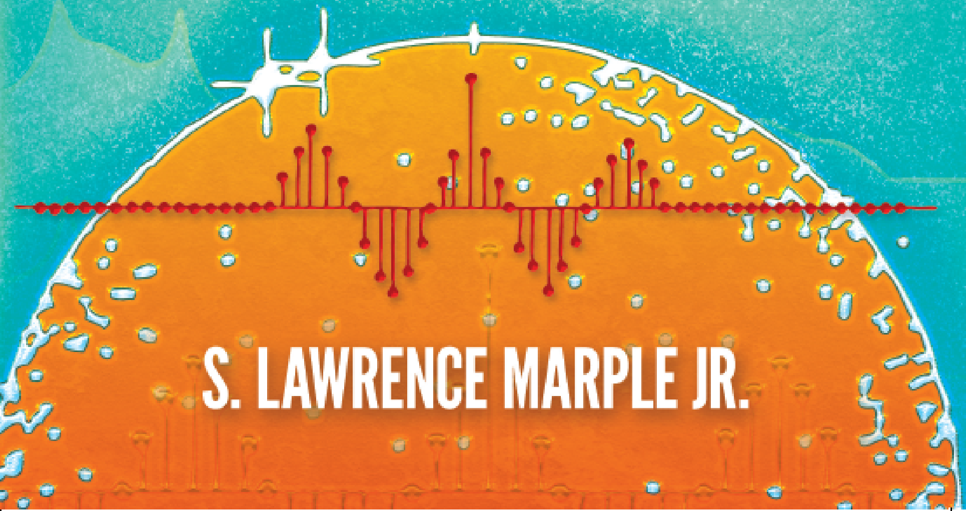
DIGITAL SPECTRAL ANALYSIS

PROBLEMS

Second Edition



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PROBLEMS FOR DIGITAL SPECTRAL ANALYSIS, 2nd Ed

Chapter 2

1. Prove that $\delta(at) = \delta(t)/|a|$, that is,

$$\int_{-\infty}^{\infty} \delta(at)\phi(t)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(t)\phi(t/a)\,dt \quad .$$

2. Prove that the complex output of the complex demodulation of a real signal need only be sampled at half the rate of the original real signal.

3. Suppose that only the first $N/2$ of the N transform output values are needed from the FFT algorithm. Show how pruning may be incorporated to reduce the computational complexity of the program. What is the amount of computational reduction?

4. Show that

$$\text{Ш}(ax) = \frac{1}{|a|}\sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{a})$$

and

$$\frac{1}{\tau} f(x)\text{Ш}(x/\tau) = \sum_{n=-\infty}^{\infty} f(n\tau)\delta(x - n\tau) \quad .$$

5. A factorization problem: given the function $g(t)$ with a transform $G(f)$ which is non-negative, find a causal function $f(t)$ such that $g(t) = f(t) \star f^*(t)$, which implies that $G(f) = |f(f)|^2$.

6. A TBP problem: define the one-sided exponential

$$x[n] = \begin{cases} A\exp(-\alpha nT) & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

where T is the sample interval and α is a positive constant. Find T_e, B_e, $\tilde{T}_e$, and $\tilde{B}_e$ and the respective TBPs for this function. Which TBP is smaller? Why?

7. Find the equivalent time width and equivalent bandwidth of the windows defined in Table 5.1.

8. Define the function

$$x(t) = \begin{cases} \exp(-.1t) + \exp(-.5t) & 0 \leq t \leq 10 \text{ sec} \\ 0 & \text{otherwise} \end{cases}$$

Compute the CTFT energy E of this function using Eq. (2.30). If $x(t)$ is sampled at intervals of T seconds starting at $t = 0$, what value of T is required so that the DTFT energy equation (2.51) approximates the CTFT energy within 1%? What value of T is required if the match is to be within .01%?

9. Let $x[n] = A\exp(-j2\pi nk/N)$. Show that Eq. (2.63) will yield the power of the complex sinusoid $x[n]$.

10. Prove the energy theorem (2.55) for the discrete-time Fourier series.

Chapter 3

1. Prove that a matrix that is both symmetric and persymmetric is also centrosymmetric, but the converse is not true. Construct such a matrix. Also prove that a matrix that possesses any two of the symmetric, persymmetric, centrosymmetric properties will also have the remaining property.

2. Prove that $(\mathbf{AB})^H = \mathbf{B}^H\mathbf{A}^H$.

3. Define the inner product relationship for block vectors.

4. If $\mathbf{H}$ is a Hankel matrix, show that $\mathbf{JH}$ or $\mathbf{HJ}$ are Toeplitz matrices.

5. Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{AB}$ are nonsingular.

6. Prove that the inverse of a diagonal matrix is a diagonal matrix with main diagonal elements $1/d[i]$.

7. If $\mathbf{A}$ is a square matrix of size $n \times n$ and $\mathbf{x}$ is a column vector of size $n \times 1$, then show that $\mathbf{x}^H \mathbf{A} \mathbf{x}$ is a non-negative scalar.

