Solutions to Problems and Additional Lessons for

# PARTIAL DIFFERENTIAL EQUATIONS for Scientists & Engineers

S. J. Farlow

SOLUTIONS TO PROBLEMS AND ADDITIONAL LESSONS

FOR

# PARTIAL DIFFERENTIAL EQUATIONS FOR SCIENTISTS AND ENGINEERS

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Related Topics to Accompany Text

## Lesson 1

1. a) This is a second order, linear, nonhomogeneous, parabolic pde with constant coefficients and two independent variables.

   b) This is a second order, linear, nonhomogeneous parabolic pde with constant coefficients with two independent variables.

   c) This is a second order, linear, nonhomogeneous, hyperbolic pde with constant coefficients with two independent variables.

   d) This is a fourth order, nonlinear pde in two independent variables (nonlinear pde are generally not classified as being parabolic, hyperbolic, or elliptic or whether they are homogeneous or have constant coefficients).

2. One of the interesting things about PDE as contrasted to ODE is that one generally doesn't find all the solutions to a PDE like one does in ODE. To find all the solutions to $u_t = u_{xx}$ requires a digression into complex variables. However if one tries to find solutions of the form $e^{ax + bt}$ one sees that $b = a^2$. That is, any exponential of the form:

$$e^{ax + a^2 t}$$

   is a solution. One may try various values of a, especially complex numbers like i to arrive at interesting solutions. After that try <u>adding</u> these solutions <u>together</u> to get still <u>more</u> solutions.

3. The answer is yes if $G = 0$ and it is very easy to prove.

4. $u_x = 0$ implies $u(x,y)$ is <u>any</u> function of y alone. Try some like $u(x,y) = \sin(y)$, $u(x,y) = c$.

5. $u_{xy} = 0$ implies $u(x,y)$ is the sum of <u>any</u> function of x plus <u>any</u> function of y. That is, like $u(x,y) = \sin(x) + y^2$, $u(x,y) = \cos(y)$.

Lesson 2

1. By using a little intuition one may guess at a temperature:

$$u(x,t) = e^{-\alpha^2 t} \sin(\pi x).$$

It is easy enough to verify that this function satisfies the heat equation and the given BC.

2. It should be clear from ones intuition that this temperature $u(x,t)$ will eventually arrive at some temperature that won't change in time (at least as $t \to 0$). Hence if we set $u_t = 0$ we arrive at an ordinary differential equation with boundary conditions in the steady state temperature $U(x)$:

$$\frac{d^2U}{dx^2} = -\frac{1}{\alpha^2} \qquad U(0) = 0 \qquad U(1) = 1$$

which has the solution:

$$U(x) = \frac{-1}{2\alpha^2}[x^2 - x] + x .$$

3. Setting $u_t = 0$ we arrive at the boundary value problem for the steady state $U(x)$:

$$\alpha^2 \frac{d^2U}{dx^2} - \beta U = 0 \qquad U(0) = 1 \qquad U(1) = 1$$

which has the solution:

$$U(x) = \frac{1}{\sinh\sqrt{\beta/\alpha^2}}[\sinh\sqrt{\beta/\alpha^2}\,(x) + \sinh\sqrt{\beta/\alpha^2}\,(1-x)].$$

Note that heat will be flowing in at each end of the rod and out the lateral sides. However at each point along the rod the net flow in and out is zero. The reader should graph this function for various values of $\alpha$ and $\beta$.

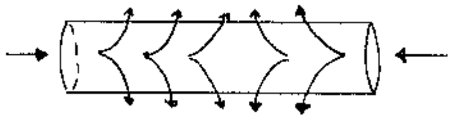


Diagram of Steady State Heat Flow

4. PDE $u_t = \alpha^2 u_{xx}$ $0 < x < 1$ $0 < t < \infty$

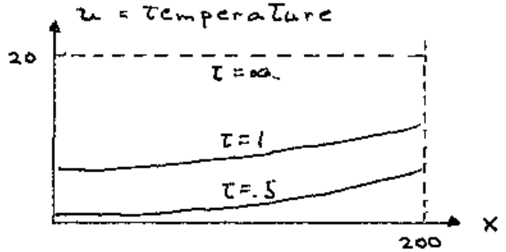
BCs $u(0,t) = 0$, $u(1,t) = 10$ $0 < t < \infty$

IC $u(x,0) = \sin(3\pi x)$ $0 \le x \le 1$ .

Note that the PDE is defined only inside the region $0 < x < 1$, $0 < t < \infty$ since the derivatives $u_t$ and $u_{xx}$ don't make sense on the boundaries. Also the BCs are defined for $t$ greater than zero or else the BCs and IC would each specify the temperature at the points $(t = 0, x = 0)$ and $(t = 0, x = 1)$.

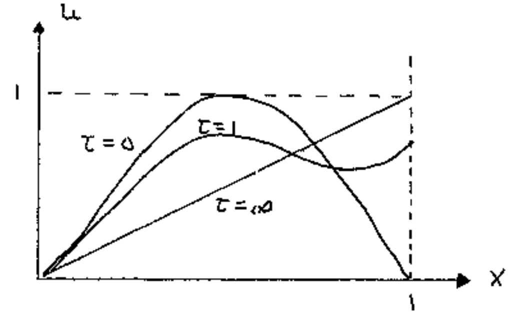
Lesson 3

1.



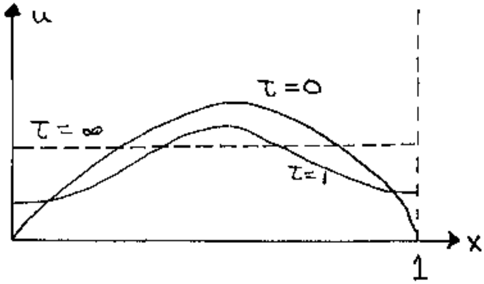

Note that the above temperature profiles are drawn so that the BCs of the problem are satisfied for all time.

2.




Heat flows into the rod at the right hand side at a constant rate while the left hand side is kept at zero degrees (zero can stand for any temperature). Keeping the left hand of the rod at zero actually means it will be a heat sink (heat flows into that point.

3.




Both ends are insulated and so the total heat energy inside the rod will eventually become distributed uniformly.

4. PDE $u_t = \alpha^2 u_{xx}$ $\qquad 0 < x < 1 \qquad 0 < t < \infty$

BCs $\begin{cases} u(0,t) = 50 \\ u_x(1,t) = -\frac{h}{k}[u(1,t) - 30] \end{cases}$ $\qquad 0 < t < \infty$

IC $u(x,0) = 20$ $\qquad 0 \le x \le 1$

Lesson 4

1. Each term has units deg/sec. The coefficient $\beta$ is a rate constant for the flow of heat across the lateral boundary and has units $sec^{-1}$.

2. Same here--each term has units deg/sec.

3. There is not much change in this derivation from the situation where $k$ is a constant.

4. The derivation is similar to the one in the lesson.

Lesson 5

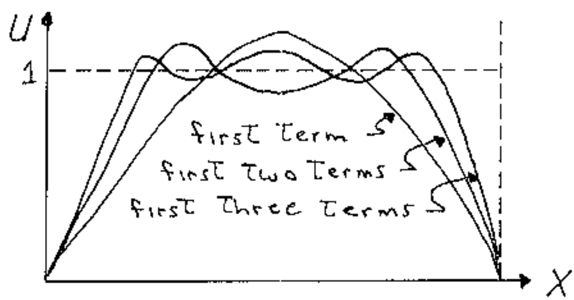
1. Straightforward substitution.

2. Straightforward substitution.

3. $\phi(x) = 1 = \Sigma A_n \sin(n\pi x)$ implies $A_n = 2\int_0^1 \sin(n\pi x)\,dx$

$$= \begin{cases} 0 & n = 0,2,4, \ldots \\ \frac{4}{n\pi} & n = 1,3,5, \ldots \end{cases}$$

Therefore $1 = \frac{4}{\pi}\left[\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x) \ldots\right]$.




4. Since the IC $u(x,0) = 1$ is expanded as

$$u(x,0) = \frac{4}{\pi}\left[\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \cdots\right]$$

the solution is:

$$u(x,t) = \frac{4}{\pi}\left[e^{-(\pi)^2 t}\sin(\pi x) + \frac{1}{3}e^{-(3\pi)^2 t}\sin(3\pi x) + \cdots\right].$$

5. $u(x,t) = e^{-(2\pi)^2 t}\sin(2\pi x) + \frac{1}{3}e^{-(4\pi)^2 t}\sin(4\pi x)$

$$+ \frac{1}{5}e^{-(6\pi)^2 t}\sin(6\pi x).$$

6. Expand $u(x,0) = x - x^2$ as a sine series $\sum_{n=1}^{\infty} A_n \sin(n\pi x)$ where $A_n = 2 \int_0^1 (x - x^2) \sin(n\pi x)\,dx$. One can either integrate this integral by <u>parts</u> or else <u>substitute</u> $y = n\pi x$ and get a new integral which one can look up from the tables:

$$\int y \sin(y)\,dy \qquad \int y^2 \sin(y)\,dy \,.$$

In either case:

$$A_n = \frac{4[1 - \cos(n\pi)]}{(n\pi)^3} \,.$$

That is:

$$x - x^2 = \frac{8}{\pi^3} \left[\frac{\sin(\pi x)}{1} + \frac{\sin(3\pi x)}{27} + \frac{\sin(5\pi x)}{125} + \cdots \right]$$

and hence the solution is:

$$u(x,t) = \frac{8}{\pi^3} \left[e^{-(\pi)^2 t} \sin(\pi x) + \frac{1}{27} e^{-(3\pi)^2 t} \sin(3\pi x) + \cdots \right].$$

Lesson 6

1. Write $u(x,t) = \underbrace{A(t)\, x + B(t)\, (1-x)}_{S(x,t) \text{ = steady state}} + \underbrace{U(x,t)}_{\text{transient}}$

and substitute $S(x,t)$ into the BCs to find $A(t)$ and $B(t)$. Doing this gives

$$B(t) = 1$$

$$A(t) = 2/(1-h)$$

Hence the solution $u(x,t)$ is

$$u(x,t) = 2x/(1-h) + (1-x) + U(x,t)$$

$$= 1 - (1+h/1-h)x + U(x,t)$$

Substituting $u(x,t)$ into the IBVP we get the transformed problem for $U(x,t)$

PDE $\quad U_t = \alpha^2 U_{xx}$

BCs $\quad U(0,t) = 0$
$\quad\quad\;\; U(1,t) = 0$

IC $\quad U(x,0) = \underbrace{\sin(2\pi x) + (1+h/1-h)x - 1}_{\phi(x)}$

which has the solution (by separation of variables)

$$U(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$$

Hence the solution of the original problem is

$$u(x,t) = 1 - (1+h/1-h)x + U(x,t)$$

2. The steady state is obviously $S(x,t) = x$ and so we write $u(x,t) = x + U(x,t)$. Substituting this into the problem gives

PDE $\quad U_t = U_{xx}$

BCs $\quad U(0,t) = 0$
$\quad\quad\;\; U(1,t) = 0$

IC $\quad U(x,0) = x^2 - x$

This is the same IBVP as Problem 6, Lesson 5 (except that the IC have opposite signs which means that the two answers have opposite signs). In that lesson we solved the problem by separation of variables. Using that result:

$$U(x,t) = \frac{-8}{\pi^3}\left[e^{-(\pi)^2 t}\sin(\pi x) + \frac{1}{27}e^{-(3\pi)^2 t}\sin(3\pi x) + \cdots\right]$$

and hence:

$$u(x,t) = \underbrace{x}_{\text{steady state}} - \frac{8}{\pi^3}\underbrace{\left[e^{-(\pi)^2 t}\sin(\pi x) + \frac{1}{27}e^{-(3\pi)^2 t}\sin(3\pi x) + \cdots\right]}_{\text{transient}}.$$

3. Try $u(x,t) = S(x,t) + U(x,t)$ where $S(x,t) = A(t)\,x + B(t)\,(1-x)$. Substituting $S(x,t)$ into the BC we get:

$$A(t) - B(t) = 0$$

$$[A(t) - B(t)] + h\,A(t) = 1\ .$$

Solving for $A(t)$, $B(t)$ gives.

$$A(t) = B(t) = 1/h$$

and hence $\quad S(x,t) = \frac{x}{h} + \frac{1}{h}(1-x) = 1/h\ .$

Lesson 7

1. The solution is.

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi/2)^2 t} \sin(n\pi x/2)$$

where (summed over the odd integers)

$$a_n = 2 \int_0^1 x \sin(n\pi x/2)\, dx .$$

2. The general solution of $X'' + \lambda X = 0$ is.

$$X(x) = c_1 \sin\sqrt{\lambda}\, x + c_2 \cos\sqrt{\lambda}\, x.$$

Substituting this expression into the BC gives:

$$X(0) = c_2 = 0$$

$$X'(1) = c_1 \sqrt{\lambda} \cos\sqrt{\lambda} = 0 \implies \cos\sqrt{\lambda} = 0$$

$$\therefore\ \sqrt{\lambda} = (n\pi/2) \quad n = 1,3,5, \cdots$$

$$\therefore\ \lambda = (n\pi/2)^2 \quad n = 1,3,5, \cdots$$

Hence the eigenvectors are.

$$X_n(x) = \sin(n\pi x/2) \qquad n = 1,3, \cdots .$$

Here $p(x) = 1$, $q(x) = 0$ and $r(x) = 1$.

3. Here the eigenvectors are $X_n(x) = \cos(n\pi x)$ $n = 0,1,2, \cdots$ (note that n begins with zero here) and hence the solution is:

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(n\pi)^2 t} \cos(n\pi x)$$

where

$$\begin{cases} a_0 = 1/2 \\ a_n = 2 \int_0^1 x \cos(n\pi x)\, dx = \dfrac{-4}{(n\pi)^2} \qquad n = 1,2, \cdots . \end{cases}$$

That is:

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-(n\pi)^2 t} \cos(n\pi x) .$$

Lesson 8

1. Substituting $u(x,t) = e^{(x-t/2)/2}\, w(x,t)$ into the original IBVP gives:

$$\begin{array}{ll} \text{PDE} & w_t = w_{xx} \\ \text{BCs} & w(0,t) = 0 \\ & w(1,t) = 0 \\ \text{IC} & w(x,0) = 1 \end{array}$$

which has the solution (be separation of variables);

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$a_n = 2\int_0^1 \sin(n\pi x)\,dx = \begin{cases} 0 & n = 2,4,\cdots \\ \frac{4}{n\pi} & n = 1,3,\cdots . \end{cases}$$

Hence:

$$u(x,t) = \frac{4\,e^{-(x-t/2)/2}}{\pi} \sum_{n=1}^{\infty} [1/(2n-1)] \sin[(2n-1)\pi x] .$$

2. Let $u(x,t) = x + U(x,t)$ and by substitution into the IBVP gives:

$$\begin{array}{ll} \text{PDE} & U_t = U_{xx} - U \\ \text{BCs} & U(0,t) = 0 \\ & U(1,t) = 0 \\ \text{IC} & U(x,0) = -x \end{array} .$$

Now let $U(x,t) = e^{-t}\, w(x,t)$ to get:

$$\begin{array}{ll} \text{PDE} & w_t = w_{xx} \\ \text{BCs} & w(0,t) = 0 \\ & w(1,t) = 0 \\ \text{IC} & w(x,0) = -x \end{array}$$

which can be solved by separation of variables:

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$a_n = -2 \int_0^1 x \sin(n\pi x)\, dx = (-1)^n \frac{2}{n\pi} \qquad n = 1,2, \cdots .$$

Hence

$$u(x,t) = x + \frac{2e^{-t}}{\pi} [\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \cdots ] .$$

One could check this answer to see if it satisfies all four conditions (the PDE, two BC and the IC). The major difficulty with this series is that it converges very slowly.

