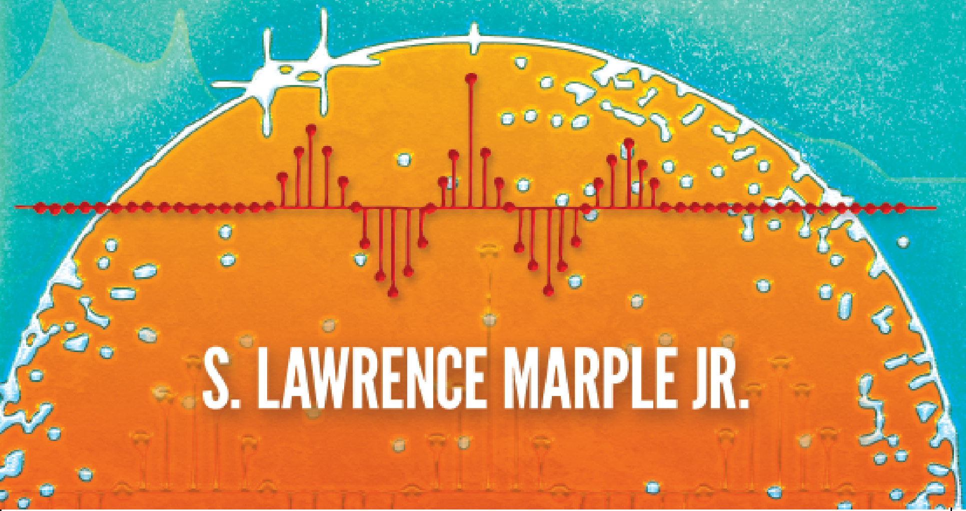
DIGITAL SPECTRAL ANALYSIS

SELECTED PROBLEM SOLUTIONS

Second Edition



S. LAWRENCE MARPLE JR.

$$E_{CT} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{10} \left[e^{-.1t} + e^{-.5t}\right]^2 dt = \int_0^{10} \left[e^{-.2t} + 2e^{-.6t} + e^{-t}\right] dt$$

$$E_{CT} = \left[\frac{1}{-.2} e^{-.2t} + \frac{2}{-.6} e^{-.6t} - e^{-t}\right]\Bigg|_0^{10} = 8.6483\ldots$$

Problem 2 Number 4

$$E_{DT} = T_s \sum_{n=0}^{N-1} \left[e^{-.1nT_s} + e^{-.5nT_s}\right]^2 \qquad \text{where } (N-1)T_s \le 10 \text{ sec} < NT_s$$

$$E_{DT} = T_s \left[\frac{1-(e^{-.2T_s})^N}{1-(e^{-.2T_s})} + 2\,\frac{1-(e^{-.6T_s})^N}{1-(e^{-.6T_s})} + \frac{1-e^{-T_sN}}{1-e^{-T_s}}\right]$$

Following MATLAB script evaluates E_{CT} and E_{DT} and finds the two T_s values that satisfy the problem criteria. The numerical analysis bisection method is used to determine the T_s values.

1% solution: $T_s = 47.43$ ms and E_dt = 8.7348
0.01% solution: $T_s = 4.70$ ms and E_dt = 8.6570

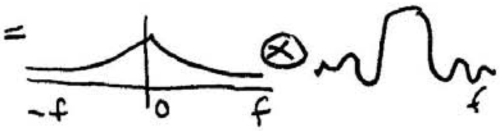
```
% Script to solve [illegible], Problem 4
% Search procedure steps by decade increments through formulas

format long

E_ct= -5*(exp(-2)-1) -(10/3)*(exp(-6)-1) -(exp(-10)-1)

% Find 1% approximation to nearest microsecond
disp('1% solution')
approx=.01;
Tstep=.1;    % starting decimal increment
Ts=1+Tstep+1.e-6;
for k=1:6    % determines number of decimal places of accuracy
    ratio=1;
    while ratio > approx
        Ts=Ts-Tstep;
        N=fix(10/Ts);
        E_dt = Ts*(((1-exp(-.2*Ts*N))/(1-exp(-.2*Ts)))+2*((1-exp(-.6*Ts*N))/(1-exp(-.6*Ts
)))+...
                 ((1-exp(-Ts*N))/(1-exp(-Ts))));
%        disp(['N = ',int2str(N),' , Ts = ',num2str(Ts),' , E_dt = ',num2str(E_dt)])
        ratio=abs((E_dt-E_ct)/E_ct);
    end
    Ts
    E_dt
    Ts=Ts+1.1*Tstep;
    Tstep=Tstep/10;
end

% Find 0.01% approximation to nearest microsecond
disp('0.01% solution')
approx=.001;
Tstep=.1;    % starting decimal increment
Ts=1+Tstep+1.e-6;
for k=1:6    % determines number of decimal places of accuracy
    ratio=1;
    while ratio > approx
        Ts=Ts-Tstep;
        N=fix(10/Ts);
        E_dt = Ts*(((1-exp(-.2*Ts*N))/(1-exp(-.2*Ts)))+2*((1-exp(-.6*Ts*N))/(1-exp(-.6*Ts
)))+...
                 ((1-exp(-Ts*N))/(1-exp(-Ts))));
%        disp(['N = ',int2str(N),' , Ts = ',num2str(Ts),' , E_dt = ',num2str(E_dt)])
        ratio=abs((E_dt-E_ct)/E_ct);
    end
    Ts
    E_dt
    Ts=Ts+1.1*Tstep;
    Tstep=Tstep/10;
end
```

This problem shows the issue of sampling signals that are not bandlimited. The signal in this problem has Fourier transform:

$$e^{-at} \cdot \Pi_{0}^{10\,\text{sec}}(t)$$

$$\longleftrightarrow \left(\frac{1}{a+j2\pi f}\right) \otimes \operatorname{sinc}(10f)$$

=



which has a nonbandlimited character. As one goes higher in frequency, both of the transforms that are convolved roll off to small values where the transform can be considerd "essentially" bandlimited. This point is determined by the % of how close to CT energy one wishes the DT energy to match.