8. Prove that $(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H$.

9. Prove that if $\mathbf{A}$ is symmetric, then $\mathbf{A}^{-1}$ is symmetric (if inverse exists). Prove that if $\mathbf{A}$ is persymmetric, then $\mathbf{A}^{-1}$ is persymmetric. Prove that if $\mathbf{A}$ is Toeplitz, then $\mathbf{A}^{-1}$ is persymmetric. Prove that if $\mathbf{A}$ is Hankel, then $\mathbf{A}^{-1}$ is symmetric. Prove that if $\mathbf{A}$ is centrosymmetric, then $\mathbf{A}^{-1}$ is centrosymmetric.

10. Show that an upper triangular matrix $\mathbf{A}$ with nonzero diagonal elements has linearly independent column vectors, and is, therefore, nonsingular.

11. Show that if $\mathbf{A}$ has rank n, then the rank of $\mathbf{A}^H$ is also n.

12. Show that the determinant of the square Vandermonde matrix

$$\mathbf{V} = \begin{pmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \ldots & x_n^{n-1} \end{pmatrix}$$

is given by

$$\det \mathbf{V} = \prod_{i \neq j} (x_j - x_i) \quad .$$

13. An $n \times n$ symmetric tridiagonal Toeplitz matrix has the banded form

$$\begin{pmatrix} b & a & 0 & \ldots & 0 \\ a & b & a & \ddots & \vdots \\ 0 & a & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a \\ 0 & \ldots & 0 & a & b \end{pmatrix} .$$

Show that it has eigenvalues $\lambda_k = a + 2b \cos(k\pi/n+1)$ for $k = 1$ to n and corresponding eigenvectors

$$\mathbf{v}_k = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin(k\pi/n+1) \\ \vdots \\ \sin(kn\pi/n+1) \end{pmatrix} .$$

14. Show how the solution $\mathbf{x}$ to the linear equation $\mathbf{C}\mathbf{x} = \mathbf{b}$, in which $\mathbf{C}$ is circulant, can be obtained with three FFT operations. How does the use of left- or right-circulant matrices change whether forward or inverse FFTs are used?

15. Show that if $\mathbf{C}$ is centrosymmetric, then $\mathbf{JC} = \mathbf{CJ}$.

16. Prove Eq. (3.160). Hint: Use the partitions

$$\mathbf{T}_M = \begin{pmatrix} t[0] & \mathbf{s}_M^T \\ \mathbf{r}_M & \mathbf{T}_{M-1} \end{pmatrix}$$

$$\mathbf{T}_M^{-1} = \mathbf{U}_M = \begin{pmatrix} h & \mathbf{g}^T \\ \mathbf{F} & \mathbf{E} \end{pmatrix}$$

and then make use of the matrix inversion lemma, Eq. (3.52). Note that $h = u[0,0] = 1/\rho_M$.

17. Investigate the properties of polynomial roots when a matrix is Hermitian Toeplitz. Show, for example, that the eigenpolynomial has complex conjugate symmetry and that the roots are of unit modulus.

18. The Levinson algorithm was shown to solve the equations

$$\mathbf{T}_M \begin{pmatrix} \mathbf{J}\mathbf{b}_M \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_M \\ \rho_M \end{pmatrix}$$

$$\mathbf{T}_M \begin{pmatrix} 1 \\ \mathbf{a}_M \end{pmatrix} = \begin{pmatrix} \rho_M \\ \mathbf{0}_M \end{pmatrix}.$$

Using Eq. (3.139), show that the following relations hold

$$t[0] + \mathbf{s}_M^T \mathbf{a}_M = t[0] + \mathbf{r}_M^T \mathbf{b}_M = \rho_M$$

$$\mathbf{r}_M + \mathbf{T}_{M-1}\mathbf{a}_M = \mathbf{J}\mathbf{s}_M + \mathbf{T}_{M-1}\mathbf{J}\mathbf{b}_M = \mathbf{0}_M.$$

Chapter 4

1. Let $x[n] = A\sin(2\pi fnT + \theta)$ be a sinusoid process. Using Eq. (4.56), determine the temporal autocorrelation in the limit as $M \to \infty$. Is the sinusoid an ergodic process, i.e., does the temporal autocorrelation yield the same result given by Eq. (4.49)?

2. Show that a wide-sense stationary process has a complex conjugate even autocorrelation, $r_{xx}(\tau) = r_{xx}^*(-\tau)$.

3. For $x(t)$ and $y(t)$ real-valued processes, prove that $\mathsf{P}_{xx}(z) = \mathsf{P}_{xx}(1/z), \mathsf{P}_{xy}(z) = \mathsf{P}^*_{yx}(1/z^*)$.

4. For a deterministic signal as described in Chap. 2, show that the energy spectral densities (ESDs) of the input and output $y(t)$ of a linear filter are related by
$$|Y(f)|^2 = |X(f)|^2|H(f)|^2 \quad .$$
Use this result and the direct definition (4.58) of the power spectral density to show that the input and output power spectral densities have the relationship
$$P_{yy}(f) = P_{xx}(f)|H(f)|^2.$$
This is an alternative proof of relationship (4.43).