3. The solution can easily be seen to be:

$$u(x,t) = e^{-t} w(x,t)$$

where $w(x,t) = e^{-(\pi)^2 t} \sin(\pi x)$. Hence by solving this problem directly by separation of variables one should arrive at:

$$u(x,t) = e^{-(\pi^2+1)t} \sin(\pi x) .$$

## Lesson 9

1. Here the eigenfunctions of the related homogeneous problem are

$$X_n(x) = \sin(n\pi x)$$

and so:

$$f_n(t) = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases} .$$

To find $u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n x)$ we must solve.

$$\begin{cases} T_1' + \pi^2 T_1 = 0 \\ T_1(0) = 1 \end{cases} \Rightarrow T_1(t) = e^{-\pi^2 t}$$

$$\begin{cases} T_3' + (3\pi)^2 T_3 = 1 \\ T_3(0) = 0 \end{cases} \Rightarrow T_3(t) = \frac{1}{(3\pi)^2} [1 - e^{-(3\pi)^2 t_3}]$$

$$\begin{cases} T_n' + (n\pi)^2 T_n = 0 \\ T_n(0) = 0 \end{cases} \Rightarrow T_n(0) = 0 \quad n = 2,4,5, \cdots .$$

Hence $u(x,t) = e^{-\pi^2 t} \sin(\pi x) + \frac{1}{(3\pi)^2} [1 - e^{-(3\pi)^2 t}] \sin(3\pi x)$.

The temperature profiles look like $\sin(\pi x)$ when $t = 0$ and gradually look more and more like:

$$\frac{1}{(3\pi)^2} \sin(3\pi x) .$$

2. Here

$$T_1(t) = \frac{1}{\pi^2} [1 - e^{-\pi^2 t}]$$

$$T_2(t) = \frac{1}{(2\pi)^2} [1 - e^{-(2\pi)^2 t}]$$

and so

$$u(x,t) = \frac{1}{\pi^2} [1 - e^{-\pi^2 t}] \sin(\pi x) + \frac{1}{(2\pi)^2} [1 - e^{-(2\pi)^2 t}] \sin(2\pi x).$$

3. Here $T_1' + \pi^2 T_1 = 1$

$$T_1(0) = a_1$$

$$T_n' + (n\pi)^2 T_n = 0 \qquad n = 2,3, \cdots$$

$$T_n(0) = a_n$$

where $a_n$ are the coefficients in the expansion for the IC

$$1 = \sum_{n=1}^{\infty} a_n \sin(n\pi x).$$

That is:

$$a_n = \begin{cases} 0 & n = 2,4, \cdots \\ \frac{4}{n\pi} & n = 1,3,5, \cdots . \end{cases}$$

Hence:

$$T_1(t) = \frac{4}{\pi} e^{-\pi^2 t} + \frac{1}{\pi^2} [1 - e^{-\pi^2 t}]$$

$$T_n(t) = \frac{4}{n\pi} e^{-(n\pi)^2 t} \qquad n = 2,3,4, \cdots .$$

Hence:

$$u(x,t) = \{\frac{4}{\pi} e^{-\pi^2 t} + \frac{1}{\pi^2} [1 - e^{-\pi^2 t}]\} \sin(\pi x) + \frac{4}{\pi} \sum_{n=3}^{\infty} \frac{1}{n} e^{-(n\pi)^2 t} \sin(n\pi x) .$$

4. Here the eigenfunctions are $X_n(x) = \sin(\lambda_n x)$ where $\lambda_n$ are the roots of $\tan(\lambda) + \lambda = 0$. Hence $f_1(t) = 1$, $f_n(t) = 0$ $n = 2,3, \cdots$ . Hence

$$u(x,t) = \frac{1}{\lambda_1^2} [1 - e^{-\lambda_1^2 t}] \sin(\lambda_1 x) .$$

5. Let $u(x,t) = S(x,t) + U(x,t)$ where $S(x,t) = A(t)x + B(t)(1 - x)$. Substituting $S(x,t)$ into the BC gives $B(t) = 0$, $A(t) = \cos(t)$ and so $u(x,t) = x\cos(t) + U(x,t)$. Substituting this into the original problem gives.

$$\text{PDE} \quad U_t = U_{xx} + x\sin(t)$$

$$\text{BCs} \quad \begin{array}{l} U(0,t) = 0 \\ U(1,t) = 0 \end{array}$$

$$\text{IC} \quad U(x,0) = 0$$

and expanding the right hand side

$$x\sin(t) = f_1(t)\sin(\pi x) + f_2(t)\sin(2\pi x) + \cdots$$

gives:

$$f_n(t) = 2\int_0^1 x\sin(t)\sin(m\pi x)\,dx$$

$$= (-1)^{n+1}\frac{2}{n\pi}\sin(t) \qquad n = 1,2,3,\cdots$$

From this the solution is:

$$U(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$

where

$$\begin{cases} T_n' + (n\pi)^2\, T_n = (-1)^n \dfrac{2}{n\pi}\sin(t) \\ T_n(0) = 0 \end{cases}$$

Solving this gives:

$$T_n(t) = (-1)^n \frac{2}{n\pi}\int_0^1 e^{-(n\pi)^2(t-\tau)}\sin(\tau)\,d\tau$$

= this can be evaluated

and thus:

$$u(x,t) = x\cos(t) + \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) .$$

*Lesson 10*

1. Straightforward integration by parts.

2. Multiply each side of the equation by the integration factor

$$\mu(t) = e^{\omega^2\alpha^2 t}$$

and then integrate the equation.

3. Taking the cosine transform (the sine transform doesn't work for this problem because of the derivative BC) of the PBVP we have

$$\frac{dU}{dt} = \alpha^2[-\frac{2}{\pi} u_x(0,t) - \omega^2 U(t)] = -\omega^2\alpha^2 U(t)$$

$$U(0) = \frac{2}{\pi}\int_0^\infty H(1-x)\cos(\omega x)\,dx = \frac{2}{\pi}\int_0^1 \cos(\omega x)\,dx$$

$$= \frac{2\sin\omega}{\pi\omega}$$

where $U(t) = \mathfrak{F}_c\{u\}$. Hence we must solve

$$\frac{dU}{dt} + \alpha^2\omega^2 U = 0$$

$$U(0) = 2\sin\omega/\pi\omega$$

which has the solution

$$U(t) = \frac{2}{\pi\omega}\sin\omega\, e^{-(\alpha\omega)^2 t}$$

and so the solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega}\sin\omega\, e^{-(\alpha\omega)^2 t}\sin(\omega x)\,d\omega$$

One must either find the inverse cosine transform of U(t) from the tables or else evaluate the integral for u(x,t) numerically on a computer.

Lesson 11

1. $a_n = 0 \qquad b_n = \begin{cases} 0 & n = 0,2,4,\cdots \\ \frac{4}{n\pi} & n = 1,3,5,\cdots \end{cases}$

and hence $f(x) = \frac{4}{\pi}\left[\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x) + \cdots\right].$

2. The sawtooth curve is represented by:

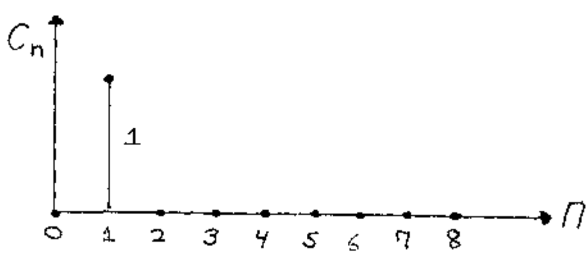
$$f(x) = \frac{2L}{\pi}\left[\sin(\pi x/L) - \frac{1}{2}\sin(2\pi x/L) + \frac{1}{3}\sin(3\pi x/L) - \cdots\right].$$

Differentiating each side of this expression gives.

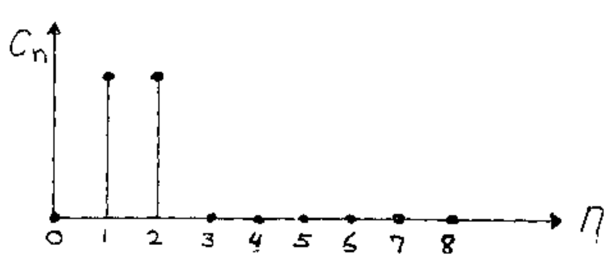
$$f'(x) = 2\left[\cos(\pi x/L) - \cos(2\pi x/L) + \cos(3\pi x/L) - \cdots\right].$$

Unfortunately this equation doesn't make any sense because $f'(x)$ (being the derivative of the sawtooth curve) is zero except at the jump points while the right hand side (the cosine series) will not converge to anything. In other words it is possible to differentiate an infinite series which represents a function term by term and get a new series that doesn't represent the derivative $f'(x)$.

3. a)




b)



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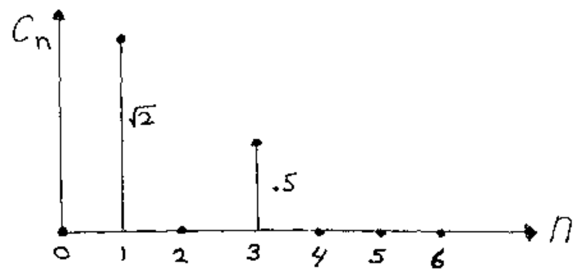



4. $|F(\xi)| = \sqrt{\frac{2}{\pi}} \left|\frac{\sin(\xi)}{\xi}\right|$, $F(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi}$

5. $F(\xi) = \frac{1}{1 + i\xi} = \frac{1 - i\xi}{(1 + i\xi)(1 - i\xi)} = \frac{1 - i\xi}{1 + \xi^2} \Rightarrow |F(\xi)| = \sqrt{\frac{1}{1 + \xi^2}}$.

Lesson 12

1. Straightforward integration.

2. Elementary properties of integrals.

3. The transformed problem is

$$\frac{dU}{dt} = -\alpha^2\xi^2 U$$
$$U(0) = \frac{1}{\sqrt{2}} e^{-(\xi/2)^2}$$

which has the solution:

$$U(\xi,t) = \frac{1}{\sqrt{2}} e^{-(\xi/2)^2} e^{-\alpha^2\xi^2 t}$$
$$= \frac{1}{\sqrt{2}} e^{-(\alpha^2 t + .25)\xi^2}$$

which has the inverse transform:

$$u(x,t) = \frac{1}{\sqrt{4\alpha^2 t + 1}} e^{-[x^2/(4\alpha^2 t + 1)]} .$$

One can check the answer.

4. Use integration by parts.

Lesson 13

1. Integrate by parts.

2. Taking the Laplace transform of the PDE we have:

$$sU - \sin(x) = \alpha^2 \frac{d^2U}{dx^2}$$

which has the general solution:

$$U(x) = c_1 e^{\sqrt{s}\, x/\alpha} + c_2 e^{-\sqrt{s}\, x/\alpha} + \frac{\sin(x)}{s + \alpha^2} .$$

Since $U(x)$ must be bounded (or else $u(x,t)$ will be unbounded as $|x| \to \infty$) we have $c_1 = c_2 = 0$ and so

$$U(x) = \frac{\sin(x)}{s + \alpha^2} .$$

Finding the inverse gives.

$$u(x,t) = \mathcal{L}^{-1}\{U(x,s)\} = \sin(x)\, \mathcal{L}^{-1}\left\{\frac{1}{s + \alpha^2}\right\}$$

$$= \sin(x)\, e^{-\alpha^2 t} .$$

One should check this answer.

3. The transformed problem is.

$$\frac{d^2U}{dx^2} - sU = 0$$

$$U(0) = \frac{1}{s^2 + 1}$$

which has the solution:

$$U(x,s) = \frac{1}{s^2 + 1} e^{-x\sqrt{s}} .$$

Taking the inverse Laplace transform (using the convolution principle):

$$u(x,t) = \sin(t) * \frac{\alpha x}{2\sqrt{\pi t^3}} e^{-\alpha^2 x/4t} .$$

4. $U(s,x) = \frac{A}{s}\left\{\frac{\cosh \sqrt{s}\, x}{\cosh \sqrt{s}} - 1\right\}$

Lesson 14

1. Proof is exactly like the one in the lesson except one leaves out the step of multiplying and dividing by s.

2. It is clear that when we differentiate the series term by term with respect to t we get a series whose terms get larger and larger

3. Duhamel's Principle says [w(x,t) is the response due to BC u(1,t) = δ(t)]:

$$u(x,t) = \int_0^t w(x,t-\tau)\, f(\tau)\, d\tau$$

hence the temperature response at the point $u(.5, t_n)$ can be approximated by some numerical rule (say the trapezoidal rule). That is:

$$u(.5, t_n) = \int_0^{t_n} w(.5, t_n-\tau)\, f(\tau)\, d\tau$$

$$= \frac{t_1}{2}\,[w_n f(t_1) + 2w_{n-1} f(t_2) + \cdots + 2w_2 f(t_{n-1}) + w_1 f(t_n)].$$

4. By Duhamel's Principle

$$u(x,t) = \int_0^t w(x,t-\tau)\cos(\tau)\, d\tau$$

where

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, e^{-(n\pi)^2 t}\sin(n\pi x) .$$

Lesson 15

1. The transformation $\tau = t$, $\xi = x - 2t$ gives the new problem.

$$u_\tau = u_{\xi\xi}$$

$$u(\xi,0) = \sin(\xi)$$

which has solution $u(\xi,\tau) = e^{-\tau}\sin(\xi)$. Hence the solution to our problem is:

$$u(x,t) = e^{-t}\sin(x - 2t) \; .$$

One should check the solution.

2. The transformation $u(x,t) = e^{(x-t)}\,w(x,t)$ gives the new problem:

$$w_t = w_{xx}$$

$$w(x,0) = \sin(x)$$

which has the solution:

$$w(x,t) = e^{-t}\sin(x) \; .$$

Hence $u(x,t) = e^{(x-t)}\,e^{-t}\sin(x) = e^{(x-2t)}\sin(x)$ .

3. $u(x,t) = e^{-(x-2t)^2}$

4. Evaluate the integral in the hint and substitute $\xi = x - vt$, $\tau = t$. One gets:

$$u(x,t) = \frac{1}{\sqrt{1 + 4Dt}}\, e^{-(x-Vt)^2/(1+4Dt)}$$

## Lesson 16

1. Differentiate the first equation of (16.3) with respect to $t$ and the second with respect to $x$. Multiplying the first by $L$ and subtracting gives:

$$LC\, v_{tt} + LG\, v_t - v_{xx} - R\, i_x = 0 \,.$$

Now using the first equation of (16.3) solve for $i_x$ and substitute into the above equation. This gives the desired result:

$$v_{xx} - LC\, v_{tt} + (RC + LG)\, v_t + RG\, v = 0 \,.$$

2. The wave initially looks like $\sin(\pi x)$ but vibrates to zero due to the friction term $-u_t$.

3. This problem describes the motion of a vibrating string (initially at rest) that is being given simple harmonic motion at the right hand side. The motion is complicated but one can imagine the waves moving down the string from right to left--bouncing off the left hand end (which is fixed at zero) and coming back (only to collide with waves coming from the other direction). One can solve this problem analytically by either the Laplace transform or Duhamel's principle.

4. Substituting $u(x,t) = e^{ax+bt}$ into the wave equation $u_{xx} = u_{tt}$ one has $b^2 = a^2$ or $b = \pm a$. Hence

$$u(x,t) = e^{a(x-t)} \quad \text{and} \quad u(x,t) = e^{a(x+t)}$$

are solutions for any constant $a$. The sum is also a solution and so we have more solutions:

$$u(x,t) = c_1\, e^{a(x-t)} + c_2\, e^{a(x,t)} \,.$$

Later we will see that the wave equation has more solutions than this.

Lesson 17

1. Straightforward substitution. The reader should use the general formula for differentiation where the limits are variables:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(s)\, ds = \frac{db(x)}{dx}\, g[b(x)] - \frac{da(x)}{dx}\, g[a(x)].$$

2. Straightforward functional substitution.

3. $u(x,t) = \frac{1}{2}\,[e^{-(x-t)^2} + e^{-(x+t)^2}]$.