Chapter4, #4 For a linear filter

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) \longleftrightarrow Y(f) = X(f) \cdot H(f)$$

time-domain convolution of x and h   transform domain multiplication of transforms X and H

since $\quad Y^*(f) = \left[X(f) \cdot H(f)\right]^* = X^*(f) \cdot H^*(f)$

thus $\quad |Y(f)|^2 = Y(f) \cdot Y^*(f) = \left[X(f) \cdot H(f)\right] \cdot \left[X^*(f) \cdot H^*(f)\right] = \left[X(f) X^*(f)\right] \cdot \left[H(f) \cdot H^*(f)\right]$

$$|Y(f)|^2 = |X(f)|^2 \cdot |H(f)|^2 \quad (\ddagger) \qquad \text{where } Y(f) = T \sum$$

Instead of the finite-energy continuous-time case above, in which $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$, consider the finite-power discrete-time case, which has infinite energy over the infinite interval

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \infty$$

but has finite energy over a finite interval $-M$ to M, $\quad x_M[n] = \begin{cases} x[n] & |n| \le M \\ 0 & |n| > M \end{cases}$

$$\sum_{n=-\infty}^{\infty} |x_M[n]|^2 = \sum_{n=-M}^{M} |x[n]|^2 < \infty$$

When filtered (convolved) with a finite-energy infinite impulse response $h[n]$ for $-\infty < n < \infty$

$$y_M[n] = x_M[n] * h[n] = \sum_{k=-M}^{M} h[n-k]\, x[k]$$

the resultant output sequence $y_M[n]$ is infinite duration, but finite energy. Thus,

$$P_{yy}(f) = \lim_{M \to \infty} \mathcal{E}\left\{ \frac{1}{(2M+1)T} \left| \text{DTFT}\{ y_M[n] \} \right|^2 \right\}$$

$$= \lim_{M \to \infty} \mathcal{E}\left\{ \frac{1}{(2M+1)T} \left| T \sum_{n=-\infty}^{\infty} y_M[n] \exp(-j2\pi f n T) \right|^2 \right\}$$

$$= \lim_{M \to \infty} \mathcal{E}\left\{ \frac{1}{(2M+1)T} \left| T \sum_{n=-\infty}^{\infty} \left[\sum_{k=-M}^{M} h[n-k]\, x[k] \right] \exp(-j2\pi f n T) \right|^2 \right\}$$

Change of variable, $n' = n-k$ and $-\infty < n' < \infty$, so that

$$P_{yy}(f) = \lim_{M \to \infty} \mathcal{E}\left\{ \frac{1}{(2M+1)T} \left| T \sum_{k=-M}^{M} \left(\sum_{n'=-\infty}^{\infty} h[n'] \exp(-j2\pi f [n'+k] T) \right) x[k] \right|^2 \right\}$$

$$= \lim_{M \to \infty} \mathcal{E}\left\{ \frac{1}{(2M+1)T} \left| T \sum_{k=-M}^{M} x[k] \exp(-j2\pi f k T) \cdot \sum_{n'=-\infty}^{\infty} h[n'] \exp(-j2\pi f n' T) \right|^2 \right\}$$

$$= \lim_{M \to \infty} \mathcal{E}\left\{ \frac{1}{(2M+1)T} \left| H(f) \cdot T \sum_{k=-M}^{M} x[k] \exp(-j2\pi f k T) \right|^2 \right\}$$

does not depend on M and is deterministic

$$= |H(f)|^2 \cdot \lim_{M \to \infty} \mathcal{E}\left\{ \frac{1}{(2M+1)T} \left| T \sum x[k] \exp(-j2\pi f k T) \right|^2 \right\} = |H(f)|^2 \, P_{xx}(f)$$

Chapter 4, # 6

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]$$

Change variables $n \to n'+m$:

$$y[n'+m] = \sum_{k=-\infty}^{\infty} x[n'+m-k]\,h[k]$$

Multiply by $x^*[n']$:

$$y[n'+m]x^*[n'] = \sum_{k=-\infty}^{\infty} \{x[n'+m-k]x^*[n']\}\,h[k]$$

Take expectation of both sides :

$$r_{yx}[m] = \mathcal{E}\{y[n'+m]x^*[n']\} = \mathcal{E}\left\{\sum_{k=-\infty}^{\infty} \{x[n'+m-k]x^*[n']\}\,h[k]\right\}$$

Note that text has indexing error of $k-m$!

$$= \sum_{k=-\infty}^{\infty} \mathcal{E}\{x[n'+m-k]x^*[n']\}\,h[k] = \sum_{k=-\infty}^{\infty} r_{xx}[m-k]\,h[k]$$

