{ij}}{(s - p_i)^j} + \sum_i \sum_{j=1} \frac{M_{ij}^*}{(-s - p_i)^j},$$

   in which M_{ij} are complex matrices and $\mathrm{R}_e(p_i) < 0$. Define

   $$\boldsymbol{Z} = \sum_i \sum_{j=1} \frac{M_{ij}}{(s - p_i)^j}.$$

   Since $\boldsymbol{\Phi}$ is real and in $\mathcal{RL}_\infty$, $\boldsymbol{Z}$ is real and in $\mathcal{RH}_\infty$.

   (Alternatively, let $\Phi(t)$ be the inverse Fourier transform of $\boldsymbol{\Phi}$, define

   $$Z(t) = \begin{cases} \Phi(t) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   and let $\boldsymbol{Z}$ be the Fourier transform of Z.)

   Since $\boldsymbol{\Phi}(j\omega) > 0$ for all ω, it follows that

   $$\boldsymbol{Z}(j\omega) + \boldsymbol{Z}^*(j\omega) > 0$$

   for all ω. Consequently, since $\boldsymbol{Z}$ is rational, there is an $\epsilon > 0$ such that

   $$\boldsymbol{Z}(j\omega) + \boldsymbol{Z}^*(j\omega) > 2\epsilon I.$$

   It follows that $\boldsymbol{Z}$ is strictly positive real.

3. The fact that $\boldsymbol{W} \in \mathcal{RH}_\infty$ and $\boldsymbol{W}^{-1} \in \mathcal{RH}_\infty$ follows trivially from the asymptotic stability of the matrices A and $A - BR^{-1}(C - B'P) = A - BW^{-1}L$. Verify that the Riccati equation can be written as

$$PA + A'P + L'L = 0.$$

(This shows that (A, L) is observable if and only if P is nonsingular.) Now verify

$$\begin{aligned}
\boldsymbol{W}^{\sim}\boldsymbol{W} &= W'W + B'(-sI - A')^{-1}L'W + W'L(sI - A)^{-1}B \\
&\quad + B'(-sI - A')^{-1}L'L(sI - A)^{-1}B \\
&= D + D' + B'(-sI - A')^{-1}(C' - PB) + (C - B'P)(sI - A)^{-1}B \\
&\quad + B'(-sI - A')^{-1}L'L(sI - A)^{-1}B \\
&= D + D' + B'(-sI - A')^{-1}C' + C(sI - A)^{-1}B \\
&\quad + B'(-sI - A')^{-1}\big(L'L - P(sI - A) \\
&\quad - (-sI - A')P\big)(sI - A)^{-1}B \\
&= D + D' + B'(-sI - A')^{-1}C' + C(sI - A)^{-1}B \\
&= \boldsymbol{Z} + \boldsymbol{Z}^{\sim}.
\end{aligned}$$

Solutions to Problems in Chapter 4

Solution 4.1.

1(a) The function $\boldsymbol{h} = \gamma\left(\frac{s-1}{s+1}\right)$ maps the imaginary axis $s = j\omega$ into the circle $|\boldsymbol{h}| = \gamma$. We can therefore find $\boldsymbol{w}$ by solving the equation

$$\begin{aligned}
\frac{1}{1-\boldsymbol{w}} &= \gamma\left(\frac{s-1}{s+1}\right) \\
\Rightarrow \boldsymbol{w} &= \left(\frac{s+1}{s-1}\right)\left(\frac{(s-1)-\gamma^{-1}(s+1)}{s+1}\right) \\
&= \left(\frac{s(1-\gamma^{-1})-(1+\gamma^{-1})}{s-1}\right).
\end{aligned}$$

1(b) In this case we need to solve

$$\begin{aligned}
\frac{\boldsymbol{w}}{1-\boldsymbol{w}} &= \gamma\left(\frac{s-1}{s+1}\right) \\
\Rightarrow \boldsymbol{w} &= \gamma\left(\frac{s-1}{s+1}\right)\left(1+\gamma\left(\frac{s-1}{s+1}\right)\right)^{-1} \\
&= \left(\frac{\gamma(s-1)}{s(1+\gamma)+(1-\gamma)}\right).
\end{aligned}$$

2(a) Let

$$\begin{aligned}
\frac{1}{1-\boldsymbol{q}} &= \gamma\left(\frac{s-1}{s+1}\right) \\
\Rightarrow \boldsymbol{q} &= \left(\frac{s(1-\gamma^{-1})-(1+\gamma^{-1})}{s-1}\right) \qquad \text{where } \boldsymbol{q} = \boldsymbol{gk} \\
\Rightarrow \boldsymbol{k} &= s(1-\gamma^{-1})-(1+\gamma^{-1}).
\end{aligned}$$

The Nyquist plot of $\boldsymbol{q}$ cuts the real axis at $1 \pm \gamma^{-1}$. This means that there will be one encirclement of $+1$ for all $\gamma > 0$. In order to make the controller realizable, one could use

$$\boldsymbol{k} = \frac{s(1-\gamma^{-1}) - (1+\gamma^{-1})}{\epsilon s + 1}$$

for arbitrarily small ϵ.

2(b) In this case we solve

$$\frac{\boldsymbol{q}}{1-\boldsymbol{q}} = \gamma \left(\frac{s-1}{s+1} \right)$$

to obtain

$$\boldsymbol{q} = \left(\frac{\gamma(s-1)}{s(1+\gamma) + (1-\gamma)} \right).$$

It is not hard to check that the Nyquist plot of $\boldsymbol{q}$ cuts the real axis at $\frac{\gamma}{1 \pm \gamma}$. We therefore require $\gamma > 1$ for the single required encirclement. The corresponding controller is given by

$$\boldsymbol{k} = \left(\frac{\gamma(s-1)^2}{s(1+\gamma) + (1-\gamma)} \right),$$

or

$$\boldsymbol{k} = \left(\frac{\gamma(s-1)^2}{s(1+\gamma) + (1-\gamma))(1+\epsilon s)} \right)$$

for a proper approximation.

3 Just repeat the calculations of Part (2a) using

$$\frac{1}{1-\boldsymbol{q}} = \gamma \left(\frac{s-1}{s+1} \right)^2.$$

This gives

$$\begin{aligned} \boldsymbol{q} &= \left(\frac{s^2(1-\gamma^{-1}) - 2s(1+\gamma^{-1}) + 1 - \gamma^{-1}}{(s-1)^2} \right) \\ \Rightarrow \boldsymbol{k} &= \left(\frac{s^2(1-\gamma^{-1}) - 2s(1+\gamma^{-1}) + 1 - \gamma^{-1}}{(\epsilon s+1)^2} \right). \end{aligned}$$

Solution 4.2. Since

$$\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K}_1) = \boldsymbol{P}_{11} + \boldsymbol{P}_{12}\boldsymbol{K}_1(I - \boldsymbol{P}_{22}\boldsymbol{K}_1)^{-1}\boldsymbol{P}_{21}$$

and

$$\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K}_2) = \boldsymbol{P}_{11} + \boldsymbol{P}_{12}(I - \boldsymbol{K}_2\boldsymbol{P}_{22})^{-1}\boldsymbol{K}_2\boldsymbol{P}_{21},$$

it follows that

$$\begin{aligned}
&\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K}_1) - \mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K}_2)\\
&\quad= \boldsymbol{P}_{12}(I - \boldsymbol{K}_2\boldsymbol{P}_{22})^{-1}\big((I - \boldsymbol{K}_2\boldsymbol{P}_{22})\boldsymbol{K}_1 - \boldsymbol{K}_2(I - \boldsymbol{P}_{22}\boldsymbol{K}_1)\big)\\
&\qquad\times (I - \boldsymbol{P}_{22}\boldsymbol{K}_1)^{-1}\boldsymbol{P}_{21}\\
&\quad= \boldsymbol{P}_{12}(I - \boldsymbol{K}_2\boldsymbol{P}_{22})^{-1}(\boldsymbol{K}_1 - \boldsymbol{K}_2)(I - \boldsymbol{P}_{22}\boldsymbol{K}_1)^{-1}\boldsymbol{P}_{21}.
\end{aligned}$$

The result now follows because $\boldsymbol{P}_{12}(I - \boldsymbol{K}_2\boldsymbol{P}_{22})^{-1}$ has full column rank for almost all s and $(I - \boldsymbol{P}_{22}\boldsymbol{K}_1)^{-1}\boldsymbol{P}_{21}$ has full row rank for almost all s.

Solution 4.3. To see this we observe that

$$\left.\begin{array}{r}\begin{bmatrix} z \\ y \end{bmatrix} = \boldsymbol{P}\begin{bmatrix} w \\ u \end{bmatrix}\\[2ex] u = \boldsymbol{K}y\end{array}\right\} \quad\Rightarrow\quad z = \boldsymbol{R}\,w \text{ where } \boldsymbol{R} = \mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K})$$

and

$$\left.\begin{array}{r}\begin{bmatrix} w \\ u \end{bmatrix} = \boldsymbol{P}^{-1}\begin{bmatrix} z \\ y \end{bmatrix}\\[2ex] z = \boldsymbol{R}\,w\end{array}\right\} \quad\Rightarrow\quad \boldsymbol{K} = \mathcal{F}_u(\boldsymbol{P}^{-1},\boldsymbol{R}).$$

Solution 4.4. This follows because:

$$\begin{aligned}
\boldsymbol{Z} &= (I + \boldsymbol{S})(I - \boldsymbol{S})^{-1}\\
&= (I - \boldsymbol{S} + 2\boldsymbol{S})(I - \boldsymbol{S})^{-1}\\
&= I + 2\boldsymbol{S}(I - \boldsymbol{S})^{-1}\\
&= \mathcal{F}_\ell\left(\begin{bmatrix} I & I \\ 2I & I \end{bmatrix}, \boldsymbol{S}\right).
\end{aligned}$$

Solution 4.5.

1. Let

$$\begin{aligned}\begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \\ u &= \boldsymbol{K} y \end{aligned} \tag{4.1}$$

so that $z = \mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K}) w$. Rewrite (4.1) as

$$\begin{aligned}\begin{bmatrix} z \\ w \end{bmatrix} &= \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ I & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}\end{aligned}$$

which gives

$$\begin{aligned}\begin{bmatrix} z \\ w \end{bmatrix} &= \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}^{-1} \begin{bmatrix} u \\ y \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{K} \\ I \end{bmatrix} y \\ &= \begin{bmatrix} \boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} \\ \boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{K} \\ I \end{bmatrix} y.\end{aligned}$$

Hence $y = (\boldsymbol{\Theta}_{21}\boldsymbol{K} + \boldsymbol{\Theta}_{22})^{-1} w$ and

$$z = (\boldsymbol{\Theta}_{11}\boldsymbol{K} + \boldsymbol{\Theta}_{12})(\boldsymbol{\Theta}_{21}\boldsymbol{K} + \boldsymbol{\Theta}_{22})^{-1} w.$$

We conclude that

$$\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K}) = (\boldsymbol{\Theta}_{11}\boldsymbol{K} + \boldsymbol{\Theta}_{12})(\boldsymbol{\Theta}_{21}\boldsymbol{K} + \boldsymbol{\Theta}_{22})^{-1}.$$

2.

$$\begin{aligned}&\boldsymbol{P}^\sim \boldsymbol{P} - I \\ &= \begin{bmatrix} \boldsymbol{P}_{11}^\sim & \boldsymbol{P}_{21}^\sim \\ \boldsymbol{P}_{12}^\sim & \boldsymbol{P}_{22}^\sim \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}_{11}^\sim \boldsymbol{P}_{11} + \boldsymbol{P}_{21}^\sim \boldsymbol{P}_{21} - I & \boldsymbol{P}_{11}^\sim \boldsymbol{P}_{12} + \boldsymbol{P}_{21}^\sim \boldsymbol{P}_{22} \\ \boldsymbol{P}_{12}^\sim \boldsymbol{P}_{11} + \boldsymbol{P}_{22}^\sim \boldsymbol{P}_{21} & \boldsymbol{P}_{12}^\sim \boldsymbol{P}_{12} + \boldsymbol{P}_{22}^\sim \boldsymbol{P}_{22} - I \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}_{11}^\sim & I \\ \boldsymbol{P}_{12}^\sim & 0 \end{bmatrix} J \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ I & 0 \end{bmatrix} - \begin{bmatrix} 0 & \boldsymbol{P}_{21}^\sim \\ I & \boldsymbol{P}_{22}^\sim \end{bmatrix} J \begin{bmatrix} 0 & I \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \boldsymbol{P}_{21}^\sim \\ I & \boldsymbol{P}_{22}^\sim \end{bmatrix} (\Theta^\sim J \Theta - J) \begin{bmatrix} 0 & I \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}.\end{aligned}$$

The last line follows from

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}^{-1}.$$

3.

$$\begin{bmatrix} 0 & I \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \\ \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ I & 0 \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_{22} \\ C_1 & D_{11} & D_{12} \\ 0 & I & 0 \end{array}\right].$$

It follows (see Problem 3.6) that $\boldsymbol{\Theta}$ has realization

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ C_2 & D_{21} & D_{22} \end{bmatrix}^{-1}.$$

That is,

$$\boldsymbol{\Theta} \stackrel{s}{=} \left[\begin{array}{c|cc} A - B_1 D_{21}^{-1} C_2 & B_2 - B_1 D_{21}^{-1} D_{22} & B_1 D_{21}^{-1} \\ \hline C_1 - D_{11} D_{21}^{-1} C_2 & D_{12} - D_{11} D_{21}^{-1} D_{22} & D_{11} D_{21}^{-1} \\ -D_{21}^{-1} C_2 & -D_{21}^{-1} D_{22} & D_{21}^{-1} \end{array}\right].$$

Solution 4.6. Note that

$$XA + DX + XBX + C = X(BX + A) + (DX + C).$$

The result is now immediate.

Solution 4.7. We see from the diagram that

$$z = \mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{\Xi}) w$$

where

$$\boldsymbol{\Xi} = \mathcal{F}_\ell(\boldsymbol{K}, \boldsymbol{\Phi}_{11}).$$

It is now immediate that

$$\begin{bmatrix} z \\ r \end{bmatrix} = \begin{bmatrix} \mathcal{F}_\ell(\boldsymbol{P}, \mathcal{F}_\ell(\boldsymbol{K}, \boldsymbol{\Phi}_{11})) & * \\ * & * \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}.$$

Solution 4.8.

1. From the diagram we see that

$$\begin{aligned} z &= \boldsymbol{P}_{11} w + \boldsymbol{P}_{12} u \\ y &= \boldsymbol{P}_{21} w + \boldsymbol{P}_{22} u \\ u &= \boldsymbol{K}_{11} y + \boldsymbol{K}_{12} v \\ r &= \boldsymbol{K}_{21} y + \boldsymbol{K}_{22} v. \end{aligned}$$

Eliminating y and u from these equations gives

$$\begin{aligned} z &= (\boldsymbol{P}_{11} + \boldsymbol{P}_{12}\boldsymbol{K}_{11}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11})^{-1}\boldsymbol{P}_{21})w + \boldsymbol{P}_{12}(I - \boldsymbol{K}_{11}\boldsymbol{P}_{22})^{-1}\boldsymbol{K}_{12}v \\ r &= \boldsymbol{K}_{21}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11})^{-1}\boldsymbol{P}_{21}w + (\boldsymbol{K}_{22} + \boldsymbol{K}_{21}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11})^{-1}\boldsymbol{P}_{22}\boldsymbol{K}_{12})v. \end{aligned}$$

2. Since we require

$$0 = \boldsymbol{P}_{11} + \boldsymbol{P}_{12}\boldsymbol{K}_{11}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11})^{-1}\boldsymbol{P}_{21},$$

we obtain

$$\begin{aligned} \boldsymbol{K}_{11} &= -(\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} \\ &= -\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11}(\boldsymbol{P}_{21} - \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11})^{-1}. \end{aligned}$$

Setting

$$I = \boldsymbol{K}_{21}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11})^{-1}\boldsymbol{P}_{21},$$

gives

$$\begin{aligned} \boldsymbol{K}_{21} &= \boldsymbol{P}_{21}^{-1}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11}) \\ &= \boldsymbol{P}_{21}^{-1} + \boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22}(\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} \\ &= \boldsymbol{P}_{21}^{-1}(I - \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1})^{-1} \\ &\quad \times (I - \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} + \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}) \\ &= (\boldsymbol{P}_{21} - \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11})^{-1}. \end{aligned}$$

A similar calculation starting from

$$I = \boldsymbol{P}_{12}(I - \boldsymbol{K}_{11}\boldsymbol{P}_{22})^{-1}\boldsymbol{K}_{12},$$

results in

$$\begin{aligned} \boldsymbol{K}_{12} &= (I - \boldsymbol{K}_{11}\boldsymbol{P}_{22})\boldsymbol{P}_{12}^{-1} \\ &= \boldsymbol{P}_{12}^{-1} + (\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1} \\ &= (\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1}. \end{aligned}$$

Finally,

$$0 = \boldsymbol{K}_{22} + \boldsymbol{K}_{21}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11})^{-1}\boldsymbol{P}_{22}\boldsymbol{K}_{12}$$

and $\boldsymbol{K}_{11} = -\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11}\boldsymbol{K}_{21}$ results in

$$\begin{aligned} \boldsymbol{K}_{22} &= -\boldsymbol{K}_{21}(I - \boldsymbol{P}_{22}\boldsymbol{K}_{11})^{-1}\boldsymbol{P}_{22}\boldsymbol{K}_{12} \\ &= -\boldsymbol{K}_{21}(I + \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11}\boldsymbol{K}_{21})^{-1}\boldsymbol{P}_{22}\boldsymbol{K}_{12} \\ &= -(\boldsymbol{K}_{21}^{-1} + \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11})^{-1}\boldsymbol{P}_{22}\boldsymbol{K}_{12} \\ &= -(\boldsymbol{P}_{21} - \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11} + \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11})^{-1}\boldsymbol{P}_{22}\boldsymbol{K}_{12} \\ &= -\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22}(\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1}. \end{aligned}$$

Hence

$$\boldsymbol{P}^{\#} = \begin{bmatrix} \boldsymbol{P}_{11}^{\#} & \boldsymbol{P}_{12}^{\#} \\ \boldsymbol{P}_{21}^{\#} & \boldsymbol{P}_{22}^{\#} \end{bmatrix}, \tag{4.2}$$

in which

$$\begin{aligned} \boldsymbol{P}_{11}^{\#} &= -(\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} \\ \boldsymbol{P}_{12}^{\#} &= (\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1} \\ \boldsymbol{P}_{21}^{\#} &= (\boldsymbol{P}_{21} - \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11})^{-1} \\ \boldsymbol{P}_{22}^{\#} &= -\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22}(\boldsymbol{P}_{12} - \boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1}\boldsymbol{P}_{22})^{-1}. \end{aligned}$$

It is easy to check that

$$\boldsymbol{P}^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \boldsymbol{P}^{\#} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

3. We are going to need the six equations from

$$\begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_{11}^{\sim} & \boldsymbol{P}_{21}^{\sim} \\ \boldsymbol{P}_{12}^{\sim} & \boldsymbol{P}_{22}^{\sim} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} \boldsymbol{P}_{11}^{\sim} & \boldsymbol{P}_{21}^{\sim} \\ \boldsymbol{P}_{12}^{\sim} & \boldsymbol{P}_{22}^{\sim} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Since

$$\boldsymbol{P}_{12}^{\sim}\boldsymbol{P}_{11} + \boldsymbol{P}_{22}^{\sim}\boldsymbol{P}_{21} = 0,$$

we have

$$\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} = -(\boldsymbol{P}_{12}^{\sim})^{-1}\boldsymbol{P}_{22}^{\sim}.$$

Substituting into the formula for $\boldsymbol{P}^{\#}$ gives

$$\begin{aligned} \boldsymbol{P}_{11}^{\#} &= -(\boldsymbol{P}_{12} + (\boldsymbol{P}_{12}^{\sim})^{-1}\boldsymbol{P}_{22}^{\sim}\boldsymbol{P}_{22})^{-1}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} \\ &= -(\boldsymbol{P}_{12}^{\sim}\boldsymbol{P}_{12} + \boldsymbol{P}_{22}^{\sim}\boldsymbol{P}_{22})^{-1}\boldsymbol{P}_{12}^{\sim}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} \\ &= -\boldsymbol{P}_{12}^{\sim}\boldsymbol{P}_{11}\boldsymbol{P}_{21}^{-1} \\ &= \boldsymbol{P}_{22}^{\sim} \end{aligned}$$

since $I = \boldsymbol{P}_{12}^{\sim}\boldsymbol{P}_{12} + \boldsymbol{P}_{22}^{\sim}\boldsymbol{P}_{22}$. In much the same way

$$\boldsymbol{P}_{11}^{\sim}\boldsymbol{P}_{12} + \boldsymbol{P}_{21}^{\sim}\boldsymbol{P}_{22} = 0$$

gives

$$\boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1} = -(\boldsymbol{P}_{21}^{\sim})^{-1}\boldsymbol{P}_{11}^{\sim}.$$

Substituting into the formula for $\boldsymbol{P}_{21}^{\#}$ yields

$$\begin{aligned}\boldsymbol{P}_{21}^{\#} &= (\boldsymbol{P}_{21} - \boldsymbol{P}_{22}\boldsymbol{P}_{12}^{-1}\boldsymbol{P}_{11})^{-1} \\ &= (\boldsymbol{P}_{21} + (\boldsymbol{P}_{21}^{\sim})^{-1}\boldsymbol{P}_{11}^{\sim}\boldsymbol{P}_{11})^{-1} \\ &= \boldsymbol{P}_{21}^{\sim}\end{aligned}$$

since $I = \boldsymbol{P}_{21}^{\sim}\boldsymbol{P}_{21} + \boldsymbol{P}_{11}^{\sim}\boldsymbol{P}_{11}$. The other partitions of $\boldsymbol{P}^{\#}$ follow in the same way.

Solution 4.9.

1. One has to check that $\mathcal{C}_\ell(\boldsymbol{P}_1, \boldsymbol{P}_2)$ and its $(1,2)$- and $(2,1)$-partitions are nonsingular. By referring to the general formula (4.1.9) for $\mathcal{C}_\ell(\cdot,\cdot)$, we see that this is indeed the case for any $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{P}$.

2. The best way to establish the associativity property is to transform the $\boldsymbol{P}_i$'s into $\boldsymbol{\Theta}_i$'s in a scattering framework (see Problem 4.5). We then get

$$\begin{aligned}\mathcal{C}_\ell(\mathcal{C}_\ell(\boldsymbol{P}_1, \boldsymbol{P}_2), \boldsymbol{P}_3) &= (\Theta_1\Theta_2)\Theta_3 \\ &= \Theta_1(\Theta_2\Theta_3) \\ &= \mathcal{C}_\ell(\boldsymbol{P}_1, \mathcal{C}_\ell(\boldsymbol{P}_2, \boldsymbol{P}_3))\end{aligned}$$

   in which $\boldsymbol{\Theta}_i$ are the scattering matrices associated with each $\boldsymbol{P}_i$. The associativity property comes from the fact that matrix multiplication is associative.

3. The identity is given by

$$\boldsymbol{P}_I = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

   and by referring to (4.1.9) it is easy to check that

$$\begin{aligned}\boldsymbol{P} &= \mathcal{C}_\ell(\boldsymbol{P}, \boldsymbol{P}_I) \\ &= \mathcal{C}_\ell(\boldsymbol{P}_I, \boldsymbol{P}).\end{aligned}$$

4. Again, it is a routine matter to check that $\boldsymbol{P}^{\#}$ given in (4.2) has the desired properties. The fact that $\mathcal{C}_\ell(\boldsymbol{P}, \boldsymbol{P}^{\#}) = \mathcal{C}_\ell(\boldsymbol{P}^{\#}, \boldsymbol{P})$ comes from $\boldsymbol{\Theta}^{\#}\boldsymbol{\Theta} = \boldsymbol{\Theta}\boldsymbol{\Theta}^{\#} = I$.

5. The group property now follows directly from the definition.

Solution 4.10.

1. We begin by expressing s^{-1} as function of w^{-1}.

$$\begin{aligned} s &= (b-wd)(cw-a)^{-1} \\ &= (w^{-1}b-d)(c-aw^{-1})^{-1} \\ \Rightarrow s^{-1} &= (c-aw^{-1})(w^{-1}b-d)^{-1} \\ &= -(c-aw^{-1})(1-w^{-1}bd^{-1})^{-1}d^{-1} \\ &= -cd^{-1}+(a-cbd^{-1})w^{-1}(1-w^{-1}bd^{-1})^{-1}d^{-1} \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} -cd^{-1} & a-bcd^{-1} \\ d^{-1} & bd^{-1} \end{bmatrix}, w^{-1}\right). \end{aligned}$$

We therefore have

$$\begin{aligned} G(s) &= D+C(sI-A)^{-1}B \\ &= D+Cs^{-1}(I-s^{-1}A)^{-1}B \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} D & C \\ B & A \end{bmatrix}, s^{-1}\right) \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} D & C \\ B & A \end{bmatrix}, \mathcal{F}_\ell(\begin{bmatrix} -cd^{-1} & a-bcd^{-1} \\ d^{-1} & bd^{-1} \end{bmatrix}, w^{-1})\right) \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} D-cC(dI+cA)^{-1}B & C(dI+cA)^{-1}(ad-bc) \\ (dI+cA)^{-1}B & (aA+bI)(dI+cA)^{-1} \end{bmatrix}, w^{-1}\right) \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} \widehat{D} & \widehat{C} \\ \widehat{B} & \widehat{A} \end{bmatrix}, w^{-1}\right) \\ &= \widehat{\boldsymbol{G}}(w). \end{aligned}$$

2. Suppose $Ax=\lambda x$ and $Cx=0$, $x\neq 0$. Define $y=(cA+dI)x=(c\lambda+d)x\neq 0$ (since $cA+dI$ is nonsingular). Then $\widehat{C}y=0$ and $\widehat{A}y=\frac{a\lambda+b}{c\lambda+d}y$.

   Suppose $x^*B=0$ and $x^*A=\lambda x$, $x\neq 0$. Define $y^*=x^*(cA+dI)=(c\lambda+d)x^*\neq 0$. Then $y^*\widehat{B}=0$ and $y^*\widehat{A}=(c\lambda+d)(a\lambda+b)x^*(cA+dI)^{-1}=\frac{a\lambda+b}{c\lambda+d}y^*$.

   Similar arguments establish the converse implications.

Solution 4.11.

1. Writing $\boldsymbol{W}_1(I-\boldsymbol{GK})^{-1}$ as $\boldsymbol{W}_1\big(I+\boldsymbol{GK}(I-\boldsymbol{GK})^{-1}\big)$, gives

$$\begin{bmatrix} \boldsymbol{W}_1(I-\boldsymbol{GK})^{-1} \\ \boldsymbol{W}_2\boldsymbol{K}(I-\boldsymbol{GK})^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{W}_1\boldsymbol{G} \\ \boldsymbol{W}_2 \end{bmatrix}\boldsymbol{K}(I-\boldsymbol{GK})^{-1}.$$

Comparing terms with

$$\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K}) = \boldsymbol{P}_{11} + \boldsymbol{P}_{12}\boldsymbol{K}(I-\boldsymbol{P}_{22}\boldsymbol{K})^{-1}\boldsymbol{P}_{21}$$

establishes that

$$\begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{W}_1 & \boldsymbol{W}_1\boldsymbol{G} \\ 0 & \boldsymbol{W}_2 \\ \hline I & \boldsymbol{G} \end{array}\right].$$

2. Note that

$$\begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}_1 & 0 & 0 \\ 0 & \boldsymbol{W}_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & \boldsymbol{G} \\ 0 & I \\ I & \boldsymbol{G} \end{bmatrix},$$

that

$$\begin{bmatrix} \boldsymbol{W}_1 & 0 & 0 \\ 0 & \boldsymbol{W}_2 & 0 \\ 0 & 0 & I \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|ccc} A_1 & 0 & B_1 & 0 & 0 \\ 0 & A_2 & 0 & B_2 & 0 \\ \hline C_1 & 0 & D_1 & 0 & 0 \\ 0 & C_2 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right],$$

and that

$$\begin{bmatrix} I & \boldsymbol{G} \\ 0 & I \\ I & \boldsymbol{G} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & 0 & B \\ \hline C & I & D \\ 0 & 0 & I \\ C & I & D \end{array}\right].$$

The state-space realization of $\boldsymbol{P}$ is obtained using the series connection rule (see Problem 3.6).

Solution 4.12. The solution comes from noting that

$$\left[\begin{array}{c} y \\ r-y \\ u \\ \hline d \\ r \\ y+n \end{array}\right] = \left[\begin{array}{ccc|c} \boldsymbol{G}_d & 0 & 0 & \boldsymbol{G}_t \\ -\boldsymbol{G}_d & I & 0 & -\boldsymbol{G}_t \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \boldsymbol{G}_d & 0 & I & \boldsymbol{G}_t \end{array}\right] \left[\begin{array}{c} d \\ r \\ n \\ \hline u \end{array}\right]$$

$$u = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{R} & \boldsymbol{K} \end{bmatrix} \begin{bmatrix} d \\ r \\ y+n \end{bmatrix}.$$

Solution 4.13. Follows immediately from Theorem 3.6.1. Alternatively, using the fact that $\|\boldsymbol{P}_{22}\boldsymbol{K}\|_\infty < 1$ with $\boldsymbol{P}_{22},\ \boldsymbol{K} \in \mathcal{RH}_\infty$, we observe that

$$\det(I - \epsilon \boldsymbol{P}_{22}\boldsymbol{K}) \neq 0 \qquad \text{for all } \epsilon \in [0,1] \text{ and all } s \in D_R.$$

This means that $\det(I - \boldsymbol{P}_{22}\boldsymbol{K})$ has a winding number of zero around the origin. We therefore conclude from the argument principle that $(I - \boldsymbol{P}_{22}\boldsymbol{K})^{-1}$ is stable and therefore that $\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})$ is stable.

Solution 4.14.

1. $D'D = I$ follows by calculation.
2. It is immediate that
$$\mathcal{F}_\ell(D, f) = \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix}$$
with $\|\mathcal{F}_\ell(D, f)\| = 1$ for all $|f| < 1$.
3. If $|f| > 1$, it is clear that $\|\mathcal{F}_\ell(D, f)\| > 1$.

Solution 4.15. We use Lemma 4.4.1. $D_{11} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $D_{12} = \begin{bmatrix} X \\ I \end{bmatrix}$ and $Q = F(I - XF)^{-1}$. Set $\tilde{D}_{12} = \begin{bmatrix} I \\ -X^* \end{bmatrix}$ and note that

$$\tilde{D}_{12}^* \begin{bmatrix} \tilde{D}_{12} & D_{12} \end{bmatrix} = \begin{bmatrix} I + XX^* & 0 \end{bmatrix}.$$

We therefore set $\widehat{D}_{12} = \tilde{D}_{12}(I + XX^*)^{-\frac{1}{2}}$. Now $\widehat{D}_{12}^* D_{11} = (I + XX^*)^{-\frac{1}{2}}$. Hence F exists if and only if $\gamma \geq \|(I + XX^*)^{-\frac{1}{2}}\|$. Now note that

$$\begin{aligned} \|(I + XX^*)^{-\frac{1}{2}}\| &= \overline{\sigma}((I + XX^*)^{-\frac{1}{2}}) \\ &= \sqrt{\overline{\sigma}((I + XX^*)^{-1})} \\ &= \frac{1}{\sqrt{\underline{\sigma}(I + XX^*)}}. \end{aligned}$$

To find $\widehat{F}$, set $\widehat{Q} = \Theta_{11} = X^*(I + XX^*)^{-1}$. Solving for $\widehat{F}$ we obtain $\widehat{F} = (I + 2X^*X)^{-1}X^*$.

Solution 4.16.

1. Θ has the property $\Theta'\Theta = I$. By Theorem 4.3.2, $\|\mathcal{F}_\ell(\Theta, \gamma^{-1}\boldsymbol{G})\|_\infty < 1$ if and only if $\|\gamma^{-1}\boldsymbol{G}\|_\infty < 1$. Hence $\|\boldsymbol{G}\|_\infty < \gamma$ if and only if $\|\widehat{\boldsymbol{G}}\|_\infty < \gamma$.

2. Since $\boldsymbol{G}(s) = \mathcal{F}_\ell(\begin{bmatrix} D & C \\ B & A \end{bmatrix}, s^{-1})$,

$$\begin{bmatrix} \gamma^{-1}\widehat{D} & \widehat{C} \\ \gamma^{-1}\widehat{B} & \widehat{A} \end{bmatrix} = \mathcal{C}_\ell(\Theta, \begin{bmatrix} \gamma^{-1}D & C \\ \gamma^{-1}B & A \end{bmatrix})$$

which yields (after some calculation)

$$\begin{aligned} \widehat{A} &= A + B(\gamma^2 I - D'D)^{-1}D'C, \qquad \widehat{B} = -\gamma B(\gamma^2 I - D'D)^{-1/2}, \\ \widehat{C} &= \gamma(\gamma^2 I - DD')^{-1/2}C, \qquad \widehat{D} = 0. \end{aligned}$$

3. If A is asymptotically stable and $\|\gamma^{-1}\boldsymbol{G}\|_\infty < 1$, then $\|\gamma^{-1}\widehat{\boldsymbol{G}}\|_\infty < 1$ and $\widehat{\boldsymbol{G}}$ is stable by Theorem 4.3.3. We conclude that $\widehat{A}$ is asymptotically stable, since any uncontrollable or unobservable modes of $(\widehat{A}, \widehat{B}, \widehat{C})$ are eigenvalues of A. The converse follows likewise.

Solution 4.17. Observe that

$$\begin{aligned} \boldsymbol{G}(z) &= D + C(zI - A)^{-1}B \\ &= D + Cz^{-1}(I - Az^{-1})^{-1}B \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} D & C \\ B & A \end{bmatrix}, z^{-1}\right). \end{aligned}$$

The result now follows from Theorem 4.3.2, since:

$$\left\|\begin{bmatrix} D & C \\ B & A \end{bmatrix}\right\| \le 1 \quad \text{and} \quad |z^{-1}| \le 1$$

for all $|z| \ge 1$.

Solution 4.18. The first step is to find $\widehat{\boldsymbol{G}}(s)$ from $\boldsymbol{G}(z)$ using the bilinear transformation:

$$\begin{aligned} \boldsymbol{G}(z) &= D + C(zI - A)^{-1}B \\ &= D + C(\left(\frac{s+1}{1-s}\right) I - A)^{-1}B \\ &= D + C(1-s)(sI - (I+A)^{-1}(A-I))^{-1}(I+A)^{-1}B \\ &= D + C(1-s)(sI - \widehat{A})^{-1}1/\sqrt{2}\widehat{B} \\ &= D + C(I - Is - \widehat{A} + \widehat{A})(sI - \widehat{A})^{-1}1/\sqrt{2}\widehat{B} \\ &= D - 1/\sqrt{2}C\widehat{B} + 1/\sqrt{2}C(I - \widehat{A})(sI - \widehat{A})^{-1}\widehat{B} \\ &= D - C(I+A)^{-1}B + \sqrt{2}C(I+A)^{-1}(sI - \widehat{A})^{-1}\widehat{B} \\ &= \widehat{D} + \widehat{C}(sI - \widehat{A})^{-1}\widehat{B} \\ &= \widehat{\boldsymbol{G}}(s). \end{aligned}$$

This completes the first part. To prove the second part, we substitute into the continuous bounded real equations:

$$\begin{aligned} 0 &= \widehat{A}'P + P\widehat{A} + \widehat{C}'\widehat{C} + \widehat{L}'\widehat{L} && (4.3)\\ 0 &= \widehat{D}'\widehat{C} + \widehat{B}'P + \widehat{W}'\widehat{L} && (4.4)\\ 0 &= \gamma^2 I - \widehat{D}'\widehat{D} - \widehat{W}'\widehat{W}. && (4.5)\end{aligned}$$

Substituting into (4.3) gives

$$0 = (I+A')^{-1}(A'-I)P + P(A-I)(I+A)^{-1} + 2(I+A')^{-1}\begin{bmatrix} C' & L' \end{bmatrix}\begin{bmatrix} C \\ L \end{bmatrix}(I+A)^{-1}$$

where

$$\widehat{L} = \sqrt{2}L(I+A)^{-1}.$$

Therefore

$$\begin{aligned} 0 &= \sqrt{2}(I+A')^{-1}\big(A'PA - P + C'C + L'L\big)\sqrt{2}(I+A)^{-1}\\ \Rightarrow 0 &= A'PA - P + C'C + L'L. && (4.6)\end{aligned}$$

Substituting (4.6) into (4.4) gives

$$\begin{aligned} 0 &= \sqrt{2}(I+A')^{-1}\begin{bmatrix} C' & L' \end{bmatrix}\left(\begin{bmatrix} D \\ W \end{bmatrix} - \begin{bmatrix} C \\ L \end{bmatrix}(I+A)^{-1}B\right)\\ &\quad + \sqrt{2}P(I+A)^{-1}B\\ &= C'D + L'W + (A'PA - P)(I+A)^{-1}B + (I+A')P(I+A)^{-1}B\\ &= C'D + L'W + (A'PA - P + P + A'P)(I+A)^{-1}B\\ &= C'D + L'W + A'PB. && (4.7)\end{aligned}$$

Finally, we may substitute into (4.5) using (4.6) and (4.7) to obtain

$$0 = \gamma^2 I - D'D - W'W - B'PB. \qquad (4.8)$$

Equations (4.3), (4.4) and (4.5) may be combined as

$$\begin{bmatrix} A' & C' & L' \\ B' & D' & W' \end{bmatrix}\begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \\ L & W \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}.$$

Note that the same P solves the discrete and continuous bounded real equations.

Solution 4.19. Since $\|\boldsymbol{G}\|_\infty < 1$, it has a minimal realization which satisfies the discrete bounded real equations

$$\begin{bmatrix} A' & C' & L' \\ B' & D' & W' \end{bmatrix}\begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \\ L & W \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$$

for certain matrices P, L and W. We may now select a new state-space basis such that $P = I$. This gives

$$\begin{bmatrix} A' & C' & L' \\ B' & D' & W' \end{bmatrix} \begin{bmatrix} A & B \\ C & D \\ L & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

so that

$$\left\| \begin{bmatrix} A & B \\ C & D \\ L & W \end{bmatrix} \right\| = 1.$$

We may now conclude that

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq 1.$$

Solutions to Problems in Chapter 5

Solution 5.1. Let $\begin{bmatrix} x_1' & u_1' \end{bmatrix}'$ and $\begin{bmatrix} x_2' & u_2' \end{bmatrix}'$ be any two initial condition and control input pairs, and let z_1 and z_2 denote the corresponding objectives ($z = \begin{bmatrix} Cx \\ Du \end{bmatrix}$). By linearity, the objective signal obtained by using the initial condition and control input pair

$$\begin{bmatrix} x_\alpha \\ u_\alpha \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_2 \\ u_2 \end{bmatrix}$$

is

$$z_\alpha = \alpha z_1 + (1-\alpha) z_2.$$

The cost associated with this initial condition and control input pair is

$$J_\alpha = \int_0^T z_\alpha z_\alpha \, dt.$$

Let J_1 and J_2 denote the cost associated with $\begin{bmatrix} x_1' & u_1' \end{bmatrix}'$ and $\begin{bmatrix} x_2' & u_2' \end{bmatrix}'$ respectively. To show convexity, we need to show that

$$J_\alpha \leq \alpha J_1 + (1-\alpha) J_2$$

for any $0 \leq \alpha \leq 1$. Now

$$\begin{aligned} z_\alpha' z_\alpha - \alpha z_1' z_1 - (1-\alpha) z_2' z_2 &= -\alpha(1-\alpha)(z_1 - z_2)'(z_1 - z_2) \\ &\leq 0 \end{aligned}$$

for any $0 \leq \alpha \leq 1$. Integrating from 0 to T, we obtain the desired inequality

$$J_\alpha \leq \alpha J_1 + (1-\alpha) J_2,$$

and we conclude that J is convex.

Since we are free to choose $x_2 = x_1$, J is convex in u. Similarly, by choosing $u_2 = u_1$, we see that J is convex in x_0.

Solution 5.2.

1. The optimal state trajectory satisfies

$$\dot{x^*} = Ax^* + Bu^*, \qquad x(0) = x_0.$$

If $u = u^* + \epsilon\tilde{u}$, then

$$\frac{d}{dt}(x - x^*) = A(x - x^*) + \epsilon B\tilde{u}, \qquad (x - x^*)(0) = 0.$$

Thus

$$\begin{aligned}(x - \tilde{x})(t) &= \epsilon \int_0^t \Phi(t,\tau)B\tilde{u}\, d\tau \\ &= \epsilon\tilde{x},\end{aligned}$$

in which

$$\tilde{x}(t) = \int_0^t \Phi(t,\tau)B\tilde{u}\, d\tau$$

and $\Phi(\cdot,\cdot)$ is the transition matrix corresponding to A.

2. Direct substitution of $u = u^* + \epsilon\tilde{u}$ and $x = x^* + \epsilon\tilde{x}$ into the cost function J yields the stated equation after elementary algebra.

3. Since u^* is minimizing, changing the control to $u^* + \epsilon\tilde{u}$ cannot decrease J. Therefore, as a function of ϵ, J must take on its minimum value at $\epsilon = 0$. Since the cost function is quadratic in ϵ, with a minimum at $\epsilon = 0$, the coefficient of the linear term must be zero. That is

$$\int_0^T (\tilde{x}'C'Cx^* + \tilde{u}'u^*)\, dt = 0.$$

4. Substituting the formula for $\tilde{x}$ into the above equation and interchanging the order of integration gives

$$\int_0^T \tilde{u}'(B'\lambda + u^*)dt = 0,$$

in which λ is the adjoint variable defined by

$$\lambda(t) = \int_t^T \Phi'(\tau,t)C'Cx^* d\tau.$$

Thus, $B'\lambda + u^*$ is orthogonal to every $\tilde{u} \in \mathcal{L}_2[0,T]$. Hence $B'\lambda + u^* = 0$ almost everywhere.

5. Differentiating λ with respect to t and using Leibnitz's rule, we obtain

$$\begin{aligned} \dot{\lambda}(t) &= -C'(t)C(t)x^*(t) - \int_t^T A'(t)\Phi'(\tau,t)C'(\tau)C(\tau)x^*(\tau)d\tau \\ &= -A'(t)\lambda(t) - C'(t)C(t)x^*(t). \end{aligned}$$

The fact that $\frac{d}{dt}\Phi'(\tau,t) = -A'(t)\Phi'(\tau,t)$ has been used—see Problem 3.3 for this. Evaluating $\lambda(T)$, we conclude that the terminal condition $\lambda(T) = 0$ applies.

Substituting $u^* = -B'\lambda$ into the dynamical equation for the optimal state, we obtain $\dot{x^*} = Ax^* - BB'\lambda$. Combining this with the equation for λ, one obtains the TPBVP.