4. $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} u_t(\xi,0)\, d\xi = \frac{1}{2}\,[e^{-(x-t)^2} - e^{-(x+t)^2}]$.

5. Straightforward algebra.

Lesson 18

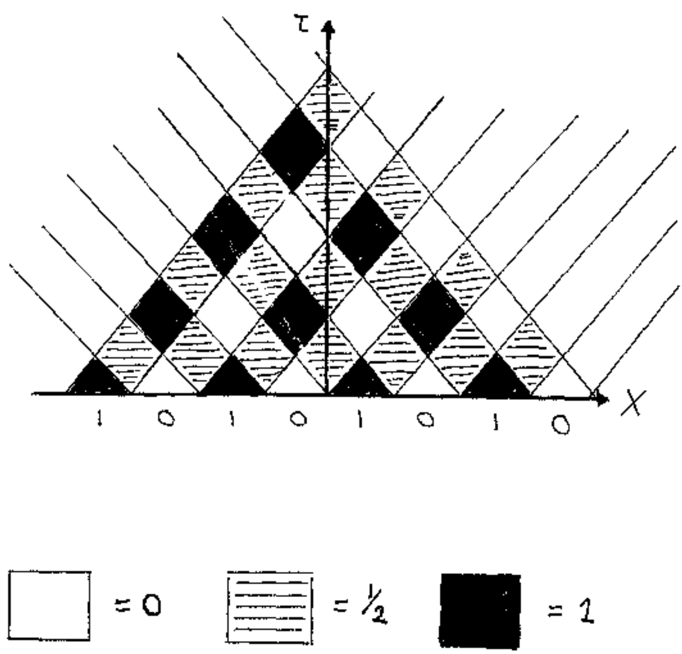
1. Apply equation (18.8) in the text with $f(x) = xe^{-x^2}$, $g(x)=0$.

2. The real wave $(x > 0)$ and the imaginary wave $(x < 0)$ will cancel each other as they pass through each other. The real interpretation is that the wave hitting the boundary at $x = 0$ will reflect and cancel with the incoming wave.

3.

$$u(x,t) = \begin{cases} \frac{1}{2}\,[f(x-ct) + f(x+ct)] & x > ct \\ \frac{1}{2}\,[f(x+ct) + f(ct-x)] & x < ct \end{cases}$$

4.



## Lesson 19

The reader can draw his/her own graphs for this lesson. The only thing to check is that all the boundary conditions of the graphs agree with the problem and that the string should have a downward force whenever the concavity is negative.

Lesson 20

1. Observe that the second term vibrates three times faster than the first term. The solution is periodic with period (fundamental frequency) $T = 2L$.

2. $u(x,t) = \left[\frac{L}{3\pi\alpha}\right] \sin(3\pi x/L) \sin(3\pi\alpha t/L)$ .

3. For $\lambda > 0$ if we substitute $X(x)$ into the BC $X(0) = X(L) = 0$ we have $X(x)$ is identically zero. Also note that the solution $T(t)$ is not periodic in time if $\lambda \geq 0$ which must be the case in our problem.

4. $u(x,t) = \sin(3\pi x/L) [\sin(3\pi\alpha t/L) + \cos(3\pi\alpha t/L)]$ .

5.
$$a_n = \begin{cases} (-1)^{n+1} \dfrac{2h}{(n\pi)^2} & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

$b_n = 0$ all $n$ .

Hence $u(x,t) = \dfrac{4h}{\pi^2} \sin(\pi x) \cos(\pi t) - \dfrac{4h}{9\pi^2} \sin(3\pi x) \sin(3\pi t)$

$$+ \frac{4h}{16\pi^2} \sin(5\pi x) \sin(5\pi t) + \cdots .$$

Note that the higher frequencies are multiples of the fundamental frequency. This will give some indication of what the string looks like for $t > 0$ .

Lesson 21

1. After separation of variables we have:

$$T'' + \lambda^2 T = 0$$

$$X^{(iv)} - \lambda^2 X = 0$$

which have solutions:

$T(t) = A \sin(\lambda t) + B \cos(\lambda t)$

$X(x) = C \sin(\sqrt{\lambda} t) + D \cos(\sqrt{\lambda} t) + E \sinh(\sqrt{\lambda} t) + F \cosh(\sqrt{\lambda} t).$

Substituting these into the BC gives.

$$u(x,t) \sum_{n=1}^{\infty} X_n(x) [a_n \sin(\lambda_n t) + b_n \cos(\lambda_n t)]$$

where

$X_n(x) = C_n \sin(\sqrt{\lambda_n} x) + D_n \cos(\sqrt{\lambda_n} x) + E_n \sinh(\sqrt{\lambda_n} x) + F_n \cosh(\sqrt{\lambda_n} x)$

and

$$C_n = -E_n = -[\cos(\sqrt{\lambda_n}) + \cosh(\sqrt{\lambda_n})]$$

$$D_n = -F_n = [\sin(\sqrt{\lambda_n}) + \sinh(\sqrt{\lambda_n})]$$

and $\lambda_n$ are the roots of the equation.

$$\begin{vmatrix} \cosh x + \cos x & \sinh x + \sin x \\ \sinh x - \sin x & \cosh x + \cos x \end{vmatrix} = 0 .$$

The $\lambda_n$ constitute a sequence of numbers converging to infinity (found numerically). Finally to find the coefficients $a_n$ and $b_n$ we substitute $u(x,t)$ into the IC to get:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n X_n(x)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \lambda_n a_n X_n(x) .$$

The Sturm-Liouville theory (extended to fourth order) says that the $X_n(x)$ are orthogonal and satisfy:

$$\int_0^1 X_m(x)\, X_n(x)\, dx \begin{cases} = 0 & m \neq n \\ \neq 0 & m = n \end{cases} .$$

Hence we can find the coefficients $a_n$, $b_n$ from the equations:

$$a_n = \frac{\frac{1}{\lambda_n}\int_0^1 g(x)\, X_n(x)\, dx}{\int_0^1 X_n^2(x)\, dx}, \quad b_n \frac{\int_0^1 f(x)\, X_n(x)\, dx}{\int_0^1 X_n^2(x)\, dx} .$$

These, of course, would have to be found numerically with a computer.

2. $u(x,t) = \sin(\pi x)\, [\cos(\pi^2 t) + \frac{1}{\pi^2}\sin(\pi^2 t)]$ .

3. $u(x,t) = \sum_{n=1}^{\infty} b_n \cos(n\pi)^2 t \sin(n\pi x)$

where

$$b_n = 2\int_0^1 (1 - x^2)\sin(n\pi x)\, dx .$$

4. The natural frequencies are the eigenvalues of:

$$X^{(iv)} - \lambda^2 X = 0$$
$$X(0) = 0$$
$$X'(0) = 0$$
$$X(1) = 0$$
$$X''(1) = 0 \quad .$$

The general solution of $X^{(iv)} - \lambda^2 X = 0$ is:

$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) + C\sinh(\sqrt{\lambda}x) + D\cosh(\sqrt{\lambda}x)$

and substitution into the IC gives:

$$\begin{vmatrix} \sin(\sqrt{\lambda}) - \sinh(\sqrt{\lambda}) & \cos(\sqrt{\lambda}) - \cosh(\sqrt{\lambda}) \\ -[\sin(\sqrt{\lambda}) + \sinh(\sqrt{\lambda})] & -[\cos(\sqrt{\lambda}) + \cosh(\sqrt{\lambda})] \end{vmatrix} = 0 .$$

Hence the natural frequencies are the roots of the above equation.

Lesson 22

1. If $\tau = (\alpha/L)t$ and $\xi = x/L$ we have.

$$u_t = u_\tau \tau_t = (\alpha/L)\, u_\tau$$
$$u_{tt} = (\alpha/L)^2\, u_{\tau\tau}$$
$$u_x = u_\xi \xi_x = (1/L)\, u_\xi$$
$$u_{xx} = (1/L^2)\, u_{\xi\xi} \quad .$$

Hence $u_{tt} = \alpha^2 u_{xx}$ becomes $u_{\tau\tau} = u_{\xi\xi}$. The BC and the IC are transformed in the same way.

2. Let $U(x,t) = \frac{1}{T_1} u(x,t)$

$$\xi = (1/L)\, x$$
$$\tau = (\alpha/L)^2 t \; .$$

From these transformations we get

| | | |
|---|---|---|
| PDE | $U_\tau = U_{\xi\xi}$ | $0 < \xi < 1 \quad 0 < \xi < \infty$ |
| BCs | $U(0,\tau) = 1$ | $0 < \tau < \infty$ |
| | $U(1,\tau) = 0$ | |
| IC | $U(\xi,0) = T_2/T_1$ | $0 \le \xi \le 1$ . |

3. Straightforward substitution.

4. With respect to the new time scale $\tau = \alpha t$ the velocity of the wave is one.

5. Try $\xi = kx$. Hence $u_x = u_\xi \xi_x = k\, u_\xi$. If we pick $k = 1/v$ we get the new equation:

$$u_t + vu_x = u_t + v\frac{1}{v}u_\xi = u_t + u_\xi = 0 \; .$$

Lesson 23

1. a) elliptic
   b) hyperbolic
   c) elliptic
   d) elliptic
   e) elliptic

2. Use the chain rule.

3. $B^2 - 4AC = 25$. The characteristic equations are:

$$\frac{dy}{dx} = 1/2$$

$$\frac{dy}{dx} = 3$$

and so the characteristic coordinates are:

$$\xi = y - \frac{x}{2}$$

$$\eta = y - 3x .$$

4. The canonical equation is $u_{\xi\eta} = 0$ .

5. The alternate canonical equation is $u_{\alpha\alpha} - u_{\beta\beta} = 0$ .

Lesson 24

1. If we differentiate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \phi(s)\,ds \quad \text{with respect to } t,$$

using Leibnitz rule (see problem 7) we get
$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$.

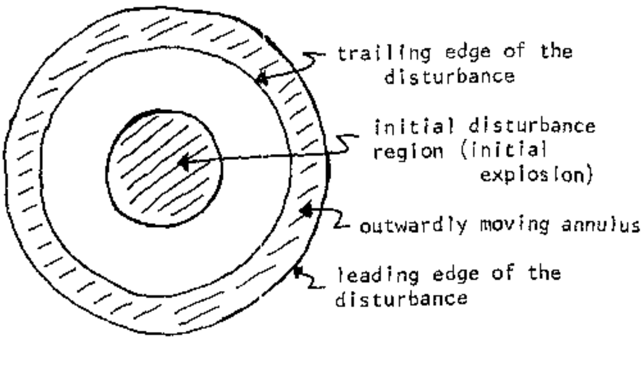
2. Interchanging the IC, the solution to

$$u_{tt} = u_{xx}$$
$$u(x,0) = 0$$
$$u_t(x,0) = x$$

is $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \xi\, d\xi = xt$. Hence if we differentiate this with respect to $t$ we get $u = x$ which is the solution to

$$u_{tt} = u_{xx}$$
$$u(x,0) = x$$
$$u_t(x,0) = 0\ .$$

3. Since the solution is $u(x,y,z,t) = t\,\bar{\psi}$ one can see the initial disturbance region (which is the unit sphere) gives rise to an outwardly propagating annulus. Each point $(x,y,z)$ outside the unit sphere will suddenly experience a distribution--will be in a disturbed state for a length of time 2/x--then $u(x,y,z,t) \equiv 0$.



4. In this case each point $(x,y)$ outside the unit circle will suddenly experience a shock and as time gets large the solution $u(x,y,t)$ gradually goes to zero. The solution would look more or less like a circular wave after dropping a pebble in a lake.

5. The solution here is the D'Alembert solution

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} u_t(s,0)\, ds$$

and has been solved in Lesson 18.

6. Imagine a single initial disturbance at the origin $(0,0)$. Since Huygen's principle doesn't hold in two dimensions this means that after a given length of time any other point $(x,y)$ will suddenly experience a shock due to this disturbance and then gradually die out as time gets large. The reason the shock gradually goes to zero is because the origin in two dimensions can physically be interpreted as an infinite line perpendicular to the xy plane and the points in the xy plane are in reality being disturbed by all the points along this line. As time increases all points along this line will be disturbing the point $(x,y)$ and as time increases it is the points further away that are doing the disturbing.

7. Leibnitz's rule tells how to differentiate an integral with respect to $t$ where the limits and the integrand can depend on $t$. Direct application of the formula gives:

$$\frac{d}{dt} \int_{x-ct}^{x+ct} \phi(s)\, ds = c\,[\phi(x+ct) + \phi(x-ct)] .$$

Lesson 25

1. Solve for $A_n = C[u]$ from.

$$\frac{dA_n(t)}{dt} + (n\pi)^2 A_n(t) = 0$$

$$A_n(0) = \begin{cases} 1 & n = 0,1 \\ 0 & n = 2,4,5,6,\cdots \\ .5 & n = 3 \end{cases}$$

Hence
$$A_0(t) = 1$$
$$A_1(t) = e^{-\pi^2 t}$$
$$A_n(t) = 0 \qquad n = 2,4,5,6,\cdots$$
$$A_3(t) = \frac{1}{2} e^{-(3\pi)^2 t}$$

Hence
$$u(x,t) = 1 + e^{-\pi^2 t} \cos(\pi x) + .5\, e^{-(3\pi)^2 t} \cos(3\pi x) .$$

2. Solve the problem(s):

$$\frac{dB_n(t)}{dt} + (n\pi\alpha)^2 B_n(t) = F_n(t)$$

$$F_n(0) = 0$$

to get:

$$B_n(t) = e^{-(n\pi\alpha)^2 t} \int_0^t e^{(n\pi\alpha)^2 s} F_n(s)\, ds .$$

Hence the solution is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin(n\pi x) .$$

3. Integrate by parts.

4.
$$B_n = \begin{cases} 1 & n = 1 \\ .5 & n = 3 \\ 0 & \text{all other } n\text{'s} \end{cases} .$$

5. $A_n = 2 \int_0^1 x \cos(n\pi x)\, dx$ = integration by parts.

6. Solve:

$$\frac{dB_n(t)}{dt} + (n\pi)^2 B_n(t) = \begin{cases} 8\pi + 1 & n = 3 \\ (-1)^{n+1} (2n\pi) & n \neq 3 \end{cases}$$

$$B_n(0) = \begin{cases} 1 & n = 1 \\ 0 & n = 2, 3, \cdots \end{cases}$$

for $B_n(t)$ and then substitute into:

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin(n\pi x) .$$

Lesson 26

1. Direct substitution.

2. The problem with the nonhomogeneous PDE (and homogeneous IC) can be solved by the finite Sine transform to get:

$$u_1(x,t) = -\frac{1}{(3\pi)^2}[\cos(3\pi t) - 1].$$

The solution of the homogeneous PDE (and nonhomogeneous IC) can be solved by Separation of Variables to get:

$$u_2(x,t) = e^{-\pi^2 t}\sin(\pi x) .$$

Hence the solution $u(x,t)$ of our problem is the sum of these two functions.

3. a) $u_1 + u_2$ is a solution (the equation is linear and homogeneous).

   b) $u_1 + u_2$ is not a solution (the equation is not homogeneous).

   c) $u_1 + u_2$ is a solution (the equation is linear and homogeneous).

   d) $u_1 + u_2$ is not a solution (the equation is not linear).