Thus

$$\boxed{r_{yx}[m] = r_{xx}[m] * h[m]}$$

Conjugate : $\quad y^*[n] = \sum_{k=-\infty}^{\infty} x^*[n-k]\,h^*[k]$

Multiply by $x[n+m]$:

$$x[n+m]y^*[n] = \sum_{k=-\infty}^{\infty} \{x[n+m]\,x^*[n-k]\}\,h^*[k]$$

Take expectations of both sides:

$$r_{xy}[m] = \mathcal{E}\{x[n+m]y^*[n]\} = \sum_{k=-\infty}^{\infty} \mathcal{E}\{x[n+m]\,x^*[n-k]\}\,h^*[k]$$

$$= \sum_{k=-\infty}^{\infty} r_{xx}[m+k]\,h^*[k]$$

Change variables $k' = -k$, then

$$r_{yx}[m] = \sum_{k'=\infty}^{-\infty} r_{xx}[m-k']\,h^*[-k'] = \boxed{r_{xx}[m] * h^*[-m] = r_{xy}[m]}\ ^{\ddagger}$$

$$y[n+m]\,y^*[n] = \left(\sum_{k=-\infty}^{\infty} x[n+m-k]\,h[k]\right) y^*[n] = \sum_{k=-\infty}^{\infty} \{x[n+m-k]\,y^*[n]\}\,h[k]$$

Take expectation of both sides :

$$r_{yy}[m] = \mathcal{E}\{y[n+m]y^*[n]\} = \sum_{k=-\infty}^{\infty} \mathcal{E}\{x[n+m-k]y^*[n]\}\,h[k] = \sum_{k=-\infty}^{\infty} r_{xy}[m-k]\,h[k]$$

$$\boxed{r_{yy}[m] = r_{xy}[m] * h[m]}$$

Substitute ‡ above to yield :

commutative law of convolution to change order of convolution

$$r_{yy}[m] = \{r_{xx}[m] * h^*[-m]\} * h[m] = r_{xx}[m] * \{h^*[-m] * h[m]\}$$

$$r_{yy}[m] = r_{xx}[m] * \left\{\sum_{-\infty}^{\infty} h[k+m]\,h^*[k]\right\}$$

Chapter 4, # 8

From class notes :

$$x[n] = \sum_{\ell=1}^{L} A_\ell \sin(2\pi f_\ell nT + \theta_\ell) + w[n]$$

$$r_{xx}[m] = \sum_{\ell=1}^{L} \left(\frac{A_\ell^2}{2}\right) \cos(2\pi f_\ell mT) + \rho_w \delta[m]$$

Euler's formula :

$$r_{xx}[m] = \sum_{\ell=1}^{L} \left(\frac{A_\ell^2}{2}\right) \left[\frac{\exp(j2\pi f_\ell mT) + \exp(-j2\pi f_\ell mT)}{2}\right] + \rho_w \delta[m]$$

$$= \sum_{\ell=1}^{L} \left(\frac{A_\ell^2}{4}\right) \exp(j2\pi f_\ell mT) + \left[\sum_{\ell=1}^{L} \left(\frac{A_\ell^2}{4}\right) \exp(j2\pi f_\ell mT)\right]^* + \rho_w \delta[m]$$

Using the results of equation (4.53) in the text :

$$\bar{R}_{xx} = \sum_{\ell=1}^{L} \left(\frac{A_\ell^2}{4}\right) \bar{e}_M(f_\ell)\bar{e}_M^H(f_\ell) + \left[\sum_{\ell=1}^{L} \left(\frac{A_\ell^2}{4}\right) \bar{e}_M(f_\ell)\bar{e}_M^H(f_\ell)\right]^* + \rho_w \bar{I}$$

and therefore

$$\bar{R}_{xx} = \sum_{\ell=1}^{L} \left(\frac{A_\ell^2}{4}\right) \left[\bar{e}_M(f_\ell)\bar{e}_M^H(f_\ell) + \bar{e}_M^*(f_\ell)\bar{e}_M^T(f_\ell)\right] + \rho_w \bar{I}$$

using the fact that conjugate of the conjugate transpose is just transpose : $(H\,*) \rightarrow T$

$\boxed{\text{Chapter 5, \#3}}$ Biased ACS estimate: $\check{r}_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-1-m} x[n+m]\, x^*[n]$

By inspection of above definition, we can see that the Toeplitz ACS matrix formed from the biased ACS estimates will have a structure of [assuming N data values from $n=0$ to $n=N-1$ are measured]:

$$\check{R}_L = \frac{1}{N} \begin{bmatrix} x[0] & \cdots & x[L] & \cdots & x[N-1] & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & & & \ddots & 0 \\ 0 & \cdots & 0 & x[0] & \cdots & x[N-L-1] & \cdots & x[N-1] \end{bmatrix} \begin{bmatrix} x^*[0] & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ x^*(L) & \cdots & \cdots & x^*[0] \\ \vdots & & & \vdots \\ x^*[N-1] & \cdots & \cdots & x^*[N-L-1] \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & x^*[N-1] \end{bmatrix}$$