5. Prove that power spectral density $P_{xx}(f)$ is a real, even, positive function if $r[m]$ is real-valued.

6. Prove the relationships given by Eq. (4.41).

7. Let the real L sinusoid-plus-noise process be
$$x[n] = \sum_{l=1}^{L} A_l \sin(2\pi f_l nT + \theta_l) + w[n] \quad ,$$
where the initial phases are all uniformly-distributed independent random variables on the interval 0 to 2π and $w[n]$ is a white noise process of variance ρ_w. Compute the statistical autocorrelation sequence. Is this random process ergodic?

8. Show that the autocorrelation matrix for the signal described in previous problem can be expressed as
$$\mathbf{R}_{xx} = \sum_{l=1}^{L} \frac{A_l^2}{4} \Big[\mathbf{e}_M(f_l)\mathbf{e}_M^H(f_l) + \mathbf{e}_M^*(f_l)\mathbf{e}_M^T(f_l) \Big] + \rho_w \mathbf{I}.$$

Chapter 5

1. Show that the Bartlett periodogram estimator [Eq. (5.35)] can be expressed in the matrix form
$$\hat{P}_{\mathrm{B}}(f) = \frac{T}{DP}\mathbf{e}_D^H(f) \left(\sum_{p=0}^{P} \mathbf{x}^{(p)} \mathbf{x}^{(p)\,H} \right) \mathbf{e}_D(f) \quad ,$$
in which $\mathbf{e}_D(f)$ is a complex sinusoid vector [see definition (3.21)] and $\mathbf{x}^{(p)} = \Big(x[pD] \quad x[pD+1] \quad \ldots \quad x[pD+D-1] \Big)^T$ is the vector of data samples for the pth segment.

2. Prove the bias result shown in the text for Eq. (5.44).

3. Show that the matrix (5.25) of the biased autocorrelation estimate may alternatively be written as

$$\check{\mathbf{R}}_L = \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{x}_L[k]\mathbf{x}_L[k]^H \quad ,$$

in which $\mathbf{x}_L[k] = \begin{pmatrix} x[k] & x[k-1] & \dots & x[k-L] \end{pmatrix}^T$ is a data vector. Assume $x[n] = 0$ for $n < 0$ or $n > N-1$.

4. Show that for real $\hat{r}_{xx}[m]$ that

$$\hat{P}_{\mathrm{BT}}(f) = T\,\hat{r}_{xx}[0] + 2T \sum_{m=1}^{M} \hat{r}_{xx}[m] \cos(2\pi f m T) \quad .$$

5. Derive the B_s and B_e bandwidths for the windows of Table 5.1, normalized to an FFT bin of $1/NT$ Hz. What is the ratio $\alpha = B_s/B_e$ for each window?

6. What is the quality ratio Q for the sample spectrum of Section 4.7? What is the QT_eB_s product for the sample spectrum?

7. The correlogram-based spectral estimator of Fig. 5.4 possessed negative PSD lobes. Choose or design a window that will yield only a positive PSD for all frequencies for the test data case. Insert this window in MATLAB function `correlogram_psd.m`. Plot the resultant spectral estimate. How does this estimate compare with the estimate of Fig. 5.4?

8. Assuming N data samples $x[0], \dots, x[N-1]$, show that the Blackman-Tukey PSD estimator

$$\check{P}_{xx}(f) = T \sum_{m=-(N-1)}^{N-1} \check{r}_{xx}[m] \exp(-j2\pi f m T) \quad ,$$

that uses the biased autocorrelation estimator for the maximum number of possible lags, and the sample spectrum

$$\tilde{P}_{xx}(f) = \frac{T}{N} \left| \sum_{n=0}^{N-1} x[n] \exp(-j2\pi f n T) \right|^2$$

are identical.

Chapter 6

1. Find an explicit relationship for $r_{xx}[m]$ in terms of $a[1]$ and $b[1]$ assuming an ARMA(1,1) process.

2. Take the inverse z-transform of Eq. (6.7) to prove the validity of Eq. (6.29).

3. Using Eq. (6.24), find matrix expressions comparable to Eqs. (6.18) and (6.19) to relate the MA(∞) parameters to the ARMA(p, q) parameters.

4. Let $\mathsf{A}(z) = 1+.7z^{-1}+.2z^{-2}$ represent the z-transform of an AR(2) process. Approximate this AR(2) by an MA(2), an MA(4), and an MA(10). Plot the results. How good is the MA approximation?

5. Prove that the autocorrelation matrix of the AR Yule-Walker equations (6.32) is positive semidefinite.

6. Suppose a stable ARMA(p, q) filter is approximated by an AR($p + q$) using the Pade approximation. Show that the AR approximation is not guaranteed to be stable.