4. In each of the four subproblems let one of the four equations be nonhomogeneous and the other three homogeneous.

5. Multiply each side of the equation by the integrating factor:

$$\mu(t) = e^{-(n\pi)^2 t}$$

to get the solution:

$$U_n(t) = e^{-(n\pi)^2 t}\int_0^t e^{(n\pi)^2 s} F_n(s)\, ds .$$

6. Yes--this can be verified by direct substitution.

Lesson 27

1. The solution is a moving cosine curve $u(x,t) = \cos(x - t)$.

2. $u(x,t) = \frac{1}{t^2} \sin(x/t)$

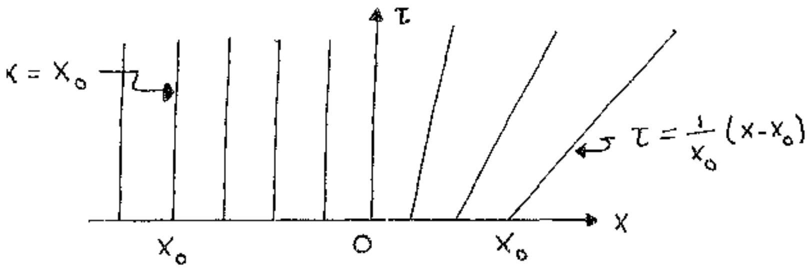
3. $u(x,y,t) = e^{-[(x-at/c)^2 + (y-bt/c)^2] - dt/c}$

4. $u(x,t) = F(x - t)\, e^{-t^2/2}$

5. $u(x,t) = F(2x-y,\ 2x-y)\, e^{-2(y-x)}$

esson 28

. It is best to draw the characteristic curve starting from each initial point $x_o$.



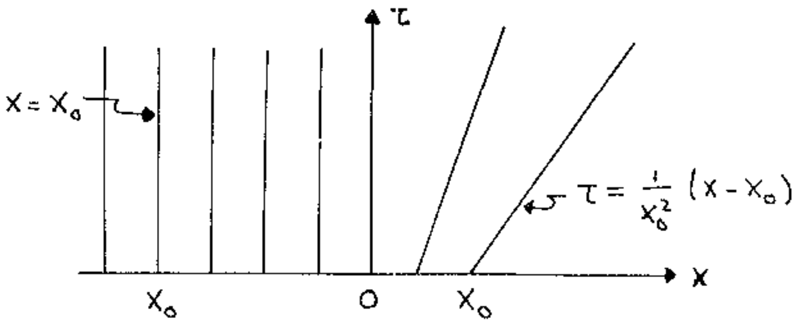

a) if $x_o < 0$, then the characteristic curve is $x - g(u)t = x - c$. To find the constant for the curve at $(x_o, 0)$ we let $\tau = 0$ and find $c = x_o$. Hence the characteristic curve starting from $x_o$ is:

$$x = x_o$$

b) if $x_o > 0$, then the characteristic curve is $x - g(u)t = x - x_o t = c$ To find the constant for the curve starting at $(x_o, 0)$ we let $t = 0$ and find $c = x_o$. Hence the characteristic curve starting from $x_o$ is.

$$t = \frac{1}{x_o}(x - x_o) \; .$$

2. $f(u) = \frac{1}{3} u^3$




3. The problem we must solve is:

$$\text{PDE} \qquad u_x + u_t + ku = 0$$

$$\text{IC} \qquad u(x,0) = \phi(x)$$

which has the solution:

$$u(x,t) = \phi(x - t)\, e^{-kt} \, .$$

4. The problem we must solve is:

$$\text{PDE} \qquad u_x + u_t + \frac{1}{t+1} = 0$$

$$\text{IC} \qquad u(x,0) = \phi(x)$$

which has the solution:

$$u(x,t) = -\ln(t+1) + \phi(x - t) \, .$$

5. Given $u = \phi(x - g(u)t)$ differentiate each side of the equation with respect to $t$ and $x$

$$u_t = \phi'[-g - g_u u_t t] \qquad u_x = \phi'[1 - g_u u_x t] \, .$$

Substituting into $u_t + g(u)u_x$ we get:

$$u_t + g(u)u_x = \phi'[-g - g_u u_t t + g - g g_u u_x t]$$

$$= -t \phi' g_u [u_t + g u_x] = 0 \, .$$

Lesson 29

1. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -c^2 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -b & 0 & -a \end{bmatrix}$

2. $\lambda_1 = 3$ $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda_2 = -1$ $X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

3. The transformation $u = Pv$ will transform $u_t + Au_x = 0$ into the equation: $v_t + \Lambda v_x = 0$ where:

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence the new equations are:

$$\frac{\partial v_1}{\partial t} + 3\frac{\partial v_1}{\partial x} = 0 \quad \Longrightarrow \quad v_1(x,t) = \phi(x - 3t)$$

$$\frac{\partial v_2}{\partial t} - \frac{\partial v_2}{\partial x} = 0 \quad \Longrightarrow \quad v_2(x,t) = \psi(x + t)$$

Multiplying $Pv$ we have:

$$u_1(x,t) = \phi(x - 3t) + \psi(x + t)$$

$$u_2(x,t) = 2\phi(x - 3t) - 2\psi(x + t) \ .$$

4. Direct substitution.

5. Direct substitution.

Lesson 30

1. Direct substitution.

2. Direct substitution.

3. The solution is:

$$u(r,t) = \sum_{m=1}^{\infty} A_m J_o(k_{om} r) \cos(k_{om} t)$$

where

$$A_m = 2 \int_0^1 r(1-r^2) J_o(k_{om} r)\, dr \,/\, J_1^2(k_{om}) .$$

These integrals would have to be evaluated numerically on a computer. The value of $J_1^2(k_{om})$ can be found in tables. For example $J_1^2(k_{o1}) = J_1^2(2.4)$ which can be found in most tables. One would not have to look very hard to find an existing computer program to evaluate these coefficients. Most computer libraries would contain a program to find the expansion:

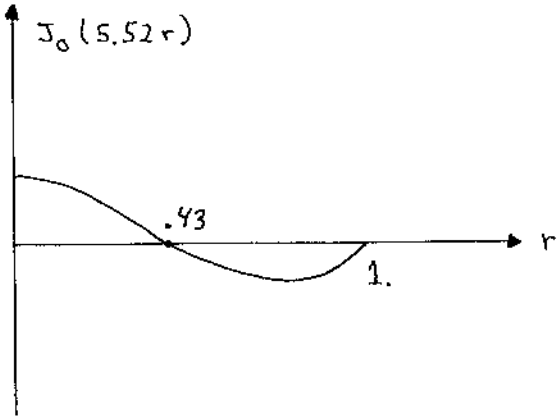
$$f(r) = \sum_{n=1}^{\infty} A_n J_o(k_{on} r)$$

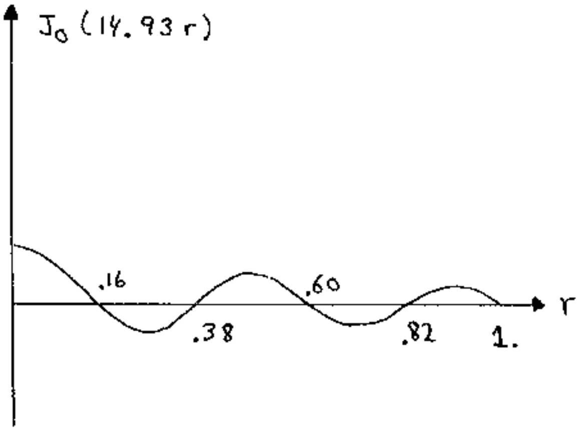
for arbitrary functions $f(r)$ $0 \le r \le 1$. The reader can consult his/her computer center and inquire about applied mathematics packages.

4. $u(r,t) = J_o(2.4r) \cos(2.4t)$

5. $u(r,t) = J_o(2.4r) \cos(2.4t) - .5\, J_o(8.65r) \cos(8.65t)$

$$+ .25\, J_o(14.93r) \cos(14.93t)$$


J₀ (5.52 r)
.43
r
1.


J₀ (14.93 r)
.16
.60
r
.38
.82
1.

Lesson 31

1. $u_{tt} = \alpha^2 [u_{rr} + \frac{2}{r} u_r]$

2. $u_{tt} = \alpha^2 [u_{rr} + \frac{1}{r} u_r]$

3. $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0$ is Laplace's equation that depends only on $r$. If we let $v = \frac{du}{dr}$ this equation becomes $\frac{dv}{dr} + \frac{1}{r} v = 0$ which has the solution $v(r) = c \frac{1}{r}$. Hence the general solution to Laplace's equation is

$$u(r) = A + B \frac{1}{r} \quad .$$

4. $u(r) = A + B/r$ .

5. Since $u_{xx} + u_{yy} = 0$ for all $x$ and $y$, each point on the surface $u(x,y) = xy$ is equal to the average of its neighbors on a circle around $(x,y)$.

6. Straightforward substitution.

Lesson 32

1. The $r$ dependence in the solution is linear and the solution turns out to be $u(r,\theta) = r\sin(\theta)$. There is no reason to believe a person should have this kind of intuition but thinking what the solution should look like is useful. The reader should imagine what this surface looks like.

2. No since the net flux is not zero.

3. Just remember the solution $u$ doesn't have to be equal to $\sin(\theta)$ on the boundary but if $u$ is less (greater) than $\sin(\theta)$ heat will flow in (out) so that the steady state solution will tend to follow $\sin(\theta)$. If $h$ is big the solution $u$ on the boundary will for all practical purposes be equal to $\sin(\theta)$.

4. Setting $u_t$ and $u_{tt}$ equal to zero we get.

$$\frac{d^2u}{dx^2} + u = 0$$

$$u(0) = 0$$

$$u(1) = 0$$

which has the solution $u(x) = 0$. This is not surprising since the PDE describes a vibrating string under the influence of a friction term $-u_t$. The solution to this problem starts at $\sin(3\pi x)$ and vibrates to zero.

5. Can we find surfaces $u(r,\theta)$ and constants $\lambda$ so that at each point $(r,\theta)$ the Laplacian (measures the degree of curvature of the surface) is proportional to $-u$? The answer is yes if we pick the proportionality $\lambda$ appropriately (roots of Bessel functions). In fact we saw what those surfaces $U_{mn}(r,\theta)$ looked like when we studied the circular vibrating membrane (Lesson 30).

6. We are trying to find the steady state solution (say temperature) inside a square that is insulated on the two lateral sides and the top is fixed at one while the bottom is in contact with a medium of two degrees (or whatever). One can imagine the direction of the flow of heat (from bottom to top).

## Lesson 33

1. Direct computation.

2. a) $u(r,\theta) = 1 + r\sin(\theta) + \frac{r}{2}\cos(\theta)$

   b) $u(r,\theta) = 2$

   c) $u(r,\theta) = r\sin(\theta)$

   d) $u(r,\theta) = r^3\sin(3\theta)$

3. $u(r,\theta) = \frac{r}{2}\sin(\theta)$

4. $u(r,\theta) = \frac{r^2}{4}\sin(2\theta)$

5. $u(r,\theta) = \sum_{n=0}^{\infty} r^n\,[a_n\cos(n\theta) + b_n\sin(n\theta)]$

   where

$$a_0 = 1/\pi$$

$$a_n = \begin{cases} 0 & n = 1,3,5,\cdots \\ \dfrac{-2}{\pi(n^2-1)} & n = 2,4,6,\cdots \end{cases}$$

$$b_n = \begin{cases} 1/2 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

   Hence

$$u(r,\theta) = \frac{r}{2}\sin(\theta) + \frac{2}{\pi}\left[\frac{1}{2} - \frac{r^2}{3}\cos(2\theta) - \frac{r^4}{15}\cos(4\theta) - \frac{r^6}{35}\cos(6\theta) - \cdots\right].$$

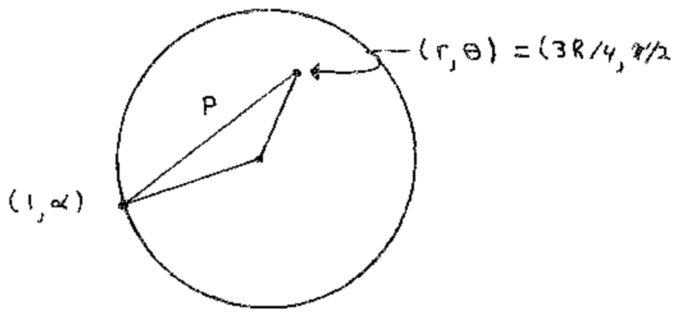
   It will be helpful to use the formulas.

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

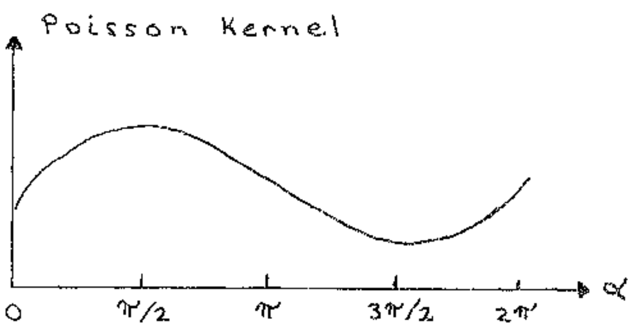
$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)].$$

6. POISSON KERNEL $= \dfrac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\alpha) + r^2} = \dfrac{7R}{16P^2}$

where $P$ is the distance from $(r,\theta)$ to $(1,\alpha)$.




Hence the graph of the Poisson kernel will look like:




7. Direct substitution of $u$ into the problem.

8. Direct substitution of $u$ into the problem.

Lesson 34

1. $u(r,\theta) = [-\frac{r}{3} + \frac{4}{3r}]\cos(\theta) + [\frac{2r}{3} - \frac{2}{3r}]\sin(\theta)$.

2. a) $u(r,\theta) = 1$

   b) $u(r,\theta) = 1 + \frac{1}{r^3}\cos(3\theta)$

   c) $u(r,\theta) = \frac{1}{r}\sin(\theta) + \frac{1}{r^3}\cos(3\theta)$

   d) $u(r,\theta) = \frac{1}{2} + \frac{2}{\pi}[\frac{1}{r}\sin(\theta) + \frac{1}{3r^3}\sin(3\theta) + \frac{1}{5r^5}\sin(5\theta) + \cdots]$

3. $u(r,\theta) = -\frac{1}{r}\sin(\theta)$

4. Direct substitution.

Lesson 35

1. Direct substitution.

2. Use the chain rule $\frac{d\Phi}{d\phi} = \frac{d\Phi}{dx}\frac{dx}{d\phi} = -\sin(\phi)\,\frac{d\Phi}{dx} \qquad x = \cos(\phi)$.

3. Direct substitution.

4. $$\cos(3\phi) = 4\cos^3(\phi) - 3\cos(\phi)$$
$$= (8/5)\,\frac{1}{2}\,[5\cos^3(\phi) - 3\cos(\phi)] - \frac{3}{5}\cos(\phi)$$
$$= \frac{8}{5}\,P_3(\cos\phi) - \frac{3}{5}\,P_1\,(\cos\phi).$$

Hence
$$u(r,\phi) = \frac{8}{5}\,r^3\,P_3(\cos\phi) - \frac{3r}{5}\,P_1(\cos\phi)$$
$$= \frac{4}{5}\,r^3\,[5\cos^3(\phi) - 3\cos(\phi)] - \frac{3r}{5}\cos(\phi)\ .$$

5. $$u(r,\phi) = \sum_{n=0}^{\infty} a_n\,r^n\,P_n(\cos\phi)$$

where
$$a_n = \frac{2n+1}{2}\int_0^{\pi/2} P_n(\cos\phi)\,\sin(\phi)\,d\phi$$
$$- \frac{2n+1}{2}\int_{\pi/2}^{\pi} P_n(\cos\phi)\,\sin(\phi)\,d\phi\ .$$

These integrals can be evaluated but one might want to use a computer to evaluate them numerically.