dimension $(L+1)$ rows $\times$ $(N+L)$ columns

dimension $(N+L)$ rows $\times$ $(L+1)$ columns

column vectors

row vectors

Note typographical error in textbook

$$\check{R}_L = \frac{1}{N} \begin{bmatrix} | & | & & | & | \\ \bar{x}_L[0] & \bar{x}_L[1] & \cdots & \bar{x}_L[N+L-2] & \bar{x}_L[N+L-1] \\ | & | & & | & | \end{bmatrix} \begin{bmatrix} -\bar{x}_L^H[0]- \\ -\bar{x}_L^H[1]- \\ \vdots \\ -\bar{x}_L^H[N+L-2]- \\ -\bar{x}_L^H[N+L-1]- \end{bmatrix} = \frac{1}{N} \sum_{k=0}^{N+L-1} \bar{x}_L[k]\, \bar{x}_L^H[k]$$

where $\bar{x}_L[k] = \begin{bmatrix} x[k] \\ \vdots \\ x[k-L] \end{bmatrix}$ and $x[k]=0$ if $k<0$ or $k \geq N$, by definition

Chapter 5, # 9

$$\tilde{P}_{xx}(f) = \frac{T}{N}\left|\sum_{n=0}^{N-1} x[n]\exp(-j2\pi f n T)\right|^2$$

$$= \frac{T}{N}\left[\sum_{n=0}^{N-1} x[n]\exp(-j2\pi f n T)\right]\left[\sum_{n'=0}^{N-1} x^*[n']\exp(+j2\pi f n' T)\right]$$

$$= \frac{T}{N}\sum_{n=0}^{N-1}\sum_{n'=0}^{N-1} x[n]x^*[n']\exp(-j2\pi f[n-n']T)$$

Change of variables: $n = m+n'$, or $m = n-n'$, and $-(N-1) \le m \le (N-1)$

$$\tilde{P}_{xx}(f) = \frac{T}{N}\sum_{m=-(N-1)}^{(N-1)}\sum_{n'=0}^{N-m-1} x[m+n']x^*[n']\exp(-j2\pi f m T)$$

$$= T\sum_{m=-(N-1)}^{(N-1)}\underbrace{\left[\frac{1}{N}\sum_{n=0}^{N-m-1} x[m+n']x^*[n']\right]}_{\check{r}_{xx}[m]}\exp(-j2\pi f m T)$$

$\boxed{\text{Chapter 6, \#1}}$ ARMA(1,1) $\longrightarrow$ $x[n] = -a[1]x[n-1] + u[n] + b[1]u[n-1]$

$$H(z) = \frac{B(z)}{A(z)} = \frac{1+b[1]z^{-1}}{1+a[1]z^{-1}} = \underset{\substack{\uparrow \\ h[0]}}{1} + \underset{\substack{\uparrow \\ h[1]}}{(b[1]-a[1])} z^{-1} + a[1]\underset{\substack{\uparrow \\ h[2]}}{(b[1]-a[1])} z^{-2} + \cdots$$

from equation (6.29) in text

$$r_{xx}[m] = \begin{cases} -a[1]r^*[1] + \rho_w\left[h^*[0] + b[1]h^*[1]\right] & m=0 \\ -a[1]r[0] + \rho_w b[1]h^*[0] & m=1 \\ -a[1]r[m-1] & m \geq 2 \end{cases}$$

Substituting for $h[0]$ and $h[1]$,

$$r_{xx}[0] = -a[1]r^*[1] + \rho_w\left[1 + b[1]\left(b^*[1] - a^*[1]\right)\right]$$

$$r_{xx}[1] = -a[1]r[0] + \rho_w b[1]$$

$$r_{xx}[m] = -a[1]r_{xx}[m-1] \qquad \text{for } m \geq 2$$

Substituting

$$r_{xx}[0] = -a[1]\Big(-a^*[1]\underbrace{r^*_{xx}[0]}_{r_{xx}[0] \text{ since real-valued by definition}} + \overset{\text{real-valued by definition}}{\rho_w} b^*[1]\Big) + \rho_w\left[1 + b[1]\left(b^*[1] - a^*[1]\right)\right]$$

$$r_{xx}[0] = \frac{\rho_w}{1-|a[1]|^2}\left[\left(1+|b[1]|^2\right) - 2\,\mathrm{Re}\{a[1]b^*[1]\}\right]$$

Similar results for $r_{xx}[1]$. All other $r_{xx}[m]$ then found recursively.

The z-transform of a sequence $x[m]$ is $X(z) = \sum_{m=-\infty}^{\infty} x[m] z^{-m}$. Given a power series of this form, the inverse z-transform may be obtained simply by inspection as $x[m]$ for $-\infty < m < \infty$. By definition

$$P_{xx}(z) = \sum_{k=-\infty}^{\infty} r_{xx}[k] z^{-k} \qquad P_{ww}(z) = \rho_w$$

$$A(z) = \sum_{k=0}^{p} a[k] z^{-k} \quad (a[0]=1) \qquad B(z) = \sum_{k=0}^{q} b[k] z^{-k} \quad (b[0]=1)$$

then

$$H^*(1/z^*) = B^*(1/z^*)/A^*(1/z^*) = \left(\sum_{k=0}^{q} b^*[k] z^k\right) \Big/ \left(\sum_{k=0}^{p} a^*[k] z^k\right)$$
$$= \sum_{k=0}^{\infty} h^*[k] z^k$$