6. The solution to the TPBVP is given by

$$\begin{bmatrix} x^* \\ \lambda \end{bmatrix}(t) = \Phi(t,T)\begin{bmatrix} x^* \\ \lambda \end{bmatrix}(T),$$

in which Φ is the transition matrix associated with the TPVBP dynamics. Imposing the boundary condition $\lambda(T) = 0$, we see that

$$\begin{bmatrix} x^* \\ \lambda \end{bmatrix}(t) = \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix}(t,T)x^*(T).$$

Thus $\lambda(t) = \Phi_{21}(t,T)\Phi_{11}^{-1}(t,T)x^*(t)$ for all time for which the inverse exits. It remains to show that $\Phi_{11}(t,T)$ is nonsingular for all $t \leq T$.

Observe that

$$\begin{aligned} \frac{d}{d\tau}\big(\Phi_{21}'(\tau,T)\Phi_{11}(\tau,T)\big) &= -\Phi_{11}'(\tau,T)C'C\Phi_{11}(\tau,T) \\ &\quad - \Phi_{21}'(\tau,T)BB'\Phi_{21}(\tau,T). \end{aligned}$$

Integrating from t to T and noting that $\Phi_{21}(T,T) = 0$ yields

$$\begin{aligned} \Phi_{21}'(t,T)\Phi_{11}(t,T) &= \int_t^T \big(\Phi_{11}'(\tau,T)C'C\Phi_{11}(\tau,T) \\ &\quad + \Phi_{21}'(\tau,T)BB'\Phi_{21}(\tau,T)\big)\, d\tau. \end{aligned}$$

Suppose $\Phi_{11}(t,T)v = 0$. Multiplying the above identity by v' on the left and v on the right, we conclude that $B'\Phi_{21}(\tau,T)v \equiv 0$ and that $C\Phi_{11}(\tau,T)v \equiv 0$ for all $\tau \in [t,T]$. Now $B'\Phi_{21}(\tau,T)v \equiv 0$ implies

$$\frac{d}{d\tau}\Phi_{11}(\tau,T)v = A\Phi_{11}(\tau,T)v.$$

Recalling that $\Phi_{11}(t,T)v = 0$, we see that $\Phi_{11}(\tau,T)v = 0$ for all τ, since linear differential equations with specified initial conditions have a unique solution. Since $\Phi_{11}(T,T) = I$, we must have $v = 0$, from which we conclude that $\Phi_{11}(t,T)$ is nonsingular.

7. That $P(t) = \Phi_{21}(t,T)\Phi_{11}^{-1}(t,T)$ is the solution to the Riccati differential equation (5.2.5) follows by direct substitution; see Problem 3.21.

Solution 5.3. Write the two Riccati equations

$$\begin{aligned} PA + A'P - PB_2B_2'P + C'C &= 0 \\ \bar{P}A + A'\bar{P} - \bar{P}B_2B_2'\bar{P} + C'C &= 0 \end{aligned}$$

and subtract them equations to get

$$(P-\bar{P})A + A'(P-\bar{P}) - PB_2B_2'P + \bar{P}B_2B_2'\bar{P} = 0.$$

Hence

$$(P-\bar{P})(A - B_2B_2'P) + (A - B_2B_2'P)'(P-\bar{P}) + (P-\bar{P})B_2B_2'(P-\bar{P}) = 0.$$

Since P is stabilizing, $A - B_2B_2'P$ is asymptotically stable and we conclude that $P - \bar{P} \geq 0$.

Suppose (A, C) is detectable and $P \geq 0$ is a solution to the Riccati equation. Write the Riccati equation as

$$P(A - B_2B_2'P) + (A - B_2B_2'P)'P + PB_2B_2'P + C'C = 0.$$

If $(A - B_2B_2'P)x = \lambda x$, then we obtain

$$(\lambda + \bar{\lambda})x'Px + \|B_2'Px\|^2 + \|Cx\|^2 = 0.$$

Hence $(\lambda + \bar{\lambda})x'Px \leq 0$. If equality holds, then $Cx = 0$ and $B_2Px = 0$, which gives $Ax = \lambda x$, $Cx = 0$ and we conclude that $\mathrm{R}_e(\lambda) < 0$ from the detectability of (A, C). If, on the other hand, $(\lambda + \bar{\lambda})x'Px < 0$, we must have $x'Px > 0$ since $P \geq 0$ and hence $\mathrm{R}_e(\lambda) < 0$. Hence P is a stabilizing solution. But the stabilizing solution is unique, so we conclude that it is the only nonnegative solution.

Solution 5.4.

1. The identity is established by noting that

$$C'C = P(sI - A) + (-sI - A')P + PB_2B_2'P,$$

which gives

$$\begin{aligned} (-sI - A')^{-1}C'C(sI - A)^{-1} &= (-sI - A')^{-1}P + P(sI - A)^{-1} \\ &\quad + (-sI - A')^{-1}PB_2B_2'P(sI - A)^{-1}, \end{aligned}$$

and

$$I + B_2(-sI - A')^{-1}C'C(sI - A)^{-1}B_2 = (I + B_2'P(sI - A)^{-1}B_2)^{\sim} (I + B_2'P(sI - A)^{-1}B_2)$$

Therefore, $\big(\boldsymbol{W}(j\omega)\big)^*\big(\boldsymbol{W}(j\omega)\big) \geq I$ for all ω, and hence $\overline{\sigma}\big(\boldsymbol{S}(j\omega)\big) \leq 1$ for all ω, which is equivalent to $\|\boldsymbol{S}\|_\infty \leq 1$.

2.

$$\begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{G} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A & B_2 \\ \hline B_2'P & I \\ C & 0 \\ 0 & I \end{array}\right].$$

It now follow from Problem 3.6 that

$$\boldsymbol{G}\boldsymbol{W}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A - B_2B_2'P & B_2 \\ \hline C & 0 \\ -B_2'P & I \end{array}\right].$$

Since the Riccati equation can be written in the form

$$P(A - B_2B_2'P) + (A - B_2B_2'P)'P + PB_2B_2'P + C'C = 0,$$

it follows from Theorem 3.2.1 that $\boldsymbol{G}\boldsymbol{W}^{-1}$ is allpass. Since P is the stabilizing solution, $A - B_2B_2'P$ is asymptotically stable, $\boldsymbol{G}\boldsymbol{W}^{-1} \in \mathcal{H}_\infty$ and hence it is contractive in the right-half plane.

3. If $u = -Kx$ is optimal w.r.t. $\int_0^\infty (x'C'Cx + u'u)\, dt$, then $K = B_2'P$, in which P is the stabilizing solution to the Riccati equation (5.2.29) and Item 1 shows that the inequality holds.

   Conversely, suppose $A - B_2K$ is asymptotically stable and

   $$\big(I + K(sI - A)^{-1}B_2\big)^{\sim}\big(I + K(sI - A)^{-1}B_2\big) \geq I.$$

   Then $\boldsymbol{S} = I - K(sI - (A - B_2K))^{-1}B_2$ satisfies $\|\boldsymbol{S}\|_\infty \leq 1$ and the equality version of the bounded real lemma ensures the existence of $P \geq 0$ and L such that

   $$\begin{aligned} P(A - B_2K) + (A - B_2K)'P + K'K &= -L'L \\ -K + B_2'P &= 0. \end{aligned}$$

   Substituting $K = B_2'P$ into the first equation and re-arranging yields (5.2.29), in which $C = L$, we and we conclude that K is the optimal controller for the performance index $\int_0^\infty (x'C'Cx + u'u)\, dt$, with $C = L$.

4. The inequality $|1+b_2'P(j\omega I - A)^{-1}b_2| \geq 1$ is immediate from the return difference equality. This inequality says that the Nyquist diagram of $-b_2'P(j\omega I - A)^{-1}b_2$ cannot enter the circle of unit radius centered at $+1$. The stated gain and phase margins then follow from the geometry of this situation.

Solution 5.5. If $\tilde{x}(t) = e^{\alpha t}x(t)$ and $\tilde{u}(t) = e^{\alpha t}u(t)$, then $J = \int_0^\infty \tilde{x}'C'C\tilde{x} + \tilde{u}'\tilde{u}\,dt$. Furthermore,

$$\dot{\tilde{x}} = (\alpha I + A)\tilde{x} + B_2\tilde{u}$$

follows from $\dot{\tilde{x}} = e^{\alpha t}\dot{x} + \alpha e^{\alpha t}x$. This is now a standard LQ problem in the variables $\tilde{x}$ and $\tilde{u}$. Hence $\tilde{u} = -B_2'P_\alpha\tilde{x}$ is the optimal controller, which is equivalent to $u = -B_2'P_\alpha x$.

The required assumptions are $(\alpha I + A, B_2)$ stabilizable and $(C, \alpha I + A)$ has no unobservable modes on the imaginary axis. Equivalently, we require that (A, B_2) has no uncontrollable modes in $\mathrm{R}_e(s) \geq -\alpha$ and that (C, A) has no unobservable modes on $\mathrm{R}_e(s) = -\alpha$.

The closed-loop dynamics are $\dot{x} = (A - B_2B_2'P_\alpha)x$; the closed-loop poles are (a subset of) the eigenvalues of $(A - B_2B_2'P_\alpha)$, which are all in $\mathrm{R}_e(s) < -\alpha$ because $\alpha I + A - B_2B_2'P_\alpha$ is asymptotically stable.

Solution 5.6.

1. Substitute $u = -Kx$ into the dynamics to obtain

$$\begin{aligned} \dot{x} &= (A - B_2K)x + B_1w \\ z &= \begin{bmatrix} C \\ -DK \end{bmatrix} x. \end{aligned}$$

   The result is now immediate from Theorem 3.3.1.

2. Elementary manipulations establish the Lyapunov equation. Theorem 3.1.1 and the asymptotic stability of $A - B_2K$ establishes that $Q - P \geq 0$.

3. $\mathrm{trace}(B_1'QB_1) = \mathrm{trace}(B_1'(Q - P)B_1) + \mathrm{trace}(B_1'PB_1)$. Hence the cost is minimized by setting $Q - P = 0$, which we do by setting $K = B_2'P$.

Solution 5.7.

$$\begin{aligned} \mathrm{trace}(QS) &= -\mathrm{trace}(QA'P + QPA) \\ &= -\mathrm{trace}(PQA' + PAQ) \\ &= \mathrm{trace}(PR). \end{aligned}$$

The main thing is to recall that $\mathrm{trace}(XY) = \mathrm{trace}(YX)$ for any square matrices X and Y.

Solution 5.8. Let

$$J(\boldsymbol{K}, x_t, T, \Delta) = \int_t^T z'z \, d\tau + x'(T)\Delta x(T).$$

For any controller, $J(\boldsymbol{K}, x_t, T, \Delta_1) \geq J(\boldsymbol{K}, x_t, T, \Delta_2)$, since $\Delta_1 \geq \Delta_2$. If we use $\boldsymbol{K} = \boldsymbol{K}^*_{\Delta_1}$, the optimal controller for the problem with terminal-state penalty matrix Δ_1, the left-hand side is equal to $x_t' P(t, T, \Delta_1) x_t$. Hence

$$\begin{aligned} x_t' P(t, T, \Delta_1) x_t &\geq J(\boldsymbol{K}^*_{\Delta_1}, x_t, T, \Delta_2) \\ &\geq \min_{\boldsymbol{K}} J(\boldsymbol{K}, x_t, T, \Delta_2) \\ &= x_t' P(t, T, \Delta_2) x_t. \end{aligned}$$

Since x_t is arbitrary, we conclude that $P(t, T, \Delta_1) \geq P(t, T, \Delta_2)$ for any $t \leq T$.

Solution 5.9. The case when

$$\Delta A + A'\Delta - \Delta B_2 B_2' \Delta + C'C \leq 0$$

is considered in the text. We therefore consider the case that

$$\Delta A + A'\Delta - \Delta B_2 B_2' \Delta + C'C \geq 0.$$

The same argument as used in the text shows that $P(t, T, \Delta)$ is monotonically non-increasing as a function of t, and $P(t, T, \Delta)$ is therefore non-decreasing as a function of T (by time-invariance). It remains to show that $P(t, T, \Delta)$ is uniformly bounded.

Let K be such that $A - B_2 K$ is asymptotically stable. Then

$$J(K, x_0, T, \Delta) \leq \int_0^\infty z'z \, dt + e^{-\alpha T} x_0' \Delta x_0$$

for some $\alpha \geq 0$. Hence

$$\begin{aligned} x_0' P(0, T, \Delta) x_0 &= \min_{\boldsymbol{K}} J(\boldsymbol{K}, x_0, T, \Delta) \\ &\leq J(K, x_0, T, \Delta) \\ &\leq \int_0^\infty z'z \, dt + e^{-\alpha T} x_0' \Delta x_0 \\ &\leq \int_0^\infty z'z \, dt + x_0' \Delta x_0, \end{aligned}$$

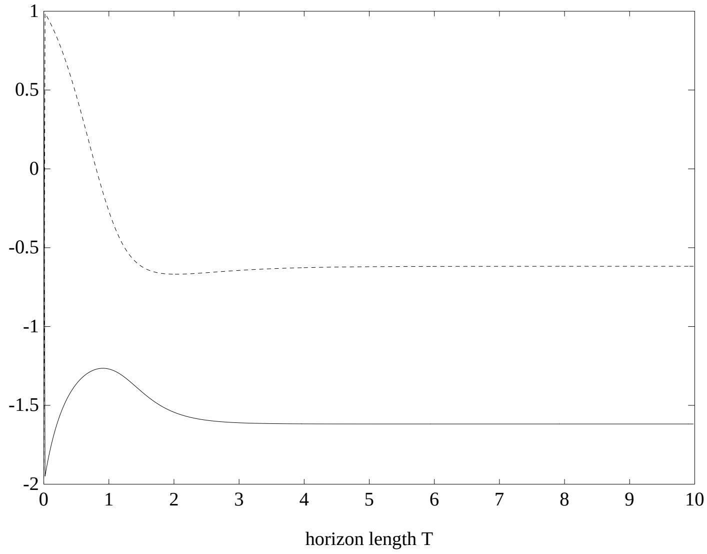
which is a uniform bound on $x_0' P(0, T, \Delta) x_0$. Thus $P(t, T, \Delta)$ is monotonic and uniformly bounded. Hence it converges to some Π as $T - t \to \infty$.

Solution 5.10.

1. By Problem 5.9, $\Pi = \lim_{T-t\to\infty} P(t,T,0)$ exists; it is a solution to the algebraic Riccati equation by virtue of time-invariance (see the argument in the text). Also, $\Pi \geq 0$ because the zero terminal condition is nonnegative definite. Thus $P(t,T,0)$ converges to a nonnegative definite solution Π to the algebraic Riccati equation. We conclude that Π is stabilizing because when (A, B_2, C) is stabilizable and detectable, the stabilizing solution is the only nonnegative definite solution.

2. Let
$$X = \Gamma A + A'\Gamma - \Gamma B_2 B_2'\Gamma + C'C.$$
Since X is symmetric, it has the form
$$X = V' \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} V.$$
Let
$$R = V' \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} V,$$
and let Δ be the stabilizing solution to the algebraic Riccati equation
$$\Delta A + A'\Delta - \Delta B_2 B_2'\Delta + C'C + R = 0,$$
which exists under the stated assumptions, since $R \geq 0$. This implies that Δ satisfies the inequality
$$\Delta A + A'\Delta - \Delta B_2 B_2'\Delta + C'C \leq 0.$$
It remains to show that $\Delta \geq \Gamma$. By subtracting the equation defining X from the Riccati equation defining Δ, we obtain
$$\begin{gathered}(\Delta - \Gamma)(A - B_2 B_2'\Delta) + (A - B_2 B_2'\Delta)'(\Delta - \Gamma) \\ +(\Delta - \Gamma)B_2 B_2'(\Delta - \Gamma) + R + X = 0.\end{gathered}$$
Since $R + X \geq 0$ and $A - B_2 B_2'\Delta$ is asymptotically stable, we conclude that $\Delta - \Gamma \geq 0$.

3. Let $\Gamma \geq 0$ be arbitrary. Let $\Delta \geq \Gamma$ be as constructed above. Then $P(t,T,0) \leq P(t,T,\Gamma) \leq P(t,T,\Delta)$ for all $t \leq T$, by Problem 5.8. Since $P(t,T,0)$ and $P(t,T,\Delta)$ both converge to the stabilizing solution to the algebraic Riccati equation, we conclude that $P(t,T,\Gamma)$ also converges to the stabilizing solution to the algebraic Riccati equation.



Solution 5.11. That the stated control law is optimal is immediate; that it is constant if the problem data are constant is also obvious, since in this case $P(t, t+T, \Delta) = P(0, T, \Delta)$, which is independent of t. A counter-example to the fallacious conjecture is

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with C arbitrary and $\Delta = \alpha I$. Then $F_T|_{T=0} = B_2'\Delta$ and

$$A - B_2 B_2' \Delta = \begin{bmatrix} -1-\alpha & 0 \\ 1 & 1 \end{bmatrix},$$

which is not asymptotically stable. The graph shows a plot of the real parts of the closed-loop poles versus the horizon length T if we take $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\alpha = 1$.

Solution 5.12. Kalman filter is

$$\dot{\widehat{x}} = A\widehat{x} + H(y - C\widehat{x}),$$

in which $H = QC'$. Hence the state estimation error equation is

$$\dot{x} - \dot{\widehat{x}} = (A - HC)(x - \widehat{x}) + \begin{bmatrix} B & -HD \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}.$$

The innovations process is

$$\eta = C(x - \widehat{x}) + Dv.$$

Therefore, the system $\boldsymbol{A}$ mapping $\begin{bmatrix} w' & v' \end{bmatrix}'$ to η is given by the realization

$$\boldsymbol{A} \overset{s}{=} \left[\begin{array}{c|c} A - HC & \begin{bmatrix} B & -HD \end{bmatrix} \\ \hline C & \begin{bmatrix} 0 & D \end{bmatrix} \end{array}\right].$$

Theorem 3.2.1 and the identity

$$(A - HC)Q + Q(A - HC)' + HH' + BB' = 0$$

shows that $\boldsymbol{A}\boldsymbol{A}^{\sim} = I$. Hence the power spectrum of η is I, which shows that η is white, with unit variance.

Solution 5.13. Combine the arbitrary filter given in the hint with the state dynamics to obtain

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ \tilde{x} \\ \xi \end{bmatrix} &= \begin{bmatrix} A & 0 & 0 \\ QC'C & A - QC'C & 0 \\ GC & 0 & F \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \\ \xi \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & QC'D \\ 0 & GD \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} \\ x - \widehat{x} &= \begin{bmatrix} I - JC & -H_1 & -H_2 \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \\ \xi \end{bmatrix} - \begin{bmatrix} 0 & JD \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}. \end{aligned}$$

Since v is white noise and $DD' = I$, $J \equiv 0$ is necessary and sufficient for $\mathcal{E}\{(\widehat{x}(t) - x(t))(\widehat{x}(t) - x(t))'\}$ to be finite. Set $J \equiv 0$ and denote the matrices in the realization above by $\tilde{A}$, $\tilde{B}$, $\tilde{C}$. We then have that

$$\mathcal{E}\{(\widehat{x}(t) - x(t))(\widehat{x}(t) - x(t))'\} = \tilde{C}(t)\tilde{P}(t)\tilde{C}',$$

in which $\tilde{P}$ is the solution to the equation

$$\dot{\tilde{P}} = \tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{B}\tilde{B}', \qquad \tilde{P}(0) = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Elementary algebra reveals that $\tilde{P}$ has the form

$$\tilde{P} = \begin{bmatrix} P & P - Q & X \\ P - Q & P - Q & X \\ X' & X' & Y \end{bmatrix},$$

in which P is the solution to

$$\dot{P} = AP + PA' + BB', \qquad P(0) = P_0,$$

and X and Y satisfy the linear matrix differential equations

$$\begin{aligned}
\dot{X} &= AX + PC'G' + XF', \qquad X(0) = 0 \\
\dot{Y} &= FY + YF' + GG' + GCX + XC'G', \qquad Y(0) = 0.
\end{aligned}$$

Since $\tilde{P} \geq 0$, we may write the Schur decomposition

$$\tilde{P} = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} I & 0 \\ Z' & I \end{bmatrix},$$

in which

$$\begin{aligned}
Z &= \begin{bmatrix} X \\ X \end{bmatrix} Y^{\#} \\
R &= \begin{bmatrix} P & P-Q \\ P-Q & P-Q \end{bmatrix} - ZYZ' = \begin{bmatrix} W+Q & W \\ W & W \end{bmatrix},
\end{aligned}$$

in which $W = P - Q - XY^{\#}X'$ and $(\cdot)^{\#}$ denotes the Moore-Penrose pseudo-inverse.[1]

It now follows that

$$\tilde{C}(t)\tilde{P}(t)\tilde{C}' = (\text{terms independent of } H_2) + (H_2 - H_2^*)Y(H_2 - H_2^*)',$$

in which $H_2^* = (I - H_1)XY^{\#}$. Now $Y \geq 0$, since $\tilde{P} \geq 0$, and it follows that an optimal choice for H_2 is H_2^*. With this choice for H_2, the cost is given by

$$\begin{aligned}
\tilde{C}(t)\tilde{P}(t)\tilde{C}' &= \begin{bmatrix} I & -H_1 \end{bmatrix} \begin{bmatrix} W+Q & W \\ W & W \end{bmatrix} \begin{bmatrix} I \\ -H_1' \end{bmatrix} \\
&= Q + (I - H_1)W(I - H_1)'.
\end{aligned}$$

From $\tilde{P} \geq 0$, it follows that $R \geq 0$ and hence that $W \geq 0$. Therefore, Q is the optimal state error covariance and $H_1^* = I$ is an optimal choice for H_1. This gives $H_2^* = 0$ as an optimal choice for H_2.

We now note that if $H_2 = 0$, the values of F and G are irrelevant, and an optimal filter is therefore $\dot{\widehat{x}} = A\widehat{x} + QC'(y - C\widehat{x})$, which is the Kalman filter.

Solution 5.14. The problem data are

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \sqrt{\sigma} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_1 = \sqrt{\rho} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad D_{11} = 0, \qquad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

[1] It is a fact that the Schur decomposition using a pseudo-inverse can always be nonnegative definite matrices. If $\begin{bmatrix} A & B \\ B' & C \end{bmatrix} \geq 0$, then $Cv = 0 \Rightarrow Bv = 0$, which is what makes it work.

$$C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D_{22} = 0.$$

Since $D'_{12}C_1 = 0$, there are no cross-terms in the control Riccati equation, which is

$$XA + A'X - XB_2B'_2X + C'_1C_1 = 0.$$

The stabilizing solution is easily verified to be

$$X = \alpha \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus $F = B'_2X = \alpha \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Similarly, since $B_1D'_{21} = 0$, the measurement and process noise are uncorrelated and the Kalman filter Riccati equation is

$$AY + YA' - YC'_2C_2Y + B_1B'_1 = 0.$$

It is easy to check that the stabilizing solution is

$$Y = \beta \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Hence $H = YC'_2 = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The optimal controller is therefore given by

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + B_2u + H(y - C_2\widehat{x}) \\ u &= -F\widehat{x}. \end{aligned}$$

Rewriting this, we obtain

$$\boldsymbol{K}^* \stackrel{s}{=} \left[\begin{array}{c|c} A - B_2F - HC_2 & H \\ \hline -F & 0 \end{array}\right].$$

Now

$$A - B_2F - HC_2 = \begin{bmatrix} 1-\beta & 1 \\ -(\alpha+\beta) & 1-\alpha \end{bmatrix}.$$

Evaluating $-F(sI - (A - B_2F - HC_2)^{-1}H$, we obtain

$$\boldsymbol{K}^* = \frac{\alpha\beta(1-2s)}{s^2 + (\alpha+\beta-2)s + 1 + \alpha\beta}.$$

The optimal cost is given by $\sqrt{\text{trace}(B'_1XB_1) + \text{trace}(FYF')} = \sqrt{5\alpha(\sigma+\alpha\beta)}$. The optimal cost is monotonically increasing in both ρ and σ.

Solutions to Problems in Chapter 6

Solution 6.1.

1. This follows by replacing u with $\tilde{u}$ and elementary algebra.
2. This follows from Theorems 6.2.1 and 6.2.4.
3. A direct application of (6.3.25) gives

$$\begin{bmatrix} \dot{\widehat{x}} \\ \tilde{u} \\ \begin{bmatrix} w - w^* \\ x - \widehat{x} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \tilde{A} - B_2 B_2' P & \begin{bmatrix} 0 & B_1 \end{bmatrix} & B_2 \\ 0 & \begin{bmatrix} -B_2' P & 0 \end{bmatrix} & I \\ \begin{bmatrix} 0 \\ -I \end{bmatrix} & \begin{bmatrix} -\gamma^{-2} B_1' P & I \\ I & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \begin{bmatrix} x \\ w \end{bmatrix} \\ r \end{bmatrix}$$

$$r = \begin{bmatrix} \boldsymbol{U} & \boldsymbol{V} \end{bmatrix} \begin{bmatrix} w - w^* \\ x - \widehat{x} \end{bmatrix}.$$

Since $u = \tilde{u} - D_{12}' C_1 x$, we obtain

$$\begin{bmatrix} \dot{\widehat{x}} \\ u \\ \begin{bmatrix} w - w^* \\ x - \widehat{x} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A - B_2 F & \begin{bmatrix} 0 & B_1 \end{bmatrix} & B_2 \\ 0 & \begin{bmatrix} -F & 0 \end{bmatrix} & I \\ \begin{bmatrix} 0 \\ -I \end{bmatrix} & \begin{bmatrix} -\gamma^{-2} B_1' P & I \\ I & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \begin{bmatrix} x \\ w \end{bmatrix} \\ r \end{bmatrix}$$

$$r = \begin{bmatrix} \boldsymbol{U} & \boldsymbol{V} \end{bmatrix} \begin{bmatrix} w - w^* \\ x - \widehat{x} \end{bmatrix},$$

in which $F = D_{12}' C_1 + B_2' P$.

4. See Section 5.2.3.
5. There exists an X_∞ satisfying

$$\tilde{A} X_\infty + \tilde{A}' X_\infty - X_\infty (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\infty + \tilde{C}' \tilde{C} = 0$$

such that $\tilde{A} - (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\infty$ is asymptotically stable and $X_\infty \geq 0$.

Solution 6.2. Using the properties of Φ (see Problem 3.3),

$$\begin{aligned}\dot{\lambda}(t) &= -B(t)u(t) + \int_t^T \frac{d}{dt}\big(\Phi'(\sigma,t)B(\sigma)u(\sigma)\big)\,d\sigma - A'(t)\Phi'(T,t)\lambda_T \\ &= -B(t)u(t) - \int_t^T A'(t)\Phi'(\sigma,t)B(\sigma)u(\sigma)\,d\sigma - A'(t)\Phi'(T,t)\lambda_T \\ &= -B(t)u(t) - A'(t)\lambda(t).\end{aligned}$$

Setting $B = C'C$ and $\lambda_T = \Delta x^*(T)$ shows that (6.2.7) satisfies (6.2.14).

Solution 6.3. For any vector z and any real number α, we have the identity

$$\begin{aligned}&\big(\alpha z_1 + (1-\alpha)z_2\big)'\big(\alpha z_1 + (1-\alpha)z_2\big) - \alpha z_1'z_1 - (1-\alpha)z_2'z_2 \\ &\qquad = -\alpha(1-\alpha)(z_1-z_2)'(z_1-z_2). \qquad (6.1)\end{aligned}$$

1. Suppose z is the response to inputs u and w, and $\tilde{z}$ is the response to inputs $\tilde{u}$ and w. The response to inputs $\alpha u + (1-\alpha)\tilde{u}$ and w is $z_\alpha = \alpha z + (1-\alpha)\tilde{z}$. Hence, for any $\alpha \in [0,1]$,

$$\begin{aligned}J(\alpha u + (1-\alpha)u, w) &= \int_0^T z_\alpha' z_\alpha - \gamma^2 w'w\,dt \\ &\le \int_0^T \alpha z'z + (1-\alpha)\tilde{z}'\tilde{z} - \gamma^2 w'w\,dt, \text{ by (6.1)} \\ &= \alpha\int_0^T z'z - \gamma^2 w'w\,dt + (1-\alpha)\int_0^T \tilde{z}'\tilde{z} - \gamma^2 w'w\,dt \\ &= \alpha J(u,w) + (1-\alpha)J(\tilde{u},w).\end{aligned}$$

   That is, J is convex in u.

2. Set $u = u^* = -B_2'Px = \boldsymbol{K}^*x$. Then $J(\boldsymbol{K}^*, w) = -\gamma^2\int_0^T (w-w^*)'(w-w^*)\,dt$, in which $w^* = \gamma^{-2}B_1'Px$. Let $\boldsymbol{W}$ be the closed-loop map from w to $w - w^*$, which is linear. Let $w \mapsto (w - w^*)$ and $\tilde{w} \mapsto (\tilde{w} - \tilde{w}^*)$. That is, w^* is produced by the input w, with the controller $\boldsymbol{K}^*$ in place, and $\tilde{w}^*$ is produced by the input $\tilde{w}$, with with the controller $\boldsymbol{K}^*$ in place. Then $\alpha w + (1-\alpha)\tilde{w} \mapsto \alpha(w-w^*) + (1-\alpha)(\tilde{w}-\tilde{w}^*)$. Thus

$$\begin{aligned}&J(\boldsymbol{K}^*, \alpha w + (1-\alpha)\tilde{w}) \\ &= -\gamma^2\int_0^T \Big(\big(\alpha(w-w^*) + (1-\alpha)(\tilde{w}-\tilde{w}^*)\big)' \\ &\qquad \times \big(\alpha(w-w^*) + (1-\alpha)(\tilde{w}-\tilde{w}^*)\big)\Big)\,dt \\ &\ge -\gamma^2\int_0^T \alpha(w-w^*)'(w-w^*) + (1-\alpha)(\tilde{w}-\tilde{w}^*)'(\tilde{w}-\tilde{w}^*)\,dt \\ &= \alpha J(u^*,w) + (1-\alpha)J(u^*,\tilde{w}).\end{aligned}$$

That is, J is concave in w.

3. Suppose $\boldsymbol{K}$ is a linear, full-information controller that makes J strictly concave in w. That is,

$$J(\boldsymbol{K}, \alpha w + (1-\alpha)\tilde{w}) > \alpha J(\boldsymbol{K}, w) + (1-\alpha)J(\boldsymbol{K}, \tilde{w}) \tag{6.2}$$

for all $w \neq \tilde{w}$ and all $\alpha \in (0,1)$. Let $w \neq \tilde{w}$, $\alpha \in (0,1)$ and let z and $\tilde{z}$ be the responses of the closed-loop system $\boldsymbol{R}_{zw}$ for inputs w and $\tilde{w}$ respectively. From (6.2) and the identity (6.1), we conclude that

$$-\alpha(1-\alpha)\int_0^T \big((z-\tilde{z})'(z-\tilde{z}) - \gamma^2(w-\tilde{w})'(w-\tilde{w})\big)\,dt > 0.$$

Taking $\tilde{w} = 0$, which implies $\tilde{z} = 0$, we see that

$$\int_0^T z'z - \gamma^2 w'w\,dt < 0 \qquad \text{for all } w \neq 0.$$

Therefore, $\|\boldsymbol{R}\|_{[0,T]} < \gamma$ and we conclude that $P(t)$ exists on $[0,T]$.

Solution 6.4. Since $\boldsymbol{R}_{zw}$ is causal and linear, $\gamma(\boldsymbol{R}_{zw}) < \gamma$ if and only if $\|\boldsymbol{R}_{zw}\|_{[0,T]} < \gamma$ for all finite T. Hence, there exists a controller such that $\gamma(\boldsymbol{R}_{zw}) < \gamma$ if and only if the Riccati differential equation

$$-\dot{P} = PA + A'P - P(B_2B_2' - \gamma^{-2}B_1B_1')P + C'C, \qquad P(T) = 0$$

has a solution on $[0,T]$ for all finite T.

Solution 6.5.

1. $P(t,T,\Delta) = \Psi_2(t)\Psi_1^{-1}(t)$ is a solution to the Riccati equation provided

$$\frac{d}{dt}\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = H\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \qquad \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}(T) = \begin{bmatrix} I \\ \Delta \end{bmatrix}. \tag{6.3}$$

(see Section 6.2.3 for details.) We therefore verify that the given formulas for Ψ_1 and Ψ_2 do indeed satisfy this linear differential equation. We are going to find the solutions Ψ_i and Ψ_2 via the change of variables

$$\begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} = Z^{-1}\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix},$$

since $Z^{-1}HZ$ block-diagonalizes H.

Let $\tilde{\Psi}_1(t)$ and $\tilde{\Psi}_2(t)$ be the solutions to

$$\frac{d}{dt}\begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}\begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix}(T) = Z^{-1}\begin{bmatrix} I \\ \Delta \end{bmatrix}.$$

That is,

$$\begin{aligned} \tilde{\Psi}_1(t) &= e^{\Lambda(t-T)}\tilde{\Psi}_1(T) \\ \tilde{\Psi}_2(t) &= e^{-\Lambda(t-T)}\tilde{\Psi}_2(T). \end{aligned}$$

The boundary condition can be written as

$$Z\begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix}(T) = \begin{bmatrix} I \\ \Delta \end{bmatrix},$$

from which we obtain

$$\begin{aligned} Z_{11}\tilde{\Psi}_1(T) + Z_{12}\tilde{\Psi}_2(T) &= I \\ Z_{21}\tilde{\Psi}_1(T) + Z_{22}\tilde{\Psi}_2(T) &= \Delta. \end{aligned}$$

Multiplying the first equation by Δ and subtracting from the second, we obtain

$$(Z_{21} - \Delta Z_{11})\tilde{\Psi}_1(T) + (Z_{22} - \Delta Z_{12})\tilde{\Psi}_2(T) = 0.$$

Therefore,

$$\begin{aligned} \tilde{\Psi}_2(T) &= -(Z_{22} - \Delta Z_{12})^{-1}(Z_{21} - \Delta Z_{11})\tilde{\Psi}_1(T) \\ &= X\tilde{\Psi}_1(T). \end{aligned}$$

Hence

$$\begin{aligned} \tilde{\Psi}_2(t) &= e^{-\Lambda(t-T)}\tilde{\Psi}_2(T) \\ &= e^{-\Lambda(t-T)}X\tilde{\Psi}_1(T) \\ &= e^{\Lambda(T-t)}Xe^{\Lambda(T-t)}\tilde{\Psi}_1(t). \end{aligned}$$

Then

$$\begin{aligned} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} &= Z\begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} \\ &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I \\ e^{\Lambda(T-t)}Xe^{\Lambda(T-t)} \end{bmatrix}\tilde{\Psi}_1(t) \end{aligned}$$

is the solution to (6.3).

2. From $HZ = Z \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$, we obtain

$$C'CZ_{12} + A'Z_{22} = Z_{22}\Lambda. \tag{6.4}$$

Hence

$$Z_{12}'C'CZ_{12} + Z_{12}'A'Z_{22} = Z_{12}'Z_{22}\Lambda = Z_{22}'Z_{12}\Lambda.$$

Suppose $Z_{22}x = 0$. Multiplying the above equation on the left by x' and on the right by x we see that $CZ_{12}x = 0$. Now multiply (6.4) on the right by x to obtain $Z_{22}\Lambda x = 0$. We can now multiply on the left by $x'\Lambda'$ and on the right by Λx, and so on, to obtain

$$Z_{22}x = 0 \Rightarrow \begin{bmatrix} Z_{22} \\ CZ_{12} \end{bmatrix} \Lambda^k x = 0,\ \ k = 0, 1, 2, \ldots.$$

We first prove that Z_{22} is nonsingular if (A, C) is detectable. Suppose, to obtain a contradiction, that $x \neq 0$ and $Z_{22}x = 0$. Then, by the above reasoning, $\left(\Lambda, \begin{bmatrix} Z_{22} \\ CZ_{12} \end{bmatrix}\right)$ is not observable. Therefore, there exists a $y \neq 0$ such that

$$\begin{bmatrix} \Lambda - \lambda I \\ Z_{22} \\ CZ_{12} \end{bmatrix} y = 0.$$

Note that $\mathrm{R}_e(\lambda) \leq 0$ because $\mathrm{R}_e(\lambda_i(\Lambda)) \leq 0$. From the $(1, 1)$-partition of $HZ = Z \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$, we obtain $AZ_{12}y = -\lambda Z_{12}y$. Therefore,

$$\begin{aligned} (\lambda I + A)Z_{12}y &= 0 \\ CZ_{12}y &= 0. \end{aligned}$$

Hence $Z_{12}y = 0$, since (A, C) is detectable. This gives $\begin{bmatrix} Z_{12} \\ Z_{22} \end{bmatrix} y = 0$, which implies $y = 0$, since Z is nonsingular. This contradicts the hypothesis that there exists an $x \neq 0$ such that $Z_{22}x = 0$ and we conclude that Z_{22} is nonsingular.

We now prove the rank defect of Z_{22} (the dimension of $\ker(Z_{22})$) is equal to the number of undetectable modes of (A, C). Let the columns of V be a basis for $\ker(Z_{22})$. Arguments parallel to those above show that

$$\begin{bmatrix} Z_{22} \\ CZ_{12} \end{bmatrix} \Lambda^k V = 0,\ \ k = 0, 1, 2, \ldots.$$

Furthermore $AZ_{12}V\alpha = -Z_{12}\Lambda V\alpha = -Z_{12}V\beta$ (for some β), so $Z_{12}V$ is an unstable A-invariant subspace contained in $\ker(C)$. That is, $Z_{12}V$ is a subset of the undetectable subspace of (A, C). Since Z is nonsingular, $\text{rank} Z_{12}V = \text{rank} V = \dim\ker(Z_{22})$. Thus (A, C) has at least as many undetectable modes as the rank defect of Z_{22}. For the converse, suppose W is a basis for the undetectable subspace of (A, C). Then

$$\begin{aligned} AW &= W\Sigma, \ \text{R}_e\lambda_i(\Sigma) > 0, \\ CW &= 0. \end{aligned}$$

(Strict inequality holds because we assume that (A, C) has no unobservable modes on the imaginary axis.) Let

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Sigma &= Z^{-1} H \begin{bmatrix} W \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} Z^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \Lambda X_1 \\ -\Lambda X_2 \end{bmatrix}. \end{aligned}$$

Since $\text{R}_e\lambda_i(\Sigma) > 0$ and $\text{R}_e\lambda_i(\Lambda) \leq 0$, $X_1 = 0$. From

$$\begin{bmatrix} W \\ 0 \end{bmatrix} = Z \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = Z \begin{bmatrix} 0 \\ X_2 \end{bmatrix} = \begin{bmatrix} Z_{12}X_2 \\ Z_{22}X_2 \end{bmatrix},$$

we see that $Z_{22}X_2 = 0$ and that $W = Z_{12}X_2$. We conclude that the dimension of the $\ker Z_{22}$ is (at least) the number of undetectable modes of (A, C).

3. Multiply $HZ = Z \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$ on the right by $\begin{bmatrix} Z_{11}^{-1} \\ 0 \end{bmatrix}$. The upper block of the resulting equation is $A - (B_2B_2' - \gamma^{-2}B_1B_1')\Pi = Z_{11}\Lambda Z_{11}^{-1}$.

Solution 6.6.

1. Suppose $Px = 0$, $x \neq 0$. Multiplying the Riccati equation by x^* on the left and by x on the right reveals that $Cx = 0$. Now multiplying by x on the right reveals that $PAx = 0$. Thus $\ker(P)$ is an A-invariant subspace. Hence, there exists a $y \neq 0$ such that $Py = 0$, $Cy = 0$ and $Ay = \lambda y$, which contradicts the assumed observability of (A, C). Thus (A, C) observable implies P is nonsingular.

2. $(A - SP)$ is asymptotically stable.

   Suppose (A, C) has no stable unobservable modes. If $Px = 0$ for some $x \neq 0$, then (as in part 1), there exists a $y \neq 0$ such that $Py = 0$, $Cy = 0$ and $Ay = \lambda y$. Thus $(A - SP)y = Ay = \lambda y$, which implies $\mathrm{R}_e(\lambda) < 0$, since $A - SP$ is asymptotically stable. This contradicts the assumption that (A, C) has no stable unobservable modes and we conclude that P is nonsingular.

   Conversely, suppose P is nonsingular. If $Ax = \lambda x$, $Cx = 0$ for some $x \neq 0$, then multiplying the Riccati equation on the right by x results in $(A - SP)'Px = -\lambda Px$. Since $(A - SP)$ is asymptotically stable and P is nonsingular, we have $\mathrm{R}_e(\lambda) > 0$. This shows that any unobservable mode is in the right-half plane, which is equivalent to the proposition that (A, C) has no unobservable modes that are stable.

3. The fact that the given P satisfies the Riccati equation is easily verified. Also, since

$$A - SP = \begin{bmatrix} A_{11} - S_{11}P_1 & 0 \\ A_{21} - S_{12}'P_1 & A_{22} \end{bmatrix},$$

   we see that $A - SP$ is asymptotically stable.

Solution 6.7. Since (A, C) is observable, X and Y are nonsingular (see Problem 6.6). Therefore,

$$\begin{aligned} AX^{-1} + X^{-1}A' - S + X^{-1}C'CX^{-1} &= 0 \\ AY^{-1} + Y^{-1}A' - S + Y^{-1}C'CY^{-1} &= 0. \end{aligned}$$

Subtract these to obtain

$$AZ + ZA' + X^{-1}C'CX^{-1} - Y^{-1}C'CY^{-1} = 0$$

in which $Z = X^{-1} - Y^{-1}$. Some elementary algebra reveals that this equation can be re-written as

$$-(A + X^{-1}C'C)Z - Z(A + X^{-1}C'C)' + ZC'CZ = 0.$$

Now $-(A + X^{-1}C'C)' = X(A - SX)X^{-1}$, so $-(A + X^{-1}C'C)$ is asymptotically stable. Hence, $Z \geq 0$, which is to say $X^{-1} \geq Y^{-1}$. Since $X \geq 0$ and $Y \geq 0$, this is equivalent to $Y \geq X$.

This problem shows that the stabilizing solution is the smallest of any nonnegative definite solutions.

Solution 6.8.

1.

$$\begin{aligned}
&\big(I + B_2'(-sI - A')^{-1}PB_2\big)\big(I + B_2'P(sI - A)^{-1}B_2\big)\\
&= \; I + B_2'P(sI - A)^{-1}B_2 + B_2'(-sI - A')^{-1}PB_2\\
&\quad +B_2'(-sI - A')^{-1}\big(-P(sI - A) - (-sI - A')P\\
&\quad\quad +C'C + \gamma^{-2}PB_1B_1'P\big)(sI - A)^{-1}B_2\\
&= \; I + B_2'(-sI - A')^{-1}\big(C'C + \gamma^{-2}PB_1B_1'P\big)(sI - A)^{-1}B_2.
\end{aligned}$$

2. Immediate from setting $B_2 = b_2$ in Part 1.
3. The Nyquist diagram of $-b_2'P(sI - A)^{-1}b_2$ cannot enter the circle of radius one, centered on $s = 1$. The stated gain and phase margins follow from this fact. (see Problem 5.4 for more details.)