7. Show that if z_i is a root of $\mathsf{A}(z)$, the polynomial defined by Eq. (6.4), then $(1/z_i)^*$ is a root of $\mathsf{A}^*(1/z^*)$.

Chapter 7

1. Show that any set of numbers $\{\rho_p, \mathsf{k}_1, \ldots, \mathsf{k}_p\}$ such that $\rho_p > 0$ and $|\mathsf{k}_i| < 1$ will uniquely determine a valid autocorrelation sequence.

2. Assume that $\{r_{xx}[0], \ldots, r_{xx}[m]\}$ is a valid autocorrelation sequence. Show that the Toeplitz autocorrelation matrices are related by

$$\det \mathbf{R}_{m+1} = -(\det \mathbf{R}_{m-1}) r^2[m+1] + \beta r_{xx}[m+1] + \alpha \quad ,$$

   where β and α are functions of $\det \mathbf{R}_{m-1}$ and $\sum a_m[i] r_{xx}[m+1-i]$. Show that as a function of $r[m+1]$, for given $\{r_{xx}[0], \ldots, r_{xx}[m]\}$, $\det \mathbf{R}_{m+1}$ has a single maximum and, therefore, that the admissible amplitude range of $r_{xx}[m+1]$ is $2\rho_m^f$, which is a non-increasing function of m. Show that by choosing $r_{xx}[m+1]$ as the midpoint of the admissible range

$$r_{xx}[m+1] = -\sum_{i=1}^{m} a_m[i] r_{xx}[m+1-i]$$

   yields $\mathsf{k}_{m+1} = 0$ and $\rho_{m+1}^f = \rho_m^f$. Show that this maximizes $\det \mathbf{R}_{m+1}$.

3. Prove that $|\mathsf{k}_m| \leq 1$ using Eq. (7.22).

4. Prove the recursive order relationship between the forward and backward linear prediction errors given by Eqs. (7.26) and (7.27).

5. Use Eq. (7.36) to prove Eq. (7.37).

6. Find the mapping from the autoregressive parameter sequence to the autocorrelation sequence, as shown on Fig. 7.3.

7. The Levinson recursion Eq. (7.17) can be viewed as a mapping of a set of p reflection coefficients k_i into a set of p linear prediction filter coefficients $a[i]$.

   (a) Prove that such a mapping is one-to-one.

   (b) Derive the descending-order (step-down) Levinson recursion that maps a set of linear prediction filter coefficients into a set of reflection coefficients.

   (c) State a stability test for all-pole filters in terms of the reflection coefficients based on the stability test for the linear prediction coefficients.

   (d) Is the filter

   $$H(z) = \frac{1}{1 - 2z^{-1} - 6z^{-2} + z^{-3} - 2z^{-4}}$$

   stable?

8. Show that the lattice filter is an orthogonalizing filter by proving that the backward linear prediction errors are orthogonal to each other in the lattice, i.e.,

   $$\mathcal{E}\{e^b[m]e^{b\,*}[n]\} = \rho_w \delta[m-n] \quad .$$

   Hint: use Eq. (7.27).

9. The analysis of an AR(p) spectrum of a process consisting of M complex sinusoids in additive white noise may be simplified whenever $p > M$ by creating a reduced-order equation set [Satorius and Zeidler, 1978; Kay, 1987]. Using the analytic expression, Eq. (4.52), for the autocorrelation function of M complex sinusoids in additive white noise, show that the autoregressive parameters satisfy the relationship

   $$a_p[k] = \sum_{i=1}^{M} \gamma_i \exp(j2\pi f_i[k-1]T)$$

   for $1 \leq k \leq p$ and $p > M$, where

   $$\gamma_m + \sum_{\substack{n=1 \\ n \neq m}}^{M} c_{mn}\gamma_n = -\frac{P_m}{pP_m + \rho_w} \exp(j2\pi f_m T)$$

and

$$c_{mn} = \frac{P_m}{pP_m + \rho_w} \left(\frac{1 - \exp(j2\pi[f_n - f_m]pT)}{1 - \exp(j2\pi[f_n - f_m]T)} \right) \quad .$$

Also show that

$$\rho_p = \rho_w \left[1 - \sum_{i=1}^{M} \gamma_i \exp(-j2\pi f_i T) \right] \quad .$$

[Hint: Substitute the vector form of the autocorrelation sequence into the Yule-Walker equations.] This procedure replaces the p Yule-Walker equations with a smaller set of M equations in the γ_i coefficients.

10. Find the step-down recursion that relates the variance ρ_{m-1} and AR parameters $a_{m-1}[k]$ at order $m-1$ to the order m variance ρ_m and AR parameters $a_m[k]$.