6. $$u(r,\phi) = \frac{1}{r} + \frac{1}{r^2}\cos(\phi).$$

Lesson 36

1. The total outward flux across the sphere of radius $r$ is $-4\pi r^2 u_r$. Hence if we set $-4\pi r^2 u_r = q$ we have

$$u_r = \frac{-q}{4\pi r^2}$$

and so we have

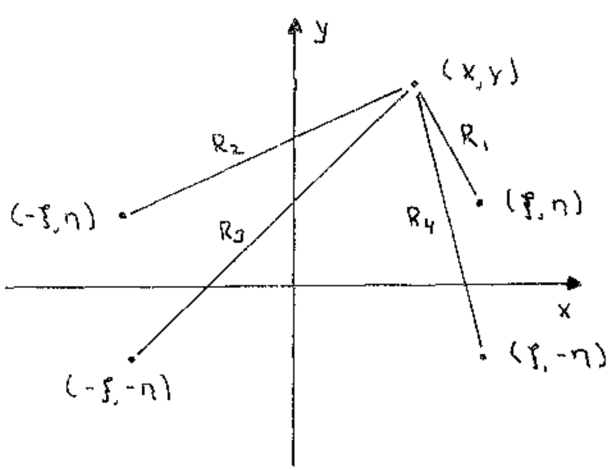
$$u(r) = \frac{q}{4\pi r} \quad .$$

2. $G(x,y,\xi,\eta) = \frac{1}{2\pi} \ln(\frac{1}{R}) - \frac{1}{2\pi} \ln(\frac{1}{\bar{R}}) - \frac{1}{2\pi} \ln(\frac{\bar{R}}{R})$.

3. $u(x,y) = \frac{-k}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \ln(\frac{\bar{R}}{R})\, d\xi\, d\eta$ .

4. Place positive charges at $(\xi,\eta)$ and $(-\xi,-\eta)$ and negative ones at $(-\xi,\eta)$ and $(\xi,-\eta)$ so that Green's function will be:

$$G(x,y,\xi,\eta) = \frac{1}{2\pi} \ln\left[\frac{R_2 \; R_4}{R_1 \; R_3}\right] \quad .$$



5. Try $u_p(r) = Ar^2$. Substituting this into the PDE we get $u_p(r) = \frac{1}{4} r^2$. If we now let $u = w + \frac{1}{4} r^2$ we see that $w$ will satisfy the problem:

$$\nabla^2 w = 0$$

$$w(1,\theta) = -\frac{1}{4} + \sin(\theta)$$

and hence $w(r,\theta) = -\frac{1}{4} + r\sin(\theta)$. Hence the solution:

$$u(r,\theta) = \frac{1}{4}(r^2 - 1) + r\sin(\theta).$$

## Lesson 37

1. Write
$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2} f''(x)$$
$$f(x-h) = f(x) - h\, f'(x) + \frac{h^2}{2} f''(x)\ .$$

   Adding these two equations and solving for $f''(x)$ we get:
$$f''(x) = \frac{1}{h^2}\,[f(x+h) - 2\, f(x) + f(x-h)]\ .$$

2.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | | | 0 | 0 | | | 0 | 0 | |
| 0 | .216 | .216 | 0 | 0 | .108 | .108 | 0 | 0 | .107 | .307 | 0 |
| 0 | .216 | .216 | 0 | 0 | .32 | .32 | 0 | 0 | .323 | .323 | 0 |
| | .866 | .866 | | | 866 | .866 | | | .866 | .866 | |
| initialize interior grid points to be average of the BC | | | | first iteration | | | | second iteration | | | |

   The numbers are converging fairly fast--but one should realize they are <u>not</u> converging to the solution of the <u>PDE</u> at those points but to the solution of the <u>difference equation</u> that <u>approximates</u> the PDE. If we want a better solution we must use more grid points. It would be a good experiment to double the number of grid points and see if the numbers converge to the same values as they do there.

3. $u_{i,j} = \frac{1}{2}\,[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}] - \frac{h^2}{4} f_{i,j}$

   where $f_{i,j} = f(x,y)$.

4. $u_{i,j} = \frac{-1}{2(h-2)}\,[u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j}]\ .$

5. Use the usual replacement formula for $\nabla^2 u = 0$ to estimate the interior grid points and the finite difference formula:
$$u_{i,n} = u_{i,n-1} + h\, g_i$$
   to find the boundary potentials.

Lesson 38

1. Straightforward computation.

2. $u(x,t) = e^{-\pi^2 t} \sin(\pi x)$.

3. More or less the same as the parabolic flow diagram.

4. Direct computation. The necessary equations have been derived in the lesson.

Lesson 39

1. Straightforward computation.

2. The finite difference formula for the PDE will be the same as in the lesson. The BC becomes:

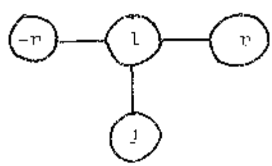
$$u_{i,1} = 0$$

$$\frac{u_{i,n} - u_{i,n-1}}{h} + u_{i,n} = g_i$$

where $g_i = g(ik)$. For each value of time we would solve $n-1$ equations for $u_{i,2}, u_{i,3}, \cdots, u_{i,n}$. The BC at $x = 1$ would give us an extra equation.

3. Same as the problem in the lesson except that the difference equation would be slightly different.

4. The molecular form is:




5. Direct use of the formulas in the lesson.

Lesson 40

1. Note that as time increases we can approximate the solution fairly accurately from the first two terms:

$$u(x,t) = \frac{2}{\pi}\left[e^{-(\pi\alpha)^2 t}\sin(\pi x) + \frac{1}{3}e^{-(3\pi\alpha)^2 t}\sin(3\pi x)\right] .$$

Hence if $x = .5$ we have:

$$u(.5,t) = \frac{2}{\pi}\left[e^{-(\pi\alpha)^2 t} - \frac{1}{3}e^{-(3\pi\alpha)^2 t}\right] .$$

We can now find $\alpha$ that will fit this curve to the observed data points. This would be a least squares problem (see problem 2). There are computer programs available that fit curves like this to arbitrary data points.

2. Differentiate $\frac{\partial(SS)}{\partial a} = 0$ $\frac{\partial(SS)}{\partial b} = 0$ and solve for a and b.

3 One can design an experiment to measure the concentration at different values of $r$ after it has come to a steady state (some sort of optical device). We then fit (least squares) the theoretical solution $u(r,\infty)$ (which is known) to this data by picking the proper choice of D. We then solve for the molecular weight algebraically from this value of D.

Lesson 41

1. a) parabolic--canonical

   b) hyperbolic--not canonical

   c) elliptic--not canonical

   d) parabolic--canonical

2. The new coordinates are $\xi = y - 3x$ and $\eta = y$ and the transformed equation is:

$$16\, u_{\eta\eta} + u = 2 \; .$$

3. The new coordinates are $\xi = y$ and $\eta = x^2/2$ and the transformed equation is:

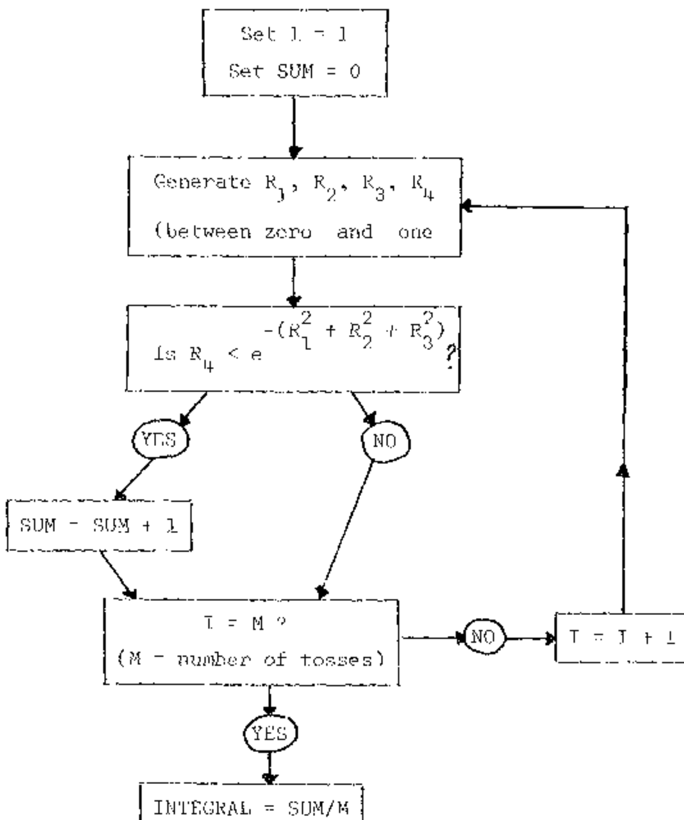
$$u_{\xi\xi} + u_{\eta\eta} + \frac{1 + 2\eta}{2\eta}\, u_{\eta} = e^{-\eta} \; .$$

*Lesson 42*

1. Apply the technique in the lesson directly to the function $f(x) = e^{\sin(x)}$ (note this function is positive for $0 \leq x < 1$). The method in this lesson assumes $f(x) \geq 0$ but it can be modified to take care of more general functions.

2. $r_{i+1} = 3r_i + 4 \bmod 7$. The sequence of random integers starting from zero is $0,4,2,3,6,1,0 \rightarrow$ repeats.

3.



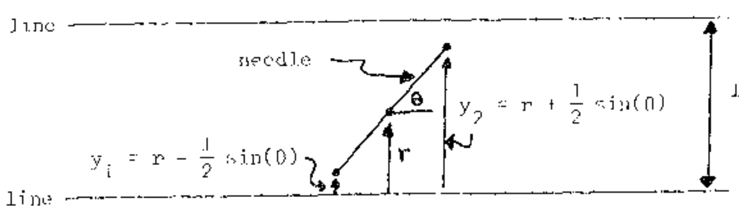
One can change the value of $M$ and find new estimates for the integral. The value of the integral will stabilize as $M$ gets larger and larger.

4. Generate $R_1$, $R_2$ random numbers between zero and one. If $R_1 + R_2 < 1$ we keep the point, otherwise generate another two numbers. Continue this process.

5. Generate a random number $r$ between zero and one. Set:

$$X = \begin{cases} 0 & 0 \le r \le .25 \\ 1 & .25 < r < .75 \\ 2 & .75 < r \le 1 \end{cases} .$$

6. Generate two random numbers $0 \le r \le 1$ and $0 \le \theta \le \pi/2$ and if $y_1 = r - \frac{1}{2}\sin(\theta) < 0$ or $y_2 = r + \frac{1}{2}\sin(\theta) > 1$ in the diagram below consider the needle crossing one of the lines. Compute the fraction of tosses $\hat{p}$ that the needle crosses a line and then estimate $\pi$ from the equation $\frac{2}{\pi} \cong \hat{p}$. That is, $\pi \cong \frac{2}{\hat{p}}$ .



Lesson 43

3. The finite difference approximation becomes (after solving for $u_{i,j}$):

$$u_{i,j} = \frac{u_{i,j+1} + u_{i,j-1} + x_j^2 [u_{i+1,j} + u_{i-1,j}]}{2 [1 + x_j^2]} .$$

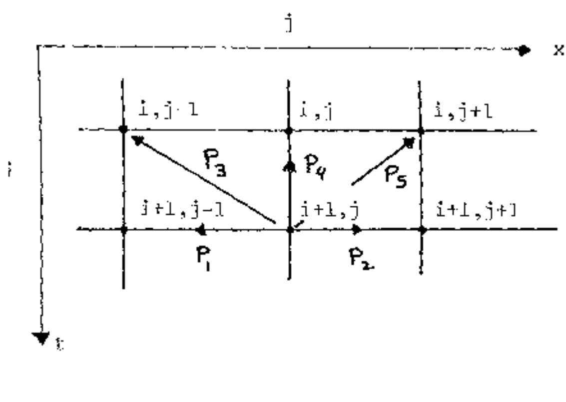
Hence the problem of going from $u_{i,j}$ to one of its four neighbors is the coefficients of the corresponding terms. Note that the coefficients are nonnegative and sum to one.

4. The BCs $u_x(1,t) = 0$ gives us the difference equation $u_{i,n} - u_{i,n-1} = 0$ and hence the average reward of the wino starting at $(i,n)$ should be equal to the average reward starting at $(i,n-1)$. This gives the strategy of a reflecting boundary. That is, if the wino hits the line $x = 1$ he automatically bounces back.

5. The Crank-Nicolson method gives us the difference equations:

$$u_{i+1,j} = P_1 u_{i+1,j-1} + P_2 u_{i+1,j+1} + P_3 u_{i,j-1} + P_4 u_{i,j} + P_5 u_{i,j+1}$$

where the $P_i^s$ are numbers computed from the coefficients in the PDE and represent the probabilities of going to the corresponding points. One could modify the BCs of this problem to solve problems with flux across the boundaries (derivative BC).



Lesson 44

1. $F(x,y,y') = \sqrt{1 + y'^2}$ and so the Euler-Lagrange equation is:

$$F_y - \frac{d}{dx} F_{y'} = - \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = 0 \ .$$

   Hence solving this ODE along with the BCs $y(0) = 0$ and $y(1) = 1$ we get $\bar{y}(x) = x$. Since $J[y]$ is the functional for the length of the curve between two points and since the end points here are $(0,0)$ and $(1,1)$ this answer is not surprising. The functional evaluated at this minimizing curve should be $\sqrt{2}$.

2. $m\dot{y} + k\ y = 0$

3. The Euler-Lagrange equation here is $-2y - 2y'' = 0$ which simplifies to $y'' + y = 0$ and along with the BCs $y(0) = 0$ and $y(\pi/2) = 1$ we get: $\bar{y}(x) = \sin(x)$. The functional evaluated at this minimizing curve is $J[\sin x] = 0$.

4. Directly analogous to the single integral problem in the lesson. Now, however, all functions depend on two independent variables.

Lesson 45

1. $J[u] = \int_0^1 \int_0^1 [u_x^2 + u_y^2 + 2u]\, dx\, dy.$

2. Let $z(t) = (1 - x)\, y(t)$. Hence $z' = -y + (1 - x)\, y'$ and so we find $\bar{z}$ that minimizes the transformed integral:

$$J[z] = \int_0^1 \left\{ \left[\frac{z}{1 - x}\right]^2 + \frac{z'}{1 - x} + \frac{z}{(1 - x)^2} \right\} dx$$

and then let $\bar{y} = \frac{\bar{z}}{1 - x}$. The point is that we can use the method of Ritz on the function $z(t)$ since it satisfies $z(0) = z(1) = 0$.

3. The reader can do this.

4. This proof is exactly like the proof in the last lesson except that single integrals are replaced by double integrals.

5. To solve the Dirichlet problem by the finite sine transform (on x) we get:

$$\frac{d^2U_n(y)}{dy^2} - (n\pi)^2 U_n(y) = \begin{cases} 1 & n = 1 \\ 0 & n = 2, 3, \cdots \end{cases}$$

$$U_n(0) = 0$$

$$U_n(1) = 0 \; .$$

Solving this we get $U_1(y) = A\, e^{\pi y} + B\, e^{-\pi y} - \frac{1}{\pi^2}$

$U_n(0) = 0 \qquad n = 2, 3, \cdots$

and so taking the inverse sine transform we have:

$$u(x,y) = U_1(y) \sin(\pi x) \; .$$

Now that we know the solution we can find the potential energy from the equation:

$$J[u] = \int_0^1 \int_0^1 \{u_x^2 + u_y^2 + 2\, u \sin(\pi x)\}\, dx\, dy \ .$$

Most likely this integral will have to be evaluated numerically.

Lesson #6

1. Set the coefficient of $1, \epsilon, \epsilon^2, \cdots$ equal to each other on both sides of the PDE and BC.

2. Same as in problem 1.

3. If we substitute

$$u_0 + u_1 = r\cos(\theta) + \frac{(r^4 - 1)}{32} - \frac{(r^4 - r^2)}{24}\cos(2\theta)$$

into $\nabla^2 u + u^2$ and $u(1,\theta)$ we will see that $u(1,\theta) = 0$ and that the function $\nabla^2 u + u^2$ is very small inside the unit circle $0 < r < 1$. The term $u^2$ will contain *nine* terms but can be evaluated nevertheless.