Solution 6.9. Substituting $u = -px$ and using $x = (1 - \gamma^{-2})q$, we have

$$\begin{aligned}
(1 - \gamma^{-2})\dot{q} &= -\big(\gamma^{-2} + \sqrt{1 + c^2(1 - \gamma^{-2})}\big)q + w\\
z &= \begin{bmatrix} c(1 - \gamma^{-2}) \\ -(1 + \sqrt{1 + c^2(1 - \gamma^{-2})}) \end{bmatrix} q.
\end{aligned}$$

When $\gamma \to 1$, $q = w/2$ and $z = \begin{bmatrix} 0 \\ -w \end{bmatrix}$.

Solution 6.10. $X(t) = \Pi - P(t, T, P_2)$ satisfies the equation

$$-\dot{X} = X\widehat{A} + \widehat{A}'X + X(B_2B_2' - \gamma^{-2}B_1B_1')X, \qquad X(T) = 0. \tag{6.5}$$

in which $\widehat{A} = A - (B_2B_2' - \gamma^{-2}B_1B_1')\Pi$. Suppose $\big(\Pi - P(t^*)\big)x = 0$ for some t^*. Then $x'(6.5)x$ yields $x'\dot{X}(t^*)x = 0$ and it follows that $\dot{X}(t^*)x = 0$, since $\dot{X} \le 0$ by the monotonicity property of $P(t, T, P_2)$. Now $(6.5)x$ yields $X(t^*)\widehat{A}x = 0$ and we conclude that $\ker X(t^*)$ is an $\widehat{A}$ invariant subspace. Therefore, there exists a y such that $X(t^*)y = 0$ and $\widehat{A}y = \lambda y$. (It follows as before that $\dot{X}y = 0$.) Hence $X(t)y$ is a solution to $\dot{\alpha} = 0$, $\alpha(t^*) = 0$ and we conclude that $X(t)y = 0$ for all t. Therefore, without loss of generality (by a change of state-space variables if necessary), $\Pi - P(t)$ has the form

$$\Pi - P(t) = \begin{bmatrix} \Pi_1 - P_1(t) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{for all } t \le T, \tag{6.6}$$

in which $\Pi_1 - P_1(t)$ is nonsingular for all t. Furthermore, since $\ker(\Pi - P(t))$ is $\widehat{A}$ invariant

$$\widehat{A} = \begin{bmatrix} \widehat{A}_{11} & 0 \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}.$$

The asymptotic stability of the $\widehat{A}_{11}$ follows as in the text. We therefore need to establish the asymptotic stability of $\widehat{A}_{22}$. Setting $t = T$ in (6.6), we see that

$$\Pi - P_2 = \begin{bmatrix} (\Pi - P_2)_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

From equation (6.3.20), $(P - P_2)y = 0$ implies $B_1'Py = 0$. Therefore, $\widehat{A}_{22} = (A - B_2B_2'P)_{22}$, which is asymptotically stable by Lemma 6.3.4.

Solution 6.11.

1. From the Riccati equation, we have $-(A + P_2^{-1}C'C) = P_2^{-1}(A - B_2B_2'P_2)'P_2$, so $-(A + P_2^{-1}C'C)$ is asymptotically stable.
2. Suppose a stabilizing, nonnegative definite P exists. Since $P \geq P_2 > 0$, P is nonsingular. Therefore, we may write

$$\begin{aligned} AP_2^{-1} + P_2^{-1}A' - B_2B_2' + P_2^{-1}C'CP_2^{-1} &= 0 \\ AP^{-1} + P^{-1}A' - B_2B_2' + \gamma^{-2}B_1B_1' + P^{-1}C'CP^{-1} &= 0. \end{aligned}$$

Subtracting these equations and a little algebra yields

$$\begin{aligned} 0 = \ & (A + P_2^{-1}C'C)(P_2^{-1} - P^{-1}) + (P_2^{-1} - P^{-1})(A + P_2^{-1}C'C)' \\ & -\gamma^{-2}B_1B_1' - (P_2^{-1} - P^{-1})C'C(P_2^{-1} - P^{-1}). \end{aligned}$$

Multiplying by γ^2 and defining $Y = \gamma^2(P_2^{-1} - P^{-1})$ yields the given Riccati equation for Y. From the Riccati equation for P, we see that

$$\begin{aligned} -(A + P_2^{-1}C'C) + \gamma^{-2}YC'C &= -(A + P^{-1}C'C) \\ &= P^{-1}\big(A - (B_2B_2' - \gamma^{-2}B_1B_1')P\big)'P. \end{aligned}$$

Hence Y is the stabilizing solution. Since $\gamma^2 I - P_2Y = \gamma^2 P_2P^{-1}$ and P and P_2 are positive definite, we conclude that $\gamma^2 > \rho(P_2Y)$.

Conversely, suppose Y is a stabilizing solution to the stated Riccati equation and $\gamma^2 > \rho(P_2Y)$. Then $X = P_2^{-1} - \gamma^{-2}Y$ is positive definite and satisfies

$$AX + XA' - B_2B_2' + \gamma^{-2}B_1B_1' + XC'CX = 0.$$

Hence $P = X^{-1}$ satisfies the Riccati equation (6.3.5). Furthermore,

$$\begin{aligned} -(A + P_2^{-1}C'C) + \gamma^{-2}YC'C &= -(A + P^{-1}C'C) \\ &= P^{-1}\big(A - (B_2B_2' - \gamma^{-2}B_1B_1')P\big)'P, \end{aligned}$$

so P is the stabilizing solution.

3. A suitable controller exists if and only if a stabilizing, nonnegative definite solution to the Riccati equation (6.3.5) exists, which we have just shown is equivalent to the existence of a stabilizing solution Y such that $\rho(P_2Y) < \gamma^2$. Since $C = 0$ and $-A$ is asymptotically stable, Y is the controllability gramian of $(-A, B_1)$. Therefore, a suitable controller exists if and only if $\gamma^2 > \rho(P_2Y)$.

   In the case that $C = 0$, this result gives a formula for the optimal performance level.

Solution 6.12.

1.

$$\begin{aligned}
\|z\|_2^2 - \gamma^2\|w\|_2^2 &= \int_0^\infty (z'z - \gamma^2 w'w)\, dt \\
&= \int_0^\infty \begin{bmatrix} z \\ w \end{bmatrix}' \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} dt \\
&= \int_0^\infty \begin{bmatrix} w \\ u \end{bmatrix}' \boldsymbol{G}' J \boldsymbol{G} \begin{bmatrix} w \\ u \end{bmatrix} dt \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \begin{bmatrix} w \\ u \end{bmatrix}' \boldsymbol{G}^* J \boldsymbol{G} \begin{bmatrix} w \\ u \end{bmatrix} d\omega
\end{aligned}$$

by Parseval's theorem.

2.

$$\begin{aligned}
\boldsymbol{G}^\sim J \boldsymbol{G} &\stackrel{s}{=} \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ -C'C & -A' & 0 & 0 \\ \hline 0 & B_1' & -\gamma^2 I & 0 \\ 0 & B_2' & 0 & I \end{array}\right] \\
&\stackrel{s}{=} \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ -C'C - PA - A'P & -A' & -PB_1 & -PB_2 \\ \hline B_1'P & B_1' & -\gamma^2 I & 0 \\ B_2'P & B_2' & 0 & I \end{array}\right] \\
&\stackrel{s}{=} \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ -P(B_2B_2' - \gamma^{-2}B_1B_1')P & -A' & -PB_1 & -PB_2 \\ \hline B_1'P & B_1' & -\gamma^2 I & 0 \\ B_2'P & B_2' & 0 & I \end{array}\right] \\
&= \boldsymbol{W}^\sim \tilde{J} \boldsymbol{W}.
\end{aligned}$$

In the above,

$$\tilde{J} = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

The dimension of the $-\gamma^2 I$-block is the same as the dimension of w and the dimension of the I-block is the dimension of u. (In J, the dimension of the I-block is the dimension of z.)

3. The A-matrix of $\boldsymbol{W}^{-1}$ is $\widehat{A} = A-(B_2B_2'-\gamma^{-2}B_1B_1')P$ which is asymptotically stable if P is the stabilizing solution. Hence $\boldsymbol{W}^{-1} \in \mathcal{H}_\infty$. Using Problem 3.6, we obtain

$$\boldsymbol{G}\boldsymbol{W}^{-1} \stackrel{s}{=} \left[\begin{array}{c|cc} \widehat{A} & B_2 & B_1 \\ \hline C & 0 & 0 \\ -DB_2'P & D & 0 \\ \gamma^{-2}B_1'P & 0 & I \end{array}\right],$$

Hence $\boldsymbol{G}\boldsymbol{W}^{-1} \in \mathcal{H}_\infty$.

4. By direct evaluation,

$$\begin{aligned}
&(\boldsymbol{G}\boldsymbol{W}^{-1})^* J(\boldsymbol{G}\boldsymbol{W}^{-1}) \\
&= \tilde{J} + \begin{bmatrix} B_2' \\ B_1' \end{bmatrix} (\bar{s}I - \widehat{A}')^{-1}\big(C'C + P(B_2B_2' - \gamma^{-2}B_1B_1')P \\
&\qquad -(\bar{s}I - \widehat{A}')P - P(sI - \widehat{A})\big)(sI - \widehat{A})^{-1} \begin{bmatrix} B_2 & B_1 \end{bmatrix} \\
&= \tilde{J} - (s+\bar{s}) \begin{bmatrix} B_2' \\ B_1' \end{bmatrix} (\bar{s}I - \widehat{A}')^{-1} P (sI - \widehat{A})^{-1} \begin{bmatrix} B_2 & B_1 \end{bmatrix} \\
&\leq \tilde{J}, \qquad \text{for } (s+\bar{s})P \geq 0.
\end{aligned}$$

The equation $\begin{bmatrix} u - u^* \\ w - w^* \end{bmatrix} = \boldsymbol{W} \begin{bmatrix} w \\ u \end{bmatrix}$ follows immediately from $u^* = -B_2'Px$ and $w^* = \gamma^{-2}B_1'Px$.

If $u = u^*$, then $\begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \end{bmatrix} \begin{bmatrix} w \\ u^* \end{bmatrix} = 0$. Hence $u^* = -\boldsymbol{W}_{12}^{-1}\boldsymbol{W}_{11}w$. (Note: It can be shown that the J-lossless property implies that $\boldsymbol{W}_{12}^{-1}$ is nonsingular.)

Solution 6.13.

1. Suppose there exists a measurement feedback controller $u = \boldsymbol{K}y$ that achieves the objective. Then

$$u = \boldsymbol{K} \begin{bmatrix} C_2 & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

is a full-information controller that achieves the objective. Therefore, $P(t)$ exists.

Conversely, if $P(t)$ exists, then $u = -B_2'Px$ is a controller that achieves the objective. Now consider the measurement feedback controller defined by

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + B_1(y - C_2\widehat{x}) + B_2u, \qquad x(0) = 0 \\ u &= -B_2'P\widehat{x}. \end{aligned}$$

Note that $\widehat{x}$ is a copy of x, since

$$\frac{d}{dt}(\widehat{x} - x) = (A - B_1C_2)(\widehat{x} - x), \qquad \widehat{x}(0) - x(0) = 0.$$

Consequently, the measurement feedback controller $u = -B_2'P\widehat{x}$ generates the same control signal and hence the same closed loop as the controller $u = -B_2'Px$. Therefore, it is a measurement feedback controller that achieves the objective.

The generator of all controllers is obtained by noting that all closed-loops generated by full-information controllers are generated by $u - u^* = \boldsymbol{U}(w - w^*)$, in which $u^* = -B_2'Px$ and $w^* = \gamma^{-2}B_1'Px$. Replacing x with $\widehat{x}$ and replacing w by $y - C_2\widehat{x}$ results in the LFT

$$\begin{aligned} \begin{bmatrix} \dot{\widehat{x}} \\ u \\ w - w^* \end{bmatrix} &= \begin{bmatrix} A - B_1C_2 - B_2B_2'P & B_1 & B_2 \\ -B_2'P & 0 & I \\ -(C_2 + \gamma^{-2}B_1'P) & I & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ y \\ r \end{bmatrix}, \\ r &= \boldsymbol{U}(w - w^*). \end{aligned}$$

2. Since the measurement feedback controller $u = -B_2'\widehat{x}$ generates the same closed-loop as the controller $u = -B_2'x$, the closed loop generated by $u = -B_2'\widehat{x}$ is stable. To conclude internal stability, we need to show that no unstable cancellations occur. The cancellations which occur are at the eigenvalues of $A - B_1C_2$, since these are uncontrollable modes (see the error dynamics equation in the preceding part). Thus, the measurement feedback controller is internally stabilizing if and only if $A - B_1C_2$ is asymptotically stable.

   (You may like to connect the controller generator $\boldsymbol{K}_a$ to the generalized plant using the inter-connection formula of Lemma 4.1.1 and verify that the modes associated with $A - B_1C_2$ are uncontrollable).

Solution 6.14. Let $A = \text{diag}(s_i)$, and let G^* and H^* be the matrices with rows g_i^* and h_i^*. The dynamics are therefore

$$\dot{x} = Ax - H^*w + G^*u.$$

1. Recall that any control signal (and hence any closed-loop transfer function matrix $\boldsymbol{R}$) that can be generated by a stabilizing full-information controller can be generated by

$$u = -Fx + \boldsymbol{U}w,$$

in which F is a stabilizing state feedback and $\boldsymbol{U} \in \mathcal{H}_\infty$ (see Section 4.2.2). Taking the Laplace transform of the dynamics and substituting the control law $u = -Fx + \boldsymbol{U}w$, we obtain

$$x = \big(sI - (A - G^*F)\big)^{-1}(G^*\boldsymbol{U} - H^*)w.$$

The closed-loop $\boldsymbol{R}$ maps w to u. Hence

$$\boldsymbol{R} = \boldsymbol{U} - F\big(sI - (A - G^*F)\big)^{-1}(G^*\boldsymbol{U} - H^*). \tag{6.7}$$

Now note the identity

$$\begin{aligned} I &= \big(sI - (A - G^*F)\big)\big(sI - (A - G^*F)\big)^{-1} \\ &= (sI - A)\big(sI - (A - G^*F)\big)^{-1} + G^*F\big(sI - (A - G^*F)\big)^{-1}. \end{aligned}$$

Therefore $G^*\boldsymbol{R}$ is given by

$$G^*\boldsymbol{R} = H^* + (sI - A)\big(sI - (A - G^*F)\big)^{-1}(G^*\boldsymbol{U} - H^*). \tag{6.8}$$

Suppose $\boldsymbol{R}$ is a closed-loop system generated by a stabilizing, full-information controller. Then $G^*\boldsymbol{R}$ is given by (6.8) for some $\boldsymbol{U} \in \mathcal{H}_\infty$ and some F such that $A - G^*F$ is asymptotically stable. This implies that the zeros of $sI - A$, which are in the right-half plane, cannot be cancelled by the poles of $\big(sI - (A - G^*F)\big)^{-1}(G^*\boldsymbol{U} - H^*)$, which are in the left-half plane. Hence, since the i^{th} row of $sI - A$ is zero for $s = s_i$, we obtain the interpolation equation $g_i^*\boldsymbol{R} = h_i$.

Conversely, suppose $\boldsymbol{R}$ satisfies the interpolation constraints. We want $\boldsymbol{R}$ to be the map from w to u for some stabilizing controller. We therefore back solve for $\boldsymbol{U}$ and then show that the satisfaction of the interpolation constraints ensures that $\boldsymbol{U}$ is stable. To back solve for $\boldsymbol{U}$, simply note that $\boldsymbol{R} : w \mapsto u$ implies that $sx = Ax + (G^*\boldsymbol{R} - H^*)w$. Hence $u + Fx$ is given by $u + Fx = \big(\boldsymbol{R} + F(sI - A)^{-1}(G^*\boldsymbol{R} - H^*)\big)w$. Therefore, $\boldsymbol{U}$, the map from w to $u + Fx$, is given by

$$\boldsymbol{U} = \boldsymbol{R} + F(sI - A)^{-1}(G^*\boldsymbol{R} - H^*).$$

Since the i^{th} row of $G^*\boldsymbol{R} - H^*$ whenever i^{th} row of $(sI - A)$ is also zero, namely when $s = s_i$, we conclude that $\boldsymbol{U} \in \mathcal{H}_\infty$.

2. Using the result of Item 1, we can determine the existence of an interpolating $\boldsymbol{R}$ such that $\|\boldsymbol{R}\|_\infty < \gamma$ by invoking our full-information $\mathcal{H}_\infty$ control results. Thus, $\boldsymbol{R}$ exists if and only if there exists a stabilizing solution to the Riccati equation
$$PA + A^*P - P(G^*G - \gamma^{-2}H^*H)P = 0$$
such that $P \geq 0$.

Suppose $\boldsymbol{R}$ exists. Then a stabilizing $P \geq 0$ exists. Because all the eigenvalues of A are in the right-half plane, we must have $P > 0$ (see Problem 6.6). Define $M = P^{-1}$. Then $M > 0$ satisfies the equation
$$AM + MA^* - G^*G + \gamma^{-2}H^*H = 0,$$
from which it follows that
$$M_{ij} = \frac{g_i^* g_j - \gamma^{-2} h_i^* h_j}{s_i + \bar{s}_j}.$$
Conversely, suppose M given by this formula is positive definite and define $P = M^{-1} > 0$. Then $A - (G^*G - \gamma^{-2}H^*H)P = -P^{-1}A^*P$, which is asymptotically stable. Thus $P = M^{-1}$ is a positive definite, stabilizing solution to the Riccati equation and we conclude that $\boldsymbol{R}$ exists.

3. Substituting into the generator of all closed-loops for the full-information problem, we see that all solutions are generated by the LFT $\boldsymbol{R} = \mathcal{F}_\ell(\boldsymbol{R}_a, \boldsymbol{U})$, in which $\boldsymbol{U} \in \mathcal{H}_\infty$, $\|\boldsymbol{U}\|_\infty < \gamma$ and
$$\boldsymbol{R}_a \stackrel{s}{=} \left[\begin{array}{c|cc} A - G^*GP & -H^* & G^* \\ \hline -GP & 0 & I \\ \gamma^{-2}HP & I & 0 \end{array}\right].$$
For example, solutions to the scalar interpolation problem $\boldsymbol{r}(1) = 1$ such that $\|\boldsymbol{r}\|_\infty < \gamma$ exist when $(1-\gamma^{-2})/(1+1) > 0$, *i.e.*, $\gamma > 1$. All solutions are generated by $\mathcal{F}_\ell(\boldsymbol{r}_a, \boldsymbol{u})$, in which
$$\boldsymbol{r}_a \stackrel{s}{=} \left[\begin{array}{c|cc} -(1+\gamma^{-2})/(1-\gamma^{-2}) & -1 & 1 \\ \hline -2/(1-\gamma^{-2}) & 0 & 1 \\ 2\gamma^{-2}/(1-\gamma^{-2}) & 1 & 0 \end{array}\right].$$
It is easy to verify that the $\boldsymbol{r}_{a11} = \frac{2}{s(1-\gamma^{-2})+1+\gamma^{-2}}$, which clearly interpolates the data and $\|\boldsymbol{r}_{a11}\|_\infty = 2/(1+\gamma^{-2})$, which is less than γ provided $\gamma > 1$. It is also easy to see that $\boldsymbol{r}_{a12}(1) = 0$, from which it follows that $\mathcal{F}_\ell(\boldsymbol{r}_a, \boldsymbol{u})(1) = \boldsymbol{r}_{a11}(1) = 1$. Note that since $\boldsymbol{r}(1) = 1$ is a one-point interpolation problem, $\gamma_{opt} \geq 1$ follows from the maximum modulus principle, and consideration of the constant interpolating function $\boldsymbol{r} = 1$ shows that $\gamma_{opt} = 1$. Can you explain what happens in the parametrization of all suboptimal ($\gamma > 1$) solutions above as $\gamma \to 1$?

4. Let $-AX - XA^* + G^*G = 0$ and $-AY - YA^* + H^*H = 0$. Note that $X \geq 0$ and $Y \geq 0$, since $-A$ is asymptotically stable. Then $M = X - \gamma^{-2}Y = \gamma^{-2}X(\gamma^2 I - X^{-1}Y)$. Therefore, $\gamma_{opt} = \sqrt{\lambda_{max}(X^{-1}Y)}$. Note that X is nonsingular provided none of the g_i's is zero.

Solution 6.15.

1. Stability of the loop is an immediate consequence of the small gain theorem.
2. Immediate from the full-information synthesis results.

Solution 6.16.

1. Completing the square with the Riccati equation yields

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 = x_0'Px_0 + \|u - u^*\|_2^2 - \gamma^2\|w - w^*\|_2^2.$$

Setting $u = u^*$ and $w \equiv 0$, we see that $\|z\|_2^2 = x_0'Px_0 - \gamma^2\|w^*\|_2^2 \leq x_0'Px_0$.

2. The closed-loop $\boldsymbol{R}_{zw}$ with $u = u^*$ is given by

$$\begin{aligned} \dot{x} &= (A - B_2B_2'P)x + B_1w \\ z &= \begin{bmatrix} C \\ -DB_2'P \end{bmatrix} x. \end{aligned}$$

Hence, by Theorem 3.3.1, $\|\boldsymbol{R}_{zw}\|_2^2 = \text{trace}(B_1'QB_1)$, in which Q is the observability gramian, which satisfies

$$Q(A - B_2B_2'P) + (A - B_2B_2'P)'Q + PB_2B_2'P + C'C = 0.$$

The Riccati equation for P can be written as

$$P(A - B_2B_2'P) + (A - B_2B_2'P)'P + P(B_2B_2' + \gamma^{-2}B_1B_1')P + C'C = 0.$$

Subtracting these results in

$$(P - Q)(A - B_2B_2'P) + (A - B_2B_2'P)'(P - Q) + \gamma^{-2}PB_1B_1'P = 0.$$

Since $A - B_2B_2'P$ is asymptotically stable, we have $P - Q \geq 0$. Hence $\|\boldsymbol{R}_{zw}\|_2^2 = \text{trace}(B_1'QB_1) \leq \text{trace}(B_1'PB_1)$.

Solution 6.17. Suppose there exists a stabilizing controller $\boldsymbol{K}$ such that $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$. Then by the argument of Section 6.3.4, there exists an L such that $\boldsymbol{K}$ stabilizes the plant

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z_a &= \begin{bmatrix} Cx \\ Lx \\ u \end{bmatrix}, \end{aligned}$$

$(A, \begin{bmatrix} C' & L' \end{bmatrix}')$ has no unobservable mode on the imaginary axis and $\|\boldsymbol{R}_{z_a w}\|_\infty < \gamma$. Hence there exists a solution to the Riccati equation

$$PA + A'P - P(B_2B_2' - \gamma^{-2}B_1B_1')P + C'C + L'L = 0 \tag{6.9}$$

such that $A - (B_2B_2' - \gamma^{-2}B_1B_1')P$ is asymptotically stable and $P \geq 0$. Clearly, P is a solution to the stated Riccati inequality.

Conversely, suppose $P \geq 0$ is a stabilizing solution to the Riccati inequality. Let L be a Cholesky factor such that

$$L'L = -(PA + A'P - P(B_2B_2' - \gamma^{-2}B_1B_1')P + C'C).$$

Thus, the Riccati equation (6.9) holds. Hence, by Theorem 6.3.1, the controller $u = -B_2'Px$ stabilizes the augmented system, and $\|\boldsymbol{R}_{z_a w}\|_\infty < \gamma$. Hence, since z consists of the upper components of z_a, the controller also stabilizes the system and satisfies $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$

Solution 6.18.

1. Suppose $\boldsymbol{K}$ stabilizes the augmented system and $\|\boldsymbol{R}_{z_a w}\|_\infty < \gamma$. Then the system is stabilized by $\boldsymbol{K}$ and

$$\|z\|_2^2 + \epsilon^2\|u\|_2^2 - \gamma^2\|w\|_2^2 \leq -\mu\|w\|_2^2 \tag{6.10}$$

for some $\mu > 0$. Hence

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 \leq -\mu\|w\|_2^2$$

and we conclude that $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$.

Conversely, suppose $\boldsymbol{K}$ stabilizes the system and $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$. Then the closed-loop system mapping w to u has finite infinity norm M, say. This follows from the definition of internal stability for linear fractional transformations. This observation shows that $\boldsymbol{K}$ stabilizes the augmented system. Set $0 < \epsilon \leq \sqrt{\frac{1}{2M^2}(\gamma^2 - \|\boldsymbol{R}_{zw}\|_\infty^2)}$. Then

$$\begin{aligned} \|z\|_2^2 + \epsilon^2\|u\|_2^2 - \gamma^2\|w\|_2^2 &\leq \|z\|_2^2 + (\epsilon^2M^2 - \gamma^2)\|w\|_2^2 \\ &\leq (\|\boldsymbol{R}_{zw}\|_\infty^2 + \epsilon^2M^2 - \gamma^2)\|w\|_2^2 \\ &\leq -\frac{1}{2}(\gamma^2 - \|\boldsymbol{R}_{zw}\|_\infty^2)\|w\|_2^2 \\ &= -\mu\|w\|_2^2, \end{aligned}$$

for $\mu = (\gamma^2 - \|\boldsymbol{R}_{zw}\|_\infty^2)/2 > 0$. Hence $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$.

2. Suppose P_ϵ exists. Then $u = -B_2' P_\epsilon x$ is stabilizing for z_a, hence also for z, and $\|\boldsymbol{R}_{z_a w}\|_\infty < \gamma$, hence also $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$. Conversely, if a stabilizing controller $\boldsymbol{K}$ satisfies $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$, then it also stabilizes an augmented objective system of the form given in Item 1 and $\|\boldsymbol{R}_{z_a w}\|_\infty < \gamma$. This augmented system satisfies the standard assumptions (modulo scaling by $\sqrt{D'D + \epsilon^2 I}$), which is nonsingular since $\epsilon > 0$. Consequently, the stated Riccati equation has a stabilizing, nonnegative definite solution.

Solution 6.19. This problem is simply a combination of the previous two. If $\boldsymbol{K}$ stabilizes the system and $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$, we can choose $\epsilon > 0$ and an L such that $(A, \begin{bmatrix} C' & L' \end{bmatrix}')$ has no unobservable mode on the imaginary axis, and $\boldsymbol{K}$ stabilizes the augmented system with

$$z_a = \begin{bmatrix} Cx \\ Lx \\ u \\ \epsilon u \end{bmatrix}$$

and $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$. Consequently, the stated Riccati inequality has a stabilizing, nonnegative definite solution. Conversely, if the stated Riccati inequality has a stabilizing, nonnegative definite solution, then the controller $u = -B_2' P_\epsilon x$ stabilizes the augmented system for

$$L'L = -(P_\epsilon A + A' P_\epsilon - P_\epsilon (B_2 R_\epsilon^{-1} B_2' - \gamma^{-2} B_1 B_1') P_\epsilon + C'C),$$

and $\|\boldsymbol{R}_{z_a w}\|_\infty < \gamma$. Since z consists of components of z_a, we conclude that $u = -B_2 P_\epsilon x$ stabilizes the system and $\|\boldsymbol{R}_{zw}\|_\infty < \gamma$.

Solutions to Problems in Chapter 7

Solution 7.1. Subtracting (7.2.17) from (7.2.16) gives

$$(\dot{\bar{Q}} - \dot{\widehat{Q}}) = (A - QC'C)(\bar{Q} - \widehat{Q}) + (\bar{Q} - \widehat{Q})(A - QC'C)' + (Q - \widehat{Q})C'C(Q - \widehat{Q}),$$

which has a nonnegative solution, since $(\bar{Q} - \widehat{Q})(0) = 0$. This proves that $\bar{Q}(t) \geq \widehat{Q}(t)$. Subtracting (7.2.16) from (7.2.13) gives

$$(\dot{Q} - \dot{\bar{Q}}) = (A - QC'C)(Q - \bar{Q}) + (Q - \bar{Q})(A - QC'C)' + \gamma^{-2}QL'LQ,$$

which also has a nonnegative solution, since $(Q - \bar{Q})(0) = 0$. We therefore conclude that $Q(t) \geq \bar{Q}(t) \geq \widehat{Q}(t)$ as required.

Solution 7.2. Firstly, we replace the message generating differential equation

$$\dot{x} = Ax + Bw$$

with

$$\dot{x} = Ax + BQ^{1/2}\tilde{w}.$$

Next, we replace the observations equation

$$y = Cx + v$$

with

$$\begin{aligned} \tilde{y} &= R^{-1/2}y \\ &= R^{-1/2}Cx + \tilde{v}. \end{aligned}$$

From the general theory presented in the text, we see that the Riccati differential equation associated with $\tilde{w}$, $\tilde{v}$ and $\tilde{y}$ is

$$\dot{Q} = AQ + QA' - Q(C'R^{-1}C - \gamma^{-2}L'L)Q + BQB', \qquad Q(0) = 0$$

and that the observer gain (for the scaled variables) is $\tilde{H} = QC'R^{-1/2}$. The filter for the scaled variables is

$$\begin{aligned}\dot{\widehat{x}} &= (A - QC'R^{-1}C)\widehat{x} + QC'R^{-1/2}\tilde{y} \\ \widehat{z} &= L\widehat{x},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\dot{\widehat{x}} &= (A - QC'R^{-1}C)\widehat{x} + QC'R^{-1}y \\ \widehat{z} &= L\widehat{x}.\end{aligned}$$

This shows that the observer gain (for the original variables) is given by $H = QC'R^{-1}$.

Solution 7.3. It is easy to verify that

$$Q = \begin{bmatrix} \widehat{Q} & 0 \\ 0 & 0 \end{bmatrix}$$

satisfies

$$AQ + QA' - Q(C'C - \gamma^{-2}L'L)Q + BB' = 0$$

if $\widehat{Q}$ satisfies

$$A_{11}\widehat{Q} + \widehat{Q}A_{11}' - \widehat{Q}(C_1'C_1 - \gamma^{-2}L_1'L_1)\widehat{Q} + B_1B_1' = 0.$$

We can now complete the calculation by observing that

$$\begin{aligned}&A - Q(C'C - \gamma^{-2}L'L) \\ &= \begin{bmatrix} A_{11} - \widehat{Q}(C_1'C_1 - \gamma^{-2}L_1'L_1) & A_{12} - \widehat{Q}(C_1'C_2 - \gamma^{-2}L_1'L_2) \\ 0 & A_{22} \end{bmatrix}\end{aligned}$$

is stable because A_{22} and $A_{11} - \widehat{Q}(C_1'C_1 - \gamma^{-2}L_1'L_1)$ are stable.

Solution 7.4. Consider the given signal generator

$$\boldsymbol{G} : \begin{cases} \dot{x} = Ax + Bw \\ y = Cx + Dw, \quad DD' = I. \end{cases}$$

The filter

$$\widehat{z} = \boldsymbol{F}y$$

is to provide an estimate of $z = Lx$. In order to remove the cross coupling between the input and observations disturbances, we introduce the observations preprocessor

$$\boldsymbol{P} : \begin{cases} \dot{\tilde{x}} = (A - BD'C)\tilde{x} + BD'y \\ \tilde{y} = -C\tilde{x} + y. \end{cases}$$

Thus, $\tilde{y}$ is generated from w by the system

$$\begin{aligned}(\dot{x}-\dot{\tilde{x}}) &= (A-BD'C)(x-\tilde{x})+B(I-D'D)w\\ \tilde{y} &= C(x-\tilde{x})+Dw.\end{aligned}$$

Since the pre-processor is an invertible system, there is a one-to-one correspondence between the original and the modified filtering problems. Since $(I-D'D)D'=0$, the modified problem fits into the standard theory, which we can use to find an estimator $\tilde{\boldsymbol{F}}$ of $L(x-\tilde{x})$ given the "observations" $\tilde{y}$. We obtain a filter $\boldsymbol{F}$ for our original problem from $\tilde{\boldsymbol{F}}$ by noting that

$$\begin{aligned}L\widehat{x} &= L(\widehat{x}-\tilde{x})+L\tilde{x}\\ &= \tilde{\boldsymbol{F}}\tilde{y}+L\big(sI-(A-BD'C)\big)^{-1}BD'y\\ &= \Big(\tilde{\boldsymbol{F}}\boldsymbol{P}+L\big(sI-(A-BD'C)\big)^{-1}BD'\Big)y.\end{aligned}$$

We use (7.2.10) and (7.2.11) to give the generator of all estimators of $L(x-\tilde{x})$, given the information $\tilde{y}$:

$$\tilde{\boldsymbol{F}}_a \overset{s}{=} \left[\begin{array}{c|cc} A-(BD'+QC')C & QC' & -\gamma^{-2}QL' \\ \hline L & 0 & I \\ -C & I & 0\end{array}\right],$$

where $Q(t)$ satisfies

$$\dot{Q} = (A-BD'C)Q+Q(A-BD'C)'-Q(C'C-\gamma^{-2}L'L)Q+B(I-D'D)B'$$

with initial condition $Q(0)=0$. Since $\tilde{\boldsymbol{F}}_a$ generates all estimators of $L(x-\tilde{x})$ (and not Lx), all the estimators of Lx will be given by

$$\begin{aligned}\boldsymbol{F}_a &\overset{s}{=} \left[\begin{array}{c|cc} A-BD'C & BD' & 0 \\ \hline L & 0 & 0 \\ 0 & 0 & 0\end{array}\right]\\ &\quad + \left[\begin{array}{c|cc} A-(BD'+QC')C & QC' & -\gamma^{-2}QL' \\ \hline L & 0 & I \\ -C & I & 0\end{array}\right]\left[\begin{array}{c|cc} A-BD'C & BD' & 0 \\ \hline -C & I & 0 \\ 0 & 0 & I\end{array}\right]\\ &\overset{s}{=} \left[\begin{array}{c|cc} A-BD'C & BD' & 0 \\ \hline L & 0 & 0 \\ 0 & 0 & 0\end{array}\right]\\ &\quad + \left[\begin{array}{cc|cc} A-(BD'+QC')C & -QC'C & QC' & -\gamma^{-2}QL' \\ 0 & A-BD'C & BD' & 0 \\ \hline L & 0 & 0 & I \\ -C & -C & I & 0\end{array}\right]\end{aligned}$$

$$
\begin{aligned}
&\stackrel{s}{=} \left[\begin{array}{c|cc} A-BD'C & BD' & 0 \\ \hline L & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \\
&\quad + \left[\begin{array}{cc|cc} A-(BD'+QC')C & 0 & BD'+QC' & -\gamma^{-2}QL' \\ 0 & A-BD'C & BD' & 0 \\ \hline L & -L & 0 & I \\ -C & 0 & I & 0 \end{array}\right] \\
&\stackrel{s}{=} \left[\begin{array}{c|cc} A-(BD'+QC')C & BD'+QC' & -\gamma^{-2}QL' \\ \hline L & 0 & I \\ -C & I & 0 \end{array}\right].
\end{aligned}
$$

That is,

$$
\begin{aligned}
\dot{\widehat{x}} &= A\widehat{x} + (BD'+QC')(y - C\widehat{x}) - \gamma^{-2}QL'\phi \\
\widehat{z} &= L\widehat{x} + \phi \\
\eta &= y - C\widehat{x}.
\end{aligned}
$$

The filters are obtained by closing the loop with $\phi = \boldsymbol{U}\eta$.

Solution 7.5.

$$
\begin{aligned}
\boldsymbol{G}(I+\boldsymbol{G}^{\sim}\boldsymbol{G})^{-1}\boldsymbol{G}^{\sim} &\leq \gamma^2 I \\
\Leftrightarrow (I+\boldsymbol{G}\boldsymbol{G}^{\sim})^{-1}\boldsymbol{G}\boldsymbol{G}^{\sim} &\leq \gamma^2 I \\
\Leftrightarrow \boldsymbol{G}\boldsymbol{G}^{\sim} &\leq \gamma^2(I+\boldsymbol{G}\boldsymbol{G}^{\sim}) \\
\Leftrightarrow (1-\gamma^2)\boldsymbol{G}\boldsymbol{G}^{\sim} &\leq \gamma^2 I \\
\Leftrightarrow (1-\gamma^2)\delta^2 &\leq \gamma^2, \text{ in which } \delta = \|\boldsymbol{G}\|_\infty \\
\Leftrightarrow \gamma^2 &\geq \frac{\delta^2}{1+\delta^2} \\
\Leftrightarrow \gamma &\geq \frac{1}{\sqrt{1+\delta^{-2}}}.
\end{aligned}
$$

Solution 7.6. We deal with the case in which $w(t)$ is frequency weighted first. Suppose

$$
\begin{aligned}
\dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{w} \\
w &= \tilde{C}\tilde{x} + \tilde{D}\tilde{w}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{x} &= Ax + Bw \\
y &= Cx + v.
\end{aligned}
$$

Combining these equations gives

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} &= \begin{bmatrix}A & B\tilde{C}\\ 0 & \tilde{A}\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}+\begin{bmatrix}B\tilde{D}\\ \tilde{B}\end{bmatrix}\tilde{w}\\ y &= \begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}+v,\end{aligned}$$

which is the standard form.

The case in which $v(t)$ is frequency weighted may be handled in much the same way. Suppose

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{A}\tilde{x}+\tilde{B}\tilde{v}\\ v &= \tilde{C}\tilde{x}+\tilde{D}\tilde{v}.\end{aligned}$$

Then

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} &= \begin{bmatrix}A & 0\\ 0 & \tilde{A}\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}+\begin{bmatrix}B & 0\\ 0 & \tilde{B}\end{bmatrix}\begin{bmatrix}w\\ \tilde{v}\end{bmatrix}\\ y &= \begin{bmatrix}C & \tilde{C}\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}+\begin{bmatrix}0 & \tilde{D}\end{bmatrix}\begin{bmatrix}w\\ \tilde{v}\end{bmatrix},\end{aligned}$$

which contains cross coupling in the disturbance input—*i.e.*, it is of the form in Problem 7.4.

Solution 7.7. The equations describing the message generating system as drawn in Figure 7.12 are

$$\begin{aligned}\dot{x} &= Ax+Bw\\ y &= Cx+v\end{aligned}$$

and

$$\begin{aligned}\dot{x}_w &= A_w x_w + B_w L x\\ \delta &= C_w x_w + D_w L x,\end{aligned}$$

which may be combined as

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{x}_w\end{bmatrix} &= \begin{bmatrix}A & 0\\ B_w L & A_w\end{bmatrix}\begin{bmatrix}x\\ x_w\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}w\\ \begin{bmatrix}y\\ \delta\end{bmatrix} &= \begin{bmatrix}C & 0\\ D_w L & C_w\end{bmatrix}\begin{bmatrix}x\\ x_w\end{bmatrix}+\begin{bmatrix}I\\ 0\end{bmatrix}v.\end{aligned}$$

Substituting this realization into the general filtering formulas and partitioning the Riccati equation solution as

$$Q=\begin{bmatrix}Q_1 & Q_2\\ Q_2' & Q_3\end{bmatrix}$$

gives

$$\tilde{\boldsymbol{F}}_a \stackrel{s}{=} \left[\begin{array}{cc|cc} A - Q_1 C'C & 0 & Q_1 C' & -\gamma^{-2}(Q_1 L' D_w' + Q_2 C_w') \\ B_w L - Q_2' C'C & A_w & Q_2' C' & -\gamma^{-2}(Q_2 L' D_w' + Q_3 C_w') \\ \hline D_w L & C_w & 0 & I \\ -C & 0 & I & 0 \end{array}\right].$$

The generator of all filters can now be found from

$$\begin{aligned} \boldsymbol{F}_a &= \begin{bmatrix} \boldsymbol{W}^{-1} & 0 \\ 0 & I \end{bmatrix} \tilde{\boldsymbol{F}}_a \\ &= \left[\begin{array}{c|cc} A_w - B_w D_w^{-1} C_w & B_w D_w^{-1} & 0 \\ \hline -D_w^{-1} C_w & D_w^{-1} & 0 \\ 0 & 0 & I \end{array}\right] \\ &\quad \times \left[\begin{array}{cc|cc} A - Q_1 C'C & 0 & Q_1 C' & -\gamma^{-2}(Q_1 L' D_w' + Q_2 C_w') \\ B_w L - Q_2' C'C & A_w & Q_2' C' & -\gamma^{-2}(Q_2 L' D_w' + Q_3 C_w') \\ \hline D_w L & C_w & 0 & I \\ -C & 0 & I & 0 \end{array}\right]. \end{aligned}$$

After the removal of the unobservable modes we get the filter generator $\boldsymbol{F}_a$ defined by the realization

$$\left[\begin{array}{cc|cc} A_w - B_w D_w^{-1} C_w & Q_2' C'C & -Q_2' C' & B_w D_w^{-1} + \gamma^{-2}(Q_2 L' D_w' + Q_3 C_w') \\ 0 & A - Q_1 C'C & Q_1 C' & -\gamma^{-2}(Q_1 L' D_w' + Q_2 C_w') \\ \hline -D_w^{-1} C_w & L & 0 & D_w^{-1} \\ 0 & -C & I & 0 \end{array}\right],$$

which is free of degree inflation.

Solution 7.8.