11. Prove relationships (7.26) and (7.27) by showing that the z-transform between input and output of each stage of the lattice filter satisfies

$$\begin{aligned} E_m^f(z) &= E_{m-1}^f(z) + \mathsf{k}_m z^{-1} E_{m-1}^b(z) \\ E_m^b(z) &= \mathsf{k}_m^* E_{m-1}^f(z) + z^{-1} E_{m-1}^b(z) \end{aligned}$$

in which $E_m^f(z) = Z\{e_m^f[n]\}$, $E_m^b(z) = Z\{e_m^b[n]\}$, and $E_0^f(z) = E_0^b(z) = X(z)$.

Chapter 8

1. Incorporate the FPE order selection criterion into the four autoregressive estimation algorithms shown in Fig. 8.1. Run the 64-point test sequence and note the order selected by each algorithm. How do these compare? Why are there differences? Now incorporate the CAT order selection criterion. What orders are selected by the CAT?

2. Derive a *descending*-order recursion for the Burg algorithm; that is, given the order p solution of the AR coefficients, find the order $p-1$ solution.

3. Show how to incorporate a correlation lag estimate within the Burg order-recursive algorithm. Hint: Use Eq. (8.3).

4. Rewrite the Burg algorithm within Matlab function `lattice.m` to save only the reflection coefficients and omitting the AR parameters. From the FPE criteria, find the best order. Then run the Levinson recursion to get the AR parameters for the best order. Show that this can be done in two steps without additional computational cost.

5. Prove Eq. (8.3).

6. Given the reflection coefficient sequence, find the recursion that yields the correlation sequence.

7. In the Burg algorithm implementation in Matlab function `lattice.m`, why is the statement DEN=P*2 used, rather than DEN=P?

8. In the covariance algorithm, prove the following:

$$\rho_p^f = \mathbf{a}_p^H \mathbf{R}_p \mathbf{a}_p = \rho_{p-1}^{f\prime}(1 - a_p[p]b_p[p])$$

$$\rho_p^b = \mathbf{b}_p^H \mathbf{R}_p \mathbf{b}_p = \rho_{p-1}^{b\prime\prime}(1 - b_[p]a_p[p])$$

$$[e_p^f(p+1)]^* = -\mathbf{r}_p^H \mathbf{d}_{p-1}'' + x^*(p+1)$$

$$[e_p^b(N)]^* = -\mathbf{s}_p^H \mathbf{c}_{p-1}' + x^*(N-p) \quad .$$

9. Prove that $\nabla_p = \Delta_p^*$ in the covariance linear prediction algorithm.

10. Show that $\delta_p, \gamma_p \to 0$ with increasing order p in the covariance algorithm.

11. Both the Burg and the modified covariance algorithms use the sum of the forward and backward linear prediction squared errors. In what sense is the Burg algorithm a constrained least squares problem and the modified covariance method an unconstrained least squares problem?

12. Use Eqs. (8.60) and (8.112) to prove that the $\mathbf{R}_p$ matrices of the covariance and modified covariance methods are positive semidefinite.

13. Prove Eq. (8.16).

14. Express $\mathbf{R}_p$ of Eq. (8.28) in terms of $\mathbf{T}_p$, $\mathbf{L}_p$, and $\mathbf{U}_p$.

15. Prove that matrix $\mathbf{R}_p$ in Eq. (8.28) for the autocorrelation method is positive semidefinite.

Chapter 9

1. Prove Eq. (9.29).

2. Prove that prewindowed method yields a stable filter, as long as $\mathbf{R}_{p,N}$ is invertible.

3. Prove that $0 \leq \gamma_{p,N} \leq 1$ in the fast RLS algorithm, and that $\gamma_{p,N}$ is real. Also show that similar results hold for $\gamma_{p,N-1}$ and $\gamma_{p-1,N+1}$.

4. Prove that

$$\gamma_{p,N+1} = \frac{\det \mathbf{R}_{p,N+1}}{\det \mathbf{R}_{p,N}}, \rho^f_{p,N+1} = \frac{\det \mathbf{R}_{p+1,N+1}}{\det \mathbf{R}_{p,N}}, \rho^b_{p,N+1} = \frac{\det \mathbf{R}_{p+1,N+1}}{\det \mathbf{R}_{p,N+1}}.$$

This means γ,ρ^f,$\rho^b > 0$ guarantee the invertibility of $\mathbf{R}_{p,N}$.

5. Define $\psi_{p,N} = \gamma^{-1}_{p,N}$. Then show that alternative updates to Eqs. (9.58) and (9.60) have the form

$$\psi_{p,N+1} = \left(\frac{\omega \rho^f_{p,N}}{\rho^f_{p,N+1}} \right) \psi_{p-1,N}$$

$$\begin{aligned} \psi_{p-1,N+1} \quad &= \psi_{p,N+1} \left(1 - \psi_{p,N+1} c_{p,N+1}[p] e^b_{p,N}[N+1]\right)^{-1} \\ &= \psi_{p,N+1} \left(2 - \frac{\rho^b_{p,N+1}}{\omega \rho^b_{p,N}} \right) \quad . \end{aligned}$$

Show how to change the implementation of Matlab function `fastrls.m` to use PSI rather than GAMMA.