4. To solve the BVP

$$\nabla^2 u = 0 \qquad 0 < r < 1$$

$$u(1,\theta) = -\sin(\theta)\cos(\theta) \qquad 0 \le \theta \le 2\pi$$

we *first* transform this problem to one that has zero BC by making the following transformation:

$$u(r,\theta) = -\sin(\theta)\cos(\theta) + U(r,\theta)\ .$$

Hence the new function $U(r,\theta)$ will satisfy the BVP:

$$\nabla^2 u = \frac{1}{r^2}\cos(2\theta) \qquad 0 < r < 1$$

$$u(1,\theta) = 0 \qquad 0 \le \theta \le 2\pi\ .$$

(we've used a trig identity here)

We now solve this problem by writing $U(r,\theta) = U_h(r,\theta) + U_p(r,\theta)$ where $U_h(r,\theta)$ is a general form of the solution of the homogeneous equation $\nabla^2 u = 0$ and $U_p(r,\theta)$ is a particular solution of the nonhomogeneous equation. A general form of the homogeneous equation (Laplace's equation) is:

$$U_h(r,\theta) = \sum_{n=0}^{\infty} r^n\,[a_n\cos(n\theta) + b_n\sin(n\theta)].$$

One can easily verify that this sum satisfies Laplace's equa-
for any choice of the constants $a_n$s and $b_n$s. To find $U_p(r,\theta)$ we try:

$$U_p(r,\theta) = A \cos(2\theta)$$

(we generally increase the power of $r$ by two) and see $A = -\frac{1}{4}$. Plugging $U_h$ and $U_p$ into the equation $U = U_h + U_p$ we have:

$$U(r,\theta) = \sum_{n=0}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] - \frac{1}{4}\cos(2\theta)$$

and by substituting this into the BC $U(1,\theta) = 0$ we see $a_2 = 1/4$ and all the other $a_n$s and $b_n$s are <u>zero</u>. Hence we have the solution $U(r,\theta)$:

$$U(r,\theta) = \frac{r^2}{4}\cos(2\theta) - \frac{1}{4}\cos(2\theta)$$

and so finally the solution $u(r,\theta)$

$$u(r,\theta) = -\sin(\theta)\cos(\theta) + \frac{r^2}{4}\cos(2\theta) - \frac{1}{4}\cos(2\theta).$$

Hence the first two terms in the approximation are one fourth of the above term plus $r\cos(\theta)$. One would have to use a computer to see how well this function approximates the <u>deformed</u> boundary problem

$$\nabla^2 u = 0 \qquad 0 < r < 1 + \frac{1}{4}\sin(\theta)$$

$$u\left(1 + \frac{1}{4}\sin(\theta),\ \theta\right) = \cos(\theta).$$

Lesson 47

$\frac{dw}{dz} = \frac{2}{z^2 - 1}$ and hence the mapping is conformal except at $z = \pm 1$. If one writes the mapping in real form:

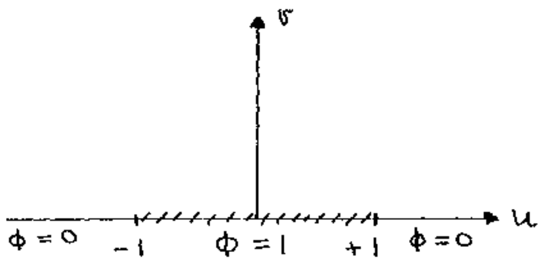
$$u = \log \left|\frac{z - 1}{z + 1}\right|$$

$$u = \tan^{-1} \left|\frac{z - 1}{z + 1}\right|$$

one can show that the upper half plane maps into the given region.

Write $w = z^3 = [r\, e^{i\theta}]^3 = r^3\, e^{3i\theta}$ and so the first quadrant in the $z$ plane maps into the first three quadrants of the $w$ plane.

3. The transformed problem is:



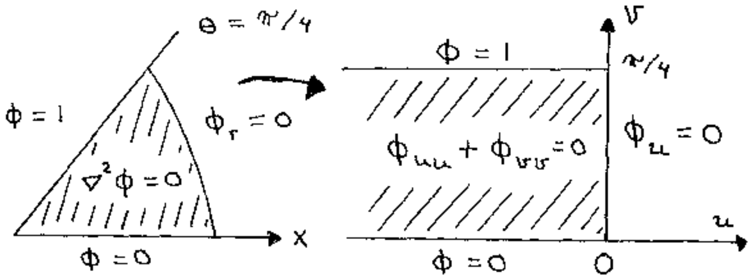

which has the solution:

$$u(u,v) = \frac{1}{\pi} \tan^{-1} \left\{\frac{2v}{u^2 + v^2 - 1}\right\}.$$

Writing the transformation $w = z^2$ in real form $u = x^2 - y^2$, $v = 2xy$ one has the solution of the original problem.

$$\phi(x,y) = \frac{1}{\pi} \tan^{-1} \left\{\frac{4xy}{(x^2-y^2)^2 + 4x^2y^2 - 1}\right\}.$$

4. The mapping $w = \text{Log}(z)$ maps the wedge in our problem into the region:




It is clear that the transformed solution is $\phi(x,v) = \frac{4v}{\pi}$ and so if we write the transformation $w = \log(z)$ in real form $u = \log |z|$, $v = \tan^{-1}(y/x) = \theta$ we get the solution:

$$\phi(r,\theta) = \frac{4\theta}{\pi} .$$

## Additional Lesson A: Control Problems in PDE

PURPOSE OF LESSON: TO SHOW HOW CONTROL THEORY ENTERS INTO PDE THEORY AND, IN PARTICULAR, TO SHOW HOW THE TEMPERATURE OF AN INSULATED ROD CAN BE CONTROLLED BY PROPER TEMPERATURE CONTROL OF ONE OF THE ENDS OF THE ROD. MORE SPECIFICALLY, WE WILL FIND THE BOUNDARY CONTROL $f(t)$ IN THE IBVP

PDE $\quad u_t = u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$

BC $\quad u_x(0,t) = 0$ (left end insulated)

$\qquad u(1,t) = f(t)$ (right end controlled)

IC $\quad u(x,0) = 1 \qquad 0 \le x \le 1$

THAT MAKES THE TEMPERATURE $u(x,t)$ GO TO ZERO IN MINIMUM TIME. WE DO THIS BY USING THE LAPLACE TRANSFORM AND A FEW IDEAS FROM CONTROL THEORY.

## Lesson A: Control Problems in PDE

One of the recent developments in PDE theory has been determining boundary conditions in initial boundary-value problems so that the solution of the problem performs in some desirable manner. One should refer to reference 2 of the recommended reading list for additional information. Typical problems of this kind arise in the steel industry where the boundary temperature of the ingots is controlled so that the steel inside the ingots is heated rapidly but uniformly. Another example would be instilling, by some means, small vibrations in a telescopic mirror so that the effect of atmospheric disturbances on incoming light is decreased.

Minimum Time Problem: One interesting problem that we present in this lesson is determining the boundary temperature of a rod so that the initial temperature goes to zero in minimum time. *Mathematically, this corresponds to finding the function* $f(t)$ *(control function) that makes the temperature* $u(x,t)$, *described by*

$$
(1)\quad
\begin{array}{lll}
\text{PDE} & u_t = u_{xx} & 0 < x < 1 \qquad 0 < t < \infty \\
\text{BC} & u_x(0,t) = 0 & \text{(left end is insulated)} \\
 & u(1,t) = f(t) & \text{(right end is controlled)} \\
\text{IC} & u(x,0) = 1 & 0 \le x \le 1
\end{array}
$$

go to zero $[u(x,t) \equiv 0]$ in minimum time.

The solution of this problem is found in reference 3 of the recommended readings, and the following discussion continues along the lines of that article.

We start by transforming the problem into an ODE by means of the Laplace transform

$$U(x,s) = \mathcal{L}[u(x,t)] = \int_0^\infty u(x,t)\, e^{-st}\, dt$$

This transform was discussed in Lesson 13 and converts the original problem (1) to the new BVP

$$
(2)\quad
\begin{array}{lll}
\text{ODE} & \dfrac{d^2U}{dx^2} = sU - 1 & 0 < x < 1 \\
\text{BCs} & \dfrac{dU(0)}{dx} = 0 & \\
 & U(1) = F(s) &
\end{array}
$$

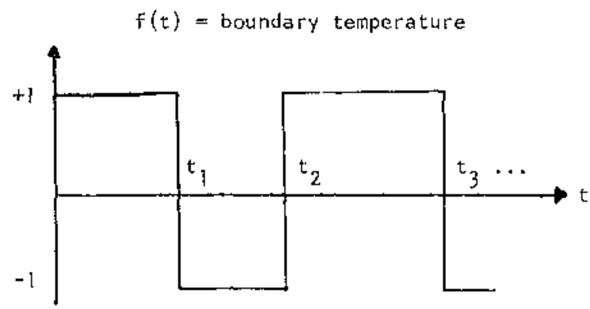
This problem can be solved very easily; the reader can verify that the solution is given by

$$(3) \qquad U(s,x) = \frac{\cosh\sqrt{s} + [sF(s) - 1]\cosh\sqrt{s}\,x}{s\cosh\sqrt{s}}$$

Generally, the next stop is to find the inverse Laplace transform and, hence, the solution. However, since our goal here is to find the boundary function $f(t)$, a short digression follows.

## CONTROL THEORY

The function $f(t)$ is called the control function and is always restricted to a given class of functions (like smooth functions, integrable functions, piecewise continuous functions and so forth. In this problem, suppose we pick the control from the class of bang-bang controls. That is, we assume the control function $f(t)$ can take on only two extreme values (which we pick as $+1$ and $-1$). In other words, our control action will look something like the graph in Figure 1.




Example of bang-bang control

Figure 1

Therefore, in order to control the rod temperature so that it goes to zero in minimum time, we must find the switching times $t_1$, $t_2$, $t_3$, ... $t_n$ of the bang-bang control. This will tell us when the boundary temperature should change from +1 to -1 (or vice versa).

The idea for finding $t_1, t_2, \ldots t_n$ is based on a simple principle of the Laplace transform which we now present.

Property of the Laplace Transform

If $g(t)$, $0 \le t < \infty$ is a function that is zero for all $t$ greater than some fixed number $T$, then the Laplace transform $G(s)$ of $g(t)$ is a bounded function of $s$ in the complex plane.

Examples 1. $g(t) = 1 \quad 0 \le t < \infty$ $\qquad G(s) = 1/s$

[not zero for $t \ge T$] [not bounded]

2. $g(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & 1 < t < \infty \end{cases}$ $\qquad G(s) = \frac{1}{s}[1-e^{-s}]$

[zero for $t > T$] [$G(s)$ is bounded]

By using this property one can present the following argument. Since we want the solution $u(x,t)$ to be zero for $t$ greater than some $T$ ($T$ as small as possible), one picks the switching times $t_1, t_2, \ldots$ so that $U(x,s)$ is bounded. The interesting point here is that the denominator $s \cosh\sqrt{s}$ has zeros at the values

$$s_k = -(k - \tfrac{1}{2})^2 \pi^2 \qquad k = 1,2,3, \ldots$$

and, hence, in order for $U(x,s)$ to remain bounded, the numerator must also be zero at these same points. That is,

$$(4) \qquad s_k F(s_k) - 1 = 0 \qquad k = 1, 2, \ldots$$

which give us necessary equations for the infinite number of switching times. We approximate this control function $f(t)$ by finding the first $n$ switches $t_1, t_2, \ldots, t_n$ and then turning the control $f(t)$ to zero. The temperature $u(x,t)$ will hopefully be close to zero if we pick $n$ large.

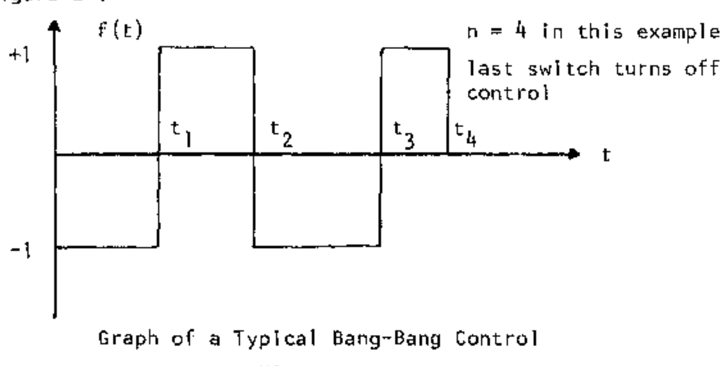
To find the expression for $F(s)$, remember the control $f(t)$ is assumed to be bang-bang (it's obvious the control starts at $-1$ when $t = 0$ for our problem), and, hence

$$f(t) = -E(t) + 2\,E(t-t_1) - 2\,E(t-t_2) + \cdots + (-1)^n\,E(t-t_{n-1}) + (-1)^{n+1}\,E(t-t_n)$$

where $E(t)$ is the simple off-on switch at zero

$$E(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

See Figure 2 .




Graph of a Typical Bang-Bang Control

Figure 2

*Since the Laplace transform of* $E(t-a)$ *is*

$$\mathcal{L}\{E(t-a)\} = e^{-sa}$$

we have

$$(5) \quad F(s) = \mathcal{L}[f] = -\frac{1}{s} + \frac{2}{s}e^{-st_1} - \frac{2}{s}e^{-st_2} + \cdots + (-1)^n \frac{2}{s}e^{-st_{n-1}} + (-1)^{n+1}\frac{1}{s}e^{-st_n}$$

Substituting this expression in equation (4), we get the desired set of equations for $t_1, t_2, \ldots, t_n$ :

$$(6) \quad \begin{aligned} 2 - 2e^{-s_1t_1} + 2e^{-s_1t_2} - \cdots + (-1)^n e^{-s_1t_n} &= 0 \\ 2 - 2e^{-s_2t_1} + 2e^{-s_2t_2} - \cdots + (-1)^n e^{-s_2t_n} &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ 2 - 2e^{-s_nt_1} + 2e^{-s_nt_2} - \cdots + (-1)^n e^{-s_nt_n} &= 0 \end{aligned}$$

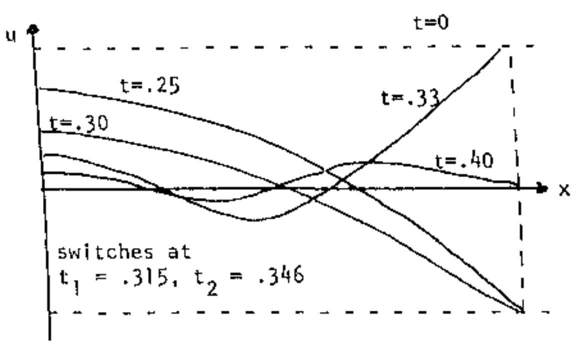
The authors in reference 3 of Other Reading have solved this system of equations numerically for various values of n, and some of the results are given in Table I .

| n | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ |
|---|---|---|---|---|---|---|
| 1 | 0.281 | | | | | |
| 2 | 0.315 | 0.346 | | | | |
| 3 | 0.318 | 0.362 | 0.372 | | | |
| 4 | 0.319 | 0.365 | 0.382 | 0.387 | | |
| 5 | 0.319 | 0.366 | 0.385 | 0.393 | 0.395 | |
| 6 | 0.319 | 0.367 | 0.386 | 0.395 | 0.400 | 0.402 |

Optimal switching times for a different number of switches

Table I

The temperature corresponding to these controls can be found either by finding the inverse Laplace transform of $U(x,s)$ or by solving the original problem numerically using our optimal control. The authors in reference 3 of the recommended reading found the analytic solution from a table of inverse Laplace transforms, and the results are given in Figure 3 for the simple control with $n = 2$ switches. Figure 3 gives the temperature for various values of time.




Controlled Solution for Two Switches

Figure 3

One can surmise from Table 1 and the above Figure 3 that the bang-bang control with switches at $t_1 = .319$, $t_2 = .367$, $\cdots$ $t_6 = .402$ will do a good job in driving the solution to zero. The final time $t_6 = .402$ corresponds to the time when the right hand control $f(t)$ is set to zero. It is also true that the values of the switches, $t_n$, get closer together as we add more and more switches.

Notes

1. If the right end of the rod were held at temperature zero, the rod would also go to zero, but not nearly so fast. Note, too, that after the last switch, the right end of the rod is

held at zero.

2. We might try to use this same approach to control the vibrations of a vibrating string. That is, vary the position of the right end of the string so that the string comes to rest in minimum time.