1. Collecting all the relevant equations gives

$$\begin{bmatrix} \dot{x} \\ \widehat{z} - Lx \\ y \end{bmatrix} = \begin{bmatrix} A & \begin{bmatrix} B & 0 \end{bmatrix} & B_2 \\ -L & \begin{bmatrix} 0 & 0 \end{bmatrix} & I \\ C & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} x \\ \begin{bmatrix} w \\ v \end{bmatrix} \\ \widehat{z} \end{bmatrix},$$

which has the associated adjoint system

$$\boldsymbol{P}^{\sim} = \left[\begin{array}{c|cc} A' & -L' & C' \\ \hline \begin{bmatrix} B' \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ B_2' & I & 0 \end{array}\right].$$

This problem is now of the form of the special measurement feedback control problem considered in Problem 6.13. The results of this problem show that a solution exists if and only if the Riccati equation

$$\dot{Q} = AQ + QA' - Q(C'C - \gamma^{-2}L'L)Q + BB', \qquad Q(0) = 0.$$

has a solution on $[0, T]$, in which case all $\boldsymbol{F}^\sim$'s are generated by $\mathcal{F}_\ell(\boldsymbol{F}_a^\sim, \boldsymbol{U}^\sim)$, in which

$$\boldsymbol{F}_a^\sim \stackrel{s}{=} \left[\begin{array}{c|cc} A' + L'B_2' - C'CQ & -L' & C' \\ \hline -CQ & 0 & I \\ -(B_2' - \gamma^{-2}LQ) & I & 0 \end{array}\right].$$

Taking the adjoint, we see that all filters are generated by $\mathcal{F}_\ell(\boldsymbol{F}_a, \boldsymbol{U})$, in which

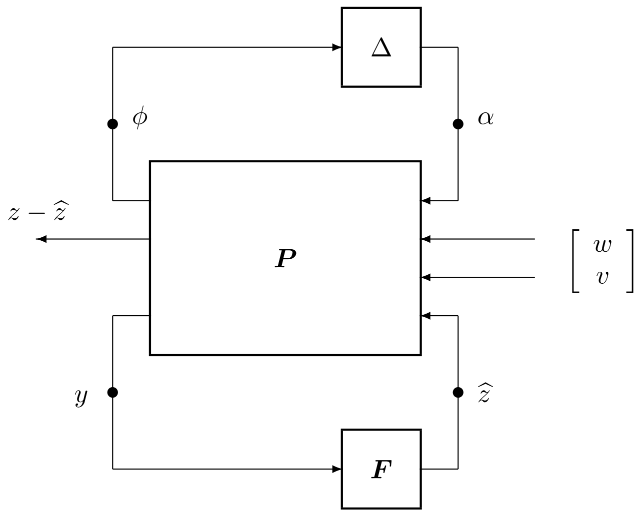
$$\boldsymbol{F}_a \stackrel{s}{=} \left[\begin{array}{c|cc} A + B_2L - QC'C & QC' & B_2 - \gamma^{-2}QL' \\ \hline L & 0 & I \\ -C & I & 0 \end{array}\right].$$

(We have adjusted the signs of the input and output matrices so that the filter state is $\widehat{x}$, an estimate of x, rather than $-\widehat{x}$.) Note that the "central" filter can be written

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + QC'(y - C\widehat{x}) + B_2\widehat{z} \\ \widehat{z} &= L\widehat{x}. \end{aligned}$$

2. The internal stability results are immediate from the corresponding results in Problem 6.13.

Solution 7.9. The generalized regulator configuration we are interested in given by the diagram:



in which $\phi = Ex$. Writing down the appropriate equations gives

$$\begin{bmatrix} \dot{x} \\ \begin{bmatrix} \phi \\ z - \widehat{z} \end{bmatrix} \\ y \end{bmatrix} = \begin{bmatrix} A & \begin{bmatrix} H_1 & B & 0 \end{bmatrix} & 0 \\ E & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 0 \\ L & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & -I \\ C & \begin{bmatrix} H_2 & 0 & I \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} x \\ \begin{bmatrix} \alpha \\ w \\ v \end{bmatrix} \\ \widehat{z} \end{bmatrix}.$$

A filter $\boldsymbol{F}$ with the desired properties exists if and only if there exists a "controller" $\boldsymbol{F}$ such that the map $\boldsymbol{R}$, which maps $\begin{bmatrix} \alpha' & w' & v' \end{bmatrix}' \to \begin{bmatrix} \phi' & (z - \widehat{z})' \end{bmatrix}'$, has the property $\|\boldsymbol{R}\|_\infty < 1$. Note that the solution of two Riccati equations will be required. The reader may like to study Chapter 8 before returning to this problem.

Solutions to Problems in Chapter 8

Solution 8.1.

1. Since $\lambda_i(I-A) = 1-\lambda_i(A)$, we conclude that $I-A$ is nonsingular if $\rho(A) < 1$.
2. First observe that $\lambda_i\big(A(I+A)^{-1}\big) = \lambda_i(A)/\big(1+\lambda_i(A)\big)$. Since $\lambda_i(A)$ are real and nonnegative and $\lambda_i(A) \geq \lambda_j(A)$ implies that $\lambda_i(A)/(1+\lambda_i(A)) \geq \lambda_j(A)/(1+\lambda_j(A))$, the result follows. You might like to check for yourself that $x/(1+x)$ is monotonically increasing for all real nonnegative values of x.

Solution 8.2.

1. Using the definition of the H_Y Hamiltonian we have that

$$
\begin{aligned}
&\begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} H_Y \begin{bmatrix} I & \gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}' & -(C_2'C_2 - \gamma^{-2}C_1'C_1) \\ -\bar{B}\bar{B}' & -\bar{A} \end{bmatrix} \begin{bmatrix} I & \gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \bar{A}' + \gamma^{-2}X_\infty\bar{B}\bar{B}' & \Phi \\ -\bar{B}\bar{B}' & -\bar{A} - \gamma^{-2}\bar{B}\bar{B}'X_\infty \end{bmatrix} \\
&= \begin{bmatrix} A_z' & \Phi \\ -\bar{B}\bar{B}' & -A_z \end{bmatrix},
\end{aligned}
$$

in which

$$
\begin{aligned}
\Phi &= -(C_2'C_2 - \gamma^{-2}C_1'C_1) + \gamma^{-2}(X_\infty\bar{A} + (\bar{A}' + \gamma^{-2}X_\infty\bar{B}\bar{B}')X_\infty) \\
&= -(C_2'C_2 - \gamma^{-2}C_1'C_1) + \gamma^{-2}X_\infty(A - B_1D_{21}'C_2) \\
&\quad + \gamma^{-2}(A - B_1D_{21}'C_2)'X_\infty + \gamma^{-4}X_\infty B_1(I - D_{21}'D_{21})B_1'X_\infty \\
&= -(C_2 + \gamma^{-2}D_{21}B_1'X_\infty)'(C_2 + \gamma^{-2}D_{21}B_1'X_\infty) \\
&\quad + \gamma^{-2}(D_{12}'C_1 + B_2'X_\infty)'(D_{12}'C_1 + B_2'X_\infty) \\
&= -C_{2z}'C_{2z} + \gamma^{-2}F_\infty'F_\infty.
\end{aligned}
$$

2. If Y_∞, the stabilizing solution to (8.3.12), exists, we have

$$H_Y \begin{bmatrix} I & 0 \\ Y_\infty & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ Y_\infty & I \end{bmatrix} \begin{bmatrix} \bar{A}' - (C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty & * \\ 0 & * \end{bmatrix},$$

in which $\mathrm{R}_e\lambda_i(\bar{A}' - (C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty) < 0$. Substituting from Part 1 gives

$$\begin{aligned} & H_Z \begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Y_\infty & I \end{bmatrix} \\ = & \begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Y_\infty & I \end{bmatrix} \begin{bmatrix} \bar{A}' - (C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty & * \\ 0 & * \end{bmatrix}, \end{aligned}$$

which implies that

$$\begin{aligned} & H_Z \begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty & -\gamma^{-2}X_\infty \\ Y_\infty & I \end{bmatrix} \\ = & \begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty & -\gamma^{-2}X_\infty \\ Y_\infty & I \end{bmatrix} \begin{bmatrix} \bar{A}' - (C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty & * \\ 0 & * \end{bmatrix}. \end{aligned}$$

If $\rho(X_\infty Y_\infty) < \gamma^2$, it follows that $(I - \gamma^{-2}X_\infty Y_\infty)^{-1}$ exists. It is now immediate that $Z_\infty = Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1}$ satisfies (8.3.9) and that it is stabilizing.

3. Multiplying

$$H_Y \begin{bmatrix} I & \gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & \gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} H_Z$$

on the right by $\begin{bmatrix} I & Z_\infty \end{bmatrix}'$ gives

$$\begin{aligned} & H_Y \begin{bmatrix} I + \gamma^{-2}X_\infty Z_\infty \\ Z_\infty \end{bmatrix} \\ = & \begin{bmatrix} I & \gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} \begin{bmatrix} A_z' - (C_{2z}'C_{2z} - \gamma^{-2}F_\infty'F_\infty)Z_\infty \\ -\bar{B}\bar{B}' - A_z Z_\infty \end{bmatrix}. \end{aligned}$$

Expanding the $(1,1)$-partition of this equation and using the Z_∞ Riccati equation yields

$$\begin{aligned} & \bar{A}'(I + \gamma^{-2}X_\infty Z_\infty) - (C_2'C_2 - \gamma^{-2}C_1'C_1)Z_\infty \\ = & \; A_z' - (C_{2z}'C_{2z} - \gamma^{-2}F_\infty'F_\infty)Z_\infty \\ & + \gamma^{-2}X_\infty(Z_\infty A_z' - Z_\infty(C_{2z}'C_{2z} - \gamma^{-2}F_\infty'F_\infty)Z_\infty). \end{aligned}$$

Since $Y_\infty(I + \gamma^{-2}X_\infty Z_\infty) = Z_\infty$, we obtain

$$\begin{aligned} & (\bar{A}' - (C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty)(I + \gamma^{-2}X_\infty Z_\infty) \\ = & \; (I + \gamma^{-2}X_\infty Z_\infty)\big(A_z' - (C_{2z}'C_{2z} - \gamma^{-2}F_\infty'F_\infty)Z_\infty\big). \end{aligned}$$

Hence

$$\begin{aligned}&\bar{A} - Y_\infty(C_2'C_2 - \gamma^{-2}C_1'C_1)\\ &\quad = \quad (I+\gamma^{-2}Z_\infty X_\infty)^{-1}\big(A_z - Z_\infty(C_{2z}'C_{2z} - \gamma^{-2}F_\infty'F_\infty)\big)(I+\gamma^{-2}Z_\infty X_\infty)\end{aligned}$$

as required.

Solution 8.3. Suppose (8.6.2) has solution $P(t)$ with $P(0) = M$. From this it is immediate that

$$\begin{bmatrix} I \\ P \end{bmatrix}(A - DP) - \begin{bmatrix} A & -D \\ -Q & -A' \end{bmatrix}\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{P} \end{bmatrix},$$

so that $\begin{bmatrix} I & P \end{bmatrix}'$ is a solution to (8.6.3) with the correct boundary conditions.

Now suppose that (8.6.3) has a solution with $P_1(t)$ nonsingular for all $t \in [0, T]$ and with $P_2(0)P_1^{-1}(0) = M$. This gives

$$\begin{aligned}\begin{bmatrix} P & -I \end{bmatrix}\begin{bmatrix} A & -D \\ -Q & -A' \end{bmatrix}\begin{bmatrix} I \\ P \end{bmatrix} &= \begin{bmatrix} P & -I \end{bmatrix}\left(\begin{bmatrix} I \\ P \end{bmatrix}P_1 X - \begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \end{bmatrix}\right)P_1^{-1}\\ &= \dot{P}_2 P_1^{-1} - P_2 P_1^{-1}\dot{P}_1 P_1^{-1}\\ &= \dot{P}\end{aligned}$$

as required.

Solution 8.4.

1. It follows from (8.2.4) and

$$\begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} H_Y \begin{bmatrix} I & \gamma^{-2}X_\infty \\ 0 & I \end{bmatrix} = H_Z + \begin{bmatrix} 0 & \gamma^{-2}\dot{X}_\infty \\ 0 & 0 \end{bmatrix}$$

that

$$\begin{aligned}&H_Y \begin{bmatrix} I+\gamma^{-2}X_\infty Z_\infty \\ Z_\infty \end{bmatrix}\\ &\quad = \begin{bmatrix} I+\gamma^{-2}X_\infty Z_\infty \\ Z_\infty \end{bmatrix}\big(A_z - Z_\infty(C_{2z}'C_{2z} - \gamma^{-2}F_\infty'F_\infty)\big)'\\ &\qquad - \frac{d}{dt}\begin{bmatrix} I+\gamma^{-2}X_\infty Z_\infty \\ Z_\infty \end{bmatrix}.\end{aligned}$$

Since $X_\infty, Z_\infty \geq 0$, it follows that $(I+\gamma^{-2}X_\infty Z_\infty)$ is nonsingular for all $t \in [0, T]$. Also $Z_\infty(I+\gamma^{-2}X_\infty Z_\infty)^{-1}(0) = 0$. We can now use Problem 8.3 to show that $Y_\infty = Z_\infty(I+\gamma^{-2}X_\infty Z_\infty)^{-1}$ is a solution to (8.2.8). Also

$$\rho(X_\infty Y_\infty) = \rho\big(X_\infty Z_\infty(I+\gamma^{-2}X_\infty Z_\infty)^{-1}\big) = \frac{\gamma^2\rho(X_\infty Z_\infty)}{\gamma^2 + \rho(X_\infty Z_\infty)} < \gamma^2.$$

2. If Y_∞ exists and $\rho(X_\infty Y_\infty) < \gamma^2$, then $I - \gamma^{-2}X_\infty Y_\infty$ is nonsingular on $[0, T]$, $Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1}(0) = 0$, and from (8.6.1) we get

$$\begin{aligned} & H_Z \begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty \\ Y_\infty \end{bmatrix} \\ &= \begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty \\ Y_\infty \end{bmatrix} (\bar{A} - Y_\infty(C_2'C_2 - \gamma^{-2}C_1'C_1))' \\ &\quad - \frac{d}{dt}\begin{bmatrix} I - \gamma^{-2}X_\infty Y_\infty \\ Y_\infty \end{bmatrix}. \end{aligned}$$

It now follows from Problem 8.3 that $Z_\infty = Y_\infty(I-\gamma^{-2}X_\infty Y_\infty)^{-1}$ is a solution to (8.2.15).

Solution 8.5. A direct application of the composition formula for LFTs (see Lemma 4.1.2) gives

$$\begin{aligned} A_{PK} &= \begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \\ B_{PK} &= \begin{bmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} A_{GK} &= \begin{bmatrix} A + \gamma^{-2}B_1B_1'X_\infty + B_2D_K(C_2 + \gamma^{-2}D_{21}B_1'X_\infty) & B_2C_K \\ B_K(C_2 + \gamma^{-2}D_{21}B_1'X_\infty) & A_K \end{bmatrix} \\ B_{GK} &= \begin{bmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{bmatrix}. \end{aligned}$$

It is now easy to see that

$$\begin{bmatrix} A_{GK} - \lambda I & B_{GK} \end{bmatrix} = \begin{bmatrix} A_{PK} - \lambda I & B_{PK} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \gamma^{-2}B_1'X_\infty & 0 & I \end{bmatrix}.$$

Solution 8.6.

1. Eliminating y and u from the five given equations yields

$$\begin{aligned} \dot{x} &= Ax - B_2C_1\widehat{x} + B_1w \\ \dot{\widehat{x}} &= (A - B_2C_1)\widehat{x} + B_1(C_2x + w - C_2\widehat{x}) \\ z &= C_1x - C_1\widehat{x}. \end{aligned}$$

This means that

$$\dot{x} - \dot{\widehat{x}} = (A - B_1C_2)(x - \widehat{x}).$$

It follows that $(x - \widehat{x})(t) = 0$ for all t, since $x(0) = 0$ and $\widehat{x} = 0$. It also follows that $z(t) = 0$.

In the same way, we eliminate y and u from the six given equations to obtain

$$\begin{aligned} \dot{x} &= Ax - B_2C_1\widehat{x} + B_1w + B_2(u - u^*) \\ \dot{\widehat{x}} &= (A - B_1C_2 - B_2C_1)\widehat{x} + B_1(C_2x + w) + B_2(u - u^*) \\ \dot{x} - \dot{\widehat{x}} &= (A - B_1C_2)(x - \widehat{x}) \end{aligned}$$

and

$$\begin{aligned} z &= C_1(x - \widehat{x}) + (u - u^*) \\ \widehat{w} &= C_2(x - \widehat{x}) + w. \end{aligned}$$

Hence $\widehat{x} = x$, $\widehat{w} = w$ and $z = u - u^*$. Since $u - u^* = \boldsymbol{U}\widehat{w}$, it follows that $z = \boldsymbol{U}w$ and the result is proved.

2. Just choose a $\|\boldsymbol{U}\|_\infty \le \gamma$ and assemble the corresponding controller.

3. We need note two things. Firstly,

$$\begin{bmatrix} z \\ \widehat{w} \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} w \\ u - u^* \end{bmatrix}$$

and secondly that all the internal cancellations occur at $\lambda_i(A - B_1C_2)$ and $\lambda_i(A - B_2C_1)$ (see Lemma 4.1.2). We can now select a stable $\boldsymbol{U}$ such that $\|\boldsymbol{U}\|_\infty \le \gamma$.

Solution 8.7.

1. Consider the diagram and observe that

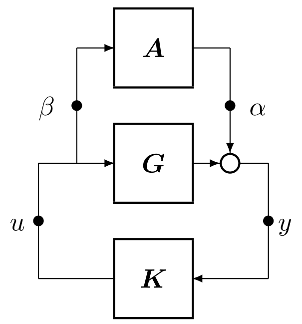
$$\beta = \boldsymbol{K}(I - \boldsymbol{G}\boldsymbol{K})^{-1}\alpha.$$

If

$$\boldsymbol{P} = \begin{bmatrix} 0 & I \\ I & \boldsymbol{G} \end{bmatrix},$$

then $\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K}) = \boldsymbol{K}(I - \boldsymbol{G}\boldsymbol{K})^{-1}$ as required. (Alternatively, observe that

$$\begin{bmatrix} \beta \\ y \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \alpha \\ u \end{bmatrix}.)$$



A state-space realization for $\boldsymbol{P}$ is

$$\boldsymbol{P} \stackrel{s}{=} \left[\begin{array}{c|cc} A & 0 & B \\ \hline 0 & 0 & I \\ C & I & 0 \end{array}\right].$$

2. If

$$A_- P + P A_-' + B_- B_-' = 0$$

and

$$A_-' Q + Q A_- + C_-' C_- = 0,$$

then

$$A_-' P^{-1} + P^{-1} A_- + P^{-1} B_- B_-' P^{-1} = 0$$

and

$$A_- Q^{-1} + Q^{-1} A_-' + Q^{-1} C_-' C_- Q^{-1} = 0.$$

Next, we see that the equation defining X_∞ is given by

$$\left[\begin{array}{cc} A_-' & 0 \\ 0 & A_+' \end{array}\right] X_\infty + X_\infty \left[\begin{array}{cc} A_- & 0 \\ 0 & A_+ \end{array}\right] - X_\infty \left[\begin{array}{c} B_- \\ B_+ \end{array}\right] \left[\begin{array}{cc} B_-' & B_+' \end{array}\right] X_\infty = 0.$$

One nonnegative solution is clearly

$$X_\infty = \left[\begin{array}{cc} -P^{-1} & 0 \\ 0 & 0 \end{array}\right].$$

To show that this solution is stabilizing, we observe that

$$\begin{aligned} &\left[\begin{array}{cc} A_- + B_- B_-' P^{-1} & 0 \\ B_+ B_-' P^{-1} & A_+ \end{array}\right] \\ &= \left[\begin{array}{cc} A_- & 0 \\ 0 & A_+ \end{array}\right] - \left[\begin{array}{c} B_- \\ B_+ \end{array}\right] \left[\begin{array}{cc} B_-' & B_+' \end{array}\right] \left[\begin{array}{cc} -P^{-1} & 0 \\ 0 & 0 \end{array}\right]. \end{aligned}$$

Since

$$A_- + B_- B_-' P^{-1} = -P A_-' P^{-1},$$

with $R_e \lambda_i(A_-) > 0$, the solution is indeed stabilizing. A parallel set of arguments may be developed for Y_∞.

3. The smallest achievable value of γ is determined by

$$\begin{aligned} \gamma^2 &\geq \rho(X_\infty Y_\infty) \\ &= \rho(P^{-1} Q^{-1}) \\ &= \frac{1}{\lambda_{\min}(PQ)}. \end{aligned}$$

Hence

$$\gamma(\boldsymbol{A}) \leq \nu = \gamma^{-1} = \sqrt{\lambda_{\min}(PQ)}.$$

4. Direct substitution into the formulas of (8.3.11) gives

$$\begin{aligned} A_k &= A - BB' X_\infty - Z_\infty C_2' C_2 \\ C_{k1} &= -B' X_\infty \\ B_{k1} &= Z_\infty C_2' \\ Z_\infty &= (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty. \end{aligned}$$

Solution 8.8.

1. Substitution into

$$A' X_\infty + X_\infty A - X_\infty B B' X_\infty = 0$$

gives

$$\begin{aligned} 0 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ & \quad - \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 - x_2^2 & x_1 + 2x_2 - x_2 x_3 \\ x_1 + 2x_2 - x_2 x_3 & 2(x_2 + x_3) - x_3^2 \end{bmatrix} \end{aligned}$$

Consider $x_3 = x_2$ and $x_1 = 2x_2$. This gives

$$0 = \begin{bmatrix} 4x_2 - x_2^2 & 4x_2 - x_2^2 \\ 4x_2 - x_2^2 & 4x_2 - x_2^2 \end{bmatrix}$$

and therefore $x_2 = 0$ or $x_2 = 4$. It is clear that $X_\infty = 0$ is not stabilizing, while $x_2 = 4$ gives

$$X_\infty = 4\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

which is both positive definite and stabilizing. A parallel set of arguments leads to

$$Y_\infty = 4\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

2. To find the optimal value of γ, we solve the equation

$$\begin{aligned} \gamma_{opt}^2 &= \rho(X_\infty Y_\infty) \\ &= 16\rho\left(\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\right). \end{aligned}$$

The eigenvalues of $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ are given by the roots of $\lambda^2 - 6\lambda + 1 = 0$. That is

$$\begin{aligned} \lambda &= \frac{6 \pm \sqrt{36-4}}{2} \\ &= 3 \pm 2\sqrt{2}. \end{aligned}$$

Thus

$$\gamma_{opt} = 4\sqrt{3 + 2\sqrt{2}}.$$

Solution 8.9.

1. This requires the facts

$$\mathcal{F}_\ell\left(\begin{bmatrix} 0 & \boldsymbol{G} \\ I & \boldsymbol{G} \end{bmatrix}, \boldsymbol{K}\right) = \boldsymbol{G}\boldsymbol{K}(I - \boldsymbol{G}\boldsymbol{K})^{-1}$$

and

$$\begin{bmatrix} 0 & \boldsymbol{G} \\ I & \boldsymbol{G} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & 0 & B \\ \hline C & 0 & D \\ C & I & D \end{array}\right].$$

2. (A, B_2, C_2) stabilizable and detectable requires (A, B, C) stabilizable and detectable.

$$\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \text{ full column rank}$$

requires $A - BD^{-1}C - j\omega I$ full rank, since

$$\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C - j\omega I & B \\ 0 & D \end{bmatrix}.$$

Finally

$$\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \text{ full row rank}$$

requires $A - j\omega I$ full rank.

3. After scaling we get

$$\tilde{\boldsymbol{P}} \stackrel{s}{=} \left[\begin{array}{c|cc} A & 0 & BD^{-1} \\ \hline C & 0 & I \\ C & I & I \end{array}\right],$$

and then direct substitution yields:

$$\begin{aligned} 0 &= (A - BD^{-1}C)'X + X(A - BD^{-1}C) - X(BD^{-1})(BD^{-1})'X \\ 0 &= AY + YA' - YC'CY. \end{aligned}$$

4. Substituting into the general $\mathcal{H}_\infty$ synthesis Riccati equations gives

$$\begin{aligned} 0 &= (A - BD^{-1}C)'X_\infty + X_\infty(A - BD^{-1}C) - X_\infty(BD^{-1})(BD^{-1})'X_\infty \\ 0 &= AY_\infty + Y_\infty A' - (1 - \gamma^{-2})Y_\infty C'CY_\infty. \end{aligned}$$

Comparing terms with the LQG equations now yields $X_\infty = X$ and $Y_\infty = (1 - \gamma^{-2})^{-1}Y$.

5. When $\boldsymbol{G}$ is stable, $\boldsymbol{K} = 0$ is a stabilizing controller, in which case $\gamma_{opt} = 0$.

6. When $\boldsymbol{G}$ has a right-half-plane pole, $Y \neq 0$. Hence $\gamma > 1$ is necessary (and sufficient) to ensure that Y_∞ exists and $Y_\infty \geq 0$. The spectral radius condition gives

$$\begin{aligned} \gamma^2 &\geq \rho(X_\infty Y_\infty) \\ &= (1 - \gamma^{-2})^{-1}\rho(XY) \\ \Rightarrow \gamma^2 &\geq 1 + \rho(XY) \\ \Rightarrow \gamma &\geq \sqrt{1 + \rho(XY)}. \end{aligned}$$

Since $\sqrt{1 + \rho(XY)} \geq 1$ for any X and Y, we see that all the conditions are satisfied for any $\gamma > \sqrt{1 + \rho(XY)}$ and we conclude that $\gamma_{opt} = \sqrt{1 + \rho(XY)}$.

7. Immediate from the above result, since in this case $X = 0$. Note that as $\gamma \downarrow 1$, Y_∞ becomes unbounded—this needs to be dealt with carefully.

Solution 8.10.

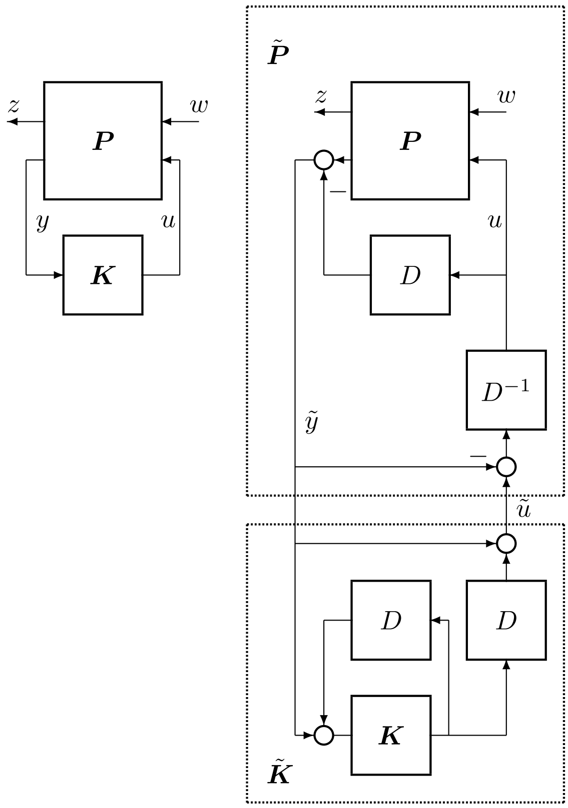
1. This follows from

$$\begin{aligned}(I - \boldsymbol{GK})^{-1} &= I + \boldsymbol{GK}(I - \boldsymbol{GK})^{-1} \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} I & \boldsymbol{G} \\ I & \boldsymbol{G} \end{bmatrix}, \boldsymbol{K}\right)\end{aligned}$$

and

$$\begin{bmatrix} I & \boldsymbol{G} \\ I & \boldsymbol{G} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & 0 & B \\ \hline C & I & D \\ C & I & D \end{array}\right].$$

2. We will prove this from the equivalence of the following two diagrams.



The second of these figures yields

$$u = D^{-1}(\tilde{u} - \tilde{y}) \text{ and } \tilde{y} = y - Du.$$

Therefore

$$\begin{aligned}\dot{x} &= Ax + Bu\\ z &= Cx + w + Du\\ y &= Cx + w + Du\end{aligned}$$

becomes

$$\begin{aligned}\dot{x} &= Ax + BD^{-1}(\tilde{u} - \tilde{y})\\ &= Ax + BD^{-1}(\tilde{u} - Cx - w)\\ &= (A - BD^{-1}C)x - BD^{-1}w + BD^{-1}\tilde{u},\end{aligned}$$

together with

$$\begin{aligned}z &= Cx + w + \tilde{u} - Cx - w\\ &= \tilde{u}\\ \tilde{y} &= Cx + w.\end{aligned}$$

Hence

$$\tilde{\boldsymbol{P}} = \left[\begin{array}{c|cc} A - BD^{-1}C & -BD^{-1} & BD^{-1}\\ \hline 0 & 0 & I\\ C & I & 0\end{array}\right].$$

Also,

$$\begin{aligned}\tilde{\boldsymbol{K}} &= I + D\boldsymbol{K}(I - D\boldsymbol{K})^{-1}\\ &= (I - D\boldsymbol{K})^{-1}.\end{aligned}$$

3.

$$\begin{aligned}(A, B_2) \text{ stabilizable} &\Rightarrow (A - BD^{-1}C, BD^{-1}) \text{ stabilizable}\\ &\Rightarrow (A, B) \text{ stabilizable.}\\ (A, C_2) \text{ detectable} &\Rightarrow (A - BD^{-1}C, C) \text{ detectable}\\ &\Rightarrow (A, C) \text{ detectable.}\end{aligned}$$

$$\begin{aligned}&\left[\begin{array}{cc} A - j\omega I & B_2\\ C_1 & D_{12}\end{array}\right] \text{ full column rank}\\ \Rightarrow\ &\left[\begin{array}{cc} A - BD^{-1}C - j\omega I & BD^{-1}\\ 0 & I\end{array}\right] \text{ full column rank}\\ \Rightarrow\ &A - BD^{-1}C - j\omega I \text{ full rank.}\end{aligned}$$

$$
\begin{array}{rl}
& \left[\begin{array}{cc} A - j\omega I & B_1 \\ C_2 & D_{21} \end{array}\right] \text{full row rank} \\
\Rightarrow & \left[\begin{array}{cc} A - BD^{-1}C - j\omega I & -BD^{-1} \\ C & I \end{array}\right] \text{ full row rank} \\
\Rightarrow & A - j\omega I \text{ full rank (take Schur complements).}
\end{array}
$$

4. By direct substitution into the general formulas we get

$$
\begin{aligned}
0 &= (A - BD^{-1}C)'X + X(A - BD^{-1}C) - X(BD^{-1})(BD^{-1})'X \\
0 &= AY + YA' - YC'CY.
\end{aligned}
$$

5. Again, direct substitution into the general formulas yields

$$
\begin{aligned}
0 &= (A - BD^{-1}C)'X_\infty + X_\infty(A - BD^{-1}C) \\
&\quad - (1 - \gamma^{-2})X_\infty(BD^{-1})(BD^{-1})'X_\infty \\
0 &= AY_\infty + Y_\infty A' - Y_\infty C'CY_\infty.
\end{aligned}
$$

Comparing terms yields $X_\infty = (1 - \gamma^{-2})^{-1}X$ and $Y_\infty = Y$. It is easy to see that these are the stabilizing solutions.

6. When $\boldsymbol{G}^{-1} \in \mathcal{RH}_\infty$, we have $\mathrm{R}_e\lambda_i(A - BD^{-1}C) < 0 \Rightarrow X_\infty = 0$. This together with $\rho(X_\infty Y_\infty) = \gamma_{opt}^2$ implies that $\gamma_{opt} = 0$, offering one explanation. Alternatively, if $\boldsymbol{G}^{-1} \in \mathcal{RH}_\infty$, we can use an infinitely high, stabilizing feedback gain to reduce the sensitivity. This together with the fact that $\lim_{k\to\infty}(I - k\boldsymbol{G})^{-1} = 0$ provides a second explanation. (Remember $\boldsymbol{G}(\infty)$ is nonsingular.)

7.

$$
\begin{aligned}
\rho(X_\infty Y_\infty) &= \gamma_{opt}^2 \\
\Rightarrow (1 - \gamma_{opt}^{-2})^{-1}\rho(XY) &= \gamma_{opt}^2 \\
\Rightarrow \rho(XY) &= \gamma_{opt}^2 - 1 \text{ provided } \gamma_{opt} \neq 0 \\
\Rightarrow \gamma_{opt} &= \sqrt{1 + \rho(XY)}.
\end{aligned}
$$

Solution 8.11.

1. The four given Riccati equations follow by substitution into their general counterparts.

2. Problem 3.23 implies that X is nonsingular if and only if $-\tilde{A}$ is asymptotically stable.

   Suppose we add
$$\tilde{A}X^{-1} + X^{-1}\tilde{A}' - B_2B_2' = 0$$
   and
$$-\gamma^{-2}\tilde{A}W - \gamma^{-2}W\tilde{A}' + \gamma^{-2}B_1B_1' = 0$$
   to get
$$\tilde{A}(X^{-1} - \gamma^{-2}W) + (X^{-1} - \gamma^{-2}W)\tilde{A}' - (B_2B_2' - \gamma^{-2}B_1B_1') = 0.$$
   Now if $\rho(XW) < \gamma^2$, $(X^{-1} - \gamma^{-2}W)^{-1}$ exists, is nonnegative definite and satisfies
$$\begin{aligned} 0 \;=\; & \tilde{A}'(X^{-1} - \gamma^{-2}W)^{-1} + (X^{-1} - \gamma^{-2}W)^{-1}\tilde{A} \\ & - (X^{-1} - \gamma^{-2}W)^{-1}(B_2B_2' - \gamma^{-2}B_1B_1')(X^{-1} - \gamma^{-2}W)^{-1}. \end{aligned}$$
   Since
$$-(X^{-1} - \gamma^{-2}W)\tilde{A}'(X^{-1} - \gamma^{-2}W) = A - (B_2B_2' - \gamma^{-2}B_1B_1')(X^{-1} - \gamma^{-2}W)^{-1},$$
   this solution is stabilizing. We may therefore set $X_\infty = (X^{-1} - \gamma^{-2}W)^{-1}$. A parallel set of arguments leads to $Y_\infty = (Y^{-1} - \gamma^{-2}V)^{-1}$.

3. We know from the general theory that a stabilizing controller exists if and only if $X_\infty \geq 0$, $Y_\infty \geq 0$ and $\rho(X_\infty Y_\infty) \leq \gamma^2$. Since $X_\infty > 0 \Leftrightarrow X_\infty^{-1} > 0$, $Y_\infty > 0 \Leftrightarrow Y_\infty^{-1} > 0$ and $\rho(X_\infty Y_\infty) \leq \gamma^2 \Leftrightarrow Y_\infty^{-1} - \gamma^{-2}X_\infty \geq 0$, we can check these three necessary and sufficient conditions via the positivity of
$$\Pi(\gamma) = \begin{bmatrix} Y^{-1} - \gamma^{-2}V & \gamma^{-1}I \\ \gamma^{-1}I & X^{-1} - \gamma^{-2}W \end{bmatrix}.$$
   It is now easy to verify that
$$\begin{bmatrix} \gamma^2 & -\gamma X \\ 0 & \gamma X \end{bmatrix} \Pi(\gamma) \begin{bmatrix} Y & 0 \\ 0 & \gamma I \end{bmatrix} = \gamma^2 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} VY + XY & -XW \\ -XY & XW \end{bmatrix}.$$
   This means that the smallest value of γ for which a solution exists is the largest value of γ for which $\Pi(\gamma)$ is singular, or in other words the largest eigenvalue of $\begin{bmatrix} VY + XY & -XW \\ -XY & XW \end{bmatrix}$.

Solution 8.12.

1. Since

$$y = \begin{bmatrix} \boldsymbol{G} & I & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} w \\ v \\ u \end{bmatrix},$$

we conclude that the generalized plant for this problem is given by

$$\begin{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \\ y \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{G} & I \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \boldsymbol{G} \\ I \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{G} & I \end{bmatrix} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \\ u \end{bmatrix}.$$

It follows from the diagram in the question that

$$\begin{aligned} y &= v + \boldsymbol{GK}y + \boldsymbol{G}w \\ \Rightarrow y &= (I - \boldsymbol{GK})^{-1} \begin{bmatrix} \boldsymbol{G} & I \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} u &= \boldsymbol{K}(v + \boldsymbol{G}w + \boldsymbol{G}u) \\ \Rightarrow u &= \boldsymbol{K}(I - \boldsymbol{GK})^{-1} \begin{bmatrix} \boldsymbol{G} & I \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}. \end{aligned}$$

Combining these equations yields

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} I \\ \boldsymbol{K} \end{bmatrix} (I - \boldsymbol{GK})^{-1} \begin{bmatrix} \boldsymbol{G} & I \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}.$$

Substituting $\boldsymbol{G} = C(sI - A)^{-1}B$ yields the realization of this generalized plant:

$$\begin{bmatrix} \begin{bmatrix} \boldsymbol{G} & I \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \boldsymbol{G} \\ I \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{G} & I \end{bmatrix} & \boldsymbol{G} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & \begin{bmatrix} B & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} C \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right].$$

2.

$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} K \begin{bmatrix} 0 & I \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & K \end{bmatrix}.$$

Therefore $\|\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})(\infty)\| \geq 1$, which shows that $\|\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})\|_\infty \geq 1$. We also see that $\|\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})\|_2 = \infty$, since $\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})$ is not strictly proper.

3.

Step 1: Since $\|D_{11}\| = 1$, $\widehat{D}_{12} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and

$$\begin{aligned} \|\widehat{D}_{12}' D_{11}\| &= \| \begin{bmatrix} I & 0 \end{bmatrix} \| \\ &= 1, \end{aligned}$$

we set $F = 0$ because the norm of $\bar{D}_{11}$ cannot be reduce further (by feedback).

Step 2: From the definition of Θ in (4.6.5) we get

$$\begin{aligned} \Theta_{11} &= \begin{bmatrix} 0 & \gamma^{-2} I \\ 0 & 0 \end{bmatrix} \\ \Theta_{12} &= \begin{bmatrix} \gamma^{-1}(1-\gamma^{-2})^{1/2} I & 0 \\ 0 & \gamma^{-1} I \end{bmatrix} \\ \Theta_{21} &= \begin{bmatrix} -\gamma^{-1} I & 0 \\ 0 & -\gamma^{-1}(1-\gamma^{-2})^{1/2} I \end{bmatrix} \\ \Theta_{22} &= \begin{bmatrix} 0 & 0 \\ \gamma^{-2} I & 0 \end{bmatrix}. \end{aligned}$$

Now

$$B_1 \Theta_{22} = \begin{bmatrix} B & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \gamma^{-2} I & 0 \end{bmatrix} = 0.$$

Next, we observe that

$$\begin{aligned} (I - \Theta_{22} D_{11})^{-1} &= \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \gamma^{-2} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} I & 0 \\ 0 & \frac{1}{1-\gamma^{-2}} I \end{bmatrix} \\ (I - D_{11} \Theta_{22})^{-1} &= \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \gamma^{-2} I & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \frac{1}{1-\gamma^{-2}} I & 0 \\ 0 & I \end{bmatrix}, \end{aligned}$$

which results in

$$\begin{aligned} \widehat{A} &= A, \text{ since } B_1 = 0 \\ \widehat{B}_1 &= B_1 (I - \Theta_{22} D_{11})^{-1} \Theta_{21} \\ &= \begin{bmatrix} B & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{1-\gamma^{-2}} I \end{bmatrix} \begin{bmatrix} -\gamma^{-1} I & 0 \\ 0 & -\gamma^{-1}\sqrt{1-\gamma^{-2}} I \end{bmatrix} \\ &= \begin{bmatrix} -\gamma^{-1} B & 0 \end{bmatrix} \\ \widehat{B}_2 &= B_2 = B \text{ since } B_1 = 0. \end{aligned}$$

In much the same way:

$$\begin{aligned}
\widehat{C}_1 &= \Theta_{12}(I - D_{11}\Theta_{22})^{-1}C_1 \\
&= \gamma^{-1}\begin{bmatrix} \sqrt{1-\gamma^{-2}}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \frac{1}{1-\gamma^{-2}}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} C \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\gamma^{-1}}{\sqrt{1-\gamma^{-2}}}C \\ 0 \end{bmatrix}, \\
\widehat{C}_2 &= C_2 + D_{21}\Theta_{22}(I - D_{11}\Theta_{22})^{-1}C_1 \\
&= C + \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \gamma^{-2}I & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{1-\gamma^{-2}}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} C \\ 0 \end{bmatrix} \\
&= C + \frac{\gamma^{-2}C}{1-\gamma^{-2}} \\
&= \frac{1}{1-\gamma^{-2}}C.
\end{aligned}$$

The whole point of the construction is to ensure $\widehat{D}_{11} = 0$. It only remains for us to evaluate the remaining partitions of the $\widehat{D}$ matrix.

$$\begin{aligned}
\widehat{D}_{12} &= \Theta_{12}(I - D_{11}\Theta_{22})^{-1}D_{12} \\
&= \gamma^{-1}\begin{bmatrix} \sqrt{1-\gamma^{-2}}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \frac{1}{1-\gamma^{-2}}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \gamma^{-1}I \end{bmatrix} \\
\widehat{D}_{21} &= D_{21}(I - \Theta_{22}D_{11})^{-1}\Theta_{21} \\
&= \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \frac{1}{1-\gamma^{-2}}I \end{bmatrix}\begin{bmatrix} -\gamma^{-1}I & 0 \\ 0 & -\gamma^{-1}\sqrt{1-\gamma^{-2}}I \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{\gamma^{-1}}{\sqrt{1-\gamma^{-2}}}I \end{bmatrix} \\
\widehat{D}_{22} &= D_{22} + D_{21}\Theta_{22}(I - D_{11}\Theta_{22})^{-1}D_{21} \\
&= \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \gamma^{-2}I & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{1-\gamma^{-2}}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= 0
\end{aligned}$$

Combining these results now yields:

$$\widehat{\boldsymbol{P}} \stackrel{s}{=} \left[\begin{array}{c|ccc} A & -\gamma^{-1}B & 0 & B \\ \hline \frac{\gamma^{-1}}{\sqrt{1-\gamma^{-2}}}C & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma^{-1}I \\ \frac{1}{1-\gamma^{-2}}C & 0 & -\frac{\gamma^{-1}}{\sqrt{1-\gamma^{-2}}}I & 0 \end{array}\right]$$

Since $\|\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K})\|_\infty \le \gamma \Leftrightarrow \|\mathcal{F}_\ell(\widehat{\boldsymbol{P}},\boldsymbol{K})\|_\infty \le \gamma^{-1}$, we can rescale $\widehat{\boldsymbol{P}}$ so that $\|\mathcal{F}_\ell(\widehat{\boldsymbol{P}},\boldsymbol{K})\|_\infty \le \gamma$ with

$$\widehat{\boldsymbol{P}} \stackrel{s}{=} \left[\begin{array}{c|ccc} A & -B & 0 & B \\ \hline \frac{1}{\sqrt{1-\gamma^{-2}}}C & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \frac{1}{1-\gamma^{-2}}C & 0 & -\frac{1}{\sqrt{1-\gamma^{-2}}}I & 0 \end{array}\right]$$

Step 3: Scale D_{12} and D_{21}. D_{12} requires no scaling, while scaling D_{21} yields

$$\widehat{\boldsymbol{P}} \stackrel{s}{=} \left[\begin{array}{c|ccc} A & -B & 0 & B \\ \hline \frac{1}{\sqrt{1-\gamma^{-2}}}C & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \frac{1}{\sqrt{1-\gamma^{-2}}}C & 0 & -I & 0 \end{array}\right]$$

4. Since (A, C_2) must be detectable, we require (A, C) detectable. To establish that this condition suffices for one of the imaginary axis conditions we argue as follows:

$$\begin{aligned} \begin{bmatrix} j\omega I - A & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= 0 \\ \Leftrightarrow \begin{bmatrix} j\omega I - A & B \\ \frac{1}{\sqrt{1-\gamma^{-2}}}C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= 0 \\ \Leftrightarrow w_2 = 0 \text{ and } \begin{bmatrix} j\omega I - A \\ C \end{bmatrix} w_1 &= 0 \end{aligned}$$

It is clear that the second condition is equivalent to (A, C) having no unobservable modes on the imaginary axis and this is implied by the detectability of (A, C). A dual argument may be used to establish the required stabilizability of (A, B).