Chapter 10

1. Prove Eq. (10.6).

2. How is a MA PSD estimator like the correlogram method PSD estimator of Chap. 5?

3. Show by counterexample that the modified Yule-Walker equations do not generally produce minimum phase AR parameter sequences, even when the ACS is exactly known.

4. Compute the AIC$[p, q]$ for $0 \leq p, q \leq 20$ using Matlab function `arma_psd.m` and the test sequence `test1987.dat`. Where is the AIC minimum? Plot the ARMA spectrum for the orders (p, q) at the minimum.

5. Substitute Matlab function `covariance_lp.m` for Matlab function `yule_walker.m` in function `ma.m`. Compute and plot an MA(15) PSD estimate using the `test1987.dat`. What differences are found between the spectral estimate of Fig. 10.2 and the plot made using `covariance_lp.m` in function `ma.m`? Why are there differences?

Chapter 11

1. Prove that $\mathbf{Z}^H\mathbf{Z}$ in Eq. (11.26) is positive semidefinite.

2. If a multiple zero is encountered when factoring the polynomial, this gives rise to the product of a polynomial with the exponential. How can the Prony method be adjusted to handle this?

3. Show that the AR and ARMA processes have autocorrelation sequences that are representable as sums of damped exponentials. Show how to apply Prony's method to the *autocorrelation sequence* (not the data sequence) to get the MA and AR coefficients.

4. Prove that the sum of p exponentials may also be generated using backward linear prediction if the time direction is reversed. How do the roots of the characteristic polynomial relate to the exponents of the p exponentials?

5. Define the exponential signal $x[n] = \exp(-.2nT)\sin(2\pi .05nT)$. Let $T = 1.$ and choose 20 samples for $0 \leq n \leq 19$. Compute the least squares Prony estimate of the damping. Now let $T = .5$ and choose 40 samples for $0 \leq n \leq 39$. Recompute the Prony estimate. How do the two estimates differ? Why? (see Kulp [1981] for more details of the effect of sampling rate on the accuracy of Prony's method).

6. Using Eq. (11.49), show that the inverse transform of $\hat{\mathsf{X}}_1(z)$ is an ARMA model with order $p = q$.

7. Prove matrix expression (11.44).

8. Let $\hat{x}[n] = a\exp[sn]$ be a single-exponential approximation for a real a and a real $s = \alpha T$, such that $\alpha < 0$, and defined over the index range 0 to $N-1$. Form $e[n] = x[n] - \hat{x}[n]$. Minimize

$$\sum_{n=0}^{N-1} e^2[n]$$

   by setting the derivative with respect to a and s to zero. Find an analytic solution for both a and s. (This example shows how nonlinear and difficult the general minimization problem is, even for a real data case.)

9. Show that for real data, Eq. (11.91) reduces to two linear equations in two real unknowns α_2 and α_3. Find explicit solutions for α_2 and α_3.

10. In this problem, the periodogram method of spectral estimation will be viewed as a special case of the Prony method in which a harmonic model

of preselected frequencies is used to fit to the data. Assume that the time-series model is

$$\hat{x}(nT) = \hat{x}[n] = \sum_{m=0}^{M-1} a_m \exp(j2\pi f_m nT)$$

for the data samples $n = 0, \ldots, N-1$, such that $M < N$. Using the preselected harmonic frequencies $f_m = m/NT$ for $0 \leq m \leq M-1$ and Eq. (11.26), show that the least squares solution for the complex sinusoidal amplitudes a_m is given by

$$a_m = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp(-j2\pi mn/N)$$

for $m = 0, \ldots, M-1$. This, of course, is the usual discrete-time Fourier series when $M = N$ and $T = 1$.

Chapter 12

1. Prove Eqs. (12.20) and (12.25).

2. Prove that

$$\frac{1}{P_{\mathrm{AR}}(p,f)} = \frac{1}{P_{\mathrm{MV}}(p,f)} - \frac{1}{P_{\mathrm{MV}}(p-1,f)} \quad .$$

3. Substitute Matlab function `yulewalker.m` for function `lattice.m` in Matlab function `minimum_variance_psd.m`. Compute a spectral estimate with this modified minimum variance PSD program using the `test1987.dat`. How does the resulting spectral plot compare with that of Fig. 12.2? Account for any differences.

4. Using the analytic form of the autocorrelation sequence up to lag p for a single sinusoid in white noise [Eq. (12.12)], develop the analytic form of the solution for (12.8) for the minimum variance filter coefficients. Plot the filter response for the case $p = 10$. Note the sidelobes. How could a weighting of the autocorrelation sequence be used to suppress the sidelobes?

Chapter 13

1. Using the case of M noiseless complex sinusoids $x[n] = \sum_{i=1}^{M} \sin(2\pi f_i nT)$ and an order $p = M$, prove that the zeros of the polynomial formed from the prediction error filter of the modified covariance data matrix (8.48) will be on the unit circle at the sinusoid frequencies.