3. The reader shouldn't get the impression that the minimum time problem is the only problem we try to solve (or that the control is always bang-bang). Reference 1 of the recommended reading list gives many other objective functions that occur in applications.

4. We could also change the control problem that we solved in this lesson, so that the boundary condition on the right-hand side of the rod (x=1) was

$$u_x(1,t) = f(t)$$

In this way, we will be controlling the amount of heat flux at x=1 instead of the actual temperature. The authors in reference 3 have also solved this problem.

Problems

1. Graph the optimal boundary-control $u(t)$ for n=4 using the data of Table 1. Draw rough graphs of the controlled solution for $t = 0, 0.1, 0.25, 0.37, 0.40$ .

2. Show that the solution of the BVP (2) is given by

$$U(x,s) = \frac{\cosh\sqrt{s} + [sF(s) - 1]\cosh\sqrt{s}}{s\cosh\sqrt{s}}$$

3. Solve the system of equations (6) for $t = t_1$ when n=1. Does this answer agree with Table 1 ? What would the graph of this control function $f(t)$ look like ?

4. What system of algebraic equations would you solve to find the bang-bang control $f(t)$ that sends the solution to the following problem to zero in minimum time ?

PDE $\quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$

BCs $\quad \begin{array}{l} u(0,t) = 0 \\ u_x(1,t) = f(t) \end{array} \quad 0 < t < \infty$

IC $\quad u(x,0) = x \quad 0 \le x \le 1$

Other Reading

1. Operational Mathematics by R. V. Churchill. McGraw Hill, 1972. One of the best applied books on integral transforms, contains many worked examples.

4. Distributed Parameter Systems edited by W. H. Ray and D. G. Lainiotis. Dekker, 1978. A comprehensive review of modern control theory in PDE.

3. Time Optimal Control of a Linear Diffusion Process by R. M. Goldwyn, K. P. Sriham, and M. Graham. Society of Industrial and Applied Mathematics, Journal of Control (5) No. 2, 1967, p 295.

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Additional Lesson B    Potential Theory (rewriting BVPs as integral equations)

PURPOSE OF LESSON    TO SHOW HOW THE FOLLOWING BVPs:

(1)
$$\nabla^2 u = 0 \quad \text{inside a region D in 2 dimensions}$$
$$u = f \quad \text{on the boundary C of D} \qquad \text{(Dirichlet Problem)}$$

(2)
$$\nabla^2 u = 0 \quad \text{inside a region D in 2 dimensions}$$
$$\frac{\partial u}{\partial n} = g \quad \text{on the boundary C of D} \qquad \text{(Neumann Problem)}$$

CAN BE SOLVED IN FAIRLY GENERAL DOMAINS D BY LOOKING FOR THE SOLUTION IN TERMS OF THE SURFACE POTENTIALS:

(3) $$W(x,y) = \int_C \mu(s) \frac{\partial}{\partial n}\left(\ln \frac{1}{r}\right) ds \quad \text{(double-layer potential)}$$

(4) $$V(x,y) = \int_C \sigma(s) \ln \frac{1}{r}\, ds \quad \text{(single-layer potential)}$$

THE IDEA IN SOLVING PROBLEMS 1 AND 2 IS THAT BOTH 3 AND 4 SATISFY $\nabla^2 u = 0$, BUT NOW WE MUST FIND THE SURFACE POTENTIAL DENSITIES $\mu(s)$ AND $\sigma(s)$ (in the integrals 3 and 4) SO THAT THE BCs $u = f$, $\partial u/\partial n = g$ ARE SATISFIED. BY SUBSTITUTING 3 AND 4 IN 1 AND 2, RESPECTIVELY, WE ARRIVE AT TWO FREDHOLM INTEGRAL EQUATIONS FROM WHICH WE CAN SOLVE FOR $\mu(s)$ AND $\sigma(s)$ .

## Lesson B: Potential Theory (rewriting BVPs as integral equations)

In previous lessons, we solved Laplace's equation with boundary conditions

(1)
$$u = f \quad \text{on the boundary} \qquad \text{(Dirichlet BC)}$$
$$\frac{\partial u}{\partial n} = g \quad \text{on the boundary} \qquad \text{(Neumann BC)}$$

in regions D that were very simple (inside a circle and in the upper half plane). There are methods for solving these problems in more general regions of space; chapter 7 in reference 2 of the recommended reading discusses the method of Balayage, the Peron-Remak method, integral equations, Dirichlet principle, finite-difference method, and conformal mapping. In this book, we have already discussed conformal mapping, finite differences, and the Dirichlet principle. We will now discuss the method of integral equations that consists of converting Dirichlet and Neumann problems into integral equations (of Fredholm type) and then solving these problems. The actual solution of our integral equations will be left to the next lesson (Lesson C).

To begin, however, we must introduce the concept of surface potentials and discuss their properties. It turns out that the solution to the Neumann Problem will be a single-layer potential.

## Kinds of Surface Potentials

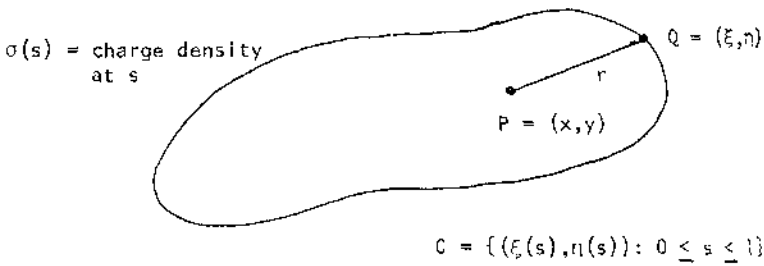
Surface potentials come in many forms. We will discuss two of them, the single-layer and the double-layer (most common kind). First, the single-layer potential.

Single-Layer Surface Potential Suppose we have a single charge (we restrict ourselves to two dimensions) $+q$ at some point $Q = (\xi,\eta)$ . From what we already know, the potential at any other point $P = (x,y)$ due to this single charge is given by

$$u(x,y) = q \ln\left(\frac{1}{r}\right) \quad \text{(logarithmic potential)}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$ .

Now, suppose instead of a single charge located at one point, we have a continuum of charges with charge density $\sigma(s)$ along a curve C (Figure 1).




Single-Layer Surface Potential

*Figure 1*

We can interpret $\sigma(s)$ as the charge in a very small line segment (centered as s) divided by the length of the segment. If we pass to the limit by letting the length of the line segment go to zero, we would arrive at the exact value of $\sigma(s)$. Now, to find the potential at any point $(x,y)$ due to the <u>entire charge</u> on C, we merely sum the potentials due to each charge on the curve to get

$$V(x,y) = \int_C \sigma(s) \ln\left(\frac{1}{r}\right) ds \qquad (2)$$

See Figure 1 . This is the single-layer potential due to the charge density on C, and later we will see how the solution to the Neumann problem

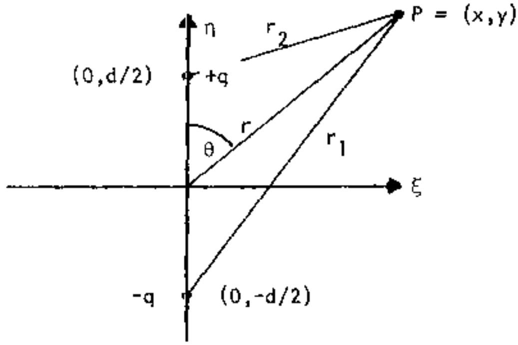
$$\nabla^2 u = 0 \qquad \text{in the region } D$$

$$\frac{\partial u}{\partial n} = g \qquad \text{on the boundary of } D$$

can be represented in terms of this potential. The general idea is that the single-layer potential satisfies Laplace's equation for any $\sigma(s)$, but we must find $\sigma(s)$ that satisfies the BC $\partial u/\partial n = g$. If we substitute the single-layer potential in the

Neumann BC, we will arrive at a Fredholm integral equation from which we can solve for $\sigma(s)$ . Now we will discuss the second surface potential.

Double-Layer Potential The double-layer surface potential is a little more complicated. We start with two charges, a positive charge +q and a negative charge -q located at a distance d apart. See Figure 2 .




Two Charges Giving Rise to a Dipole Potential

Figure 2

If we want to find the potential field due to these two charges and if we are located far from the charges in comparison to the distance d between the charges and calling $\mu = qd$ (dipole moment), we can write

$$W(x,y) = q \lim_{d\to 0} [\ln(\frac{1}{r_2}) - \ln(\frac{1}{r_1})]$$

$$= \frac{\mu}{d} \lim [\ln(\frac{1}{r_2}) - \ln(\frac{1}{r_1})]$$

$$= \mu \frac{\partial}{\partial \eta} \ln(\frac{1}{r}) \Bigg|_{\eta=0} \qquad \text{(definition of derivative)}$$

where $r^2 = x^2 + (y-\eta)^2$ .

We can find an alternative form of the dipole potential by evaluating this derivative to get

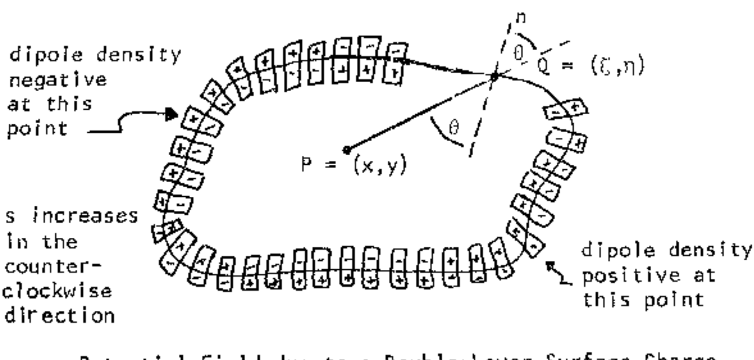
$$W(x,y) = \frac{\mu \cos \theta}{r}$$

where $r$ is now the distance from the dipole center (the origin in Figure 2 to the point P). That is

$$r = (x^2 + y^2)^{1/2} .$$

The reader may find it useful to graph this dipole potential in three dimensions.

We now place dipoles along a curve $C$ and allow the dipole moment density $\mu(s)$ to vary. See Figure 3. The dipole density $\mu(s)$ can be interpreted as placing dipoles along the curve, each with its own dipole moment, and then finding the density of these dipoles in a manner similar to the single-layer density. We will let the sign of $\mu(s)$ be positive if the inner charge is positive and negative if the inner charge is negative. Hence, a uniform density of $\mu(s) = 1$ on the boundary means all inner charges are $+1$ and all outer charges are $-1$.




Potential Field due to a Double-Layer Surface Charge

Figure 3

To find the potential due to this double-layer density along $C$, we merely sum the individual potentials due to each dipole to get

$$W(x,y) = \int_C \mu(s) \frac{\partial}{\partial n} \ln\left(\frac{1}{r}\right) ds$$

$$= \int_C \frac{\mu(s) \cos\theta}{r} ds$$

This completes our introduction to single and double-layer potentials. We will now (without any proofs) list some properties of these two potentials and describe what these surfaces look like. We then show how these properties can be used to convert the Dirichlet and Neumann problems to integral equations.

Properties of the Single-Layer Potential $V(x,y) = \int_C \sigma(s) \ln\left(\frac{1}{r}\right) ds$

Suppose we have a smooth curve $C$ containing a region $D$ and suppose we have a single-layer density $\sigma(s)$ distributed along $C$. The following are a few properties of the single-layer potential $V(x,y)$:

1. $V(x,y)$ satisfies Laplace's equation $\nabla^2 u = 0$ inside and outside the curve $C$.
2. The potential surface $V(x,y)$ is continuous across $C$.
3. The normal derivative of $V(x,y)$ has a jump discontinuity of $2\pi\sigma(\bar{x},\bar{y})$ across $C$ at the point $(\bar{x},\bar{y})$.
4. The property

$$\lim \frac{\partial V(x,y)}{\partial n} \to \frac{\partial V(\bar{x},\bar{y})}{\partial n} - \pi\sigma(\bar{x},\bar{y})$$

as $(x,y) \to (\bar{x},\bar{y})$, holds for points $(x,y)$ approaching boundary points $(\bar{x},\bar{y})$ from the inside.

## Properties of the Double-Layer Potential $W(x,y) = \int_C \mu(s)\frac{\partial}{\partial n}\ln(\frac{1}{r})ds$

We now list four important properties of the double-layer potential.

1. $W(x,y)$ satisfies Laplace's equation $\nabla^2 u = 0$ inside and outside the curve $C$.
2. The potential surface $W(x,y)$ has a jump discontinuity of $2\pi\mu(\bar{x},\bar{y})$ across $C$ at the point $(\bar{x},\bar{y})$.
3. The normal derivative of $W(x,y)$ is continuous across the curve $C$.
4. The property

$$W(x,y) \to W(\bar{x},\bar{y}) + \pi\mu(\bar{x},\bar{y})$$

as $(x,y) \to (\bar{x},\bar{y})$ holds for points $(x,y)$ approaching any boundary point $(\bar{x},\bar{y})$ from the inside.

For example, in the simple case, if the double-layer density is $\mu(\xi,\eta) = 1$, then the double-layer potential is

$$W(x,y) = \begin{cases} 2\pi & (x,y) \text{ inside } C \\ \pi & (x,y) \text{ on } C \\ 0 & (x,y) \text{ outside } C \end{cases}$$

giving a jump of $2\pi$. We now get to the major task of the lesson, converting BVPs to integral equations.

## Converting the Dirichlet Problem to an Integral Equation

Suppose we wish to solve the BVP

$$\nabla^2 u = 0 \qquad \text{inside } D$$

$$u = f \qquad \text{on the boundary of } D$$

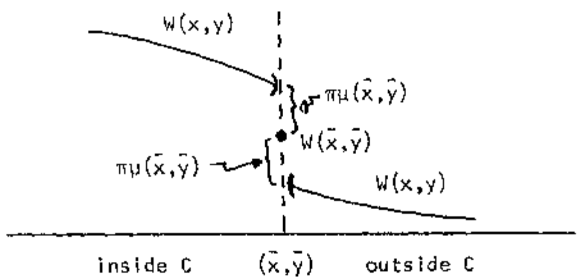
Let us try to find the solution as a double-layer potential

$$u(x,y) \;=\; W(x,y) = \int_C \mu(s) \frac{\partial}{\partial n} \ln(\tfrac{1}{r})\, ds$$

Since we know this expression satisfies Laplace's equation for any density $\mu(s)$, we try to find the density $\mu(s)$ that will satisfy the BC

$$W(x,y) \;=\; f(x,y) \quad \text{on the boundary.}$$

We must be careful here due to the fact that $W(x,y)$ has a jump discontinuity across the boundary C (Figure 4).




Slice of W(x,y) Across the Boundary

Figure 4

Since we want the solution $u(x,y) = W(x,y)$ to be a continuous surface out to the boundary we replace our BC

$$u(x,y) = f(x,y)$$

by

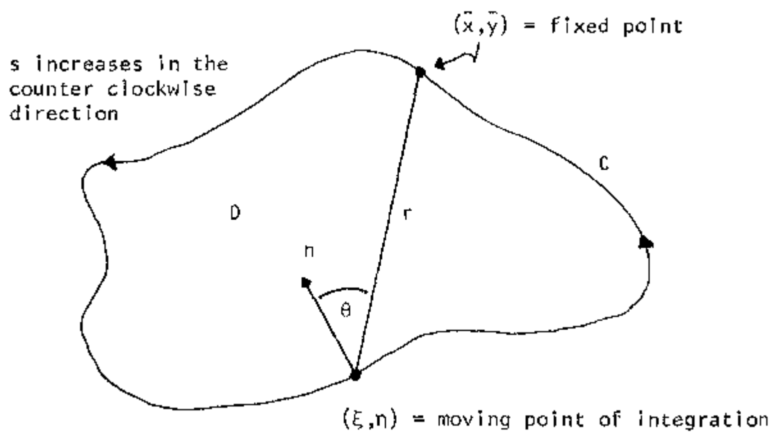
$$\lim_{(x,y)\to(\bar{x},\bar{y})} W(x,y) \;=\; f(\bar{x},\bar{y})$$

or, from Figure 4, $W(\bar{x},\bar{y}) + \pi\mu(\bar{x},\bar{y}) = f(\bar{x},\bar{y})$ .