Note that there is no cross coupling in the generalized plant $\widehat{\boldsymbol{P}}$. It is immediate that the X_∞ and Y_∞ Riccati equations are as stated in the question.

5. The generalized plant for this problem is given by

$$\begin{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix} \\ y \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{G} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \boldsymbol{G} \\ I \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{G} & I \end{bmatrix} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \\ u \end{bmatrix}.$$

The realization of this generalized plant is

$$\begin{bmatrix} \begin{bmatrix} \boldsymbol{G} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \boldsymbol{G} \\ I \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{G} & I \end{bmatrix} & \boldsymbol{G} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & \begin{bmatrix} B & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} C \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right].$$

Substituting into the general formulas now gives

$$\begin{aligned} A'X + XA - XBB'X + C'C &= 0 \\ AY + YA' - YC'CY + BB' &= 0. \end{aligned}$$

6. Substituting $X_\infty = (1-\gamma^{-2})^{-1}X$ into the X_∞ Riccati equation gives

$$A'X + XA - XBB'X + C'C = 0$$

which shows that $X_\infty = (1-\gamma^{-2})^{-1}X$ is indeed a solution. If $A - BB'X$ is stable, then so is $A - BB'(1-\gamma^{-2})X_\infty$. It is clear that $X_\infty \geq 0$ for all $\gamma > 1$. The fact that $Y_\infty = Y$ is trivial.

The spectral radius condition gives:

$$\begin{aligned} \gamma^2 &\geq \rho(X_\infty Y_\infty) \\ &= (1-\gamma^{-2})^{-1}\rho(XY) \\ \Rightarrow \gamma &\geq \sqrt{1+\rho(XY)}. \end{aligned}$$

Hence $\gamma > \sqrt{1+\rho(XY)}$ implies that all the conditions are met and we conclude that $\gamma_{opt} = \sqrt{1+\rho(XY)}$.

7. Substitution into the various definitions in the text gives:

$$\begin{aligned} F_\infty &= B'X_\infty = (1-\gamma^{-2})^{-1}B'X \\ C_{2z} &= \frac{1}{\sqrt{1-\gamma^{-2}}}C \\ Z_\infty &= (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty = \gamma^2(1-\gamma^{-2})W^{-1}Y. \end{aligned}$$

From (8.3.11) we obtain the central controller $\widehat{\boldsymbol{K}} \stackrel{s}{=} (A_k, B_{k1}, C_{k1})$ for the scaled system $\widehat{\boldsymbol{P}}$:

$$\begin{aligned} B_{k1} &= Z_\infty C_{2z}' = \gamma^2\sqrt{1-\gamma^{-2}}W^{-1}YC' \\ C_{k1} &= -(1-\gamma^{-2})^{-1}B'X \\ A_k &= A + \gamma^{-2}BB'X_\infty - BB'X_\infty - Z_\infty \frac{1}{\sqrt{1-\gamma^{-2}}}C' \frac{1}{\sqrt{1-\gamma^{-2}}}C \\ &= A - BB'X - \gamma^2 W^{-1}YC'C. \end{aligned}$$

Recalling the D_{21}-scaling, which means $\widehat{\boldsymbol{K}} = \frac{1}{\sqrt{1-\gamma^{-2}}}\boldsymbol{K}$, we set

$$\boldsymbol{K} = \sqrt{1-\gamma^{-2}}\widehat{\boldsymbol{K}}$$

by multiplying B_{k1} by $\sqrt{1-\gamma^{-2}}$. This gives the controller

$$\begin{aligned} \dot{\tilde{x}} &= (A - BB'X - \gamma^2 W^{-1}YC'C)\tilde{x} + (\gamma^2 - 1)W^{-1}YC'y \\ u &= -(1-\gamma^{-2})^{-1}B'X\tilde{x}. \end{aligned}$$

Defining $\widehat{x} = (1-\gamma^{-2})^{-1}\tilde{x}$ yields

$$\begin{aligned} \dot{\widehat{x}} &= (A - BB'X - \gamma^2 W^{-1}YC'C)\widehat{x} + \gamma^2 W^{-1}YC'y \\ u &= -B'X\widehat{x}. \end{aligned}$$

Finally, multiplying by W yields

$$\begin{aligned} W\dot{\widehat{x}} &= \big(W(A - BB'X) - \gamma^2 YC'C\big)\widehat{x} + \gamma^2 YC'y \\ u &= -B'X\widehat{x}, \end{aligned}$$

which is the desired controller.

Notice that for suboptimal γ, W is nonsingular and we have

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + Bu + \gamma^2 W^{-1}YC'(y - C\widehat{x}) \\ u &= -B'X\widehat{x}, \end{aligned}$$

which is an observer and state-estimate feedback. The LQG optimal controller for the problem in Part 5 is

$$\begin{aligned} \dot{\widehat{x}}_{LQG} &= A\widehat{x}_{LQG} + Bu + YC'(y - C\widehat{x}_{LQG}) \\ u &= -B'X\widehat{x}_{LQG}. \end{aligned}$$

The $\mathcal{H}_\infty$ controller therefore uses a different observer gain matrix, but the same feedback gain matrix.

8. Using

$$\boldsymbol{g} \stackrel{s}{=} \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right],$$

we obtain $X = 1$ and $Y = 1$. Therefore $\gamma_{opt} = \sqrt{1+\rho(XY)} = \sqrt{2}$.

Setting $\boldsymbol{k} = -1$ gives a closed-loop transfer function matrix of

$$\begin{aligned} \begin{bmatrix} 1 \\ \boldsymbol{k} \end{bmatrix} (1 - \boldsymbol{g}\boldsymbol{k})^{-1} \begin{bmatrix} \boldsymbol{g} & 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{s+1} \begin{bmatrix} 1 & s \end{bmatrix} \\ &= \frac{1}{s+1} \begin{bmatrix} 1 & s \\ -1 & -s \end{bmatrix}. \end{aligned}$$

The singular values of this matrix are the square roots of the eigenvalues of

$$\frac{1}{j\omega+1}\begin{bmatrix} 1 & j\omega \\ -1 & -j\omega \end{bmatrix} \frac{1}{-j\omega+1}\begin{bmatrix} 1 & -1 \\ -j\omega & j\omega \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Now $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has eigenvalues 2 and 0, and so $\sigma_1 = \sqrt{2}$ and $\sigma_2 = 0$.

(It is instructive to examine the general controller formula for this case. We have $X = 1$ and $Y = 1$, so $W = \gamma^2 - 2$. Substitution into the controller formulas give

$$u = -\frac{\gamma^2}{(\gamma^2-2)s + 2(\gamma^2-1)} y.$$

For $\gamma^2 = 2$, this gives the optimal controller $u = -y$. The LQG controller is $-\frac{1}{s+2}$.)

Solution 8.13. We show how the optimal controllers given in Examples 8.4 and 2.4.2 may be obtained using The Robust Control Toolbox and Matlab, version 4.0.[1] Note that Chiang and Safonov, authors of the Robust Control Toolbox [35], consider the synthesis problem $\|\gamma^{-1}\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})\|_\infty < 1$.

Servomechanism of Section 8.4:

```
>> J1=1;
>> J2=2;
>> D1=.01;
>> D2=.02;
>> K=30;
>> A=[-D1/J1,0,-K/J1;0,-D2/J2,K/J2;1,-1,0];
>> B=[40/J1;0;0];
>> C=[0,1,0];
>> D=0;
>> B1=[B, zeros(3,1)];
>> B2=-B;
>> C1=[zeros(1,3);C];
>> C2=C;
>> D11=[1,0;D,1];
>> D12=[-1;D];
>> D21=[D,1];
>> D22=-D;
>> genplant=mksys(A,B1,B2,C1,C2,D11,D12,D21,D22,'tss');
```

[1]MATLAB is a registered trademark of The MathWorks, Inc.

```
>> GOPT=[1;2];
>> aux=[1e-12,1/4,1/3];
>> [gopt,ss_cp,ss_cl]=hinfopt(genplant,GOPT,aux);

                   << H-Infinity Optimal Control Synthesis >>
```

Information about $\mathcal{H}_\infty$ optimization displayed.

```
>> 1/gopt

ans =

    3.8856

>> [Ak,Bk,Ck,Dk]=branch(ss_cp);
>> [NUMk,DENk]=ss2tf(Ak,Bk,Ck,Dk)

NUMk =

    3.6283    6.8528   88.0362


DENk =

    1.0000   25.3218  343.0775
```

Example 2.4.2:

Additive robustness problem

```
>> num=1;
>> den=[1,-2,1];
>> [A,B,C,D]=tf2ss(num,den);
>> genplant=mksys(A,zeros(2,1),B,zeros(1,2),C,0,1,1,0,'tss');
>> gamopt=4*sqrt(3+2*sqrt(2))

gamopt =

    9.6569

>> aux=[1e-12,1/9,1/10];
>>  GOPT=1;
>> [gopt,ss_cp,ss_cl]=hinfopt(genplant,GOPT,aux);
```

```
                    << H-Infinity Optimal Control Synthesis >>
```

Information about $\mathcal{H}_\infty$ optimization displayed.

```
>>  1/gopt

ans =

    9.6569

>> [Ak,Bk,Ck,Dk]=branch(ss_cp);
>> [NUMk,DENk]=ss2tf(Ak,Bk,Ck,Dk)

NUMk =

   -9.6569    4.0000


DENk =

    1.0000    4.4142
```

The combined additive/multiplicative problem

```
>> num=1; den=[1,-2,1];
>> [A,B,C,D]=tf2ss(num,den);
>> Z12=zeros(1,2);
>> Z21=zeros(2,1);
>> genplant=mksys(A,Z21,B,[C;Z12],C,Z21,[D;1/10],1,D,'tss');
>> GOPT=[1;2];
>> aux=[1e-12,1/2,1/3];
>> [gopt,ss_cp,ss_cl]=hinfopt(genplant,GOPT,aux);

                    << H-Infinity Optimal Control Synthesis >>
```

Information about $\mathcal{H}_\infty$ optimization displayed.

```
>> 1/gopt

ans =

    2.3818
```

```
>> [Ak,Bk,Ck,Dk]=branch(ss_cp);
>> [NUMk,DENk]=ss2tf(Ak,Bk,Ck,Dk)

NUMk =

  -23.8181    4.8560


DENk =

    1.0000    6.9049
```

Solution 8.14. Suppose

$$\boldsymbol{P} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

Since (A, B_2) is stabilizable, there exists an F such that the eigenvalues of $A - B_2F$ are not on the imaginary axis. Similarly, since (A, C_2) is detectable, there exists an H such that the eigenvalues of $A - HC_2$ are not on the imaginary axis.

Now consider the dilated plant

$$\boldsymbol{P}_a \stackrel{s}{=} \left[\begin{array}{c|cc} A & \left[\begin{array}{cc} B_1 & \epsilon H \end{array}\right] & B_2 \\ \hline \left[\begin{array}{c} C_1 \\ \epsilon F \end{array}\right] & \left[\begin{array}{cc} D_{11} & 0 \\ 0 & 0 \end{array}\right] & \left[\begin{array}{c} D_{12} \\ \epsilon I \end{array}\right] \\ C_2 & \left[\begin{array}{cc} D_{21} & \epsilon I \end{array}\right] & D_{22} \end{array}\right].$$

Firstly, we show that

$$\operatorname{rank} \left[\begin{array}{cc} A - j\omega I & B_2 \\ C_1 & D_{12} \\ \epsilon F & \epsilon I \end{array}\right] = m + n$$

for any $\epsilon \neq 0$.

This follows since

$$\left[\begin{array}{ccc} I & 0 & -\epsilon^{-1}B_2 \\ 0 & I & -\epsilon^{-1}D_{12} \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} A - j\omega I & B_2 \\ C_1 & D_{12} \\ \epsilon F & \epsilon I \end{array}\right] = \left[\begin{array}{cc} A - B_2F - j\omega I & 0 \\ C_1 - D_{12}F & 0 \\ \epsilon F & \epsilon I \end{array}\right]$$

in which $A - B_2F$ may be chosen with no eigenvalues on the imaginary axis.

A parallel argument proves that

$$\operatorname{rank}\begin{bmatrix} A - j\omega I & B_1 & \epsilon H \\ C_2 & D_{21} & \epsilon I \end{bmatrix} = q + n.$$

To prove the first direction suppose we select $\boldsymbol{K}$ such that $\mathcal{F}_\ell(\boldsymbol{P}_a, \boldsymbol{K})$ is internally stable and such that $\|\mathcal{F}_\ell(\boldsymbol{P}_a, \boldsymbol{K})\|_\infty < \gamma$. Since the dilation process has no effect on internal stability, $\mathcal{F}_\ell(\boldsymbol{P}_a, \boldsymbol{K})$ is stable. In addition, $\|\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})\|_\infty < \gamma$ since removing the dilation must be norm non-increasing.

Conversely, if $\boldsymbol{K}$ is stabilizing and satisfies $\|\mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K})\|_\infty < \gamma$, the closed loop mapping from w to $\begin{bmatrix} x' & u' \end{bmatrix}'$ has finite norm. Thus there exists an $\epsilon > 0$ such that $\|\mathcal{F}_\ell(\boldsymbol{P}_a, \boldsymbol{K})\|_\infty < \gamma$.

Solutions to Problems in Chapter 9

Solution 9.1.

1. Let C have SVD $C = U_1 \Sigma V_1^*$, in which

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \det(\Sigma_1) \neq 0.$$

Then $BB^* = C^*C = V_1 \Sigma^2 V_1^*$, so B has SVD

$$B = V_1 \bar{\Sigma} U_2^*, \qquad \bar{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

($\bar{\Sigma}$ has fewer columns and Σ.) Therefore,

$$\begin{aligned} B &= V_1 \Sigma U_1^* U_1 \begin{bmatrix} U_2^* \\ 0 \end{bmatrix} \\ &= -C^* U / \sigma \end{aligned}$$

in which

$$U = -\sigma U_1 \begin{bmatrix} U_2^* \\ 0 \end{bmatrix}.$$

2. The point here is that C is no longer assumed to have at least as many rows as B has columns. To overcome this, augment C with zero rows:

$$\tilde{C} = \begin{bmatrix} C \\ 0 \end{bmatrix},$$

with the number of zero rows of $\tilde{C}$ chosen so that $\tilde{C}$ has at least as many rows as B has columns. Therefore, there exists a $\tilde{U}$ such that $\tilde{U}^*\tilde{U} = \sigma^2 I$ and $\sigma B + \tilde{C}^*\tilde{U} = 0$. Partition $\tilde{U}$ conformably with $\tilde{C}$:

$$\tilde{U} = \begin{bmatrix} U \\ U_2 \end{bmatrix}.$$

Then $U^*U = \sigma^2 I - U_2^* U_2 \leq \sigma^2 I$ and $\sigma B + C^* U = 0$. Furthermore, U has SVD

$$U = \begin{bmatrix} \sigma I & 0 \end{bmatrix} (\tilde{U}/\sigma),$$

so the singular values of U are either σ or zero.

Solution 9.2. In order for

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

to be the controllability/observability gramian of the combined system, we require that A_{21} and A_{12} satisfy

$$\begin{aligned} A_{12}\Sigma_2 + \Sigma_1 A_{21}' + B_1 B_2' &= 0 \\ \Sigma_1 A_{12} + A_{21}'\Sigma_2 + C_1' C_2 &= 0. \end{aligned}$$

Hence

$$\begin{aligned} A_{12}\Sigma_2^2 + \Sigma_1 A_{21}'\Sigma_2 + B_1 B_2'\Sigma_2 &= 0 \\ \Sigma_1^2 A_{12} + \Sigma_1 A_{21}'\Sigma_2 + \Sigma_1 C_1' C_2 &= 0, \end{aligned}$$

giving

$$\Sigma_1^2 A_{12} - A_{12}\Sigma_2^2 + \Sigma_1 C_1' C_2 - B_1 B_2' \Sigma_2 = 0, \tag{9.1}$$

which has a unique solution A_{12} provided that no eigenvalue of Σ_1 is also an eigenvalue of Σ_2. This is a standard result from the theory of linear matrix equations. To prove it, let

$$\begin{bmatrix} \Sigma_1^2 & \Sigma_1 C_1' C_2 - B_1 B_2' \Sigma_2 \\ 0 & \Sigma_2^2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Sigma_2^2 \tag{9.2}$$

in which $\begin{bmatrix} V_1' & V_2' \end{bmatrix}'$ has full column rank. (This can be obtained from an eigenvalue decomposition.) Provided V_2 is nonsingular, it is easy to check that $A_{12} = V_1 V_2^{-1}$ is a solution to (9.1). To show that V_2 is nonsingular, suppose that $V_2 x = 0$. Multiplying (9.2) by x, we see that $V_2 \Sigma_2^2 x = 0$ and we conclude that there exists a y such that $V_2 y = 0$ and $\Sigma_2^2 y = \sigma^2 y$. Now multiplying (9.2) by y we obtain $\Sigma_1^2 V_1 y = \sigma^2 V_1 y$. Since Σ_1 and Σ_2 have no eigenvalues in common, we conclude that $V_1 y = 0$. Thus $\begin{bmatrix} V_1' & V_2' \end{bmatrix}' y = 0$, which implies $y = 0$ and we conclude that V_2 is nonsingular. Conversely, if A_{12} is a solution to (9.1), then $V_1 = A_{12}$ and $V_2 = I$ is a solution to (9.2). Thus, V_1, V_2 satisfy (9.2) if and only if

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} A_{12} \\ I \end{bmatrix} V_2, \qquad \det V_2 \neq 0.$$

Since the eigenvalue decomposition ensures that we can always find a V_1 and V_2 to satisfy (9.2), we conclude that $A_{12} = V_1 V_2^{-1}$ exists and is unique.

The matrix A_{21} may be similarly obtained as the unique solution to the linear equation

$$A_{21}\Sigma_1^2 - \Sigma_2^2 A_{21} + B_2 B_1' \Sigma_1 - \Sigma_2 C_2' C_1 = 0.$$

Solution 9.3. Pick any ω and let $A = \tilde{A}(j\omega)$, $B = \tilde{B}(j\omega)$ and $C = \tilde{C}(j\omega)$. Suppose $Ax = \lambda x$, $x \neq 0$. From $\sigma(A + A^*) + BB^* = 0$, we conclude that $\sigma x^* x(\lambda + \bar{\lambda}) + \|B^* x\|^2 = 0$. Since $\sigma x^* x \neq 0$, we have that $(\lambda + \bar{\lambda}) \leq 0$. Equality cannot hold (by assumption) and hence A is asymptotically stable. The bounded real lemma says that $\|C(sI - A)^{-1}B\|_\infty < \gamma$ if and only if there exists a $P \geq 0$ such that

$$PA + A^*P + \gamma^{-2}PBB^*P + C^*C = 0,$$

with $A + \gamma^{-2}BB^*P$ is asymptotically stable. If we hypothesize a solution of the form $P = \mu I$ and substitute for C^*C and BB^*, we see that $P = \mu I$ is a solution provided μ satisfies the quadratic equation

$$\sigma\mu^2/\gamma^2 - \mu + \sigma = 0.$$

The condition for a real solution is $\gamma \geq 2\sigma$. In this case, both the solutions are nonnegative. It remains to show that one of these solution is stabilizing (*i.e.*, $A + \gamma^{-2}BB^*\mu$ asymptotically stable). Assume that $\gamma > 2\sigma$, let μ_1 be the smaller of the two solutions, and let μ_2 be the larger of the two solutions. Note that both $P = \mu_1 I$ and $P = \mu_2 I$ are solutions of the Riccati equation. Subtracting these two Riccati equations and re-arranging terms yields

$$(\mu_2 - \mu_1)(A + \gamma^{-2}BB^*\mu_1) + (A + \gamma^{-2}BB^*\mu_1)^*(\mu_2 - \mu_1) + \gamma^{-2}(\mu_2 - \mu_1)^2 BB^* = 0$$

and it follows that $(A + \gamma^{-2}BB^*\mu_1)$ is asymptotically stable. Thus, for any $\gamma > 2\sigma$, there exists a stabilizing nonnegative definite solution to the bounded real lemma Riccati equation and we conclude that $\|C(sI - A)^{-1}B\|_\infty < \gamma$ for any $\gamma > 2\sigma$. In particular, we have $\overline{\sigma}\big(\tilde{C}(j\omega)(j\omega I - \tilde{A}(j\omega))^{-1}\tilde{B}(j\omega)\big) < \gamma$ for any $\gamma > 2\sigma$. Since ω was arbitrary, we conclude that

$$\sup_\omega \overline{\sigma}\big(\tilde{C}(j\omega)(j\omega I - \tilde{A}(j\omega))^{-1}\tilde{B}(j\omega)\big) \leq 2\sigma.$$

Solution 9.4. Since $\alpha > 0$ and $\alpha \neq 1$, the poles are obviously simple and located at $s = -\alpha^i < 0$. To see that the zeros lie between the poles, consider any term $t_i = \alpha^i/(s+\alpha^i)$ in the sum. As s moves out along the negative real axis, t_i increases monotonically from 1 (at $s = 0$) to ∞ (at $s = -\alpha^i$); it then becomes negative and decreases in magnitude to zero at $s = -\infty$. Thus the function $\boldsymbol{g}_n(s)$, which is continuous except at the poles, moves from $-\infty$ to ∞ as s moves from pole to pole. It must therefore have a zero between every pole. This accounts for the $n-1$ zeros and we conclude they are each simple and are located between each of the poles.

Considering the basic facts concerning Bode phase diagrams, we conclude from the interlacing property that the phase of $\boldsymbol{g}_n(j\omega)$ always lies between $0°$ and $-90°$. This means that $\boldsymbol{g}_n$ is positive real, and it is there the impedance function of a passive circuit.

Solution 9.5. The balanced truncation reduced-order model is given by the realization $(L'AM, L'B, CM, D)$, in which L' is the matrix consisting of the first r rows of T and M is the matrix consisting of the first r columns of T^{-1}, in which T is as defined in Lemma 9.3.1. That is, $T = \Sigma^{\frac{1}{2}}U'R^{-1}$, in which $P = RR'$ and $R'QR = U\Sigma^2U'$. We need to verify that L and M are given by the alternative expressions stated in the problem.

Since $P = RR' = U_P S_P U_P'$, we see that $R = U_P S_P^{\frac{1}{2}}$. Thus

$$\begin{aligned} R'QR &= (S_Q^{\frac{1}{2}}U_Q'U_P S_P^{\frac{1}{2}})'(S_Q^{\frac{1}{2}}U_Q'U_P S_P^{\frac{1}{2}}) \\ &= (V\Sigma U')'(V\Sigma U') \\ &= U\Sigma^2 U'. \end{aligned}$$

Hence

$$\begin{aligned} T &= \Sigma^{\frac{1}{2}}U'R^{-1} \\ &= \Sigma^{\frac{1}{2}}U'S_P^{-\frac{1}{2}}U_P' \\ &= \Sigma^{\frac{1}{2}}U'U\Sigma^{-1}V'S_Q^{\frac{1}{2}}U_Q' \\ &= \Sigma^{-\frac{1}{2}}V'S_Q^{\frac{1}{2}}U_Q'. \end{aligned} \tag{9.3}$$

In the above, the fact that $U\Sigma^{-1}V' = (S_Q^{\frac{1}{2}}U_Q'U_P S_P^{\frac{1}{2}})^{-1}$ has been used. From the partions of V and Σ, it is now easily seen that L', the first r rows of T, is given by $L' = \Sigma_1^{-\frac{1}{2}}V_1'S_Q^{\frac{1}{2}}U_Q'$ as required.

Now take the inverse of the formula (9.3) for T to give $T^{-1} = U_P S_P^{\frac{1}{2}}U\Sigma^{-\frac{1}{2}}$, and from the partitions of U and Σ we see that M, the first r columns of T^{-1}, is given by $M = U_P S_P^{\frac{1}{2}}U_1\Sigma_1^{-\frac{1}{2}}$.

Solution 9.6.

1. Proof is identical to that of Lemma 9.3.1
2. From the $(1,1)$-block of the observability gramian equation, we have

$$A_{11}'\Sigma_1 A_{11} - \Sigma_1 + A_{21}'\Sigma_2 A_{21} + C_1'C_1 = 0.$$

Suppose $A_{11}x = \lambda x$. Then

$$x^*\Sigma_1 x(|\lambda|^2 - 1) + x^*(A_{21}'\Sigma_2 A_{21} + C_1'C_1)x = 0.$$

Since $\Sigma_1 > 0$ and $\Sigma_2 > 0$, we conclude that either (a) $|\lambda| < 1$, or that (b) $A_{21}x = 0$, $C_1 x = 0$. If case (b) holds, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0 \end{bmatrix}$$

which implies that $|\lambda| < 1$ or $x = 0$, since A is asymptotically stable. Thus $A_{11}x = \lambda x$ implies that $|\lambda| < 1$ or $x = 0$, which means that A_{11} is asymptotically stable.

3. Follows by direct calculation. Let $\boldsymbol{\Psi} = e^{j\theta} - A_{11}$, so that $\tilde{\boldsymbol{A}}(\theta) = A_{22} + A_{21}\boldsymbol{\Psi}^{-1}A_{12}$. The calculations required to show

$$\tilde{\boldsymbol{A}}(\theta)\Sigma_2\tilde{\boldsymbol{A}}^*(\theta) - \Sigma_2 + \tilde{\boldsymbol{B}}(\theta)\tilde{\boldsymbol{B}}^*(\theta) = 0$$

are facilitated by the identities

$$\begin{aligned} A_{12}\Sigma_2 A_{12}' &= \Sigma_1 - A_{11}\Sigma_1 A_{11}' - B_1 B_1' \\ &= \boldsymbol{\Psi}\Sigma_1\boldsymbol{\Psi}^* + A_{11}\Sigma_1\boldsymbol{\Psi}^* + \boldsymbol{\Psi}\Sigma_1 A_{11}' - B_1 B_1' \end{aligned}$$

and

$$A_{11}\Sigma_1 A_{21}' + A_{12}\Sigma_2 A_{22}' + B_1 B_2' = 0.$$

Similarly, the identities

$$\begin{aligned} A_{21}'\Sigma_2 A_{21} &= \Sigma_1 - A_{11}'\Sigma_1 A_{11} - C_1'C_1 \\ &= \boldsymbol{\Psi}^*\Sigma_1\boldsymbol{\Psi} + A_{11}'\Sigma_1\boldsymbol{\Psi} + \boldsymbol{\Psi}^*\Sigma_1 A_{11} - C_1'C_1 \end{aligned}$$

and

$$A_{11}'\Sigma_1 A_{12} + A_{21}'\Sigma_2 A_{22} + C_1 C_2' = 0$$

may be used to establish

$$\tilde{\boldsymbol{A}}^*(\theta)\Sigma_2\tilde{\boldsymbol{A}}(\theta) - \Sigma_2 + \tilde{\boldsymbol{C}}^*(\theta)\tilde{\boldsymbol{C}}(\theta) = 0.$$

4. Since A is asymptotically stable and $BB' = I - AA'$, B has full row rank. Similarly, C has full column rank. Therefore, by introducing unitary changes of coordinates to the input and output spaces B and C may be written as

$$B = \begin{bmatrix} \bar{B} & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} \bar{C} \\ 0 \end{bmatrix}$$

in which $\bar{B}$ and $\bar{C}$ are nonsingular. Now define $\bar{D} = -\bar{C}'^{-1}A'\bar{B}$ and

$$U = \begin{bmatrix} A & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}.$$

A calculation shows that $U'U = I$, so U is orthogonal. We therefore have

$$\begin{aligned}
&\bar{B}'(e^{-j\theta}I - A')^{-1}\bar{C}'\bar{C}(e^{j\theta}I - A)^{-1}\bar{B} \\
&= \bar{B}'(e^{-j\theta}I - A')^{-1}(I - A'A)(e^{j\theta}I - A)^{-1}\bar{B} \\
&= \bar{B}'(e^{-j\theta}I - A')^{-1}\big((e^{-j\theta}I - A')(e^{j\theta}I - A) \\
&\quad +(e^{-j\theta}I - A')A + A'(e^{j\theta}I - A)\big)(e^{j\theta}I - A)^{-1}\bar{B} \\
&= \bar{B}'\big(I + (e^{-j\theta}I - A')^{-1}A' + A(e^{j\theta}I - A)^{-1}\big)\bar{B}.
\end{aligned}$$

By direct calculation, $A'\bar{B} + \bar{C}'\bar{D} = 0$ and $\bar{D}'\bar{D} + \bar{B}'\bar{B} = I$, we see that

$$\big(\bar{D} + \bar{C}(e^{j\theta}I - A)^{-1}\bar{B}\big)^*\big(\bar{D} + \bar{C}(e^{j\theta}I - A)^{-1}\bar{B}\big) = I.$$

Since $\bar{B}$ is nonsingular, it follows from $U'U = I$ that that $\|\bar{D}\| < 1$. Hence

$$\begin{aligned}
\overline{\sigma}\big(C(e^{j\theta}I - A)B\big) &= \overline{\sigma}\big(\bar{C}(e^{j\theta}I - A)\bar{B}\big) \\
&= \overline{\sigma}\big(\bar{D} + \bar{C}(e^{j\theta}I - A)\bar{B} - \bar{D}\big) \\
&\leq \overline{\sigma}\big(\bar{D} + \bar{C}(e^{j\theta}I - A)\bar{B}\big) + \|\bar{D}\| \\
&< 1 + 1 = 2.
\end{aligned}$$

5. Obvious from

$$\begin{aligned}
A_{11}\Sigma_1 A'_{11} - \Sigma_1 + A_{12}\Sigma_2 A'_{12} + B_1 B'_1 &= 0 \\
A'_{11}\Sigma_1 A_{11} - \Sigma_1 + A'_{21}\Sigma_2 A_{21} + C'_1 C_1 &= 0.
\end{aligned}$$

6. Delete the states associated with σ_m to give $\boldsymbol{G}_1$, with error $\boldsymbol{E}_1 = \tilde{\boldsymbol{C}}(\theta)\big(e^{j\theta}I - \tilde{\boldsymbol{A}}(\theta)\big)^{-1}\tilde{\boldsymbol{B}}(\theta)$, in which

$$\begin{aligned}
\sigma_m(\tilde{\boldsymbol{A}}(\theta)\tilde{\boldsymbol{A}}^*(\theta) - I) + \tilde{\boldsymbol{B}}(\theta)\tilde{\boldsymbol{B}}^*(\theta) &= 0 \\
\sigma_m(\tilde{\boldsymbol{A}}^*(\theta)\tilde{\boldsymbol{A}}(\theta) - I) + \tilde{\boldsymbol{C}}^*(\theta)\tilde{\boldsymbol{C}}(\theta) &= 0.
\end{aligned}$$

Thus $\overline{\sigma}\big(\boldsymbol{E}_i\big) < 2\sigma_n$. Now consider

$$\widehat{\boldsymbol{G}}_1 = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} A_{21m}\Sigma_{2m}^{\frac{1}{2}} \\ C_{1m} \end{bmatrix}(zI - A_{11m})^{-1}\begin{bmatrix} \Sigma_{2m}^{\frac{1}{2}}A_{12m} & B_{1m} \end{bmatrix}$$

in which A_{11m}, A_{21m}, A_{12m}, B_{1m} C_{1m} and Σ_{2m} come from the partitioning that is associated with the truncation of the states associated with σ_m.

Now truncate this realization of $\widehat{\boldsymbol{G}}_1$, incur an error bounded by $2\sigma_{m-1}$ and embed in a still larger system which is a balanced realization. Continue this process until the desired r^{th}-order truncated model is sitting in the bottom-right-hand corner. The final bound follows from the triangle inequality, together with the fact that the infinity norm of a submatrix can never exceed the norm of the matrix it forms part of. (Each step i incurs an augmented error that is less than $2\sigma_{m-i}$; the error we are actually interested in is the $(2,2)$-corner of the augmented error, which must also be less than $2\sigma_{m-i}$.

This proof generalizes that given in [5], which is limited to the case when the σ_i deleted are each of unit multiplicity.

When the state(s) associated with one σ_i are truncated in discrete-time, the actual error is strictly smaller that $2\sigma_i$, whereas in the continuous-time case, equality holds. This effect is compounded as further states are removed. For this reason, the discrete-time algorithm offers superior performance (and the bound is correspondingly weaker) than its continuous-time counterpart.

Solution 9.7.

1. The dynamics are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u.$$

   Replacing $\dot{x}_2$ with αx_2 yields

$$x_2 = (\alpha I - A_{22})^{-1}(A_{21}x_1 + B_2 u).$$

   Replacing x_2 in the dynamical equation for x_1 and in the output equation yields the GSPA approximant.

2. Use the Schur decomposition

$$\begin{aligned} \alpha I - A \quad = & \begin{bmatrix} I & -A_{12}(\alpha I - A_{22})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha I - \widehat{A} & 0 \\ 0 & \alpha I - A_{22} \end{bmatrix} \\ & \times \begin{bmatrix} I & 0 \\ -(\alpha I - A_{22})^{-1}A_{21} & I \end{bmatrix}, \end{aligned}$$

   in which $\widehat{A} = A_{11} - A_{12}(\alpha I - A_{22})^{-1}A_{21}$.

3. Problem 4.10 gives a formula for the realization of a system obtained via an arbitrary linear fractional transformation of the complex variable. Substitution into this formula gives the result. Alternatively, let

$$\boldsymbol{F}(z) = \boldsymbol{G}\left(\frac{\alpha(z-1)}{z+1}\right) = \tilde{D} + \tilde{C}(zI - \tilde{A})^{-1}\tilde{B}$$

denote the discrete-time equivalent transfer function matrix. Now re-arrange $(\frac{\alpha(z-1)}{z+1}I - A)^{-1}$ as follows:

$$\begin{aligned}\left(\frac{\alpha(z-1)}{z+1}I - A\right)^{-1} &= (z+1)\big(\alpha(z-1)I - (z+1)A\big)^{-1}\\ &= (z+1)\big(z(\alpha I - A) - (\alpha I + A)\big)^{-1}\\ &= (\alpha I - A)^{-1}\big(zI - (\alpha I + A)(\alpha I - A)^{-1}\big)^{-1}(z+1).\end{aligned}$$

(Note that $\alpha I - A$ is nonsingular for $\alpha > 0$, since A has no eigenvalue in the closed-right-half plane.) Define $\tilde{A} = (\alpha I + A)(\alpha I - A)^{-1}$ and note that

$$(z+1)I = zI - \tilde{A} + 2\alpha(\alpha I - A)^{-1},$$

from which we see that

$$\left(\frac{\alpha(z-1)}{z+1}I - A\right)^{-1} = (\alpha I - A)^{-1} + 2\alpha(\alpha I - A)^{-1}(zI - \tilde{A})^{-1}(\alpha I - A)^{-1}.$$

Therefore, we have that

$$\begin{aligned}\tilde{A} &= (\alpha I + A)(\alpha I - A)^{-1} = (\alpha I - A)^{-1}(\alpha I + A)\\ \tilde{B} &= \sqrt{2\alpha}(\alpha I - A)^{-1}B\\ \tilde{C} &= \sqrt{2\alpha}C(\alpha I - A)^{-1}\\ \tilde{D} &= D + C(\alpha I - A)^{-1}B,\end{aligned}$$

is the realization of $\boldsymbol{F}(z)$.

We now show that this realization is a balanced discrete-time realization. Suppose Σ is the controllability/observability gramian of the balanced realization (A, B, C, D). Then

$$\begin{aligned}&\tilde{A}'\Sigma\tilde{A} - \Sigma + \tilde{C}'\tilde{C}\\ &\quad= (\alpha I - A')^{-1}\big((\alpha I + A')\Sigma(\alpha I + A) - (\alpha I - A')\Sigma(\alpha I - A)\\ &\qquad + 2\alpha C'C\big)(\alpha I - A)^{-1}\\ &\quad= 2\alpha(\alpha I - A')^{-1}\big(\Sigma A + A'\Sigma + C'C\big)(\alpha I - A)^{-1}\\ &\quad= 0.\end{aligned}$$

Similarly, we have

$$\tilde{A}\Sigma\tilde{A}' - \Sigma + \tilde{B}\tilde{B}' = 0.$$

It remains to show that $\tilde{A}$ has all its eigenvalues inside the unit circle—there are several ways of doing this. One possibility is to suppose that $\tilde{A}x = \lambda x$ and show that this implies $Ax = \frac{\alpha(\lambda-1)}{\lambda+1}x$. Since A is asymptotically stable, we must have $R_e \frac{\alpha(\lambda-1)}{\lambda+1} < 0$ and we conclude from this that $|\lambda| < 1$. An alternative is to suppose that $\tilde{A}x = \lambda x$ and use the discrete-time observability gramian equation to obtain

$$x^*\Sigma x(|\lambda|^2 - 1) + \|\tilde{C}x\| = 0.$$

Since $\Sigma > 0$, this shows that either: (a) $|\lambda| < 1$ or (b) $\tilde{C}x = 0$. If case (b) holds, we see that $Ax = \frac{\alpha(\lambda-1)}{\lambda+1}x$ and $Cx = 0$, which implies $x = 0$ because (A, C) is observable.

4. Use the Schur decomposition

$$\begin{aligned}\alpha I - A &= \begin{bmatrix} I & -A_{12}(\alpha I - A_{22})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha I - \widehat{A} & 0 \\ 0 & \alpha I - A_{22} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 \\ -(\alpha I - A_{22})^{-1}A_{21} & I \end{bmatrix}\end{aligned}$$

to obtain

$$\begin{aligned}(\alpha I - A)^{-1} &= \begin{bmatrix} I & 0 \\ (\alpha I - A_{22})^{-1}A_{21} & I \end{bmatrix} \begin{bmatrix} (\alpha I - \widehat{A})^{-1} & 0 \\ 0 & (\alpha I - A_{22})^{-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & A_{12}(\alpha I - A_{22})^{-1} \\ 0 & I \end{bmatrix}\end{aligned}$$

It is now easy to see that

$$\begin{aligned}\tilde{A}_{11} &= (\alpha I + \widehat{A})(\alpha I - \widehat{A})^{-1} \\ \tilde{B}_1 &= \sqrt{2\alpha}(\alpha I - \widehat{A})^{-1}\widehat{B} \\ \tilde{C}_1 &= \sqrt{2\alpha}\widehat{C}(\alpha I - \widehat{A})^{-1} \\ \tilde{D} &= \widehat{D} + \widehat{C}(\alpha I - \widehat{A})^{-1}\widehat{B},\end{aligned}$$

which shows that $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \tilde{D})$ is indeed the discrete-time equivalent of the GSPA approximant.

5. For $\alpha = \infty$, the GSPA approximant is the balanced truncation approximant. For $\alpha = 0$, the GSPA approximant is the SPA approximant. Both these algorithms have already been shown to preserve stability and to satisfy the twice-the-sum-of-the-tail error bound. For the case $0 < \alpha < \infty$, we use the discrete-time balanced truncation results of Problem 9.6 and the facts proved in this problem. Since $\tilde{A}_{11}$ has all its eigenvalues inside the unit circle, $\widehat{A}$ has all its eigenvalues in the open-left-half plane, and since the unit circle is mapped to the imaginary axis by $s = \frac{\alpha(z-1)}{z+1}$, the unit circle infinity norm bound for the discrete-time truncation provides and infinity norm bound for the GSPA algorithm.

As a final remark, we not that $\alpha = 0$ gives exact matching at steady-state ($s = 0$), $\alpha = \infty$ gives exact matching at infinite frequency. For α between 0 and ∞, exact matching occurs at the point $s = \alpha$ on the positive real axis. Varying α can be used to adjust the emphasis the reduction procedure gives to high and low frequencies. A low value of α emphasizes low frequencies and a high value of α emphasizes high frequencies.

Solutions to Problems in Chapter 10

Solution 10.1.

1. Let $\boldsymbol{w} \in \mathcal{H}_2^-$ and $\boldsymbol{z} = \boldsymbol{F}\boldsymbol{w}$. Then $\boldsymbol{z}$ is analytic in $\mathrm{R}_e(s) < 0$ and

$$\begin{aligned} \|z\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{z}(j\omega)^*\boldsymbol{z}(j\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{w}(j\omega)^*\boldsymbol{F}(j\omega)^*\boldsymbol{F}(j\omega)\boldsymbol{w}(j\omega)\,d\omega \\ &\le \sup_{\omega}\overline{\sigma}[\boldsymbol{F}(j\omega)]^2\frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{w}(j\omega)^*\boldsymbol{w}(j\omega)\,d\omega \\ &= \|\boldsymbol{F}\|_\infty^2\|\boldsymbol{w}\|_2^2. \end{aligned}$$

   Hence $z \in \mathcal{H}_2^-$.

   By the Paley-Wiener theorem, $\boldsymbol{F}\mathcal{L}_2(-\infty, 0] \subset \mathcal{L}_2(-\infty, 0]$, which, since $\boldsymbol{F}$ is time-invariant, implies that $\boldsymbol{F}$ is anticausal.

2. Suppose $\boldsymbol{G} = \boldsymbol{G}_+ + \boldsymbol{G}_-$ with $\boldsymbol{G}_+ \in \mathcal{RH}_\infty$ and $\boldsymbol{G}_- \in \mathcal{RH}_\infty^-$ (conceptually, this may be done by considering partial fraction expansions of the entries of $\boldsymbol{G}$, or alternatively we may use the state-space algorithm below). By the Paley-Wiener theorem, the inverse Laplace transform of $\boldsymbol{G}_+$ is a function $G_+(t)$ that is zero for $t < 0$ and the inverse Laplace transform of $\boldsymbol{G}_-$ is a function $G_-(t)$ that is zero for $t > 0$. Hence the system $\boldsymbol{G}$, which is represented by the convolution

$$y(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)\,d\tau$$

   may be decomposed as

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} G_+(t-\tau)u(\tau)\,d\tau + \int_{-\infty}^{\infty} G_-(t-\tau)u(\tau)\,d\tau \\ &= \int_{-\infty}^{t} G_+(t-\tau)u(\tau)\,d\tau + \int_{t}^{\infty} G_-(t-\tau)u(\tau)\,d\tau, \end{aligned}$$

which is a causal/anticausal decomposition.

The converse follows similarly by using the Paley-Wiener theorem.