Note that $h[k] = \emptyset$ for $k < 0$ (i.e., causal) due to the definitions of $A(z)$ and $B(z)$, and also that $h[0] = b[0] = 1$. Equation (6.7) can also be written as

$$P_{xx}(z) A(z) = \rho_w B(z) H^*(1/z^*) \qquad \boxed{A}$$

Using the definitions for the z-transforms $P_{xx}(z)$ and $A(z)$, then

$$P_{xx}(z) A(z) = \left(\sum_{k=-\infty}^{\infty} r_{xx}[k] z^{-k}\right)\left(\sum_{k=0}^{p} a[k] z^{-k}\right) = \sum_{m=-\infty}^{\infty} \left(\sum_{k=0}^{p} a[k] r_{xx}[m-k]\right) z^{-m}$$

so that the inverse z-transform is obtained simply by inspection of the power series to be the sequence

$$\sum_{k=0}^{p} a[k] r_{xx}[m-k] = r_{xx}[m] + \sum_{k=1}^{p} a[k] r_{xx}[m-k] \qquad \text{for} \quad -\infty < m < \infty \qquad \boxed{B}$$

which is simply the convolution of the finite sequence $a[k]$ with the infinite ACS $r_{xx}[k]$. Similarly

$$\rho_w B(z) H^*(1/z^*) = \rho_w \left(\sum_{k=0}^{q} b[k] z^{-k}\right)\left(\sum_{k=0}^{\infty} h^*[k] z^k\right)$$
$$= \rho_w \left[\sum_{m=-\infty}^{-1} \left(\sum_{k=0}^{q} b[k] h^*[k-m]\right) z^{-m} + \sum_{m=0}^{q} \left(\sum_{k=m}^{q} b[k] h^*[k-m]\right) z^{-m}\right]$$

The inverse z-transform is obtained by inspection to be

$$\rho_w \sum_{k=m}^{q} b[k] h^*[k-m] \qquad \text{for} \quad 0 \le m \le q \qquad \text{and} \quad \emptyset \quad \text{for} \quad m > q \qquad \boxed{C}$$

Substituting $\boxed{B}$ and $\boxed{C}$ into $\boxed{A}$ then yields the results shown in Equation (6.29).

Chapter 6, # 4 $\qquad A(z) = 1 + .7z^{-1} + .2z^{-2} \implies a[1] = .7$ and $a[2] = .2$

Let $B(z)$ be the approximating MA model polynomial.

$$\frac{1}{A(z)} = \frac{B(z)}{1}$$

$$B(z) = 1 \overbrace{-a[1]}^{b[1]} z^{-1} + \overbrace{(a^2[1] - a[2])}^{b[2]} z^{-2} + \cdots \text{ etc.}$$

polynomial division

$$1 + a[1]z^{-1} + a[2]z^{-2} \;\overline{\big)\, 1 \qquad\qquad\qquad\qquad}$$

$$\underline{1 + a[1]z^{-1} + a[2]z^{-2}}$$

$$-a[1]z^{-1} - a[2]z^{-2}$$

$$\underline{-a[1]z^{-1} - a[1]^2 z^{-2} - a[1]a[2]z^{-3}}$$

$$\left(a^2[1] - a[2]\right)z^{-2} + a[1]a[2]z^{-3}$$

$$\vdots$$

See posted student, paper for plot results. Note $MA(10)$ is a very good approximation.

$\boxed{\text{Chapter 6, \# 7}}$

$$A(z_i) = \sum_{k=0}^{p} a[k]\, z_i^{-k} = 0$$

where $a[k]$ are assumed complex-valued and $a[0] = 1$ by definition.

Taking the conjugate,

$$A^*(z_i) = \sum_{k=0}^{p} a^*[k]\, (z_i^*)^{-k} = 0$$

does not affect the zero result.

$$A^*(1/z^*) = \left(\sum_{k=0}^{p} a[k]\,(1/z^*)^{-k}\right)^* = \sum_{k=0}^{p} a^*[k]\,(1/z)^{-k} = \sum_{k=0}^{p} a^*[k]\, z^k$$

Now let $z = (1/z_i)^*$,

$$A^*(1/z^*)\Big|_{z = 1/z_i^*} = \sum_{k=0}^{p} a^*[k]\,(1/z_i^*)^k = \sum_{k=0}^{p} a^*[k]\,(z_i^*)^{-k} = 0 \text{ by above}$$

Thus, this a root of

From Equation (3.146), one may infer for Toeplitz matrices: $(\det \bar{R}_{m+1}) = \rho_{m+1} (\det \bar{R}_m)$ $\boxed{A}$

and using (3.52), $\det AB = \det A \det B$ $(\det \bar{R}_m) = \rho_m (\det \bar{R}_{m-1})$ $\boxed{B}$

From equations (3.158), (3.139), and (3.136), the following holds: $\rho_{m+1} = \rho_m (1 - a^2_{m+1}[m+1])$

(assuming only the real case so that $|a_{m+1}[m+1]|^2 = a^2_{m+1}[m+1]$) $= \rho_m - \Delta^2_{m+1} / \rho_m$ $\boxed{C}$

in which $\Delta_{m+1} = r_{xx}[m+1] + \sum_{i=1}^{m} a_m[i] r_{xx}[m+1-i]$.