2. Show for AR of Eq. (13.34) that the signal subspace version is

$$\mathbf{a}_p = -\sum_{k=1}^{M}(\alpha_k/\lambda_k)\mathbf{v}_k$$

where $\alpha_k = \mathbf{v}_k^H \mathbf{r}_p$ is the inner product of the eigenvector $\mathbf{v}_k$ with $\mathbf{r}_p = [R_{xx}[1] \quad \ldots \, r_{xx}[p]]^T$.

3. Prove that $\hat{\rho}_w = r[0] - \sum \hat{P}_i$. This can be used as a numerical check on the eigenvalue given by Matlab function `minimum_eigenvalue.m`.

4. Show how the MUSIC frequency estimator can be expressed in terms of the signal subspace, rather than the noise subspace, eigenvectors. Use Eq. (13.17) to prove this.

Chapter 15

1. Prove Eq. (15.79), i.e., show that

$$\mathbf{\Delta}_{p+1} = \mathcal{E}\{\mathbf{e}_p^f[n]\mathbf{e}_p^{b\,H}[n-1]\} \quad .$$

2. Show that the residual covariance matrices $\boldsymbol{P}_p{}^f$ and $\boldsymbol{P}_p{}^b$ satisfy the order update relationships

$$\boldsymbol{P}_p{}^f = \boldsymbol{P}_{p-1}^f - \left[(\boldsymbol{P}_{p-1}^f)^{1/2}\mathbf{\Lambda}_p\right]\left[(\boldsymbol{P}_{p-1}^f)^{1/2}\mathbf{\Lambda}_p\right]^H$$

$$\boldsymbol{P}_p{}^b = \boldsymbol{P}_{p-1}^b - \left[(\boldsymbol{P}_{p-1}^b)^{1/2}\mathbf{\Lambda}_p^H\right]\left[(\boldsymbol{P}_{p-1}^b)^{1/2}\mathbf{\Lambda}_p^H\right]^H$$

3. Show for the multichannel case that a one-to-one correspondence exists between $\{\mathbf{R}_{xx}[0], \mathbf{R}_{xx}[1], \ldots, \mathbf{R}_{xx}[p]\}$ and the set $\{\mathbf{R}_{xx}[0], \mathbf{\Lambda}_1, \ldots, \mathbf{\Lambda}_p\}$. Either sequence constitutes a parameterization of the autoregressive sequence $\left\{\boldsymbol{P}_p^f, \mathbf{A}_p[1], \ldots, \mathbf{A}_p[p]\right\}$.

4. Define the normalized residuals

$$\tilde{\mathbf{e}}_p^f[n] = (\hat{\boldsymbol{P}}_{p-1}^f{}^{1/2})^{-1}\mathbf{e}_p^f[n] = \tilde{\mathbf{e}}_{p-1}^f - \mathbf{\Lambda}_p\tilde{\mathbf{e}}_{p-1}^b$$

$$\tilde{\mathbf{e}}_p^b[n] = (\hat{\boldsymbol{P}}_{p-1}^b{}^{1/2})^{-1}\mathbf{e}_p^b[n] = \tilde{\mathbf{e}}_{p-1}^b - \mathbf{\Lambda}_p^H\tilde{\mathbf{e}}_{p-1}^f$$

that use the previous error covariance as normalizing weights. Show that minimizing the arithmetic mean of the weighted residuals

$$\mathrm{tr}\left[\sum_{n=p+1}^{N} \tilde{\mathbf{e}}_p^f[n]\tilde{\mathbf{e}}_p^{f\,H}[n] + \tilde{\mathbf{e}}_p^b[n]\tilde{\mathbf{e}}_p^{b\,H}[n]\right]$$

with respect to $\boldsymbol{\Lambda}_{p+1}$ yields the normalized partial correlation of Eq. (15.88).

5. Prove that Eqs. (15.121) and (15.122) form a valid alternative expression for the multichannel minimum variance spectral estimate. Hint: use the concepts in Sec. 12.4 and relationship (15.86) for the inverse of a block-Toeplitz matrix.

6. In the multichannel Levinson algorithm, $\mathbf{B}_p[p] \neq \mathbf{A}_p^H[p]$. However, show that
$$\det \mathbf{B}_p[p] = \det \mathbf{A}_p^H[p]$$
does hold. Also show that
$$\operatorname{tr} \mathbf{A}_p[1] = \operatorname{tr} \mathbf{B}_p^H[1]$$
and that
$$\det \mathbf{R}_p = \prod_{n=1}^{p} \det \boldsymbol{P}_n{}^f = \prod_{n=1}^{p} \det \boldsymbol{P}_n{}^b.$$

7. Develop the fast algorithm to solve the multichannel covariance linear prediction normal equations. Use the same concepts as employed with the single-channel covariance algorithm in Section 8.10.

8. Show that the selection of $\mathbf{V}_p = \mathbf{W}_p = \mathbf{I}$ in the Nuttall-Strand algorithm does not guarantee a stable correlation sequence or a positive-definite spectral estimate.