3. Start with a realization (A, B, C, D) of $\boldsymbol{G}$. Produce, via an eigenvalue decomposition for example, a realization of the form

$$\begin{aligned} A &= \begin{bmatrix} A_- & 0 & 0 \\ 0 & A_+ & 0 \\ 0 & 0 & A_0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_- \\ B_+ \\ B_0 \end{bmatrix} \\ C &= \begin{bmatrix} C_- & C_+ & C_0 \end{bmatrix} \end{aligned}$$

in which $\mathrm{R}_e[\lambda(A_-)] > 0$, $\mathrm{R}_e[\lambda(A_+)] < 0$, $\mathrm{R}_e[\lambda(A_0)] = 0$. The assumption $\boldsymbol{G} \in \mathcal{RL}_\infty$ implies that $C_0(sI - A_0)^{-1}B_0 = 0$. Therefore, (A_+, B_+, C_+, D) is a realization for the stable or causal part and $\boldsymbol{G}_+$ and $(A_-, B_-, C_-, 0)$ is a realization for the antistable or anticausal part $\boldsymbol{G}_-$.

Solution 10.2. Sufficiency follows from Theorem 3.2.1, or by direct calculation as follows:

$$\begin{aligned} \boldsymbol{G}^\sim \boldsymbol{G} &= D'D + D'C(sI-A)^{-1}B + B'(-sI-A')^{-1}C'D \\ &\quad + B'(-sI-A')^{-1}C'C(sI-A)^{-1}B \\ &= I + B'(sI-A')^{-1}\big(-Q(sI-A) - (-sI-A')Q + C'C\big)(sI-A)^{-1}B \\ &= I. \end{aligned}$$

Conversely, suppose $\boldsymbol{G}^\sim \boldsymbol{G} = I$. Since $\boldsymbol{G}$ is square, this implies that $\boldsymbol{G}^\sim = \boldsymbol{G}^{-1}$. Now $\boldsymbol{G}^\sim$ has realization

$$\boldsymbol{G}^\sim \stackrel{s}{=} \left[\begin{array}{c|c} -A' & -C' \\ \hline B' & D' \end{array}\right],$$

which is easily seen to be minimal. Also,

$$\boldsymbol{G}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right],$$

which is also minimal. From $\boldsymbol{G}^\sim = \boldsymbol{G}^{-1}$ and the uniqueness of minimal realizations, we conclude that $D' = D^{-1}$ and that there exists a nonsingular matrix Q such that

$$\begin{aligned} B'Q &= -D^{-1}C \\ -Q^{-1}A'Q &= A - BD^{-1}C. \end{aligned}$$

Elementary manipulations now yield the desired equations.

Let $P = Q^{-1}$ and note that

$$AP + PA' + BB' = 0.$$

Therefore P is the controllability gramian of $\boldsymbol{G}$. If $\boldsymbol{G}$ is stable, the Hankel singular values of $\boldsymbol{G}$ are given by $\lambda_i(PQ) = 1$ for all i. This means that approximating a stable allpass system is a fruitless exercise—one can never do better than approximating it by the zero system, since this gives an infinity norm error of one.

Solution 10.3.

$$\begin{aligned}
\boldsymbol{E}_a &\stackrel{s}{=} \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right] \\
\boldsymbol{V}_a &\stackrel{s}{=} \frac{1}{\sqrt{\sigma_{r+1}}}\left[\begin{array}{c|c} A' & M \\ \hline B_a' & 0 \end{array}\right] \\
\boldsymbol{W}_a &\stackrel{s}{=} \frac{1}{\sqrt{\sigma_{r+1}}}\left[\begin{array}{c|c} A & M \\ \hline C_a & 0 \end{array}\right],
\end{aligned}$$

in which

$$M = \begin{bmatrix} 0 \\ I_l \end{bmatrix}$$

with l being the multiplicity of σ_{r+1}.

$$\begin{aligned}
\boldsymbol{E}_a^{\sim}\boldsymbol{W}_a &\stackrel{s}{=} \frac{1}{\sqrt{\sigma_{r+1}}}\left[\begin{array}{c|c} -A_e' & C_e' \\ \hline -B_e' & D_e' \end{array}\right]\left[\begin{array}{c|c} A & M \\ \hline C_a & 0 \end{array}\right] \\
&\stackrel{s}{=} \frac{1}{\sqrt{\sigma_{r+1}}}\left[\begin{array}{cc|c} -A_e' & C_e'C_a & 0 \\ 0 & A & M \\ \hline -B_e' & D_e'C_a & 0 \end{array}\right].
\end{aligned} \tag{10.1}$$

Now from (10.3.4) and (10.3.5), we have

$$\begin{aligned}
A_e'\begin{bmatrix} Q \\ N' \end{bmatrix} + \begin{bmatrix} Q \\ N' \end{bmatrix} A + C_e'C_a &= 0 \\
D_e'C_a + B_e'\begin{bmatrix} Q \\ N' \end{bmatrix} &= 0
\end{aligned}$$

in which $N' = \begin{bmatrix} -E_1' & 0 \end{bmatrix}$. Also, from (10.4.2),

$$\begin{bmatrix} Q \\ N' \end{bmatrix} M = \sigma_{r+1}\begin{bmatrix} M \\ 0 \end{bmatrix}.$$

Applying the state transformation

$$\begin{bmatrix} I & \begin{bmatrix} Q \\ N' \end{bmatrix} \\ 0 & I \end{bmatrix}$$

to the realization in (10.1), we see that

$$\begin{aligned}
\boldsymbol{E}_a^{\sim}\boldsymbol{W}_a &\stackrel{s}{=} \sqrt{\sigma_{r+1}}\left[\begin{array}{c|c} -A_e' & \left[\begin{array}{c} M \\ 0 \end{array}\right] \\ \hline -B_e & 0 \end{array}\right] \\
&\stackrel{s}{=} \sqrt{\sigma_{r+1}}\left[\begin{array}{c|c} -A' & M \\ \hline -B_a' & 0 \end{array}\right] \\
&= -\sqrt{\sigma_{r+1}}B_a'(sI+A')^{-1}M \\
&= \sigma_{r+1}\boldsymbol{V}_a(-s)
\end{aligned}$$

The dual identity $\boldsymbol{E}_a(s)\boldsymbol{V}_a(-s) = \sigma_{r+1}\boldsymbol{W}_a(s)$ follows using as similar approach.

Solution 10.4. Let $\boldsymbol{q} \in \mathcal{RH}_\infty^-(r)$ satisfy $\|\boldsymbol{g}-\boldsymbol{q}\|_\infty = \sigma_{r+1}$. (Such a $\boldsymbol{q}$ exists by Theorem 10.4.2.) By Lemma 10.4.3 we have that $(\boldsymbol{g}-\boldsymbol{q})(s)\boldsymbol{v}_{r+1}(-s) = \sigma_{r+1}\boldsymbol{w}_{r+1}(s)$. Hence

$$\boldsymbol{q}(s) = \boldsymbol{g}(s) - \sigma_{r+1}\frac{\boldsymbol{w}_{r+1}(s)}{\boldsymbol{v}_{r+1}(-s)}.$$

Solution 10.5.

1. See Theorem A.4.4; by Lemma A.4.5, the $\boldsymbol{T}_{ij}$'s have realization

$$\left[\begin{array}{cc} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{T}_{21} & 0 \end{array}\right] \stackrel{s}{=} \left[\begin{array}{cc|cc} A-B_2F & HC_2 & H & B_2 \\ 0 & A-HC_2 & B_1-H & 0 \\ \hline C_1-F & C_1 & 0 & I \\ 0 & C_2 & I & 0 \end{array}\right].$$

2. The assumption that $A - B_2C_1$ and $A - B_1C_2$ have no eigenvalues on the imaginary axis is necessary and sufficient for the existence of stabilizing solutions X and Y to the given Riccati equations (see Chapter 5). That is, $A - B_2C_1 - B_2B_2'X = A - B_2F$ and $A - B_1C_2 - YC_2'C_2 = A - HC_2$ are asymptotically stable. Now $\boldsymbol{T}_{12}$ has realization

$$\boldsymbol{T}_{12} \stackrel{s}{=} \left[\begin{array}{c|c} A-B_2F & B_2 \\ \hline C_1-F & I \end{array}\right].$$

Obviously, T_{12} is square. Elementary algebra shows that

$$\begin{aligned}
X(A-B_2F)+(A-B_2F)'X+(C_1-F)'(C_1-F) &= 0 \\
(C_1-F)+B_2'X &= 0
\end{aligned}$$

and we conclude that $\boldsymbol{T}_{12}^{\sim}\boldsymbol{T}_{12} = I$ from Theorem 3.2.1. The assertion that $\boldsymbol{T}_{21}$ is square and allpass may be established similarly.

Since $\boldsymbol{T}_{12}$ and $\boldsymbol{T}_{21}$ are square and allpass, we have

$$\begin{aligned}\|\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K})\|_\infty &= \|\boldsymbol{T}_{11}+\boldsymbol{T}_{12}\boldsymbol{Q}\boldsymbol{T}_{21}\|_\infty\\ &= \|\boldsymbol{T}_{12}^\sim(\boldsymbol{T}_{11}+\boldsymbol{T}_{12}\boldsymbol{Q}\boldsymbol{T}_{21})\boldsymbol{T}_{21}^\sim\|_\infty\\ &= \|\boldsymbol{T}_{12}^\sim\boldsymbol{T}_{11}\boldsymbol{T}_{21}^\sim+\boldsymbol{Q}\|_\infty\\ &= \|\boldsymbol{R}+\boldsymbol{Q}\|_\infty.\end{aligned}$$

3. Note that

$$\begin{aligned}\|\boldsymbol{R}+\boldsymbol{Q}\|_\infty &= \|\boldsymbol{R}^\sim+\boldsymbol{Q}^\sim\|_\infty\\ &= \|(\boldsymbol{R}^\sim)_+ + ((\boldsymbol{R}^\sim)_- + \boldsymbol{Q}^\sim)\|_\infty\end{aligned}$$

Now $(\boldsymbol{R}^\sim)_+\in\mathcal{RH}_\infty$ and $((\boldsymbol{R}^\sim)_- + \boldsymbol{Q}^\sim)$ is an arbitrary element of $\mathcal{RH}_\infty^-$, so by Nehari's theorem the infimal norm is indeed the Hankel norm of $(\boldsymbol{R}^\sim)_+$.

We now compute a realization of $\boldsymbol{R}$. Since

$$\boldsymbol{T}_{11}\stackrel{s}{=}\left[\begin{array}{cc|c} A-B_2F & HC_2 & H\\ 0 & A-HC_2 & B_1-H\\ \hline C_1-F & C_1 & 0\end{array}\right],$$

we have

$$\begin{aligned}\boldsymbol{T}_{11}\boldsymbol{T}_{21}^\sim &\stackrel{s}{=} \left[\begin{array}{cc|c} A-B_2F & HC_2 & H\\ 0 & A-HC_2 & B_1-H\\ \hline C_1-F & C_1 & 0\end{array}\right]\left[\begin{array}{c|c} -(A-HC_2)' & -C_2'\\ \hline (B_1-H)' & I\end{array}\right]\\
&\stackrel{s}{=} \left[\begin{array}{ccc|c} A-B_2F & HC_2 & H(B_1-H)' & H\\ 0 & A-HC_2 & (B_1-H)(B_1-H)' & B_1-H\\ 0 & 0 & -(A-HC_2)' & -C_2'\\ \hline C_1-F & C_1 & 0 & 0\end{array}\right]\\
&\stackrel{s}{=} \left[\begin{array}{ccc|c} A-B_2F & HC_2 & 0 & H\\ 0 & A-HC_2 & 0 & 0\\ 0 & 0 & -(A-HC_2)' & -C_2'\\ \hline C_1-F & C_1 & C_1Y & 0\end{array}\right]\\
&\stackrel{s}{=} \left[\begin{array}{cc|c} A-B_2F & 0 & H\\ 0 & -(A-HC_2)' & -C_2'\\ \hline C_1-F & C_1Y & 0\end{array}\right].\end{aligned}$$

In the above calculation, we made use of the state transformation

$$\left[\begin{array}{ccc} I & 0 & 0\\ 0 & I & Y\\ 0 & 0 & I\end{array}\right],$$

the identity $B_1 - H + YC_2' = 0$ and the identity $(A - HC_2)Y + Y(A - HC_2)' + (B_1 - H)(B_1 - H)' = 0$.

$$
\begin{aligned}
\boldsymbol{R} &= \boldsymbol{T}_{12}^{\sim}\boldsymbol{T}_{12}\boldsymbol{T}_{21}^{\sim} \\
&\overset{s}{=} \left[\begin{array}{c|c} -(A-B_2F)' & (C_1-F)' \\ \hline -B_2' & I \end{array}\right]\left[\begin{array}{cc|c} A-B_2F & 0 & H \\ 0 & -(A-HC_2)' & -C_2' \\ \hline C_1-F & C_1Y & 0 \end{array}\right] \\
&\overset{s}{=} \left[\begin{array}{ccc|c} -(A-B_2F)' & (C_1-F)'(C_1-F) & (C_1-F)'C_1Y & 0 \\ 0 & A-B_2F & 0 & H \\ 0 & 0 & -(A-HC_2)' & -C_2' \\ \hline -B_2' & C_1-F & C_1Y & 0 \end{array}\right] \\
&\overset{s}{=} \left[\begin{array}{ccc|c} -(A-B_2F)' & 0 & (C_1-F)'C_1Y & XH \\ 0 & A-B_2F & 0 & H \\ 0 & 0 & -(A-HC_2)' & -C_2' \\ \hline -B_2' & 0 & C_1Y & 0 \end{array}\right] \\
&\overset{s}{=} \left[\begin{array}{cc|c} -(A-B_2F)' & (C_1-F)'C_1Y & XH \\ 0 & -(A-HC_2)' & -C_2' \\ \hline -B_2' & C_1Y & 0 \end{array}\right].
\end{aligned}
$$

Thus

$$
\boldsymbol{R}^{\sim} \overset{s}{=} \left[\begin{array}{cc|c} A-B_2F & 0 & B_2 \\ YC_1'(C_1-F) & A-HC_2 & -YC_1' \\ \hline H'X & -C_2 & 0 \end{array}\right].
$$

Since $A - B_2F$ and $A - HC_2$ are asymptotically stable, $\boldsymbol{R}^{\sim} \in \mathcal{RH}_\infty$, so we may take $(\boldsymbol{R}^{\sim})_+ = \boldsymbol{R}^{\sim}$.

4. Use Theorem 10.4.6 to obtain all optimal Nehari extensions of $\boldsymbol{R}^{\sim}$. This will have the linear fractional form $-\boldsymbol{Q}^{\sim} = \mathcal{F}_\ell(\boldsymbol{Q}_a^{\sim}, \boldsymbol{U}^{\sim})$, in which $\boldsymbol{Q}_a^{\sim}$ constructed from $\boldsymbol{R}^{\sim}$ following the steps in Section 10.4.1 and $\boldsymbol{U}^{\sim} \in \mathcal{RH}_\infty^-$, $\|\boldsymbol{U}\|_\infty \le \|\boldsymbol{R}^{\sim}\|_H^{-1}$. Thus $\boldsymbol{Q} = \mathcal{F}_\ell(-\boldsymbol{Q}_a, \boldsymbol{U})$, in which $\boldsymbol{U} \in \mathcal{RH}_\infty$ and $\|\boldsymbol{U}\|_\infty \le \|\boldsymbol{R}^{\sim}\|_H^{-1}$. From the parametrization of all stabilizing controllers, we have

$$
\begin{aligned}
\boldsymbol{K} &= \mathcal{F}_\ell(\boldsymbol{K}_s, \boldsymbol{Q}) \\
&= \mathcal{F}_\ell(\boldsymbol{K}_s, \mathcal{F}_\ell(-\boldsymbol{Q}_a, \boldsymbol{U})) \\
&= \mathcal{F}_\ell(\mathcal{C}_\ell(\boldsymbol{K}_s, -\boldsymbol{Q}_a), \boldsymbol{U}) \\
&= \mathcal{F}_\ell(\boldsymbol{K}_a, \boldsymbol{U})
\end{aligned}
$$

in which $\boldsymbol{K}_a = \mathcal{C}_\ell(\boldsymbol{K}_s, -\boldsymbol{Q}_a)$ is the composition of the two linear fractional transformations $\boldsymbol{K}_s$ and $-\boldsymbol{Q}_a$ and $\boldsymbol{U} \in \mathcal{RH}_\infty$ with $\|\boldsymbol{U}\|_\infty \le \|\boldsymbol{R}^{\sim}\|_H^{-1}$.

Solution 10.6.

1. If $\widehat{G} \in \mathcal{RH}_\infty^-$ has (strictly) fewer poles that G, then

$$\|G - \widehat{G}\|_\infty = \|G^\sim - \widehat{G}^\sim\|_\infty \geq \sigma_{\min}[G^\sim] = \epsilon_0.$$

Conversely, if $\epsilon \geq \epsilon_0$, let $\widehat{G}^\sim$ be the optimal Hankel norm approximant to $G^\sim$ of degree $n - l$, in which n is the degree of $G^\sim$ and l is the multiplicity of $\sigma_{\min}[G^\sim]$. Then $\widehat{G}^\sim \in \mathcal{RH}_\infty^-$, so $\widehat{G} \in \mathcal{RH}_\infty$ is of strictly lower degree than G and $\|G - \widehat{G}\|_\infty = \epsilon_0 \leq \epsilon$.

Now suppose that $\epsilon > \epsilon_0$, let $\widehat{G}$ be the optimal Hankel norm approximant as discussed above and consider the family of plants

$$G + \Delta = \widehat{G} + \frac{\delta M}{(s-1)^l}.$$

Then

$$\begin{aligned} \|\Delta\|_\infty &= \|\widehat{G} - G + \frac{\delta M}{(s-1)^l}\|_\infty \\ &\leq \|\widehat{G} - G\|_\infty + \|\frac{\delta M}{(s-1)^l}\|_\infty \\ &\leq \epsilon_0 + \delta\|M\| \\ &< \epsilon. \end{aligned}$$

Furthermore, for and $\delta > 0$, $\widehat{G} + \frac{\delta M}{(s-1)^l}$ has $n - l + l = n$ unstable poles, the same number as G.

Suppose, to obtain a contradiction, that K stabilizes the famlily of plants $G + \Delta$ as given above. For $\delta \neq 0$, $G + \Delta$ has n unstable poles. Since K stabilizes $G + \Delta$, the Nyquist diagram of $\det\big(I - (G + \Delta)K(s)\big)$ must make $n + k$ encirclements of the origin as s traverses the Nyquist contour, in which k is the number of right-half-plane poles of K. Since K must also stabilize $\widehat{G}$, the Nyquist diagram of $\det\big(I - \widehat{G}K(s)\big)$ must make $n - l + k$ encirclements of the origin as s traverses the Nyquist contour, since $\widehat{G}$ has only $n - l$ right-half-plane poles. But the Nyquist diagram of $\det\big(I - GK(s)\big)$ is a continuous function of δ, and therefore there exists a δ_0 such that the Nyquist diagram of $\det\big(I - (\widehat{G} + \frac{\delta_0 M}{(s-1)^l})K(s)\big)$ crosses the origin. This plant is therefore not stabilized by K. We conclude that no controller can stabilize the the famlily of plants $G + \Delta$ as given above.

2. Write $G = G_+ + G_-$. Then

$$\begin{aligned} K(I - GK)^{-1} &= \widehat{K}(I + G_+\widehat{K})^{-1}[I - (G_+ + G_-)\widehat{K}(I + G_+\widehat{K})^{-1}]^{-1} \\ &= \widehat{K}[I + G_+\widehat{K} - (G_+ + G_-)\widehat{K}]^{-1} \\ &= \widehat{K}(I - G_-\widehat{K})^{-1}. \end{aligned}$$

Note that

$$\begin{bmatrix} I & -\boldsymbol{K} \\ -\boldsymbol{G} & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\boldsymbol{G}_+ & I \end{bmatrix} \begin{bmatrix} I & -\widehat{\boldsymbol{K}} \\ -\boldsymbol{G}_- & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I+\boldsymbol{G}_+\widehat{\boldsymbol{K}} \end{bmatrix}^{-1},$$

and that $I+\boldsymbol{G}_+\widehat{\boldsymbol{K}} = (I-\boldsymbol{G}_+\boldsymbol{K})^{-1}$.

Suppose $\widehat{\boldsymbol{K}}$ stabilizes $\boldsymbol{G}_-$. Define $\boldsymbol{K} = \widehat{\boldsymbol{K}}(I+\boldsymbol{G}_+\widehat{\boldsymbol{K}})^{-1}$ and note that

$$\begin{bmatrix} I & -\boldsymbol{K} \\ -\boldsymbol{G} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ 0 & I+\boldsymbol{G}_+\widehat{\boldsymbol{K}} \end{bmatrix} \begin{bmatrix} I & -\widehat{\boldsymbol{K}} \\ -\boldsymbol{G}_- & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ \boldsymbol{G}_+ & I \end{bmatrix}$$

The last two matrices on the right-hand side of this identity are in $\mathcal{H}_\infty$, since $\boldsymbol{G}_+ \in \mathcal{H}_\infty$ and $\widehat{\boldsymbol{K}}$ stabilizes $\boldsymbol{G}_-$. The only poles that $I+\boldsymbol{G}_+\widehat{\boldsymbol{K}}$ can have in the right-half plane are the right-half-plane poles of $\widehat{\boldsymbol{K}}$, which must be cancelled by the zeros of $\begin{bmatrix} I & -\widehat{\boldsymbol{K}} \\ -\boldsymbol{G}_- & I \end{bmatrix}^{-1}$. Hence the matrix on the left-hand side is in $\mathcal{RH}_\infty$, which means that $\boldsymbol{K}$ stabilizes $\boldsymbol{G}$. Conversely, suppose $\boldsymbol{K}$ stabilizes $\boldsymbol{G}$. Define $\widehat{\boldsymbol{K}} = \boldsymbol{K}(I-\boldsymbol{G}_+\boldsymbol{K})^{-1}$ and note that

$$\begin{bmatrix} I & -\widehat{\boldsymbol{K}} \\ -\boldsymbol{G}_- & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ 0 & I-\boldsymbol{G}_+\boldsymbol{K} \end{bmatrix} \begin{bmatrix} I & -\boldsymbol{K} \\ -\boldsymbol{G} & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -\boldsymbol{G}_+ & I \end{bmatrix}.$$

Arguing as before, we conclude that $\widehat{\boldsymbol{K}}$ stabilizes $\boldsymbol{G}_-$.

Hence $\inf_{\boldsymbol{K}} \|\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K}^{-1}\|_\infty = \inf_{\widehat{\boldsymbol{K}}} \|\widehat{\boldsymbol{K}}(I-\boldsymbol{G}_-\widehat{\boldsymbol{K}})^{-1}\|_\infty = 1/\sigma_{\min}[\boldsymbol{G}_-^\sim]$.

Thus the optimal (maximum) stability margin for additive uncertainties is $\sigma_{\min}[\boldsymbol{G}_-^\sim]$, which means that the easier the unstable part is to approximate, the harder it is to robustly stablize. This counter-intuitive result stems from the requirement of the additive robustness theorem that all the $(\boldsymbol{G}+\boldsymbol{\Delta})$'s must have the same number of unstable poles. If the unstable part is easy to approximate, there is a system that is close to $\boldsymbol{G}$ that has fewer unstable poles. Thus, we cannot allow very large $\|\boldsymbol{\Delta}\|_\infty$.

3. Comparing $\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K})^{-1}$ with $\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K}) = \boldsymbol{P}_{11}+\boldsymbol{P}_{12}\boldsymbol{K}(I-\boldsymbol{P}_{22}\boldsymbol{K})^{-1}\boldsymbol{P}_{21}$, we see that $\boldsymbol{P}_{11}=0$, $\boldsymbol{P}_{12}=I$, $\boldsymbol{P}_{21}=I$ and $\boldsymbol{P}_{22}=\boldsymbol{G}$. Hence we need

$$\begin{aligned}\begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ I & \boldsymbol{G} \end{bmatrix} \\ &\overset{s}{=} \left[\begin{array}{c|cc} A & 0 & B \\ \hline 0 & 0 & I \\ C & I & 0 \end{array}\right].\end{aligned}$$

We can assume that $\boldsymbol{G}(\infty)=0$, without loss of generality; the system $\boldsymbol{G}$ can be replaced by $\boldsymbol{G}-\boldsymbol{G}(\infty)$ by using the controller $\tilde{\boldsymbol{K}} = \boldsymbol{K}(I-\boldsymbol{G}(\infty)\boldsymbol{K})^{-1}$.

Before beginning the detailed analysis, we present a brief plausibility argument. If $\widehat{\boldsymbol{K}}$ is invertible,

$$\widehat{\boldsymbol{K}}(I - \boldsymbol{G}_-\widehat{\boldsymbol{K}})^{-1} = (\widehat{\boldsymbol{K}}^{-1} - \boldsymbol{G}_-)^{-1}.$$

Now let $(\widehat{\boldsymbol{K}}^{\sim})^{-1}$ be an $(n-l)$th order Hankel approximation of $\boldsymbol{G}_-^{\sim}$, so that $\widehat{\boldsymbol{K}}^{-1} - \boldsymbol{G}_-$ is allpass, completely unstable and with $\|\widehat{\boldsymbol{K}}^{-1} - \boldsymbol{G}_-\|_\infty = \sigma_n(\boldsymbol{G}_-^{\sim})$. This means that $\widehat{\boldsymbol{K}}(I - \boldsymbol{G}_-\widehat{\boldsymbol{K}})^{-1}$ will be stable and $\|\widehat{\boldsymbol{K}}(I - \boldsymbol{G}_-\widehat{\boldsymbol{K}})^{-1}\|_\infty = \frac{1}{\sigma_n(\boldsymbol{G}_-^{\sim})}$.

Assume, without loss of generality, that

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}, \qquad B = \begin{bmatrix} B_+ \\ B_- \end{bmatrix}, \qquad C = \begin{bmatrix} C_+ & C_- \end{bmatrix},$$

in which $\mathrm{R}_e[\lambda(A_+)] < 0$ and $\mathrm{R}_e[\lambda(A_-)] > 0$. Thus $\boldsymbol{G}_+ = C_+(sI - A_+)^{-1}B_+$ and $\boldsymbol{G}_- = C_-(sI - A_-)^{-1}B_-$. Hence $\boldsymbol{G}_-^{\sim} = -B_-'(sI + A')^{-1}C_-'$ and the controllability and observability gramians of this realization of $\boldsymbol{G}_-^{\sim}$ satisfy

$$\begin{aligned} -A_-'Q_- - Q_-A_- + C_-'C_- &= 0 \\ -P_-A_-' - A_-P_- + B_-B_-' &= 0. \end{aligned}$$

Thus the Hankel singular values of $\boldsymbol{G}_-^{\sim}$ are equal to $\sqrt{\lambda_i(Q_-P_-)}$.

According to the solution of Problem 10.5, we need to find $\boldsymbol{R}^{\sim}$. To do this, we need to find the stabilizing solutions to the Riccati equations

$$\begin{aligned} XA + A'X - XBB'X &= 0 \\ AY + YA' - YC'CY &= 0, \end{aligned}$$

which exist provided A has no eigenvalue on the imaginary axis. Assume now, without loss of generality, that (A_-, B_-, C_-) is minimal, so Q_- and P_- are nonsingular. Then $A_- - B_-B_-'P_-^{-1} = -P_-A_-'P_-^{-1}$, which is asymptotically stable, and $A_- - Q_-^{-1}C_-'C_- = -Q_-^{-1}A_-'Q_-$, which is also asymptotically stable. Therefore, the stabilizing solutions to the Riccati equations are

$$X = \begin{bmatrix} 0 & 0 \\ 0 & P_-^{-1} \end{bmatrix}; \qquad Y = \begin{bmatrix} 0 & 0 \\ 0 & Q_-^{-1} \end{bmatrix}$$

The system $\boldsymbol{R}^{\sim}$ is therefore given by

$$\begin{aligned} \boldsymbol{R}^{\sim} &\stackrel{s}{=} \left[\begin{array}{cc|c} A - BF & 0 & B \\ 0 & A - HC & 0 \\ \hline H'X & -C & 0 \end{array}\right] \\ &\stackrel{s}{=} \left[\begin{array}{c|c} A - BF & B \\ \hline H'X & 0 \end{array}\right] \end{aligned}$$

in which $F = B'X$ and $H = YC'$. Noting that

$$H'X = CYX = \begin{bmatrix} 0 & C_- Q_-^{-1} P_-^{-1} \end{bmatrix},$$

we eliminate the unobservable modes to obtain the realization

$$\begin{aligned} \boldsymbol{R}^\sim &\stackrel{s}{=} \left[\begin{array}{c|c} A_- - B_- B_-' P_-^{-1} & B_- \\ \hline C_- Q_-^{-1} P_-^{-1} & 0 \end{array}\right] \\ &\stackrel{s}{=} \left[\begin{array}{c|c} -P_- A_-' P_-^{-1} & B_- \\ \hline C_- Q_-^{-1} P_-^{-1} & 0 \end{array}\right] \\ &\stackrel{s}{=} \left[\begin{array}{c|c} -A_-' & P_-^{-1} B_- \\ \hline C_- Q_-^{-1} & 0 \end{array}\right]. \end{aligned}$$

The controllability and observability gramians of this realization are easily seen to be P_-^{-1} and Q_-^{-1} respectively. Hence

$$\begin{aligned} \|\boldsymbol{R}^\sim\|_H &= \sqrt{\lambda_{\max}(P_-^{-1} Q_-^{-1})} \\ &= \frac{1}{\sqrt{\lambda_{\min}(Q_- P_-)}} \\ &= \frac{1}{\sigma_{\min}(\boldsymbol{G}_-^\sim)}. \end{aligned}$$

Solution 10.7.

1. If the future input is zero, *i.e.*, $u_k = 0$ for $k \geq 1$, then the future output is

$$\begin{aligned} y_n &= \sum_{k=-\infty}^{0} h_{n-k} u_k, \qquad n = 1, 2 \ldots \\ &= \sum_{m=0}^{\infty} h_{n+m} u_{-m}, \qquad n = 1, 2 \ldots \\ &= \sum_{m=1}^{\infty} h_{n+m-1} v_m, \qquad n = 1, 2 \ldots \end{aligned}$$

in which $v_m = u_{1-m}$ is the reflection of the past input. This may be written as the semi-infinite vector equation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}.$$

2. Note that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} j(\theta-\pi)e^{jk\theta}\,d\theta &= \left[\frac{(\theta-\pi)e^{jk\theta}}{2\pi k}\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{2\pi k}e^{jk\theta}\,d\theta \\ &= \frac{1}{k}.\end{aligned}$$

Therefore, by Nehari's theorem, $\|\mathbf{\Gamma}_H\| \leq \sup_{\theta\in[0,2\pi)} |j(\theta-\pi)| = \pi$. Hilbert's inequality is immediate, noting that $v^*\mathbf{\Gamma}_H u \leq \|\mathbf{\Gamma}_H\|\|v\|_2\|u\|_2 \leq \pi\|v\|_2\|u\|_2$.

3. This is a direct application of Nehari's theorem. The idea is that $\boldsymbol{e}_N = z^{N-1}(\boldsymbol{f}-\widehat{\boldsymbol{f}})$ is an extension of the "tail" function $\boldsymbol{t}_N$.

Solution 10.8. That $\mathcal{OC} = \mathbf{\Gamma}$ is obvious. To find the Hankel singular values, note that

$$\begin{aligned}\lambda(\mathbf{\Gamma}'\mathbf{\Gamma}) &= \lambda(\mathcal{C}'\mathcal{O}'\mathcal{O}\mathcal{C}) \\ &= \lambda(\mathcal{C}\mathcal{C}'\mathcal{O}'\mathcal{O}) \\ &= \lambda(PQ).\end{aligned}$$

Alternatively, consider the equations $\mathbf{\Gamma}v_i = \sigma_i w_i$ and $\mathbf{\Gamma}'w_i = \sigma_i v_i$. If $v = \mathcal{C}'P^{-1}x_0$, then $\mathbf{\Gamma}v = \mathcal{O}\mathcal{C}\mathcal{C}'P^{-1}x_0 = \mathcal{O}x_0$. And if $w = \mathcal{O}x_0$, then $\mathbf{\Gamma}'w = \mathcal{C}'\mathcal{O}'\mathcal{O}x_0 = \mathcal{C}'Qx_0$. Therefore $\mathbf{\Gamma}'\mathbf{\Gamma}v = \sigma^2 w$ if $Qx_0 = \sigma^2 P^{-1}x_0$, from which we conclude that σ^2 must be an eigenvalue of PQ.

Solution 10.9.

1. Choosing $r = n-1$ and $\gamma = \sigma_n$ in the construction of the optimal allpass embedding yields a $\boldsymbol{Q}_a \in \mathcal{RH}_\infty$ of McMillan degree $n-l$, since no Hankel singular value of $\boldsymbol{G}$ is strictly smaller than σ_n and exactly $n-l$ are strictly larger than σ_n. Now $\|\boldsymbol{G}_a - \boldsymbol{Q}_a\|_\infty = \sigma_n$, so set $\widehat{\boldsymbol{G}}$ equal to the (1,1)-block of $\boldsymbol{Q}_a$. (We may also choose $\widehat{\boldsymbol{G}}$ by using any constant U in Theorem 10.4.5. Hence $\|\boldsymbol{G}\_\widehat{\boldsymbol{G}}\|_\infty \leq \sigma_n$, and in fact equality must hold because σ_n is a lower bound on the infinity norm error incurred in approximating $\boldsymbol{G}$ by a stable system of degree less than or equal to $n-1$.

2. Suppose, without loss of generality, that $\boldsymbol{G}$ is given by a balanced realization (A, B, C, D) partitioned as

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma_n I_l \end{bmatrix}$$

with

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \end{aligned}$$

Then $\widehat{\boldsymbol{G}}$, the $(1,1)$-block of $\boldsymbol{Q}_a$, is given by

$$\widehat{\boldsymbol{G}}_{a11} = \tilde{D} + \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$$

in which

$$\begin{aligned} \tilde{A} &= E_1^{-1}(\sigma_{r+1}^2 A_{11}' + \Sigma_1 A_{11} \Sigma_1 + \sigma_n C_1' U B_1') \\ \tilde{B} &= E_1^{-1}(\Sigma_1 B_1 - \sigma_{r+1} U C_1) \\ \tilde{C} &= C_1 \Sigma_1 - B_1' \sigma_{r+1} U \\ \tilde{D} &= D + \sigma_n U \end{aligned}$$

in which $UU' \leq I$ and $B_2 = C_2' U$. Direct calculations show that the controllability gramian of this realization is $\Sigma_1 E_1^{-1}$ and the observability gramian of this realization is $\Sigma_1 E_1$ and we conclude that the Hankel singular values of $\widehat{\boldsymbol{G}}$ are the diagonal entries of Σ_1.

3. Iteration and the triangle inequality show the result.

4. The optimal allpass embedding is constructed so that $\sigma_n^{-1}(\boldsymbol{G}_a - \boldsymbol{Q}_a)$ is allpass. Deleting the states corresponding to σ_n from $\boldsymbol{G}_a$ gives $\tilde{\boldsymbol{G}}_a$ and an examination of the equations (10.3.4) to (10.3.6) and (10.4.3) shows that corresponding equations for the truncated error system can be obtained merely by truncating the rows and columns associated with $\gamma = \sigma_n$ from P_e and Q_e in (10.4.3), the product of which remains $\sigma_n I$. We conclude that $\sigma_n^{-1}(\tilde{\boldsymbol{G}}_a - \boldsymbol{Q}_a)$ is allpass by invoking Theorem 3.2.1.

   To conclude the error bound, we note that

$$\begin{aligned} \|\boldsymbol{G}_a - \tilde{\boldsymbol{G}}_a\|_\infty &\leq \|\boldsymbol{G}_a - \boldsymbol{Q}_a\| + \|\boldsymbol{Q}_a - \tilde{\boldsymbol{G}}_a\|_\infty \\ &= \sigma_n + \sigma_n = 2\sigma_n. \end{aligned}$$

   Taking the $(1,1)$-block, we obtain the desired inequality $\|\boldsymbol{G} - \tilde{\boldsymbol{G}}\|_\infty \leq 2\sigma_n$.

Solutions to Problems in Chapter 11

Solution 11.1. Consider

$$\begin{bmatrix} A & B \end{bmatrix} W W^{-1} \begin{bmatrix} C \\ D \end{bmatrix} = 0$$

where

$$W = \begin{bmatrix} I & 0 \\ -B^{-1}A & I \end{bmatrix}.$$

This gives

$$\begin{bmatrix} 0 & B \end{bmatrix} \begin{bmatrix} C \\ D + B^{-1}AC \end{bmatrix} = 0.$$

Since B is nonsingular, $D + B^{-1}AC = 0$. Because C is square and of full column rank it must be nonsingular.

Solution 11.2. By direct computation

$$\begin{aligned} X^*X &= \begin{bmatrix} X_{11}^* & X_{21}^* \\ X_{12}^* & X_{22}^* \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \\ &= \begin{bmatrix} X_{11}^*X_{11} + X_{21}^*X_{21} & X_{11}^*X_{12} + X_{21}^*X_{22} \\ X_{12}^*X_{11} + X_{22}^*X_{21} & X_{12}^*X_{12} + X_{22}^*X_{22} \end{bmatrix}, \end{aligned}$$

therefore

$$\text{trace}(X^*X) = \text{trace}(X_{11}^*X_{11}) + \text{trace}(X_{21}^*X_{21}) + \text{trace}(X_{12}^*X_{12}) + \text{trace}(X_{22}^*X_{22}).$$

Since each term is nonnegative,

$$\begin{aligned} \inf_{X_{22}} \left\| \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \right\|_2 &= \left\| \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & 0 \end{bmatrix} \right\|_2 \\ &= \text{trace}(X_{11}^*X_{11}) + \text{trace}(X_{21}^*X_{21}) + \text{trace}(X_{12}^*X_{12}). \end{aligned}$$

Solution 11.3. Substituting for $\widehat{A}$ and $\widehat{B}_{aa}$ gives

$$\begin{aligned}
&= \widehat{A}Y(Z^{-1})' + Z^{-1}Y\widehat{A}' + \widehat{B}_{aa}\widehat{B}'_{aa} \\
&= (-A' - Z^{-1}(YB_{aa} + C'_{aa}D_{aa})B'_{aa})Y(Z^{-1})' \\
&\quad + Z^{-1}Y(-A - B_{aa}(B'_{aa}Y + D'_{aa}C_{aa})(Z^{-1})') \\
&\quad + Z^{-1}(YB_{aa} + C'_{aa}D_{aa})(B'_{aa}Y + D'_{aa}C_{aa})(Z^{-1})' \\
&= -A'Y(Z^{-1})' - Z^{-1}YA - Z^{-1}YB_{aa}B'_{aa}Y(Z^{-1})' + Z^{-1}C'_{aa}C_{aa}(Z^{-1})' \\
&= Z^{-1}\big(-ZA'Y - YAZ' - YB_{aa}B'_{aa}Y + C'_{aa}C_{aa}\big)(Z^{-1})' \\
&= Z^{-1}\big(-(YX - I)A'Y - YA(XY - I) + Y(AX + XA')Y - A'Y - YA\big)Z^{-1} \\
&= 0
\end{aligned}$$

Solution 11.4. From

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ 0 & D_{42} & D_{43} & 0 \end{bmatrix} \begin{bmatrix} D'_{11} & D'_{21} & D'_{31} & 0 \\ D'_{12} & D'_{22} & D'_{32} & D'_{42} \\ D'_{13} & D'_{23} & D'_{33} & D'_{43} \\ 0 & D'_{24} & D'_{34} & 0 \end{bmatrix} = I$$

it follows that

$$\begin{aligned}
D_{42}D'_{12} + D_{43}D'_{13} &= 0 \\
\Rightarrow D_{42}^{-1}D_{43} &= -D'_{12}(D_{13}^{-1})'.
\end{aligned}$$

In the same way, we may conclude from $D'_{aa}D_{aa} = I$, that

$$\begin{aligned}
D'_{21}D_{24} + D'_{31}D_{34} &= 0 \\
\Rightarrow D_{34}D_{24}^{-1} &= -(D_{31}^{-1})'D'_{21}.
\end{aligned}$$

Solution 11.5. From the $(1,2)$-partition of

$$\begin{aligned}
0 \;=\; & \begin{bmatrix} A' & 0 \\ 0 & \widehat{A}' \end{bmatrix} \begin{bmatrix} Y & -Z \\ -Z' & XZ \end{bmatrix} + \begin{bmatrix} Y & -Z \\ -Z' & XZ \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \widehat{A} \end{bmatrix} \\
& + \begin{bmatrix} C'_{aa} \\ -\widehat{C}'_{aa} \end{bmatrix} \begin{bmatrix} C_{aa} & -\widehat{C}_{aa} \end{bmatrix},
\end{aligned}$$

we conclude that

$$\begin{aligned}
0 &= A'Z + Z\widehat{A} + C'_{aa}\widehat{C}_{aa} \\
\Rightarrow \widehat{A} &= -Z^{-1}(A'Z + C'_{aa}\widehat{C}_{aa})
\end{aligned}$$

which establishes the first part.

Substituting for C_{aa}, $\widehat{C}_{aa}$ and $\widehat{B}_4$ gives

$$\begin{aligned}\widehat{A} - \widehat{B}_4 D_{24}^{-1}\widehat{C}_2 &= -Z^{-1}\big(A'Z + C_2'\widehat{C}_2 + C_3'\widehat{C}_3 - (C_2'D_{24} + C_3'D_{34})D_{24}^{-1}\widehat{C}_2\big)\\ &= -Z^{-1}\big(A'Z + C_3'\widehat{C}_3 - C_3'D_{34}D_{24}^{-1}\widehat{C}_2\big)\\ &= -Z^{-1}\big(A'Z + C_3'(\widehat{C}_3 + (D_{31}^{-1})'D_{21}'\widehat{C}_2)\big).\end{aligned}$$

This proves the second part.

By direct calculation

$$\begin{aligned}\widehat{C}_3 + (D_{31}^{-1})'D_{21}'\widehat{C}_2 &= (D_{31}^{-1})' \begin{bmatrix} D_{21}' & D_{31}' \end{bmatrix} \begin{bmatrix} \widehat{C}_2 \\ \widehat{C}_3 \end{bmatrix}\\ &= (D_{31}^{-1})' \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} D_{aa}'\widehat{C}_{aa}\\ &= (D_{31}^{-1})' \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} D_{aa}'(C_{aa}X + D_{aa}B_{aa}')\\ &= (D_{31}^{-1})'(\begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} (D_{aa}'C_{aa}X + B_1')\\ &= (D_{31}^{-1})'([D_{11}'C_1 + D_{21}'C_2]X + D_{31}'C_3X + B_1')\\ &= (D_{31}^{-1})'(-B_1'YX + B_1')\\ &= -(D_{31}^{-1})'B_1'Z\end{aligned}$$

which proves part three.