Define $\gamma = \sum_{i=1}^{m} a_m[i] r_{xx}[m+1-i]$, then $\Delta_{m+1} = r_{xx}[m+1] + \gamma$. Substituting $\boxed{C}$ into $\boxed{A}$ yields

$$(\det \bar{R}_{m+1}) = \left[\rho_m - \Delta^2_{m+1} / \rho_m\right](\det \bar{R}_m)$$

From $\boxed{B}$, we can substitute $\rho_m = (\det \bar{R}_m)/(\det \bar{R}_{m-1})$, yielding

$$(\det \bar{R}_{m+1}) = \left[\frac{\det \bar{R}_m}{\det \bar{R}_{m-1}} - \Delta^2_{m+1} \cdot \frac{\det \bar{R}_{m-1}}{\det \bar{R}_m}\right](\det \bar{R}_m)$$

Noting that $\Delta^2_{m+1} = (r_{xx}[m+1] + \gamma)^2 = r^2_{xx}[m+1] + 2\gamma r_{xx}[m+1] + \gamma^2$, then

$$\begin{aligned}(\det \bar{R}_{m+1}) &= \frac{(\det \bar{R}_m)^2}{(\det \bar{R}_{m-1})} - (\det \bar{R}_{m-1})\left[r^2_{xx}[m+1] + 2\gamma r_{xx}[m+1] + \gamma^2\right] \\ &= \underbrace{-(\det \bar{R}_{m-1})}_{a} r^2_{xx}[m+1] \underbrace{-2\gamma(\det \bar{R}_{m-1})}_{b} r_{xx}[m+1] + \underbrace{\left[\frac{(\det \bar{R}_m)^2}{(\det \bar{R}_{m-1})} - \gamma^2(\det \bar{R}_{m-1})\right]}_{c}\end{aligned}$$

Assigning $\beta = -2\gamma(\det \bar{R}_{m-1})$ and $\alpha = \left[\frac{(\det \bar{R}_m)^2}{(\det \bar{R}_{m-1})} - \gamma^2(\det \bar{R}_{m-1})\right]$ yields the result shown in the text for this problem. Thus, $(\det \bar{R}_{m+1})$ is a function of a quadratic expression of the structure:

$$(\det \bar{R}_{m+1}) = a\, r^2_{xx}[m+1] + b\, r_{xx}[m+1] + c \qquad \boxed{D}$$

A quadratic equation can have only a single maximum [or minimum; the analysis to follow will show that it is a maximum.]. The maximum is obtained where the derivative of the quadratic with respect to the variable $r_{xx}[m+1]$ is zero,

$$\frac{\partial\left[a\, r^2_{xx}[m+1] + b\, r_{xx}[m+1] + c\right]}{\partial r_{xx}[m+1]} = 0 \Rightarrow 2a\, r_{xx}[m+1] + b = 0 \quad \text{or} \quad \underbrace{r_{xx}[m+1]}_{\text{maximizes } (\det \bar{R}_{m+1})} = -\frac{b}{2a}$$

which is

$$(r_{xx}[m+1])_{max} = -\frac{-2\gamma(\det \bar{R}_{m-1})}{-2(\det \bar{R}_{m-1})} = -\gamma = -\sum_{i=1}^{m} a_m[i]\, r_{xx}[m+1-i] \qquad \boxed{E}$$

← extension of ACS by this ↑

so that $(\det \bar{R}_{m+1})$ obtains its maximum value

$$\begin{aligned}(\det \bar{R}_{m+1})_{max} &= -(\det \bar{R}_{m-1})(-\gamma)^2 - 2\gamma(\det \bar{R}_{m-1})(-\gamma) + \left[\frac{(\det \bar{R}_m)^2}{(\det \bar{R}_{m-1})} - \gamma^2(\det \bar{R}_{m-1})\right] \\ &= \frac{(\det \bar{R}_m)^2}{(\det \bar{R}_{m-1})} = \rho_m (\det \bar{R}_m)\end{aligned}$$

because $\rho_m \geq 0$ and $(\det \bar{R}_m) \geq 0$ (Chapter 3)

Due to the fact that $(\det \bar{R}_{m+1}) \geq 0$, then the permissible values of $r_{xx}[m+1]$ must lie between the roots of $\boxed{D}$ (at the roots, $(\det \bar{R}_{m+1}) = 0$ and in-between the roots, $(\det \bar{R}_{m+1}) > 0$ and its maximum will occur midway between the roots because it is a quadratic equation $\boxed{D}$). The roots of $\boxed{D}$ are given by the usual $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ so that the range between roots is

$$\text{range of } r_{xx}[m+1] = \frac{-b+\sqrt{b^2-4ac}}{2a} - \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{\sqrt{b^2-4ac}}{a}$$

Substituting the values for a, b, and c yields

$$b^2-4ac = 4\gamma^2(\det \bar{R}_{m-1})^2 - 4(-\det \bar{R}_{m-1})\left(\frac{\det^2 \bar{R}_m}{\det \bar{R}_{m-1}} - \gamma^2 \det \bar{R}_{m-1}\right) = 4(\det \bar{R}_m)^2$$

and therefore

$$\text{range of } r_{xx}[m+1] = \frac{\sqrt{4 \det^2 \bar{R}_m}}{(-\det \bar{R}_{m-1})} = \frac{2 \det \bar{R}_m}{\det R_{m-1}} = 2\rho_m$$

because $\det \bar{R}_m = \rho_m \det \bar{R}_{m-1}$.