Now, replacing $W(\bar{x},\bar{y})$ by it's integral form we have

$$\boxed{\int_C \frac{\mu(s)\cos\theta}{r}\,ds + \pi\mu(\bar{x},\bar{y}) = f(\bar{x},\bar{y})}$$

which is a Fredholm integral equation of the second kind (Figure 5).




Variables in the Fredholm Integral Equation

Figure 5

We can now be assured that when we solve this integral equation for $\mu(\bar{x},\bar{y})$ and substitute it into

$$u(x,y) = \int_C \mu(s)\,\frac{\partial}{\partial n}\ln\left(\frac{1}{r}\right) ds$$

We will get a continuous function $u(x,y)$ that satisfies:

i) $\nabla^2 u = 0$ inside C

ii) $u(x,y) = f(x,y)$ on C .

This completes our discussion of the Dirichlet problem and how we can convert it to an integral equation. The technique of finding the solution of the Neumann problem as a single-layer potential is carried out in a similar manner. In this case, we use the property

$$\lim_{(x,y)\to(\bar{x},\bar{y})} \frac{\partial V}{\partial n}(x,y) = \frac{\partial V}{\partial n}(\bar{x},\bar{y}) - \pi\sigma(\bar{x},\bar{y})$$

to arrive at the Fredholm integral equation

$$\int_C \sigma(s) \ln\left(\frac{1}{r}\right) ds - \pi\sigma(\bar{x},\bar{y}) = g(\bar{x},\bar{y})$$

from which we can solve for the single-layer potential $\sigma(s)$. We will discuss the general nature of integral equations in Lesson C.

Notes

1. An integral equation is an equation where the unknown function occurs in the integrand (possibly other places too).

2. This method leaves a lot to be desired since it is difficult to find analytic solutions to integral equations for the general domain. We may think that all we've done is convert one hard problem (involving derivatives) to another hard problem (involving integrals). This is true in a sense, but there is another side to the story. It turns out that integral equations are better to work with from a theoretical point of view. Suppose we aren't so interested in finding the analytic solution (maybe we can't find it) as we are in knowing if there is a solution and what kind of assumptions of the boundary condition and boundary we must make in order for a solution to exist. It's certainly

not true that all BVPs even have solutions, and many times, we would like to know what physical conditions we must assume in order to get solutions to our equation. Integral equations are better to work with when these questions about existence and uniqueness of solutions have to be answered.

Problems

1. Prove the alternative form of the double-layer potential can be written

$$W(x,y) = \frac{\mu \cos \theta}{r}$$

given that we start with the first form (as given in the text).

2. Plot the dipole potential due to two charges $+q$ and $-q$ at a distance $d$ apart.

3. Derive the Fredholm integral equation for $\sigma(s)$ in the Neumann problem. Draw a picture describing the relevant variables.

Other Reading

1. Partial Differential Equations by I. G. Petrovsky. Interscience Publishers, 1957. Contains a nice section on potential theory, describing the single and double-layer potentials.

2. Partial Differential Equations by B. Epstein. McGraw-Hill, 1962. Chapter 7 of this text discusses many techniques for solving the Dirichlet problem in general domains.

Additional Lesson C: Integral Equations

PURPOSE OF LESSON: TO INTRODUCE THE BASIC FREDHOLM INTEGRAL EQUATION

$$\phi(x) - \lambda \int_a^b K(x,s)\,\phi(s)\,ds = f(x) \qquad a \le x \le b$$

WHERE THE UNKNOWN FUNCTION $\phi(x)$ OCCURS UNDER THE INTEGRAL SIGN AND TO SHOW HOW THIS EQUATION CAN BE SOLVED FOR CERTAIN KERNEL FUNCTIONS $K(x,s)$. IN PARTICULAR, IF $K(x,s)$ CAN BE WRITTEN IN THE FORM (or approximated by)

$$K(x,s) = \sum_{i=1}^{n} a_i(x)\,b_i(s)$$

THEN THE INTEGRAL EQUATION CAN BE REDUCED TO A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS.

FINALLY, A SPECIFIC DIRICHLET PROBLEM IS SOLVED BY CONVERTING IT TO AN INTEGRAL EQUATION BY MEANS OF THE POTENTIAL-THEORY TECHNIQUES DISCUSSED IN LESSON C.

Lesson C: Integral Equations

Linear integral equations can be interpreted as extensions of systems of linear algebraic equations to an infinite continuum. For example, the linear system of $n$ equations with $n$ unknowns $x_i$ :

$$\sum_{j=1}^{n} a_{ij}\,x_j = b_i \qquad i = 1,2, \ldots n$$

has the continuous analog

$$\int_a^b A(x,s)\,f(s)\,ds = b(x) \qquad a \le x \le b$$

where now the unknown $f(s)$ is a function, and the continuous variables x and s take the place of the discrete variables i and j. The equation where the functions A(x,s) and b(x) are assumed known and f(s) is unknown is called an integral equation, and it is an example of the general Fredholm integral equation

$$(1) \qquad \phi(x) - \lambda \int_a^b K(x,s)\,\phi(s)\,ds = f(x) \qquad a \le x \le b \;.$$

The general names for the variables are:

f(x) = right-hand side of the integral equation
K(x,s) = kernel of the integral equation
$\lambda$ = known constant (often an eigenvalue)
$\phi(s)$ = unknown function (everything else is known)

and a couple of examples are

$$\phi(x) - \int_0^1 \sin(xs)\,\phi(s)\,ds = x^2 \qquad 0 \le x \le 1$$

$$\phi(x) - \int_{-\infty}^{\infty} \frac{\phi(s)}{x-s}\,ds = \sin x \qquad -\infty < x < \infty \;.$$

The last example is a singular-integral equation, since the kernel and domain are unbounded.

We will now find the solution to integral equation (1) for those cases where the kernel K(x,s) can be approximated by an expression of the form

$$K(x,s) = \sum_{j=1}^{n} a_j(x)\,b_j(s) \qquad \text{(degenerate kernel)} \;.$$

## Solving Integral Equations with Degenerate Kernels

Suppose the kernel K(x,s) in the Fredholm equation

$$\phi(x) - \int_a^b K(x,s)\, \phi(s)\, ds = f(x) \qquad a \leq x \leq b$$

can be written (or approximated) as

$$K(x,s) = \sum_{j=1}^{n} a_j(x)\, b_j(s) \quad .$$

Substituting this expression in the integral equation, we get

$$\phi(x) - \sum_{j=1}^{n} a_j(x) \int_a^b b_j(s)\, \phi(s)\, ds = f(x)$$

or

$$\text{(2)} \qquad \phi(x) - \sum_{j=1}^{n} c_j\, a_j(x) = f(x)$$

where

$$c_j = \int_a^b b_j(s)\, \phi(s)\, ds \quad .$$

If we now multiply equation (2) by each of the functions $b_1(x)$, $b_2(x)$, ..., $b_n(x)$ and integrate the resulting equation with respect to $x$, we get the following system of algebraic equations:

$$\text{(3)} \qquad \int_a^b \phi(s)\, b_i(s)\, ds - \sum_{j=1}^{n} c_j \int_a^b b_i(s)\, a_j(s)\, ds = \int_a^b f(s)\, b_i(s)\, ds \qquad (i=1,2,\ldots n)$$

or in more compact form

$$\text{(4)} \qquad c_i - \sum_{j=1}^{n} k_{ij}\, c_j = f_i \qquad i=1,2, \ldots n$$

where

$$k_{ij} = \int_a^b b_i(s)\, a_j(s)\, ds$$

$$f_i = \int_a^b f(s)\, b_i(s)\, ds$$

We can now solve this system of equations (providing it is a nonsingular system having a nonzero determinant) for the numbers $c_i$ and then substitute these values into equation (2) to find the solution $\phi(x)$. That is,

$$(5) \qquad \phi(x) = f(x) + \sum_{j=1}^{n} c_j\, a_j(x)$$

Remember that $a_j(x)$ and $b_j(x)$ are assumed known [they can be computed from the kernel]. In other words, if the kernel is degenerate and if there is a solution, then it has the form

$$\boxed{\phi(x) = f(x) + \sum_{j=1}^{n} c_j\, a_j(x)}$$

where the $c_j$'s are found by solving a system of algebraic equations. We will now give an example.

Solution to an integral equation

*Suppose we have the equation*

$$\phi(x) - \int_0^1 xs\, \phi(s)\, ds = x^2 \qquad 0 \le x \le 1$$

We know from equation (5) that the solution has the form

$$\begin{aligned} \phi(x) &= f(x) + \sum_{j=1}^{n} c_j\, a_j(x) \\ &= x^2 + cx \qquad \text{(c an unknown constant)} \end{aligned}$$

and so, substituting it in the integral equation, we get

$$x^2 + cx - \int_0^1 xs\,(s^2 + cs)\, ds = x^2$$

Evaluating this integral, we arrive at $c = 3/8$, and, hence, the solution is

$$\phi(x) = x^2 + \frac{3}{8}x$$

We can easily check this answer. There are more methods for solving other types of integral equations, and a few of them can be found in reference 2 of the recommended reading. We will now use our technique to solve a problem in PDE.

Dirichlet Problem

Suppose we wish to solve the Dirichlet problem (which we have solved before)

| | | |
|---|---|---|
| PDE | $\nabla^2 u = 0$ | (inside the unit circle) |
| BC | $u(1,\theta) = f(\theta)$ | $0 \le \theta \le 2\pi$ |

We know the solution can be written as a double-layer potential

$$u(r,\theta) = \int_0^{2\pi} \frac{\partial}{\partial\rho} \ln\left(\frac{1}{R}\right) \mu(\phi)\, d\phi$$

where $R^2 = r^2 - 2r\rho\cos(\theta-\phi) + \rho^2$ and $\mu(\theta)$ satisfies the integral equation

$$(6) \qquad \mu(\theta) - \frac{1}{\pi}\int_0^{2\pi} \frac{\partial}{\partial\rho} \ln\left(\frac{1}{R}\right) \mu(\phi)\, d\phi = -\frac{1}{\pi} f(\theta) \qquad 0 \le \theta \le 2\pi$$

If we now compute the kernel (which will be evaluated on the boundary $r = \rho = 1$), we get

$$\left.\frac{\partial}{\partial\rho} \ln\left(\frac{1}{R}\right)\right|_{\rho=1} = -\frac{1}{2}$$

and so we arrive at the integral equation

$$\mu(\theta) + \frac{1}{2\pi}\int_0^{2\pi} \mu(\phi)\, d\phi = -\frac{1}{\pi} f(\theta)$$

Since this integral equation has a degenerate kernel $K(\theta,\phi) = -\frac{1}{2\pi}$ the solution will be of the form

$$\mu(\theta) = -\frac{f(\theta)}{\pi} - \frac{c}{2\pi}$$

Substituting this in the integral equation, we have

$$c = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi$$

and so

$$\mu(\theta) = -\frac{1}{\pi} f(\theta) - \frac{1}{(2\pi)^2}\int_0^{2} f(\phi)\, d\phi$$

Finally, putting this in the double-layer potential, we have the solution

$$u(r,\theta) = \int_0^{2\pi} \frac{\partial}{\partial\rho} \ln(\frac{1}{R}) \quad [-\frac{1}{\pi} f(\phi) - \frac{1}{(2\pi)^2} \int_0^{2\pi} f(\bar{\phi})\, d\bar{\phi}\ ]\ d\phi$$

$$= \text{ a few calculus manipulations}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[ \frac{1-r^2}{1 - 2r\rho\cos(\theta-\phi) + r^2} \right] d\phi$$

which is the Poisson integral formula.

If the reader would like to see more general domains attacked (which is the purpose of the integral-equation method, he or she can refer to chapter 4 in reference 1 of the recommended reading. There, the Dirichlet problem is solved in more general domains. The computations become involved, so they aren't done here.

Notes

1. Laplace's equation is not the only equation that can be solved by potential methods (method of integral equations). More general elliptic equations (Helmholtz's equation) can be solved in addition to heat and wave problems. See reference 1 (chapter 7) of the recommended reading.

Problems

1. Solve the integral equation

$$\phi(x) - \int_0^1 (1+s^2)\,\phi(s)\,ds = x^2 \qquad 0 \le x \le 1$$

Check the answer.

2. Solve

$$\phi(x) - 2 \int_0^{2\pi} (x^2 s + xs)\,\phi(s)\,ds = \sin(\pi x)$$

$$0 \le x \le \pi$$

3. Solve the Neumann problem

| | | | |
|---|---|---|---|
| PDE | $\nabla^2 u = 0$ | inside the unit circle | |
| BC | $\frac{\partial}{\partial r}(1,\theta) = \sin\theta$ | | $0 \le \theta \le 2\pi$ |

by the integral-equation method.

4. Verify the formula

$$\frac{\partial}{\partial\rho} \ln\left(\frac{1}{R}\right)\Bigg|_{\rho=1} = -\frac{1}{2}$$

where $R^2 = 1 - 2\rho\cos(\theta-\phi) + \rho^2$

<u>Other Reading</u>

1. <u>Integral Equations</u> by S. G. Mikhlin. Pergamon Press, 1957. This text discusses many applications of integral equations, in particular, the theory of elasticity, fluid dynamics, and vibration analysis. Part II of this book shows how to solve the Dirichlet problem in various domains by potential-theory methods.

2. <u>An Introduction to Mathematical Physics</u> by L. M. Jones. Benjamin/Cummings, 1979. Chapter 11 contains an excellent overview of some basic methods for solving integral equations, included are integral equations of degenerate type, Neumann and Fredholm series, Hilbert-Schmidt theory, and the Wiener-Hopf technique.

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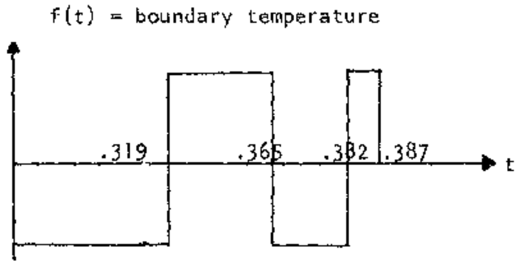
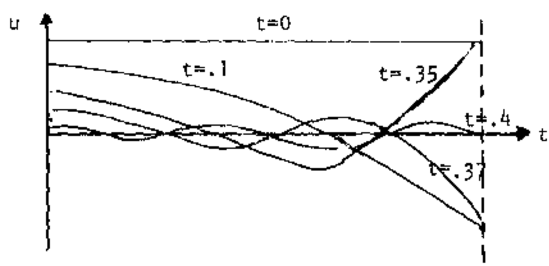
Solutions to Problems in Lesson A

1.




2. Straightforward, find the general solution of the ODE and substitute into the BC.

3. When $n = 1$ we have only one equation: $2 - e^{-s_1 t_1} = 0$ where

$$s_1 = -(n - \tfrac{1}{2})^2 \pi^2 = -2.467 .$$

Solving for $t_1$ we have $t_1 = .281$ .

4. This problem parallels the treatment in the text. The equations we have are:

$$s_n F(s_n) - 1 = 0$$

where

$$s_n = -(n - \tfrac{1}{2})^2 \pi^2$$

## Solutions to Problems in Lesson B

1. Direct computation.

2. The surface goes to $+\infty$ near the charge $+q$ and $-\infty$ in a neighborhood of the charge $-q$.

3. Analogous to the Dirichlet problem.

## Solutions to Problems in Lesson C

1. Try $\phi(x) = x + c$. One finds $c = -9/4$ .

2. Try $\phi(x) = \sin(\pi x) + c_1 x + c_2$ .

3. Similar to the Dirichlet problem.

4. Direct computation.