From parts one, two and three we have

$$\begin{aligned}\widehat{A} - \widehat{B}_4 D_{24}^{-1}\widehat{C}_2 &= -Z^{-1}\big(A'Z + C_3'(\widehat{C}_3 + (D_{31}^{-1})'D_{21}'\widehat{C}_2)\big)\\ &= -Z^{-1}\big(A'Z - C_3'(D_{31}^{-1})'B_1'Z\big)\\ &= -Z^{-1}\big(A - B_1 D_{31}^{-1}C_3\big)'Z\end{aligned}$$

as required.

Solution 11.6. Let us suppose that the generator of all controllers given in (8.3.11) is described by

$$\begin{aligned}\dot{x} &= A_k x + B_{k1}y + B_{k2}r\\ u &= C_{k1}x + r\\ s &= C_{k2}x + y.\end{aligned}$$

Multiplying the first equation on the left by $Y_{\infty 1}'(I - \gamma^{-2}Y_\infty X_\infty)$ and replacing x with $X_{\infty 1}q$ gives

$$\begin{aligned}Y_{\infty 1}'(I - \gamma^{-2}Y_\infty X_\infty)X_{\infty 1}\dot{q} &= Y_{\infty 1}'(I - \gamma^{-2}Y_\infty X_\infty)A_k X_{\infty 1}q\\ &\quad + Y_{\infty 1}'(I - \gamma^{-2}Y_\infty X_\infty)[B_{k1}y + B_{k2}r]\\ u &= C_{k1}X_{\infty 1}q + r\\ s &= C_{k2}X_{\infty 1}q + y.\end{aligned}$$

Since $Y_\infty = (Y'_{\infty 1})^{-1} Y'_{\infty 2}$ and $X_\infty = X_{\infty 2} X_{\infty 1}^{-1}$, we get

$$E_k = Y'_{\infty 1} X_{\infty 1} - \gamma^{-2} Y'_{\infty 2} X_{\infty 2}.$$

In the same way we get

$$\begin{aligned} C_{k1} &= -F_\infty X_{\infty 1} \\ &= -(D'_{12} C_1 X_{\infty 1} + B'_2 X_{\infty 2}) \end{aligned}$$

and

$$\begin{aligned} C_{k2} &= -(C_2 + \gamma^{-2} D_{21} B'_1 X_\infty) X_{\infty 1} \\ &= -(C_2 X_{\infty 1} + \gamma^{-2} D_{21} B'_1 X_{\infty 2}). \end{aligned}$$

The formulae for the B_{ki}'s are just a little more complicated, but direct calculation gives

$$\begin{aligned} B_{k1} &= Y'_{\infty 1}(I - \gamma^{-2} Y_\infty X_\infty) B_1 D'_{21} + Y'_{\infty 1} Y_\infty (C'_2 + \gamma^{-2} X_\infty B_1 D'_{21}) \\ &= Y'_{\infty 1} B_1 D'_{21} + Y'_{\infty 2} C'_2 \end{aligned}$$

and

$$\begin{aligned} B_{k2} &= Y'_{\infty 1}(I - \gamma^{-2} Y_\infty X_\infty)(B_2 + \gamma^{-2} Z_\infty F'_\infty) \\ &= Y'_{\infty 1}(I - \gamma^{-2} Y_\infty X_\infty) B_2 + \gamma^{-2} Y'_{\infty 1} Y_\infty (C'_1 D_{12} + X_\infty B_2) \\ &= Y'_{\infty 1} B_2 + \gamma^{-2} Y'_{\infty 2} C'_1 D_{12}. \end{aligned}$$

Finally, the equation for A_k comes from expanding

$$\begin{aligned} &Y'_{\infty 1}(I - \gamma^{-2} Y_\infty X_\infty)\big(A + \gamma^{-2} B_1 B'_1 X_\infty - B_2 F_\infty \\ &\quad -(B_1 D'_{21} + Z_\infty C'_{2z})(C_2 + \gamma^{-2} D_{21} B'_1 X_\infty)\big) X_{\infty 1}, \end{aligned}$$

in which

$$\begin{aligned} C_{2z} &= C_2 + \gamma^{-2} D_{21} B'_1 X_\infty \\ F_\infty &= D'_{12} C_1 + B'_2 X_\infty. \end{aligned}$$

This gives

$$\begin{aligned} A_k = {} & (Y'_{\infty 1} X_{\infty 1} - \gamma^{-2} Y'_{\infty 2} X_{\infty 2}) X_{\infty 1}^{-1}\big(A X_{\infty 1} + \gamma^{-2} B_1 B'_1 X_{\infty 2} \\ & - B_2(D'_{12} C_1 X_{\infty 1} + B'_2 X_{\infty 2})\big) \\ & -\big(-Y'_{\infty 1}(I - \gamma^{-2} Y_\infty X_\infty) B_1 D'_{21} \\ & + Y'_{\infty 2}(C'_2 + \gamma^{-2} X_\infty B_1 D_{21})\big)(C_2 X_{\infty 1} + \gamma^{-2} D_{21} B'_1 X_{\infty 2}). \end{aligned}$$

The first Hamiltonian expression in Theorem 11.5.1 gives

$$(A - B_2 D'_{12} C_1) X_{\infty 1} - (B_2 B'_2 - \gamma^{-2} B_1 B'_1) X_{\infty 2} = X_{\infty 1} T_X.$$

Substituting and re-arranging gives

$$\begin{aligned} A_k &= (Y'_{\infty 1}X_{\infty 1} - \gamma^{-2}Y'_{\infty 2}X_{\infty 2})T_X \\ &\quad - (Y'_{\infty 1}B_1D'_{21} + Y'_{\infty 2}C'_2)(C_2X_{\infty 1} + \gamma^{-2}D_{21}B'_1X_{\infty 2}) \\ &= E_kT_X + B_{k1}C_{k2}. \end{aligned}$$

Solution 11.7.

1. Since $(I - \boldsymbol{GK})^{-1} = I + \boldsymbol{GK}(I - \boldsymbol{GK})^{-1}$, we get

$$\begin{aligned} \mathcal{F}_\ell(\boldsymbol{P}, \boldsymbol{K}) &= \begin{bmatrix} \boldsymbol{GK}(I - \boldsymbol{GK})^{-1} \\ (I - \boldsymbol{GK})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ I \end{bmatrix} + \begin{bmatrix} \boldsymbol{G} \\ \boldsymbol{G} \end{bmatrix} \boldsymbol{K}(I - \boldsymbol{GK})^{-1}I \\ &= \begin{bmatrix} \boldsymbol{GK}(I - \boldsymbol{GK})^{-1} \\ (I - \boldsymbol{GK})^{-1} \end{bmatrix}, \end{aligned}$$

since $I + \boldsymbol{GK}(I - \boldsymbol{GK})^{-1} = (I - \boldsymbol{GK})^{-1}$. That $\boldsymbol{P}$ has the realization given is straight forward.

2. Replacing u with $(\sqrt{2}D)^{-1}u$ gives

$$\tilde{\boldsymbol{P}} \stackrel{s}{=} \left[\begin{array}{c|cc} A & 0 & B(\sqrt{2}D)^{-1} \\ \hline \begin{bmatrix} C \\ C \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} (\sqrt{2})^{-1}I \\ (\sqrt{2})^{-1}I \end{bmatrix} \\ C & I & (\sqrt{2})^{-1}I \end{array}\right].$$

Since

$$\begin{bmatrix} (\sqrt{2})^{-1}I & (\sqrt{2})^{-1}I \end{bmatrix} \begin{bmatrix} (\sqrt{2})^{-1}I \\ (\sqrt{2})^{-1}I \end{bmatrix} = I,$$

D_{12} has been orthogonalized as required.

3. It is easy to check that

$$(\sqrt{2})^{-1} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}$$

is orthogonal.

4. Direct substitution into the LQG Riccati equations given in Chapter 5 yields

$$0 = (A - BD^{-1}C)'X + X(A - BD^{-1}C) - \frac{1}{2}XB(D'D)^{-1}B'X$$

and

$$AY + YA' - YC'CY.$$

5. There are at least two ways of deriving the Riccati equations we require. One method uses the loop-shifting transformations given in Chapter 4, while an alternative technique employs the following $\mathcal{H}_\infty$ Riccati equations which where derived to deal with the case that $D_{11} \neq 0$. They are:

$$\begin{aligned} 0 &= (A + (B_1 D_{11}' \widehat{D}_{12} \widehat{D}_{12}' - \gamma^2 B_2 D_{12}')(\gamma^2 I - D_{11} D_{11}' \widehat{D}_{12} \widehat{D}_{12}')^{-1} C_1)' X_\infty \\ &\quad + X_\infty (A + (B_1 D_{11}' \widehat{D}_{12} \widehat{D}_{12}' - \gamma^2 B_2 D_{12}')(\gamma^2 I - D_{11} D_{11}' \widehat{D}_{12} \widehat{D}_{12}')^{-1} C_1) \\ &\quad - X_\infty (B_2 B_2' - (B_1 - B_2 D_{12}' D_{11}) R^{-1} (B_1 - B_2 D_{12}' D_{11})') X_\infty \\ &\quad + C_1' \widehat{D}_{12} (I - \gamma^{-2} \widehat{D}_{12}' D_{11} D_{11}' \widehat{D}_{12})^{-1} \widehat{D}_{12}' C_1, \end{aligned}$$

in which

$$R = (\gamma^2 I - D_{11}' \widehat{D}_{12} \widehat{D}_{12}' D_{11})$$

and

$$\begin{aligned} 0 &= (A + B_1 (\gamma^2 I - \widehat{D}_{21}' \widehat{D}_{21} D_{11}' D_{11})^{-1} (\widehat{D}_{21}' \widehat{D}_{21} D_{11}' C_1 - \gamma^2 D_{21}' C_2)) Y_\infty \\ &\quad + Y_\infty (A + B_1 (\gamma^2 I - \widehat{D}_{21}' \widehat{D}_{21} D_{11}' D_{11})^{-1} (\widehat{D}_{21}' \widehat{D}_{21} D_{11}' C_1 - \gamma^2 D_{21}' C_2))' \\ &\quad - Y_\infty (C_2' C_2 - (C_1 - D_{11} D_{21}' C_2)' \widehat{R}^{-1} (C_1 - D_{11} D_{21}' C_2)) Y_\infty \\ &\quad + B_1 \widehat{D}_{21}' (I - \gamma^{-2} \widehat{D}_{21} D_{11}' D_{11} \widehat{D}_{21}')^{-1} \widehat{D}_{21} B_1', \end{aligned}$$

in which

$$\widehat{R} = (\gamma^2 I - D_{11} \widehat{D}_{21}' \widehat{D}_{21} D_{11}').$$

Evaluating the linear term in the first equation gives

$$\begin{aligned} &(A - \frac{\gamma^2}{2} B D^{-1} \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \gamma^{-2} I & 0 \\ -\frac{\gamma^{-2}}{2}(\gamma^2 - \frac{1}{2})^{-1} I & (\gamma^2 - \frac{1}{2})^{-1} I \end{bmatrix} \begin{bmatrix} C \\ C \end{bmatrix}) X_\infty \\ &= (A - B D^{-1} \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \frac{1}{2} I \\ \frac{1}{2} I \end{bmatrix} C) X_\infty. \end{aligned}$$

The constant term is zero because $D_\perp^* C_1 = 0$. The coefficient of the quadratic term is given by

$$\begin{aligned} &\frac{1}{2} B (D'D)^{-1} B' (1 - \frac{1}{2}(\gamma^2 - \frac{1}{2})^{-1}) \\ &= \frac{1}{2} B (D'D)^{-1} B' \frac{\gamma^2 - 1}{\gamma^2 - \frac{1}{2}}. \end{aligned}$$

Combining these yields

$$0 = (A - B D^{-1} C)' X_\infty + X_\infty (A - B D^{-1} C) - \frac{\gamma^2 - 1}{2\gamma^2 - 1} X_\infty B (D'D)^{-1} B' X_\infty.$$

Turning to the second equation we see that the constant term is zero since $B_1 = 0$. The linear terms are given by $AY_\infty + Y_\infty A'$. The coefficient of the quadratic term is

$$C'C - \gamma^{-2}\left(\begin{bmatrix} C \\ C \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} C\right)'\left(\begin{bmatrix} C \\ C \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} C\right) = (1-\gamma^{-2})C'C.$$

Collecting terms then gives

$$AY_\infty + Y_\infty A' - (1-\gamma^{-2})Y_\infty C'CY_\infty.$$

6. By referring to the LQG equations, it is east to check that

$$X_\infty = \frac{\gamma^2 - \frac{1}{2}}{\gamma^2 - 1}X \text{ and } Y_\infty = (1-\gamma^{-2})^{-1}Y$$

solve the $\mathcal{H}_\infty$ Riccati equations. Check that they are the stabilizing solutions.

7. If $\boldsymbol{G}$ is stable, $Y = 0$ and $X'_{\infty 2}X_{\infty 1}$ is nonnegative when $\gamma \geq 1$. When $\boldsymbol{G}$ is minimum phase $X = 0$ and $Y'_{\infty 2}Y_{\infty 1}$ is nonnegative when $\gamma \geq 1$ (these conditions come from Theorem 11.5.1). If $\boldsymbol{G}$ is stable and nonminimum phase we see that $Y = 0$, $X \neq 0$, $\gamma_{opt} = 1$ and

$$X_\infty = \lim_{\gamma \downarrow 1} \frac{\gamma^2 - \frac{1}{2}}{\gamma^2 - 1}X$$

which is unbounded. A parallel argument may be used for Y_∞.

8. If $\boldsymbol{G}$ is stable and minimum phase, $-\rho\boldsymbol{G}^{-1}$ is a stabilizing controller because no right-half-plane cancellations occur between the plant and controller when forming $\boldsymbol{GK}$ and the resulting transfer function is constant. Next, we see that

$$\begin{bmatrix} \boldsymbol{GK}(I-\boldsymbol{GK})^{-1} \\ (I-\boldsymbol{GK})^{-1} \end{bmatrix} = \begin{bmatrix} \rho(1+\rho)^{-1} \\ (1+\rho)^{-1} \end{bmatrix},$$

which gives

$$\begin{aligned} \left\| \begin{bmatrix} \boldsymbol{GK}(I-\boldsymbol{GK})^{-1} \\ (I-\boldsymbol{GK})^{-1} \end{bmatrix} \right\|_\infty &= \sqrt{\left(\frac{\rho}{1+\rho}\right)^2 + \left(\frac{1}{1+\rho}\right)^2} \\ &= \frac{\sqrt{\rho^2+1}}{1+\rho}. \end{aligned}$$

Now

$$\frac{d}{d\rho}\left\| \begin{bmatrix} \boldsymbol{GK}(I-\boldsymbol{GK})^{-1} \\ (I-\boldsymbol{GK})^{-1} \end{bmatrix} \right\|_\infty = \frac{\rho - 1}{(1+\rho)^2\sqrt{\rho^2+1}}$$

which vanishes at $\rho = 1$. This means that

$$\left\| \begin{bmatrix} \boldsymbol{GK}(I - \boldsymbol{GK})^{-1} \\ (I - \boldsymbol{GK})^{-1} \end{bmatrix} \right\|_\infty$$

has a minimum at $\rho = 1$. Thus

$$\begin{aligned} \inf_{\boldsymbol{K}} \left\| \begin{bmatrix} \boldsymbol{GK}(I - \boldsymbol{GK})^{-1} \\ (I - \boldsymbol{GK})^{-1} \end{bmatrix} \right\|_\infty &= \left. \left(\frac{\sqrt{\rho^2 + 1}}{1 + \rho} \right) \right|_{\rho = 1} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Solution 11.8. It follows from (10.2.9) that

$$\left\| \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{N} \end{bmatrix} \right\|_H \leq \left\| \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{N} \end{bmatrix} \right\|_\infty = 1.$$

Now suppose that $\| \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{N} \end{bmatrix} \|_H = 1$. In this case there exists $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{RH}_2$ such that

$$\begin{bmatrix} \boldsymbol{D}(-s) \\ \boldsymbol{N}(-s) \end{bmatrix} \boldsymbol{g}(s) = \boldsymbol{f}(-s).$$

Since $\boldsymbol{D}, \boldsymbol{N} \in \mathcal{RH}_\infty$ are coprime, there exist $\boldsymbol{U}, \boldsymbol{V} \in \mathcal{RH}_\infty$ such that

$$\begin{aligned} \boldsymbol{VD} + \boldsymbol{UN} &= I \\ \Rightarrow \boldsymbol{V}(-s)\boldsymbol{D}(-s) + \boldsymbol{U}(-s)\boldsymbol{N}(-s) &= I. \end{aligned}$$

This gives

$$\boldsymbol{g}(s) = \begin{bmatrix} \boldsymbol{V}(-s) \\ \boldsymbol{U}(-s) \end{bmatrix} \boldsymbol{f}(-s) \notin \mathcal{RH}_2$$

which is a contradiction. We therefore conclude that

$$\| \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{N} \end{bmatrix} \|_H < 1.$$

Solution 11.9. We follow the construction given in Section 11.2.

Step 1: Construct D_{13} such that

$$\begin{bmatrix} D_{12} & D_{13} \end{bmatrix} \begin{bmatrix} D_{12}' \\ D_{13}' \end{bmatrix} = I.$$

Step 2: Construct D_{31} such that

$$\left[\begin{array}{cc} D_{21}' & D_{31}' \end{array}\right]\left[\begin{array}{c} D_{21} \\ D_{31} \end{array}\right] = I.$$

Step 3: Find $\left[\begin{array}{cc} D_{42} & D_{43} \end{array}\right]$ such that

$$\left[\begin{array}{cc} D_{12} & D_{13} \end{array}\right]\left[\begin{array}{c} D_{42}' \\ D_{43}' \end{array}\right] = 0.$$

Step 4: Find $\left[\begin{array}{cc} D_{24}' & D_{34}' \end{array}\right]$ such that

$$\left[\begin{array}{cc} D_{24}' & D_{34}' \end{array}\right]\left[\begin{array}{c} D_{21} \\ D_{31} \end{array}\right] = 0.$$

Step 5: We can now complete the construction with

$$\begin{aligned}\left[\begin{array}{cc} D_{22} & D_{23} \\ D_{32} & D_{33} \end{array}\right] &= -\left[\begin{array}{cc} D_{21}' & D_{31}' \\ D_{24}' & D_{34}' \end{array}\right]^{-1}\left[\begin{array}{cc} D_{11}' & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} D_{12} & D_{13} \\ D_{42} & D_{43} \end{array}\right] \\ &= \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]\end{aligned}$$

since $D_{11} = 0$.

Solution 11.10. The first expression comes from the $(2,2)$-partition of:

$$\begin{aligned} 0 \;=\; & \left[\begin{array}{ccc} A_1' & 0 & 0 \\ 0 & A_2' & 0 \\ 0 & 0 & A' \end{array}\right]\left[\begin{array}{ccc} Y_{11} & 0 & \epsilon Y_{13} \\ 0 & Y_{22} & 0 \\ \epsilon Y_{13}' & 0 & \epsilon^2 Y_{33} \end{array}\right] \\ & + \left[\begin{array}{ccc} Y_{11} & 0 & \epsilon Y_{13} \\ 0 & Y_{22} & 0 \\ \epsilon Y_{13}' & 0 & \epsilon^2 Y_{33} \end{array}\right]\left[\begin{array}{ccc} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A \end{array}\right] \\ & + \left[\begin{array}{ccc} 0 & C_1' & C_{31}' \\ C_2' & 0 & 0 \\ 0 & \epsilon C' & \epsilon C_{33}' \end{array}\right]\left[\begin{array}{ccc} 0 & C_2 & 0 \\ C_1 & 0 & \epsilon C \\ C_{31} & 0 & \epsilon C_{33} \end{array}\right], \end{aligned}$$

which defines the observability gramian of

$$\left[\begin{array}{ccc} 0 & \boldsymbol{R}_{12} & \boldsymbol{R}_{13} \\ \boldsymbol{R}_{21} & \gamma_{r+1}^{-1}\boldsymbol{R}_{22} & \boldsymbol{R}_{23} \\ \boldsymbol{R}_{31} & \boldsymbol{R}_{32} & \boldsymbol{R}_{33} \end{array}\right].$$

The controllability gramian is defined by

$$\begin{aligned} 0 &= \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A \end{bmatrix} \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & \epsilon X_{23} \\ 0 & X_{23}' & \epsilon^2 X_{33} \end{bmatrix} \\ &\quad + \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & \epsilon X_{23} \\ 0 & X_{23}' & \epsilon^2 X_{33} \end{bmatrix} \begin{bmatrix} A_1' & 0 & 0 \\ 0 & A_2' & 0 \\ 0 & 0 & A' \end{bmatrix} \\ &\quad + \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & B_{32} \\ 0 & \epsilon B & \epsilon B_{33} \end{bmatrix} \begin{bmatrix} B_1' & 0 & 0 \\ 0 & B_2' & \epsilon B' \\ 0 & B_{32}' & \epsilon B_{33}' \end{bmatrix}. \end{aligned}$$

The $(2,2)$-partition yields

$$A_2X_{22} + X_{22}A_2' + B_2B_2' + B_{32}B_{32}' = 0.$$

To establish the spectral radius condition, we consider

$$\begin{bmatrix} \boldsymbol{R}_{12} & \boldsymbol{R}_{13} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_2 & B_2 & B_{32} \\ \hline C_2 & D_{12} & D_{13} \end{array}\right].$$

Since

$$\boldsymbol{R}_{12}\boldsymbol{R}_{12}^{\sim} + \boldsymbol{R}_{13}\boldsymbol{R}_{13}^{\sim} = I$$

and since $\begin{bmatrix} \boldsymbol{R}_{12} & \boldsymbol{R}_{13} \end{bmatrix}$ has a stable right inverse, $(\boldsymbol{R}_{12}, \boldsymbol{R}_{13})$ are a normalized left coprime pair. This means that $\| \begin{bmatrix} \boldsymbol{R}_{12} & \boldsymbol{R}_{13} \end{bmatrix} \|_H < 1$ and therefore that $\rho(X_{22}Y_{22}) < 1$.

Solution 11.11. The realization in Theorem 11.3.3 is given by

$$\begin{aligned} \dot{x} &= \widehat{A}x + \begin{bmatrix} \widehat{B}_2 & \widehat{B}_4 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \\ \begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} \widehat{C}_2 \\ \widehat{C}_4 \end{bmatrix} x + \begin{bmatrix} D_{22} & D_{24} \\ D_{42} & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}. \end{aligned}$$

Multiplying the first equation on the left by Z gives

$$\begin{aligned} Z\dot{x} &= Z\widehat{A}x + Z\begin{bmatrix} \widehat{B}_2 & \widehat{B}_4 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \\ \begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} \widehat{C}_2 \\ \widehat{C}_4 \end{bmatrix} x + \begin{bmatrix} D_{22} & D_{24} \\ D_{42} & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}. \end{aligned}$$

Carrying out the calculations gives:

$$\begin{aligned} Z\dot{x} &= (A' + YAX - C_{aa}'D_{aa}B_{aa}')x \\ &\quad + \left(\begin{bmatrix} YB_2 & 0 \end{bmatrix} + \begin{bmatrix} C_1' & C_2' & C_3' \end{bmatrix} \begin{bmatrix} D_{12} & 0 \\ D_{22} & D_{24} \\ D_{32} & D_{34} \end{bmatrix} \right) \begin{bmatrix} w \\ u \end{bmatrix} \end{aligned}$$

which completes the verification since the second equation remains uneffected.

Solutions to Problems in Appendix A

Solution A.1. Since $\boldsymbol{N}\boldsymbol{D}^{-1} = \boldsymbol{N}_c\boldsymbol{D}_c^{-1}$, it follows that $\boldsymbol{W} = \boldsymbol{D}_c^{-1}\boldsymbol{D}$. To show that $\boldsymbol{W} \in \mathcal{RH}_\infty$, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\mathcal{RH}_\infty$ transfer function matrices such that $\boldsymbol{X}\boldsymbol{N}_c + \boldsymbol{Y}\boldsymbol{D}_c = I$, which exist because $\boldsymbol{N}_c$ and $\boldsymbol{D}_c$ are right coprime. Multiplying on the right by $\boldsymbol{W} = \boldsymbol{D}_c^{-1}\boldsymbol{D}$, we see that $\boldsymbol{W} = \boldsymbol{X}\boldsymbol{N} + \boldsymbol{Y}\boldsymbol{D} \in \mathcal{RH}_\infty$.

Solution A.2. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\mathcal{RH}_\infty$ transfer function matrices such that $\boldsymbol{X}\boldsymbol{N} + \boldsymbol{Y}\boldsymbol{D} = I$, which exist because $\boldsymbol{N}$ and $\boldsymbol{D}$ are right coprime. Now write the Bezout identity as

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{D} \end{bmatrix} = I$$

For any s in the closed-right-half plane (including infinity), $\begin{bmatrix} \boldsymbol{X} & \boldsymbol{N} \end{bmatrix}(s)$ is a finite complex matrix, and

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}(s) \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{D} \end{bmatrix}(s) = I$$

implies that

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{D} \end{bmatrix}(s)$$

is a complex matrix with full column rank.

Now write $\boldsymbol{G}\boldsymbol{D} = \boldsymbol{N}$. If s_0 is a pole of $\boldsymbol{G}$ in the CRHP, it must be a zero of $\det \boldsymbol{D}(s_0)$, since $\boldsymbol{N} \in \mathcal{RH}_\infty$. If s_0 is a zero of $\det \boldsymbol{D}(s)$, there exists an $x \neq 0$ such that

$$\boldsymbol{f} = \boldsymbol{D}\frac{x}{s - s_0} \in \mathcal{RH}_\infty.$$

Hence

$$\boldsymbol{G}\boldsymbol{f} = \boldsymbol{N}\frac{x}{s - s_0},$$

which implies that $\boldsymbol{G}$ has a pole at s_0, since $\boldsymbol{f} \in \mathcal{RH}_\infty$ and $\boldsymbol{N}(s_0)x \neq 0$, due to coprimeness.

Solution A.3. The verification of (A.2.3) is a routine application of the state-space system interconnection (or inversion) results of Problem 3.6. Since

$$\begin{bmatrix} \boldsymbol{D}_r \\ \boldsymbol{N}_r \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A - BF & B \\ \hline -F & I \\ C - DF & D \end{array}\right]$$

a direct application of Problem 3.6, part 4, yields $\boldsymbol{N}_r\boldsymbol{D}_r^{-1} = D + C(sI - A)^{-1}B = \boldsymbol{G}$. The identity $\boldsymbol{D}_l^{-1}\boldsymbol{N}_l = \boldsymbol{N}_r\boldsymbol{D}_r^{-1}$ follows from the $(2,1)$-block of (A.2.3).

Now note that

$$\begin{bmatrix} \boldsymbol{V}_r & \boldsymbol{U}_r \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - HC & B - HD & H \\ \hline F & I & 0 \end{array}\right].$$

Application of a dual version of Problem 3.6, part 4, yields

$$-\boldsymbol{V}_r^{-1}\boldsymbol{U}_r \stackrel{s}{=} \left[\begin{array}{c|c} A - HC - (B - HD)F & H \\ \hline -F & 0 \end{array}\right],$$

which we may write out as

$$\begin{aligned} \dot{\widehat{x}} &= (A - HC - (B - HD)F)\widehat{x} + Hy \\ u &= -F\widehat{x}. \end{aligned}$$

Replacing $-F\widehat{x}$ with u in the $\dot{\widehat{x}}$ equation yields (A.2.6).

Solution A.4. Let

$$\begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{N} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C}_1 & \widehat{D}_1 \\ \widehat{C}_2 & \widehat{D}_2 \end{array}\right]$$

be a minimal realization. Since $\boldsymbol{N}$ and $\boldsymbol{D}$ are in $\mathcal{RH}_\infty$, $\widehat{A}$ is asymptotically stable. Also, since $\boldsymbol{N}$, $\boldsymbol{D}$ are r.c.,

$$\begin{bmatrix} \widehat{A} - sI & \widehat{B} \\ \widehat{C}_1 & \widehat{D}_1 \\ \widehat{C}_2 & \widehat{D}_2 \end{bmatrix}$$

has full column rank for all s in the closed-right-half plane.

Using Problem 3.6, part 4, yields

$$\boldsymbol{G} = \boldsymbol{N}\boldsymbol{D}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} \widehat{A} - \widehat{B}\widehat{D}_1^{-1}\widehat{C}_1 & \widehat{B}\widehat{D}_1^{-1} \\ \hline \widehat{C}_2 - \widehat{D}_2\widehat{D}_1^{-1}\widehat{C}_1 & \widehat{D}_2\widehat{D}_1^{-1} \end{array}\right].$$

Define

$$\begin{aligned} A &= \widehat{A} - \widehat{B}\widehat{D}_1^{-1}\widehat{C}_1 \\ B &= \widehat{B}\widehat{D}_1^{-1} \\ C &= \widehat{C}_2 - \widehat{D}_2\widehat{D}_1^{-1}\widehat{C}_1 \\ D &= \widehat{D}_2\widehat{D}_1^{-1} \\ W &= \widehat{D}_1^{-1} \\ F &= -\widehat{D}_1^{-1}\widehat{C}_1. \end{aligned}$$

Then

$$\begin{aligned} BW^{-1} &= \widehat{B} \\ A - BW^{-1}F &= \widehat{A} - \widehat{B}\widehat{D}_1^{-1}\widehat{C}_1 + \widehat{B}\widehat{D}_1^{-1}\widehat{C}_1 = \widehat{A} \\ -W^{-1}F &= \widehat{C}_1 \\ W^{-1} &= \widehat{D}_1 \\ C - DW^{-1}F &= \widehat{C}_2 - \widehat{D}_2\widehat{D}_1^{-1}\widehat{C}_1 + \widehat{D}_2\widehat{D}_1^{-1}\widehat{C}_1 = \widehat{C}_2 \\ DW^{-1} &= \widehat{D}_2. \end{aligned}$$

Since $A - B(W^{-1}F) = \widehat{A}$ is asymptotically stable, (A, B) is stabilizable. To prove the detectability of (A, C), note that if

$$\begin{aligned} \begin{bmatrix} A - sI \\ C \end{bmatrix} x = 0 \quad &\Leftrightarrow \quad \begin{bmatrix} \widehat{A} - sI & \widehat{B} \\ \widehat{C}_2 & \widehat{D}_2 \end{bmatrix} \begin{bmatrix} I \\ -\widehat{D}_1^{-1}\widehat{C}_1 \end{bmatrix} x = 0 \\ &\Leftrightarrow \quad \begin{bmatrix} \widehat{A} - sI & \widehat{B} \\ \widehat{C}_1 & \widehat{D}_1 \\ \widehat{C}_2 & \widehat{D}_2 \end{bmatrix} \begin{bmatrix} I \\ -\widehat{D}_1^{-1}\widehat{C}_1 \end{bmatrix} x = 0. \end{aligned}$$

Thus, if $\mathrm{R}_e(s) \geq 0$, then $\begin{bmatrix} I \\ -\widehat{D}_1^{-1}\widehat{C}_1 \end{bmatrix} x = 0$, implying $x = 0$, and we conclude that (A, C) is detectable. Thus (A, B, C, D) is a stabilizable and detectable realization of $\boldsymbol{G}$ such that $\boldsymbol{N}$, $\boldsymbol{D}$ has state-space realization as given in the problem statement, for suitable W and F.

Solution A.5. The $(1,1)$-block was verified in the solution to Problem A.3. The $(1,2)$- and $(2,1)$-blocks are direct applications of the formula for inverting a state-space realization—see Problem 3.6. The $(2,2)$-block is a direct application of Problem 3.6, part 4.

Solution A.6. Since $\boldsymbol{G}$ is assumed stable, every stabilizing controller is given by $\boldsymbol{K} = \boldsymbol{Q}(I + \boldsymbol{G}\boldsymbol{Q})^{-1}$. Now

$$y = (I - \boldsymbol{G}\boldsymbol{K})^{-1}v$$

$$= \quad (I + \boldsymbol{GQ})v.$$

Therefore, for perfect steady-state accuracy in response to step inputs, we need $(I + \boldsymbol{GQ})(0) = 0$ (by the final value theorem of the Laplace transform). Hence all the desired controllers have the form

$$\boldsymbol{K} = \boldsymbol{Q}(I + \boldsymbol{GQ})^{-1}, \qquad \boldsymbol{Q} \in \mathcal{RH}_\infty,\ \boldsymbol{Q}(0) = -\boldsymbol{G}^{-1}(0).$$

As an example, consider $\boldsymbol{g}(s) = \frac{1}{s+1}$. Then

$$\boldsymbol{k} = -\frac{(s+1)\boldsymbol{q}}{s+1-\boldsymbol{q}}, \qquad \boldsymbol{q} \in \mathcal{RH}_\infty,\ \boldsymbol{q}(0) = 1.$$

Solution A.7. Let (A, B, C, D) be any stabilizable and detectable realization of $\boldsymbol{G}$, and suppose $\boldsymbol{D}$ and $\boldsymbol{N}$ are given by

$$\begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{N} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A - BW^{-1}F & BW^{-1} \\ \hline -W^{-1}F & W^{-1} \\ C - DW^{-1}F & DW^{-1} \end{array}\right].$$

We aim to choose F and W such that the allpass equations of Theorem 3.2.1 are satisfied and $A - BW^{-1}F$ is asymptotically stable. If we can do this, then the coprime factorization satisfies the equation

$$\begin{bmatrix} \boldsymbol{D}^\sim & \boldsymbol{N}^\sim \end{bmatrix} \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{N} \end{bmatrix} = I,$$

which defines the normalized coprime factorization. The allpass equations obtained from Theorem 3.2.1 yield

$$\begin{aligned} 0 &= X(A - BW^{-1}F) + (A - BW^{-1}F)'X + (W^{-1}F)'(W^{-1}F) \\ &\quad + (C - DW^{-1}F)'(C - DW^{-1}F) \\ 0 &= (W')^{-1}\big(-W^{-1}F + D'(C - DW^{-1}F) + B'X\big) \\ I &= (W')^{-1}(I + D'D)W^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} W'W &= I + D'D \\ F &= (W')^{-1}(D'C + B'X). \end{aligned}$$

It remains to determine X. Substitution F and W into into the observability gramian equation, we obtain (after some manipulation) the Riccati equation

$$X(A - BS^{-1}D'C) + (A - BS^{-1}D'C)'X - XBS^{-1}B'X + C'\tilde{S}^{-1}C = 0$$

in which $S = I + D'D$ and $\tilde{S} = I + DD'$. (Note that $I - DS^{-1}D' = \tilde{S}^{-1}$). By the results of Chapter 5, this Riccati equation has a stabilizing solution provided $(A - BS^{-1}D'C, BS^{-\frac{1}{2}})$ is stabilizable (which is true, since (A, B) is stabilizable) and provided $(A - BS^{-1}D'C, \tilde{S}^{-\frac{1}{2}}C)$ has no unobservable modes on the imaginary axis (which is also true, since (A, C) is detectable). The fact that the required assumptions hold follow immediately from the Popov-Belevitch-Hautus test for controllability/observability.

Thus, if X is the stabilizing solution to the above Riccati equation, and F and W are defined from X as above, the allpass equations are satisfied and $A - BW^{-1}F$ is asymptotically stable. We conclude that $\boldsymbol{N}$, $\boldsymbol{D}$ defined by the given state-space realization is a normalized right coprime factorization of $\boldsymbol{G}$.

To interpret this in the context of LQ control, consider the case $D = 0$. We then have

$$\begin{aligned} XA + A'X - XBB'X + C'C &= 0 \\ F &= B'X \\ W &= I. \end{aligned}$$

This is exactly the same as the situation that arises in minimizing

$$J = \int_0^\infty (x'C'Cx + u'u)\, dt.$$

For $D \neq 0$, the Riccati equation is that which we need to minimize

$$J = \|z\|_2^2 = \int_0^\infty z'z\, dt.$$

Now

$$\begin{aligned} z &= \begin{bmatrix} \boldsymbol{G} \\ I \end{bmatrix} u \\ &= \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{D} \end{bmatrix} \boldsymbol{D}^{-1}u. \end{aligned}$$

Since $\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{D} \end{bmatrix}$ is allpass, minimizing $\|z\|_2^2$ is the same as minimizing $\|\boldsymbol{D}^{-1}u\|_2^2$. Thus we choose $\boldsymbol{D}^{-1}u = 0$. Now

$$\boldsymbol{D}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline F & W \end{array}\right],$$

so setting $\boldsymbol{D}^{-1}u = 0$ means

$$\begin{aligned} \dot{x} &= Ax + Bu \\ 0 &= Fx + Wu. \end{aligned}$$

That is, $u = -W^{-1}Fx$. Thus, computing a normalized right coprime factorization is the frequency-domain equivalent of completing the square.

Solution A.8. From Lemma A.4.5,

$$\boldsymbol{T}_{12} \stackrel{s}{=} \left[\begin{array}{c|c} A - B_2F & B_2 \\ \hline C_1 - D_{12}F & D_{12} \end{array}\right].$$

From Theorem 3.2.1, $\boldsymbol{T}_{12}^{\sim}\boldsymbol{T}_{12} = I$ if

$$\begin{aligned} 0 &= X(A - B_2F) + (A - B_2F)'X + (C_1 - D_{12}F)'(C_1 - D_{12}F) \\ 0 &= D_{12}'(C_1 - D_{12}F) + B_2'X \\ I &= D_{12}'D_{12}. \end{aligned}$$

Since $D_{12}'D_{12} = I$ holds by assumption, we require

$$\begin{aligned} 0 &= X(A - B_2D_{12}'C_1) + (A - B_2D_{12}'C_1)'X - XB_2B_2'X \\ &\quad + C_1'(I - D_{12}D_{12}')C_1 \\ F &= D_{12}'C_1 + B_2'X, \end{aligned}$$

which is precisely the F from the (cross-coupled) LQ problem. Similarly for $\boldsymbol{T}_{21}$.

Solution A.9. $\begin{bmatrix} \widehat{\boldsymbol{T}}_{12} & \boldsymbol{T}_{12} \end{bmatrix}$ is clearly square—its D-matrix is square, since $\widehat{D}_{12}$ is an orthogonal completion of D_{12}. From Theorem 3.2.1, $\begin{bmatrix} \widehat{\boldsymbol{T}}_{12} & \boldsymbol{T}_{12} \end{bmatrix}$ is allpass provided

$$\begin{aligned} X(A - B_2F) + (A - B_2F)'X + (C_1 - D_{12}F)'(C_1 - D_{12}F) &= 0 \\ \begin{bmatrix} \widehat{D}_{12}' & D_{12}' \end{bmatrix}(C_1 - D_{12}F) + \begin{bmatrix} \widehat{B}' & B_2' \end{bmatrix}X &= 0. \end{aligned}$$

By construction, the first equation is satisfied, and the $(2,1)$-block of the second equation is also satisfied. We therefore need to confirm the $(1,1)$-block of the second equation. Now

$$\widehat{D}_{12}'(C_1 - D_{12}F) + \widehat{B}'X = \widehat{D}_{12}'C_1(I - X^{\#}X).$$

The right-hand side is zero if $\ker(X) \subset \ker(\widehat{D}_{12}'C_1)$. To show this, suppose $Xx = 0$. Then $(C_1 - D_{12}F)x = 0$ and consequently $0 = \widehat{D}_{12}'(C_1 - D_{12}F)x = \widehat{D}_{12}'C_1x$. Thus $Xx = 0$ implies $\widehat{D}_{12}'C_1x$, so $\ker(X) \subset \ker(\widehat{D}_{12}'C_1)$ and we conclude that $\widehat{D}_{12}'(C_1 - D_{12}F) + \widehat{B}'X = 0$.

The reasoning for $\begin{bmatrix} \widehat{\boldsymbol{T}}_{21} \\ \boldsymbol{T}_{21} \end{bmatrix}$ is analogous.

Solution A.10. Choose $\begin{bmatrix} \widehat{\boldsymbol{T}}_{12} & \boldsymbol{T}_{12} \end{bmatrix}$ and $\begin{bmatrix} \widehat{\boldsymbol{T}}_{21} \\ \boldsymbol{T}_{21} \end{bmatrix}$ as in the previous problem.

1.

$$\begin{aligned}\|\mathcal{F}_\ell(\boldsymbol{P},\boldsymbol{K})\|_{2,\infty} &= \left\| \begin{bmatrix} \widehat{\boldsymbol{T}}_{12}^{\sim} \\ \boldsymbol{T}_{12}^{\sim} \end{bmatrix} \boldsymbol{T}_{11} \begin{bmatrix} \widehat{\boldsymbol{T}}_{21}^{\sim} & \boldsymbol{T}_{21}^{\sim} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{Q} \end{bmatrix} \right\|_{2,\infty} \\ &= \left\| \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22}+\boldsymbol{Q} \end{bmatrix} \right\|_{2,\infty}.\end{aligned}$$

2. $\boldsymbol{R}$ is given by the realization

$$\boldsymbol{R} \stackrel{s}{=} \left[\begin{array}{c|c} A_R & B_R \\ \hline C_R & D_R \end{array}\right],$$

in which

$$\begin{aligned}&A_R \\ &= \begin{bmatrix} -(A-B_2F)' & XHD_{21}(B_1-HD_{21})' + (C_1-D_{12}F)'C_1Y + XHC_2Y \\ 0 & -(A-HC_2)' \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned} B_R &= \begin{bmatrix} -X\widehat{B}\widehat{D}_{21}' & -XB_1D_{21}' \\ \widehat{C}' & C_2' \end{bmatrix} \\ C_R &= \begin{bmatrix} \widehat{B}' & 0 \\ B_2' & 0 \end{bmatrix} \\ D_R &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Since $A-B_2F$ and $A-HC_2$ are asymptotically stable, we see that $\boldsymbol{R} \in \mathcal{RH}_\infty^-$.

3. Using the result of Problem 11.2, we have

$$\begin{aligned}\operatorname{trace}(X^*X) &= \operatorname{trace}(X_{11}^*X_{11}) + \operatorname{trace}(X_{12}^*X_{12}) \\ &\quad + \operatorname{trace}(X_{21}^*X_{21}) + \operatorname{trace}(X_{22}^*X_{22}),\end{aligned}$$

for any partitioned matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Hence

$$\left\| \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22}+\boldsymbol{Q} \end{bmatrix} \right\|_2^2 = (\text{terms independent of } \boldsymbol{Q}) + \|\boldsymbol{R}_{22}+\boldsymbol{Q}\|_2^2.$$

Hence, we need to choose $\boldsymbol{Q} \in \mathcal{RH}_\infty$ to minimize $\|\boldsymbol{R}_{22}+\boldsymbol{Q}\|_2^2$. Now $\|\boldsymbol{R}_{22}+\boldsymbol{Q}\|_2$ is finite if and only if $\boldsymbol{Q}(\infty) = -\boldsymbol{R}_{22}(\infty) = 0$. In this case,

$$\|\boldsymbol{R}_{22}+\boldsymbol{Q}\|_2^2 = \|\boldsymbol{R}_{22}\|_2^2 + \|\boldsymbol{Q}\|_2^2,$$

since $\mathcal{RH}_\infty^-$ and $\mathcal{RH}_\infty$ are orthogonal in the 2-norm. Thus the minimum norm is achieved by setting $\boldsymbol{Q}(\infty) = -\boldsymbol{R}_{22}(\infty) = 0$, and the minimum norm is $\|\boldsymbol{R}\|_2$.