If the value of $\underline{r_{xx}[m+1] \text{ that maximizes } \det \bar{R}_{m+1}}$ is selected (equation $\boxed{D}$), then

$$\Delta_{m+1} = r_{xx}[m+1] + \gamma = -\gamma + \gamma = \emptyset$$

and therefore

$$a_{m+1}[m+1] = k_{m+1} = -\Delta_{m+1}/\rho_m = -0/\rho_m = \emptyset$$

$$\rho_{m+1} = \rho_m\left(1 - |a_{m+1}[m+1]|^2\right) = \rho_m(1-\emptyset) = \rho_m .$$

Note that maximizing $(\det \bar{R}_{m+1})$ is equivalent to maximizing the entropy since

$$\text{Entropy rate} = H = \ln (2\pi e)^{(N+1/2)} \det (\bar{R}_{m+1})^{\frac{1}{2}}$$

for a gaussian process.

Chapter 7, #4

Forward linear prediction error: $e_m^f[n] = \sum_{i=0}^{m} a_m^f[k]\, x[n-i]$, $a_m^f[0]=1$ by definition
(7.2)

Backward linear prediction error: $e_m^b[n] = \sum_{i=0}^{m} a_m^b[k]\, x[n-m+i]$ $a_m^b[0]=1$ by definition
(7.9)

or $e_m^b[n] = \sum_{i=0}^{m} \left(a_m^f[k]\right)^* x[n-m+i]$ using $a^b[k] = a^{f*}[k]$ conjugation property

Principal recursion of Levinson algorithm is

$$a_m^f[i] = a_{m-1}^f[i] + a_m^f[m]\left(a_{m-1}^f[m-i]\right)^* \qquad 1 \le i \le m-1$$

or

$$a_m^f[i] = a_{m-1}^f[i] + k_m\left(a_{m-1}^f[m-i]\right)^* \qquad ☆$$

using property $a_m^f[m] = a_m[m] = k_m$ (equations 7.21 and 7.25)

Noting that forward error can be expressed as

$$e_m^f[n] = \underbrace{a_m^f[m]}_{k_m} x[n-m] + x[n] + \sum_{i=1}^{m-1} a_m^f[i]\, x[n-i]$$

and substituting ☆,

$$e_m^f[n] = k_m x[n-m] + x[n] + \sum_{i=1}^{m-1}\left[a_{m-1}^f[i] + k_m\left(a_{m-1}^f[m-i]\right)^*\right] x[n-i]$$

$$e_m^f[n] = \left[x[n] + \sum_{i=1}^{m-1} a_{m-1}^f[i]\, x[n-i]\right] + k_m\left[x[n-m] + \sum_{i=1}^{m-1}\left(a_{m-1}^f[m-i]\right)^* x[n-i]\right]$$

Using a change of variable $i' = m-i$ on the second summation,

$$e_m^f[n] = \sum_{i=0}^{m-1} a_{m-1}^f[i]\, x[n-i] + k_m \sum_{i'=0}^{m-1}\left(a_{m-1}^f[i']\right)^* x[n-m+i']$$

resulting in

$$\boxed{e_m^f[n] = e_{m-1}^f[n] + k_m\, e_{m-1}^b[n-1]}$$

In a similar manner, note that backward error can be written as

$$e_m^b[n] = x[n-m] + \underbrace{\left(a_m^f[m]\right)^*}_{k_m^*} x[n] + \sum_{i=1}^{m-1}\left(a_m^f[i]\right)^* x[n-m+i]$$

Now, substituting ☆ after conjugating yields

$$e_m^b[n] = x[n-m] + k_m^* x[n] + \sum_{i=1}^{m-1}\left[\left(a_{m-1}^f[i]\right)^* + k_m^*\, a_{m-1}^f[m-i]\right] x[n-i]$$

or

$$e_m^b[n] = \left[x[n-m] + \sum_{i=1}^{m-1}\left(a_{m-1}^f[i]\right)^* x[n-i]\right] + k_m^*\left[x[n] + \sum_{i=1}^{m-1} a_{m-1}^f[m-i]\, x[n-i]\right]$$

or

$$\boxed{e_m^b[n] = e_{m-1}^b[n-1] + k_m^*\, e_{m-1}^f[n]}$$

Based on equation (4.53), the autocorrelation matrix for the single sinusoid plus noise of equation (7.36) has the form [equation numbers from Marple text]

$$\bar{R}_{xx} = P\,\bar{e}_m(f_0)\bar{e}_m^H(f_0) + \sigma_w^2 \bar{I} \qquad \text{where} \quad \bar{e}_m(f_0) = \begin{bmatrix} 1 \\ \exp(j2\pi f_0 T_s) \\ \vdots \\ \exp(j2\pi f_0 m T_s) \end{bmatrix}$$