9. Analytically determine the pole and zero locations of the two-channel example of Eq. (15.118). Find an analytic expression for the MSC. Show that it has four poles and four zeros in the finite z-plane. What is its maximum value?

10. Prove that filtering the X and Y channels with the same filter will yield the same MSC for input and output.

11. Determine the computational operation count of the two algorithms in subroutine MCAR as a function of number of channels, number of data points, and order. How does the Nuttall-Strand algorithm compare with the operation count provided for BURG algorithm in Chap. 8?

12. Prove Eq. (15.13).

13. Why is the matrix of Eq. (15.18) not Hermitian, except for $\mathbf{R}_{xx}[0]$?

14. Prove Eqs. (15.22) and (15.23).

15. Using principles from Chap. 6, show that the multichannel Yule-Walker equations for a multichannel ARMA(p, q) are

$$\mathbf{R}_{xx}[m] = \begin{cases} -\sum_{k=1}^{p} \mathbf{A}[k]\mathbf{R}_{xx}[m-k] + \sum_{k=m}^{q} \mathbf{B}[k]\boldsymbol{P}_w\mathbf{H}^H[k-m] \\ \text{for } 0 \le m \le q \\ -\sum_{k=1}^{p} \mathbf{A}[k]\mathbf{R}_{xx}[m-k] \\ \texttt{for } m > q \quad . \end{cases}$$

What is the block-matrix structure of the modified Yule-Walker equations for the multichannel ARMA case?

Chapter 16

1. A separable 2-D sequence is one that can be expressed as the product of two independent sequences, $x[m,n] = x_1[m]x_2[n]$. Prove that the 2-D convolution of a separable sequence is also separable.

2. Prove that the 2-D DTFS is a 2-D periodic function.

3. Show that the frequency response function for a rectangular impulse function

$$h[m,n] = \begin{cases} 1 & \text{for } |m| \le a/2 \quad , \quad |n| \le b/2 \\ 0 & \text{otherwise} \end{cases}$$

is

$$H(f_1, f_2) = \text{sinc}(f_1/a), sinc(f_2/b)/ab.$$

4. Prove that the 2-D autocorrelation term $r_{xx}[0,0]$ is real and positive, and satisfies

$$r_{xx}[0,0] \ge |r_{xx}[m,n]|$$

for all m and n. Furthermore, show that the 2-D autocorrelation is Hermitian symmetric, $r^*_{xx}[m,n] = r_{xx}[-m,-n]$.

5. Show that the 2-D PSD as defined by Eq. (16.26) is real and positive. Also show that $P(f_1, f_2) = P(-f_1, -f_2)$ if the 2-D ACS is real.

6. Suppose the following samples of a 2-D ACS are known:

$$\begin{aligned} r_{xx}[0,0] &= 1 \\ r_{xx}[1,0] &= r_{xx}[-1,0] = \alpha \\ r_{xx}[0,1] &= r_{xx}[0,-1] = \beta \\ r_{xx}[1,1] &= r_{xx}[1,-1] = r_{xx}[-1,1] = r_{xx}[-1,-1] = 0 \quad , \end{aligned}$$

where α and β are real-valued parameters. Compute and plot the 2-D correlogram method PSD estimate.

7. Using the same 2-D ACS values of Problem 6, form the 4×4 autocorrelation matrix of Eq. (16.51) and analytically determine its inverse. Form the minimum variance spectral estimator, Eq. (16.91), and plot the PSD.

8. Using the same 2-D ACS values of Problem 6 and the autocorrelation matrix of Problem 7, analytically determine the first- and second-quadrant QP autoregressive parameters. Compute and plot the first-quadrant, second-quadrant, and combined-quadrant AR PSD estimates [Eqs. (16.84), (16.85), and (16.86)].

9. Explain why it requires more 2-D ACS samples than unknown 2-D AR parameters, in contrast to the 1-D case, in which the number of ACS samples was one more than the number of 1-D AR parameters.

10. Write the matrix expression for the Yule-Walker equations of the causal NSHP region of support [Jain and Ranganath, 1981].

11. Develop the normal equations for the 2-D modified covariance method of 2-D linear prediction for a first-quadrant linear prediction region of support.

12. Show that the 2-D minimum variance spectral estimator can be expressed in the form

$$\hat{P}_{\mathrm{MV}}(f_1, f_2) = \frac{T_1 T_2}{\sum_{k=-p}^{p} \sum_{l=-q}^{q} \alpha[k,l] \exp(-j2\pi[f_1 k T_1 + f_2 l T_2])}.$$

    What is the relationship of the coefficients $\alpha[k,l]$ to the elements of the inverse autocorrelation matrix?

13. Derive a 2-D version of Eq. (4.53) for the 2-D block autocorrelation matrix of M 2-D complex sinusoids in 2-D white noise. What would be the 2-D form of the MUSIC frequency estimator of Chap. 13 that would be suggested by this block autocorrelation matrix?

14. Prove the relationships of Eqs. (16.78) and (16.79).