Solutions to Problems in Appendix B

Solution B.1.

1. Since $Q(N+1) = 0$,

$$\sum_{k=0}^{N} x'_{k+1} Q(k+1) x_{k+1} - x'_k Q(k) x_k = -x'_0 Q(k) x_0.$$

Now

$$\begin{aligned}
&\sum_{k=0}^{N} z'_k z_k + x'_{k+1} Q(k+1) x_{k+1} - x'_k Q(k) x_k \\
&\quad = \sum_{k=0}^{N} x'_k \big(C'(k)C(k) + A'(k)Q(k+1)A(k) - Q(k)\big) x_k \\
&\quad = 0.
\end{aligned}$$

Hence

$$\sum_{k=0}^{N} z'_k z_k = x'_0 Q(0) x_0.$$

2.

(i)⇒(ii) Immediate from $z_k = CA^k x_0$.

(ii)⇒(iii) If $AW = WJ$, in which J is a Jordan block corresponding to an eigenvalue λ with $|\lambda| \geq 1$, then $CA^k W = CWJ^k$. Hence $CA^k W \to 0$ as $k \to \infty$ implies $CW = 0$. That is, every observable eigenspace is asymptotically stable.

(iii)⇒(iv) Uniform boundedness follows from

$$Q(k) = \sum_{i=k}^{N} (A')^{N-i} C' C A^{N-i}$$

and the asymptotic stability of every observable mode. Note that $Q(k)$ is monotonic. The convergence result is a consequence of monotonicity and uniform boundedness.

(iv)⇒(v) Set $Q = \lim_{k\to-\infty} Q(k)$.

(v)⇒(i) $X(k) = Q - Q(k)$ satisfies $X(k) = A'X(k+1)A$, $X(N+1) = Q$. Therefore

$$X(k) = A^{N+1-k} Q A^{N+1-k} \geq 0.$$

Thus $0 \leq Q(k) \leq Q$, which establishes that $\lim_{k\to-\infty} Q(k)$ is indeed the smallest solution, and

$$\|z\|^2_{2,[0,N]} = x_0' Q(0) x_0 \leq x_0' Q x_0.$$

Since this is a uniform bound on $\|z\|^2_{2,[0,N]}$, we conclude that $z \in \ell_2[0,\infty)$.

Solution B.2. The system $\boldsymbol{L}$ that maps w to $w - w^*$ when the input is u^*, which is introduced in the proof of Theorem B.2.1, has realization

$$\begin{bmatrix} x_{k+1} \\ w_k - w_k^* \end{bmatrix} = \begin{bmatrix} A - B_2 R_3^{-1} L_2 & B_1 - B_2 R_3^{-1} R_2 \\ \nabla^{-1} L_\nabla & I \end{bmatrix} (k) \begin{bmatrix} x_k \\ w_k \end{bmatrix}.$$

This is causally invertible, since its inverse is

$$\begin{bmatrix} x_{k+1} \\ w_k \end{bmatrix} = \begin{bmatrix} A - BR^{-1} L & B_1 - B_2 R_3^{-1} R_2 \\ -\nabla^{-1} L_\nabla & I \end{bmatrix} (k) \begin{bmatrix} x_k \\ w_k - w_k^* \end{bmatrix}.$$

$\|\boldsymbol{L}^{-1}\|_{[0,N]} \geq 1$ because the response to $w_0 - w_0^* = e_1$, the first standard basis vector, for $k = 0$ and $w_k - w_k^* = 0$ for all $k \neq 0$ has two-norm at least unity. (The response is $w_0 = e_1$, $w_1 = -\nabla^{-1} L_\nabla (B_1 - B_2 R_3^{-1} R_2) e_1, \ldots$.)

Solution B.3. Suppose there exists a strictly proper controller such that (B.2.10) holds (when $x_0 = 0$). Consider the input $w_i = 0$ for $i \leq k-1$. The strictly proper nature of the state dynamics and the controller implies that $u_i = 0$ and $x_i = 0$ for $i \leq k$. Hence $R_1(k) \leq -\epsilon I$. Therefore, the stated Schur decomposition in the hint exits and

$$\begin{aligned} & z_k' z_k - \gamma^2 w_k' w_k + x_{k+1}' X_\infty(k+1) x_k \\ &= \; x_k' X_\infty(k) x_k + (w_k - w_k^*)' R_1 (w_k - w_k^*) + (u_k - u_k^*)' \nabla (u_k - u_k^*) \end{aligned}$$

in which $\nabla(k) = (R_3 - R_2 R_1^{-1} R_2')(k)$ and

$$\begin{bmatrix} w_k^* \\ u_k^* \end{bmatrix} = - \begin{bmatrix} R_1^{-1} L_1 & R_1^{-1} R_2' \\ \nabla^{-1}(L_2 - R_2 R_1^{-1} L_1) & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}.$$

The Riccati equation follows as before (note that $R_3 \geq 0$ as before, so $R_3 - R_2 R_1^{-1} R_2'$ is nonsingular, implying R is nonsingular). Furthermore, by choosing $u = u_k^*$ and $w_k = 0$ we see that $X_\infty(k) \geq 0$, as before. The rest of the iterative argument is identical to that presented in the text.

Solution B.4. By completing the square using X_∞ and $\overline{X}_\infty$, we obtain

$$\begin{aligned}
\|z\|_{2,[k,N]}^2 - \gamma^2 \|w\|_{2,[k,N]}^2 + x_{N+1}' \Delta x_{N+1} &= \|r\|_{2,[k,N]}^2 - \gamma^2 \|s\|_{2,[k,N]}^2 \\
&\quad + x_k' X_\infty(k) x_k \\
\|z\|_{2,[k,N]}^2 - \gamma^2 \|w\|_{2,[k,N]}^2 + x_{N+1}' \overline{\Delta} x_{N+1} &= \|\overline{r}\|_{2,[k,N]}^2 - \gamma^2 \|\overline{s}\|_{2,[k,N]}^2 \\
&\quad + x_k' \overline{X}_\infty(k) x_k.
\end{aligned}$$

Setting $u_i = u_i^*$ and $w_i = \overline{w}_i^*$ for $i = k, \ldots, N$ gives $r_i = 0$ and $\overline{s}_i = 0$ for $i = k, \ldots, N$. Therefore

$$\begin{aligned}
x_k \big(X_\infty(k) - \overline{X}_\infty(k)\big) x_k &= x_{N+1}'(\Delta - \overline{\Delta}) x_{N+1} + \|\overline{r}\|_{2,[k,N]}^2 + \gamma^2 \|s\|_{2,[k,N]}^2 \\
&\geq x_{N+1}'(\Delta - \overline{\Delta}) x_{N+1}.
\end{aligned}$$

Since x_k may be regarded as an arbitrary initial condition, $\Delta \geq \overline{\Delta}$ implies that $X_\infty(k) \geq \overline{X}_\infty(k)$.

Solution B.5.

1. Substitute for x_{k+1} from the dynamics and use the equation defining z_k.
2. Elementary algebra verifies the "completion of square" identity. The conclusion that $X_2(k) \geq 0$ follows from the fact that the left-hand side of the completion of squares identity is nonnegative for any u_k and x_k—in particular, for $u_k = u_k^*$.
3. $X_2(N+1) \geq 0$ implies $R^{-1}(N)$ exists, which implies $X_2(N)$ is well defined and nonnegative definite. Hence $R^{-1}(N-1)$ exists, which implies $X_2(N-1)$ is well defined and nonnegative definite ....

   From the completion of squares identity in Part 2, we obtain

   $$\sum_{k=0}^{N} z_k' z_k + x_{k+1}' X_2(k+1) x_{k+1} - x_k' X_2(k) x_k = \sum_{k=0}^{N} (u_k - u_k^*)' R(k) (u_k - u_k^*).$$

   Since

   $$\sum_{k=0}^{N} x_{k+1}' X_2(k+1) x_{k+1} - x_k' X(k) x_k = x_{N+1}' X_2(N+1) x_{N+1} - x_0' X_2(0) x_0$$

and $X_2(N+1)=\Delta$), we obtain

$$\sum_{k=0}^{N} z_k' z_k + x_{k+1}'\Delta x_{k+1} = x_0' X_2(0) x_0 + \sum_{k=0}^{N} (u_k - u_k^*)' R(k)(u_k - u_k^*).$$

4. That the optimal control is $u_k^* = -R(k)^{-1}L(k)x_k$ is immediate from the preceding identity. The optimal control is unique because $R(k)>0$ for all k.

Solution B.6. Note that

$$\begin{aligned} & L(k) - R(k)(D_{12}'D_{12})^{-1}D_{12}'C_1 \\ = \;& B_2'X_2(k+1)A - B_2'X_2(k+1)B_2(D_{12}'D_{12})^{-1}D_{12}'C_1 \\ = \;& B_2'X_2(k+1)\tilde{A}. \end{aligned} \tag{B.1}$$

The time-dependence of the matrices A, B_2, C_1, D_{12}, $\tilde{A}$ and $\tilde{C}$ will be suppressed. Using the Riccati equation from Problem B.5, we have

$$\begin{aligned} & \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}' \begin{bmatrix} X_2(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \\ = \;& \begin{bmatrix} X_2(k) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L'(k) \\ R(k) \end{bmatrix} R^{-1}(k) \begin{bmatrix} L(k) & R(k) \end{bmatrix}. \end{aligned}$$

Multiply on the right by $\begin{bmatrix} I & 0 \\ -(D_{12}'D_{12})^{-1}D_{12}'C_1 & I \end{bmatrix}$ and on the left by the transpose of this matrix to obtain

$$\begin{aligned} & \begin{bmatrix} \tilde{A} & B_2 \\ \tilde{C} & D_{12} \end{bmatrix}' \begin{bmatrix} X_2(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A} & B_2 \\ \tilde{C} & D_{12} \end{bmatrix} \\ & \qquad = \begin{bmatrix} \tilde{A}'X_2(k+1)B_2R^{-1}(k)B_2'X_2(k+1)\tilde{A} & \tilde{A}'X_2(k+1)B_2 \\ B_2'X_2(k+1)\tilde{A} & R(k) \end{bmatrix} \\ & \quad + \begin{bmatrix} X_2(k) & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The $(1,1)$-block is the desired Riccati equation.

Using (B.1), we obtain

$$\begin{aligned} \tilde{A} - B_2R^{-1}(k)B_2'X_2(k+1)\tilde{A} &= \tilde{A} - B_2R^{-1}(k)(L(k) - R(k)(D_{12}'D_{12})^{-1}D_{12}'C_1) \\ &= A - B_2R^{-1}(k)L(k). \end{aligned}$$

Solution B.7.

1.
$$\begin{aligned}
&(A-B_2K)'P(A-B_2K)+(C_1-D_{12}K)'(C_1-D_{12}K)\\
&= A'PA+C_1'C_1-L_P'R_P^{-1}L_P\\
&\quad +(K-R_P^{-1}L_P)'R_P(K-R_P^{-1}L_P)
\end{aligned}$$

in which $R_P = D_{12}'D_{12}+B_2'PB_2$ and $L_P = D_{12}'C_1+B_2'PA$. Substituting this into (B.6.5), we obtain the inequality (B.6.3), with $\Delta = P$.

Now suppose
$$\begin{bmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \\ P & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Multiplying (B.6.5) on the left by x^* and on the right by x results in
$$0 = \|P^{\frac{1}{2}}(A-B_2K)x\| + \|(C_1-D_{12}F)x\|.$$
Hence $(C_1-D_{12}K)x = 0$, since both terms on the right-hand side are non-negative. Since $C_1x = -D_{12}u$, we have $D_{12}(u+Kx) = 0$ and $D_{12}'D_{12} > 0$ implies that $u = -Kx$. Hence
$$\begin{aligned} 0 &= (A-\lambda I)x + B_2u \\ &= (A-B_2K-\lambda I)x, \end{aligned}$$
which implies that $|\lambda| < 1$ or $x = 0$, since $A-B_2K$ is asymptotically stable. Thus
$$\begin{bmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \\ P & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
implies either $|\lambda| < 1$ or $x = 0$, $u = 0$ and we conclude that
$$\begin{bmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \\ P & 0 \end{bmatrix}$$
has full column rank for all $|\lambda| \geq 1$.

2. Define $u_k^* = -R(k)^{-1}L(k)x_k$. Then
$$\begin{aligned}
&x_k'X_2(k)x_k + \|R^{\frac{1}{2}}(u-u^*)\|_{2,[k,N]}^2\\
&= \|z\|_{2,[k,N]}^2 + x_{N+1}'\Delta x_{N+1}\\
&= \|z\|_{2,[k,N-1]}^2 + x_N'\Delta x_N + x_N'(A'\Delta A + C_1'C_1 - \Delta)x_N, \quad \text{if } u_N = 0,\\
&\leq \|z\|_{2,[k,N-1]}^2 + x_N'\Delta x_N\\
&= x_k'X_2(k+1)x_k + \|R^{\frac{1}{2}}(u-u^\circ)\|_{2,[k,N-1]}^2
\end{aligned}$$

in which $u_k^\circ = -R(k+1)^{-1}L(k+1)x_k$, which is the optimal control for the time-horizon $[k, N-1]$. (Remember that we are dealing with the time-invariant case, so $X_2(k, N-1, \Delta) = X_2(k+1, N, \Delta)$.) Setting $u = u^\circ$ on $[k, N-1]$ and $u_N = 0$, we obtain

$$x_k'\big(X_2(k+1) - X_2(k)\big)x_k \geq \|R^{\frac{1}{2}}(u - u^*)\|_{2,[k,N-1]}^2$$

and since x_k may be regarded as an arbitrary initial condition, we conclude that $X_2(k) \leq X_2(k+1)$.

3. $X_2(k)$ is monotonic, bounded above (by Δ) and bounded below (by 0). Hence $X_2 = \lim_{k\to-\infty} X_2(k)$ exists and satisfies the algebraic Riccati equation. The completion of squares identify is immediate from the corresponding finite horizon identity, since $\lim_{N\to\infty} x_{N+1}'\Delta x_{N+1} = 0$ for any stabilizing controller.

4. Let $X_M = X_2(N+1-M)$ and $\Gamma_M = X_2(N+1-M) - X_2(N-M)$ and $R_M = R(N-M)$. We use the "fake algebraic Riccati technique", to determine stability. Write the Riccati equation as

$$\begin{aligned} X_M &= (A - B_2F_M)'X_M(A - B_2F_M) \\ &\quad + (C_1 - D_{12}F_M)'(C_1 - D_{12}F_M) + \Gamma_M. \end{aligned} \qquad \text{(B.2)}$$

We need to show that $|\lambda_i(A - B_2F_M)| < 1$. Let $x \neq 0$ and λ satisfy

$$(A - B_2F_M)x = \lambda x.$$

Then x^*(B.2)x yields

$$(|\lambda|^2 - 1)x^*X_Mx + \|(C_1 - D_{12}F_M)x\|^2 + \|\Gamma_M^{\frac{1}{2}}x\|^2 = 0.) \qquad \text{(B.3)}$$

Since $X_M \geq 0$, we must have either

(a) $|\lambda| < 1$ or

(b) $(C_1 - D_{12}F_M)x = 0$ and $\Gamma_Mx = 0$.

Case (a) is what we want, so it remains to see what happens in case (b).

Claim: Case (b) implies $X_Mx = 0$. Suppose case (b) holds. From (B.3), if $|\lambda| \neq 1$, we have $X_Mx = 0$. On the other hand, if $|\lambda| = 1$, use (B.2) to obtain

$$\begin{aligned} &\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}' \begin{bmatrix} X_M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \\ &= \begin{bmatrix} X_M - \Gamma_M & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F_M' \\ I \end{bmatrix} R_M \begin{bmatrix} F_M & I \end{bmatrix}. \end{aligned}$$

Since $(A - B_2F_M)x = \lambda x$, $(C_1 - D_{12}F_M)x = 0$ and $\Gamma_M x = 0$, multiplying on the right by $\begin{bmatrix} I \\ -F_M \end{bmatrix} x$ results in

$$\lambda \begin{bmatrix} A' \\ B_2' \end{bmatrix} X_M x = \begin{bmatrix} X_M x \\ 0 \end{bmatrix}.$$

Therefore, since (A, B_2) is stabilizable, $|\lambda| = 1$ implies that $X_M x = 0$. Thus case (b) implies $X_M x = 0$ and the claim is established.

Claim: Case (b) implies $|\lambda| < 1$. Suppose case (b) holds. Then $X_M x = 0$. We now consider the implications of this fact.

If $M = 0$ (*i.e.*, the horizon length is zero), we have

$$\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} I \\ -F_M \end{bmatrix} x = \begin{bmatrix} \lambda x \\ 0 \\ 0 \end{bmatrix}.$$

Since $x \neq 0$, we conclude that $|\lambda| < 1$ from the assumption that (B.6.4) has full column rank for all $|\lambda| \geq 1$.

If $M \geq 1$, consider the system

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \qquad x_{N+1-M} = x.$$

Then, by completing the square,

$$\begin{aligned} &\sum_{k=N+1-M}^{N} z_k' z_k + x_{N+1} \Delta x_{N+1} \\ &= x' X_M x + \sum_{k=N+1-M}^{N} (u_k - u_k^*) R(k) (u_k - u_k^*) \\ &= \sum_{k=N+1-M}^{N} (u_k - u_k^*) R(k) (u_k - u_k^*). \end{aligned}$$

Therefore, the control strategy $u_k = u_k^* = -R(k)^{-1} L(k) x_k$ results in $z_k = 0$ for $k = N + 1 - M, \ldots, N$ and $\Delta x_{N+1} = 0$, since the left-hand side of the above identity is nonnegative and the right-hand side is zero when $u = u^*$. Since $(A - B_2F_M)x = \lambda x$ and $(C_1 - D_{12}F_M)x = 0$, the control strategy $u_k = -F_M x_k$ also results in $z_k = 0$. Since $D_{12}' D_{12} > 0$, this implies that the controls $u_k = -F_M x_k$ and $u_k = u_k^*$ are identical. Consequently, the state trajectories with $u_k = -F_M x_k$ and $u_k = u_k^*$ are identical. Since the state trajectory resulting from $u_k = -F_M x_k$ is $\lambda^{k+M-N-1} x$ and the state trajectory associated with $u_k = u_k^*$ satisfies $\Delta x_{N+1} = 0$, we conclude that $\Delta\lambda^M x = 0$. We now have

$$\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} I \\ -F_M \end{bmatrix} \lambda^M x = \lambda \begin{bmatrix} \lambda^M x \\ 0 \\ 0 \end{bmatrix}.$$

Invoking assumption that (B.6.4) has full column rank for all $|\lambda| \geq 1$, we conclude that $|\lambda| < 1$ or $\lambda^M x = 0$, which implies $\lambda = 0$ since $x \neq 0$.

This completes the proof that $A - B_2F_M$ is asymptotically stable.

The cost of the control law F_M is

$$\begin{aligned} \|z\|_2^2 &= x_0' P_M x_0 - \|\Gamma_M^{\frac{1}{2}} x_0\|_2^2 \\ &\leq x_0' P_M x_0. \end{aligned}$$

5. Since $F_M \to F = R^{-1}L$ as $M \to \infty$ and $|\lambda_i(A - B_2F_M)| < 1$ for all i and all M, we must have that $|\lambda_i(A - B_2F)| \leq 1$ for all i. To prove that strict inequality holds, we must show that $(A - B_2F)x = e^{j\theta}x$ implies $x = 0$.

   Suppose $(A - B_2F)x = e^{j\theta}x$. Write the Riccati equation as

   $$X_2 = (A - B_2F)'X_2(A - B_2F) + (C_1 - D_{12}F)'(C_1 - D_{12}F).$$

   Multiplying on the left by x^* and on the right by x we conclude that $(C_1 - D_{12}F)x = 0$. Hence

   $$\begin{bmatrix} A - e^{j\theta}I & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} I \\ -F \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   Consequently, if (B.6.7) has full column rank for all θ, we must have $x = 0$. Thus $A - B_2F$ is asymptotically stable.

   Conversely, suppose

   $$\begin{bmatrix} A - e^{j\theta}I & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{B.4}$$

   for some θ and some x, u not both zero. Write the Riccati equation as

   $$\begin{aligned} &\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}' \begin{bmatrix} X_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \\ &= \begin{bmatrix} X_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L'R^{-1} \\ I \end{bmatrix} R \begin{bmatrix} R^{-1}L & I \end{bmatrix}. \end{aligned}$$

   Multiplying on the left by $\begin{bmatrix} x^* & u^* \end{bmatrix}$ and on the right by $\begin{bmatrix} x \\ u \end{bmatrix}$, we see that

   $$x^*X_2x = x^*X_2x + (R^{-1}Lx + u)^*R(R^{-1}Lx + u)$$

   and hence $u = -R^{-1}Lx$. Substituting into (B.4) gives $(A - BR^{-1}L)x = e^{j\theta}x$, and we conclude that $A - BR^{-1}L$ has an imaginary axis eigenvalue.

6. By completing the square with X_2, the cost associated with any stabilizing controller satisfies

$$\|z\|_2^2 = x_0' X_2 x_0 + \|R^{\frac{1}{2}}(u + R^{-1}Lx)\|_2^2$$

Hence $\|z\|_2^2 \geq x_0' X_2 x_0$ for any stabilizing controller. The lower bound $x_0' X_2 x_0$ can only be achieved by the controller $u = -R^{-1}Lx$, which is stabilizing if and only if (B.6.7) has full column rank for all real θ.

Solution B.8.

1. Complete the square with X_2 and with $X_2(k, N+1, \Delta)$ to obtain

$$\begin{aligned}\sum_0^N z_k' z_k + x_{N+1}' X_2 x_{N+1} &= x_0' X_2 x_0 + \sum_0^N (u_k - u_k^*)' R (u_k - u_k^*)\\ \sum_0^N z_k' z_k + x_{N+1}' \Delta x_{N+1} &= \sum_0^N (u_k - u_k^\circ)' R (u_k - u_k^\circ)\\ &\quad + x_0' X_2(0, N+1, \Delta) x_0.\end{aligned}$$

Subtracting these gives the stated identity (B.6.8).

The minimization of the left-hand side of (B.6.8) is an LQ problem, although the terminal penalty matrix may or may not be nonnegative definite and consequently we do not know that a solution to such problems exists in general. In this particular case, a solution does exist, because the right-hand side of (B.6.8) shows that $u_k = u_k^\circ$ is the optimal control and the optimal cost is $x_0'(X_2(0, N+1, \Delta) - X_2)x_0$. The objective function and state dynamics for the problem of minimizing the left-hand side of (B.6.8) can be written as

$$\begin{bmatrix} x_{k+1} \\ R^{\frac{1}{2}}(u_k - u_k^*) \end{bmatrix} = \begin{bmatrix} A - B_2 F & B_2 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x_k \\ u_k - u_k^* \end{bmatrix},$$

in which $F = R^{-1}L$ is the optimal feedback gain, (*i.e.*, $u_k^* = -Fx_k$). Hence, the the Riccati equation associated with the minimization of the left-hand side of (B.6.8) is (B.6.9) and the minimum cost is $x_0' \Gamma(0) x_0$. Since we concluded from the right-hand side of (B.6.8) that the minimum cost is $x_0'(X_2(0, N+1, \Delta) - X_2)x_0$, it follows that $\Gamma(0) = X_2(0, N+1, \Delta) - X_2$, since x_0 is arbitrary. By time invariance, $\Gamma(k) = X_2(k, N+1, \Delta) - X_2$.

2. This can be quite tricky if you take a brute force approach, which is one reason why the preceding argument is so delightful. It also gives a clue about the manipulations, since the argument above is about optimizing something you already know the optimal control for.

Write the two Riccati equations as

$$\begin{aligned}
&\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}' \begin{bmatrix} X_2(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \\
&\quad = \begin{bmatrix} X_2(k) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L'(k) \\ R(k) \end{bmatrix} R^{-1}(k) \begin{bmatrix} L(k) & R(k) \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}' \begin{bmatrix} X_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \\
&\quad = \begin{bmatrix} X_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L' \\ R \end{bmatrix} R^{-1} \begin{bmatrix} L & R \end{bmatrix}.
\end{aligned}$$

Subtract them to obtain

$$\begin{aligned}
&\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}' \begin{bmatrix} \Gamma(k+1) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \\
&\quad = \begin{bmatrix} \Gamma(k) & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L' \\ R \end{bmatrix} R^{-1} \begin{bmatrix} L' \\ R \end{bmatrix}' + \begin{bmatrix} L'(k) \\ R(k) \end{bmatrix} R^{-1}(k) \begin{bmatrix} L'(k) \\ R(k) \end{bmatrix}'.
\end{aligned}$$

Multiply on the right by $\begin{bmatrix} I & 0 \\ -R^{-1}L & I \end{bmatrix}$ and by its transpose on the left to obtain

$$\begin{aligned}
&\begin{bmatrix} A - BR^{-1}L & B_2 \\ C_1 - D_{12}R^{-1}L & D_{12} \end{bmatrix}' \begin{bmatrix} \Gamma(k+1) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A - BR^{-1}L & B_2 \\ C_1 - D_{12}R^{-1}L & D_{12} \end{bmatrix} \\
&\quad = \begin{bmatrix} L'(k) - L'R^{-1}R(k) \\ R(k) \end{bmatrix} R^{-1}(k) \begin{bmatrix} L'(k) - L'R^{-1}R(k) \\ R(k) \end{bmatrix}' \\
&\qquad + \begin{bmatrix} \Gamma(k) & 0 \\ 0 & R \end{bmatrix}. \qquad \text{(B.5)}
\end{aligned}$$

Now $R(k) = R + B_2'\Gamma(k+1)B_2$ and

$$\begin{aligned}
L(k) - R(k)R^{-1}L &= L + B_2'\Gamma(k+1)A - (R + B_2\Gamma(k+1)B_2)R^{-1}L \\
&= B_2'\Gamma(k+1)(A - B_2R^{-1}L).
\end{aligned}$$

Therefore, the $(1,1)$-block of (B.5) is the desired Riccati equation for $\Gamma(k)$.

Solution B.9. As in the previous problem, the calculations can become horrendous if a brute force approach is adopted. The technique of the previous problem provides the remedy (which is why that problem is there).

1. Use the Schur decomposition (B.2.7) to write the Riccati equation for X_∞ as

$$X_\infty = A'X_\infty A + C_1'C_1 - L_2'R_3^{-1}L_2 - L_\nabla'\nabla^{-1}L_\nabla,$$

in which $L\nabla = L_1 - R_2'R_3^{-1}L_2$ as before. Combining this with the definitions of L and R, we may write

$$\begin{aligned}&\begin{bmatrix} A & B_2 \\ C_1 & D_{12}\end{bmatrix}'\begin{bmatrix} X_\infty & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} A & B_2 \\ C_1 & D_{12}\end{bmatrix}\\ &\quad= \begin{bmatrix} X_\infty + L_\nabla'\nabla^{-1}L_\nabla & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix} L_2' \\ R_3\end{bmatrix} R_3^{-1}\begin{bmatrix} L_2 & R_3\end{bmatrix}.\end{aligned}$$

We also have for the X_2 Riccati equation

$$\begin{aligned}&\begin{bmatrix} A & B_2 \\ C_1 & D_{12}\end{bmatrix}'\begin{bmatrix} X_2 & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} A & B_2 \\ C_1 & D_{12}\end{bmatrix}\\ &\quad= \begin{bmatrix} X_2 & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix} \widehat{L}' \\ \widehat{R}\end{bmatrix} \widehat{R}^{-1}\begin{bmatrix} \widehat{L} & \widehat{R}\end{bmatrix},\end{aligned}$$

in which $\widehat{R} = D_{12}'D_{12} + B_2'X_2B_2$ and $\widehat{L} = D_{12}'C_1 + B_2'X_2A$. Subtracting these gives

$$\begin{aligned}&\begin{bmatrix} A & B_2 \\ C_1 & D_{12}\end{bmatrix}'\begin{bmatrix} \Gamma & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} A & B_2 \\ C_1 & D_{12}\end{bmatrix}\\ &\quad= \begin{bmatrix} \Gamma + L_\nabla'\nabla^{-1}L_\nabla & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix} L_2' \\ R_3\end{bmatrix} R_3^{-1}\begin{bmatrix} L_2' \\ R_3\end{bmatrix}' - \begin{bmatrix} \widehat{L}' \\ \widehat{R}\end{bmatrix} \widehat{R}^{-1}\begin{bmatrix} \widehat{L}' \\ \widehat{R}\end{bmatrix}'.\end{aligned}$$

Multiply by $\begin{bmatrix} I & 0 \\ -R_3^{-1}L_2 & I\end{bmatrix}$ on the right, and by its transpose on the left, to obtain

$$\begin{aligned}&\begin{bmatrix} A - B_2R_3^{-1}L_2 & B_2 \\ C_1 - D_{12}R_3^{-1}L_2 & D_{12}\end{bmatrix}'\begin{bmatrix} \Gamma & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} A - B_2R_3^{-1}L_2 & B_2 \\ C_1 - D_{12}R_3^{-1}L_2 & D_{12}\end{bmatrix}\\ &\quad= \begin{bmatrix} \widehat{L}' - L_2'R_3^{-1}\widehat{R} \\ \widehat{R}\end{bmatrix} \widehat{R}^{-1}\begin{bmatrix} \widehat{L}' - L_2'R_3^{-1}\widehat{R} \\ \widehat{R}\end{bmatrix}'\\ &\qquad + \begin{bmatrix} \Gamma + L_\nabla'\nabla^{-1}L_\nabla & 0 \\ 0 & R_3\end{bmatrix}.\end{aligned} \tag{B.6}$$

Now $\widehat{R} = R_3 - B_2'\Gamma B_2$ and $\widehat{L} = L_2 - B_2'\Gamma A$. Therefore

$$\begin{aligned}\widehat{L} - \widehat{R}R_3^{-1}L_2 &= L_2 - B_2'\Gamma A - (R_3 - B_2'\Gamma B_2)R_3^{-1}L_2\\ &= -B_2'\Gamma(A - B_2R_3^{-1}L_2).\end{aligned}$$

Therefore, the $(1,1)$-block of (B.6) is the desired Riccati equation.

2. Recall, from the monotonicity property of the solution $X_\infty(k, N+1, X_2)$, that $X_\infty \geq X_2$. Therefore, $\Gamma \geq 0$. Also, from (B.2.33), $(A - B_2 R_3^{-1} L_2, L_\nabla)$ is detectable. Hence, since $\nabla < 0$, we conclude that $A - B_2 R_3^{-1} L_2$ is asymptotically stable from the equation established in Part 1.

3. If $\Gamma x = 0$, then the equation established in Part 1 gives $L_\nabla x = 0$. From (B.2.33), we therefore conclude that $(A - BR^{-1}L)x = (A - B_2 R_3^{-1} L_2)x$. (Use equation (B.2.33).)

Solution B.10.

1. Write the two Riccati equations as

$$\begin{aligned}
&\begin{bmatrix} A & B \\ \overline{C} & \overline{D} \end{bmatrix}' \begin{bmatrix} X_\infty(k+1) & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} A & B \\ \overline{C} & \overline{D} \end{bmatrix} \\
&\quad = \begin{bmatrix} X_\infty(k) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L'(k) \\ R(k) \end{bmatrix} R^{-1}(k) \begin{bmatrix} L(k) & R(k) \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\begin{bmatrix} A & B \\ \overline{C} & \overline{D} \end{bmatrix}' \begin{bmatrix} X_\infty & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} A & B \\ \overline{C} & \overline{D} \end{bmatrix} \\
&\quad = \begin{bmatrix} X_\infty & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L' \\ R \end{bmatrix} R^{-1} \begin{bmatrix} L & R \end{bmatrix}.
\end{aligned}$$

Subtract them to obtain

$$\begin{aligned}
&\begin{bmatrix} A & B \\ \overline{C} & \overline{D} \end{bmatrix}' \begin{bmatrix} \Gamma(k+1) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ \overline{C} & \overline{D} \end{bmatrix} \\
&\quad = \begin{bmatrix} \Gamma(k) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L' \\ R \end{bmatrix} R^{-1} \begin{bmatrix} L' \\ R \end{bmatrix}' - \begin{bmatrix} L'(k) \\ R(k) \end{bmatrix} R^{-1}(k) \begin{bmatrix} L'(k) \\ R(k) \end{bmatrix}'
\end{aligned}$$

Multiply on the right by $\begin{bmatrix} I & 0 \\ -R^{-1}L & I \end{bmatrix}$ and by its transpose on the left to obtain

$$\begin{aligned}
&\begin{bmatrix} A - BR^{-1}L & B_2 \\ \overline{C} - \overline{D}R^{-1}L & \overline{D} \end{bmatrix}' \begin{bmatrix} \Gamma(k+1) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A - BR^{-1}L & B \\ \overline{C} - \overline{D}R^{-1}L & \overline{D} \end{bmatrix} \\
&\quad = -\begin{bmatrix} L'(k) - L'R^{-1}R(k) \\ R(k) \end{bmatrix} R^{-1}(k) \begin{bmatrix} L'(k) - L'R^{-1}R(k) \\ R(k) \end{bmatrix}' \\
&\qquad + \begin{bmatrix} \Gamma(k) & 0 \\ 0 & R \end{bmatrix}. \qquad \text{(B.7)}
\end{aligned}$$

Now $R(k) = R - B'\Gamma(k+1)B$ and

$$\begin{aligned} L(k) - R(k)R^{-1}L &= L - B'\Gamma(k+1)A - (R - B\Gamma(k+1)B)R^{-1}L \\ &= -B'\Gamma(k+1)(A - BR^{-1}L). \end{aligned}$$

Therefore, the $(1,1)$-block of (B.7) is the desired Riccati equation for Γ

2. This is an intricate argument similar to that used in the solution of Problem B.7. It involves figuring out what happens at the terminal time $N+1$ from something that happens at time k and using the properties of the terminal condition.

   Suppose $x \neq 0$ and $(X_\infty - X(k, N+1, X_2))x = 0$. That is, $\Gamma(k)x = 0$. Let the initial condition for the dynamics be $x_k = x$. Complete the square with X_∞ to obtain

$$\begin{aligned} \|z\|^2_{2,[k,N]} - \gamma^2\|w\|^2_{2,[k,N]} + x_{N+1}X_\infty x_{N+1} &= \|r\|^2_{2,[k,N]} - \gamma^2\|s\|^2_{2,[k,N]} \\ &\quad + x'X_\infty x. \end{aligned}$$

   Also, complete the square with $X_\infty(k, N+1, X_2)$ to obtain

$$\begin{aligned} \|z\|^2_{2,[k,N]} - \gamma^2\|w\|^2_{2,[k,N]} + x_{N+1}X_2 x_{N+1} &= \|\overline{r}\|^2_{2,[k,N]} - \gamma^2\|\overline{s}\|^2_{2,[k,N]} \\ &\quad + x'X(k, N+1, X_2)x. \end{aligned}$$

   Now subtract to obtain

$$x_{N+1}(X_\infty - X_2)x_{N+1} = \|r\|^2_{2,[k,N]} + \gamma^2\|\overline{s}\|^2_{2,[k,N]} - \gamma^2\|s\|^2_{2,[k,N]} - \|\overline{r}\|^2_{2,[k,N]}.$$

   If we choose u such that $r = 0$ (*i.e.*, $u = u^*$ for the infinite-horizon problem) and w such that $\overline{s} = 0$ (*i.e.*, $w = w^*_{[k,N]}$, the worst w for the finite-horizon problem), then we have

$$x_{N+1}(X_\infty - X_2)x_{N+1} = -\gamma^2\|s\|^2_{2,[k,N]} - \|\overline{r}\|^2_{2,[k,N]}.$$

   Since $X_\infty - X_2 \geq 0$, we must have $\overline{r} = 0$ and $s = 0$. This implies that the optimal controls for the finite- and infinite-horizon problems are identical on $[k, N]$. We also see that $(X_\infty - X_2)x_{N+1} = 0$. Since $u = u^*$ and $w = w^*$, the state dynamics reduce to

$$x_{i+1} = (A - BR^{-1}L)x_i, \qquad x_k = x.$$

   Multiplying the equation for $X_\infty - X_2$, which is given in (B.2.39), on the left by x'_{N+1} and on the right by x_{N+1} we conclude that $(A - B_2R_3^{-1}L_2)x_{N+1}$ is also in the kernel of $X_\infty - X_2$ and that $L_\nabla x_{N+1} = 0$, because the terms on the right-hand side are all nonnegative and the left-hand side is zero. From (B.2.33), we see that $(A - B_2R_3^{-1}L_2)x_{N+1} = (A - BR^{-1}L)x_{N+1}$. Therefore, there is an eigenvalue of $A-BR^{-1}L$ in the kernel of $X_\infty - X_2$. This eigenvalue must be asymptotically stable because $A - B_2R_3^{-1}L_2$ is.

Conclusion: If $x \in \ker\big(X_\infty - X_\infty(k, N+1, X_2)\big)$, the application of the dynamics $A-BR^{-1}L$ leads to x_{N+1} being in an invariant subspace of $\ker(X_\infty - X_2)$ after a finite number of steps. This invariant subspace is asymptotically stable by the stability of $A - B_2R_3^{-1}L_2$. Hence the subspace corresponding to $\ker\big(X_\infty - X_\infty(k, N+1, X_2)\big)$ is asymptotically stable.

It follows that if

$$X_\infty - X_2 = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

in which Γ_1 is nonsingular, then

$$X_\infty - X_\infty(k, N+1, X_2) = \begin{bmatrix} \Gamma_1(k) & 0 \\ 0 & 0 \end{bmatrix}$$

in which $\Gamma_1(k)$ is nonsingular for all k. Furthermore,

$$A - BR^{-1}L = \begin{bmatrix} \widehat{A}_{11} & 0 \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}.$$

The argument of the text shows that $\widehat{A}_{11}$ is asymptotically stable, while $\widehat{A}_{22}$ is asymptotically stable due to the stabilizing properties of X_2.

Solution B.11. The $\ell_2[0, N]$ adjoint of $\boldsymbol{G}$ is the system $\boldsymbol{G}^\sim$ that has the property $\langle u, \boldsymbol{G}w\rangle = \langle \boldsymbol{G}^\sim u, w\rangle$ for all w, u. The inner product is defined by

$$\langle w, z\rangle = \sum_0^N z_k' w_k.$$

Suppose $z = \boldsymbol{G}w$ is generated by

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}(k) \begin{bmatrix} x_k \\ w_k \end{bmatrix}, \qquad x_0 = 0,$$

and p_k is an arbitrary $\ell_2[0, N]$ sequence such that $p_N = 0$. Then

$$\begin{aligned}
\langle u, z\rangle &= \sum_0^N z_k' u_k + x_{k+1}' p_k - x_k' p_{k-1} \\
&= \sum_0^N \begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix}' \begin{bmatrix} p_k \\ u_k \end{bmatrix} - x_k' p_{k-1} \\
&= \sum_0^N \begin{bmatrix} x_k \\ w_k \end{bmatrix}' \begin{bmatrix} A & B \\ C & D \end{bmatrix}'(k) \begin{bmatrix} p_k \\ u_k \end{bmatrix} - x_k' p_{k-1}
\end{aligned}$$

Therefore, if we choose the sequence p_k such that $p_{k-1} = A'(k)p_k + C'(k)u_k$ and define y_k by

$$\begin{bmatrix} p_{k-1} \\ y_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}'(k) \begin{bmatrix} p_k \\ u_k \end{bmatrix}, \qquad p_N = 0,$$

then

$$\begin{aligned} \langle u, z\rangle &= \sum_0^N \begin{bmatrix} x_k \\ w_k \end{bmatrix}' \begin{bmatrix} p_{k-1} \\ y_k \end{bmatrix} - x_k' p_{k-1} \\ &= \sum_0^N {w_k}' y_k \\ &= \langle y, w\rangle. \end{aligned}$$

Hence $y = \boldsymbol{G}^{\sim} u$.

Solution B.12.

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 w, \\ y_k &= C_2 x_k + D_{21} w_k, \\ \widehat{x}_{k+1} &= A\widehat{x}_k + H(y_k - C_2 \widehat{x}_k), \end{aligned}$$

in which $H = M_2 S_3^{-1}$. Subtracting these equations results in the error dynamics

$$\widehat{x}_{k+1} - x_{k+1} = (A - HC_2)(\widehat{x}_k - x_k) + (HD_{21} - B_1)w_k.$$

By standard results on linear systems driven by white noise processes (see, for example, [12]),

$$\mathcal{E}\{(\widehat{x}_{k+1} - x_{k+1})(\widehat{x}_{k+1} - x_{k+1})'\} = P(k)$$

in which $P(k)$ is the solution to the linear matrix equation

$$\begin{aligned} P(k+1) &= (A - HC_2)(k)P(k)(A - HC_2)'(k) \\ &\quad + (HD_{21} - B_1)(k)(HD_{21} - B_1)'(k), \qquad P(0) = 0. \end{aligned}$$

(We have $P(0) = 0$ because the initial state is assumed to be know; otherwise, we set $P(0)$ to the initial error covariance.)

Now

$$Y_\infty(k+1) = B_1 B_1' + AY_\infty(k)A' - M_2 S_3^{-1} M_2' - M_\nabla \nabla^{-1} M_\nabla'$$

in which

$$\begin{aligned} \nabla(k) &= S_1(k) - S_2(k)S_3^{-1}(k)S_2'(k) \\ M_\nabla(k) &= M_1(k) - S_2(k)S_3^{-1}(k)M_2(k). \end{aligned}$$

Re-write the Riccati equation as

$$\begin{aligned} Y(k+1) &= (A - HC_2)Y_\infty(k)(A - HC_2)' + M_2 S_3^{-1} M_2' \\ &\quad - M_\nabla \nabla^{-1} M_\nabla' + (HD_{21} - B_1)(HD_{21} - B_1)'. \end{aligned}$$

Hence

$$\begin{aligned} Y_\infty(k+1) - P(k+1) &= (A - HC_2)(Y_\infty(k) - P(k))(A - HC_2)' \\ &\quad + M_2 S_3^{-1} M_2' - M_\nabla \nabla^{-1} M_\nabla'. \end{aligned}$$

Since $\nabla < 0$ and $Y_\infty(0) - P(0) = 0$, we have $Y_\infty(k) - P(k) \geq 0$ for all k. Hence

$$\mathcal{E}\{(\widehat{x}_{k+1} - x_{k+1})(\widehat{x}_{k+1} - x_{k+1})'\} = P(k) \leq Y_\infty(k).$$