Based on equation (6.14), the m-th order AR PSD has the form

$$P_{AR}(f) = \frac{T_s \rho_m}{\bar{e}_m^H(f)\bar{a}_m \bar{a}_m^H \bar{e}_m(f)} = \frac{T_s \rho_m}{|\bar{e}_m^H(f)\bar{a}_m|^2} \qquad \text{where} \quad \bar{a}_m = \begin{bmatrix} 1 \\ a_m[1] \\ \vdots \\ a_m[m] \end{bmatrix}$$

and ρ_m is the m-th order "white noise" process driving the AR process (not the same as the white noise σ_w^2 of the sinusoid process). An alternative representation is obtained by dividing both the numerator and denominator by ρ_m^2 to yield

$$P_{AR}(f) = \frac{T(1/\rho_m)}{|\bar{e}_m^H(f)\,\bar{a}_m/\rho_m|^2} = \frac{T\alpha_0}{|\bar{e}_m^H(f)\,\bar{\alpha}_m|^2} \qquad \text{where} \quad \bar{\alpha}_m = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 1/\rho_m \\ a_m[1]/\rho_m \\ \vdots \\ a_m[m]/\rho_m \end{bmatrix}$$

The Yule-Walker equations that relate the ACS to the AR parameters can be expressed in matrix form as

$$\bar{R}_{xx}\,\bar{a}_m = \begin{bmatrix} \rho_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{or} \quad \bar{R}_{xx}\bar{\alpha}_m = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \bar{\alpha}_m = (\bar{R}_{xx})^{-1}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using the augmented matrix inversion lemma (equation 3.53) and assigning $\bar{A} = \sigma_w^2 \bar{I}$, $\bar{B} = \bar{e}_m(f_0)$, $\bar{C} = P$ (a scalar, 1x1 matrix), $\bar{D} = \bar{e}_m^H(f_0)$, then $\bar{A}^{-1} = \frac{1}{\sigma_w^2}\bar{I}$ and $\bar{C}^{-1} = \frac{1}{P}$

$$(\bar{R}_{xx})^{-1} = \frac{1}{\sigma_w^2}\bar{I} - \frac{1}{\sigma_w^2}\bar{I}\,\bar{e}_m(f_0)\underbrace{\left[\bar{e}_m^H(f_0)\frac{1}{\sigma_w^2}\bar{I}\,\bar{e}_m(f_0) + \frac{1}{P}\right]}_{\text{a scalar}}^{-1}\bar{e}_m^H(f_0)\frac{1}{\sigma_w^2}\bar{I}$$

$$= \frac{1}{\sigma_w^2}\left[\bar{I} - \bar{e}_m(f_0)\bar{e}_m^H(f_0)/\beta\right] \qquad \text{where } \beta = \frac{\rho_w}{P} + \bar{e}_m^H(f_0)\bar{e}_m(f_0) = \overset{\frac{1}{SNR}}{\frac{\sigma_w^2}{P}} + (m+1)$$

$$\text{and} \quad 1/\beta = P/(\sigma_w^2 + [m+1]P) = \left(1/[\sigma_w^2/P + (m+1)]\right)$$

The solution vector $\bar{\alpha}_m$ is therefore

$$\bar{\alpha}_m = \overbrace{\frac{1}{\sigma_w^2}\left[\bar{I} - \bar{e}_m(f_0)\bar{e}_m^H(f_0)/\beta\right]}^{(\bar{R}_{xx})^{-1}}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sigma_w^2}\left\{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \bar{e}_m(f_0)/\beta\right\}.$$

Note that $\quad \alpha_0 = (1 - 1/\beta)/\sigma_w^2 = \frac{1}{\sigma_w^2}\left[1 - \frac{P}{(\sigma_w^2 + [m+1]P)}\right]$

and

$$\bar{e}_m^H(f)\bar{\alpha}_m = \frac{1}{\sigma_w^2}\left\{\bar{e}_m^H(f)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{1}{\beta}\bar{e}_m^H(f)\bar{e}_m(f_0)\right\}$$

$$= \frac{1}{\sigma_w^2}\left\{1 - \frac{1}{\beta}\sum_{k=0}^{m}\exp\left(-j2\pi k[f-f_0]T_s'\right)\right\}$$

and therefoe

$$P_{AR}(f) = \frac{T\alpha_0}{\left|e_m^H(f)\bar{\alpha}_m\right|^2} = \frac{1/\sigma_w^2\, T_s'\,(1-1/\beta)}{\left(1/\sigma_w^2\right)^2\left|1-\frac{1}{\beta}\sum_{k=0}^{m}\exp(-j2\pi k[f-f_0]T_s\right|^2}$$

$$= \frac{T\sigma_{(w)}^2\left(1-\left[\frac{P}{\sigma_w^2+[m+1]P}\right]\right)}{\left|1-\left(\frac{P}{\sigma_w^2+[m+1]P}\right)\sum_{k=0}^{m}\exp(-j2\pi k[f-f_0]T'\right|^2}$$

(c) A linear prediction error filter is stable if all the poles are inside the unit z-plane circle. The equivalent lattice filter (which uses the reflection coefficients) is stable if each reflection coefficient $|k_m| < 1$ for $m = 1$ to p.

(d) To determine if the filter is stable, one must factor the denominator polynomial in z. It has the factors

$$H(z) = \frac{1}{1 - 2z^{-1} - 6z^{-2} + z^{3} - 2z^{-4}} = \frac{1}{(1 - az^{-1})(1 - bz^{-1})(1 - cz^{-1})(1 - dz^{-1})}$$

If all factors a, b, c, d are within the unit z-plane circle, then the filter is stable.

The backward linear prediction error at order m and time index n can be expressed as

$$e_m^b[k] = \sum_{i=0}^{m} a_m^*[i]\, x[k-m+i] = \bar{a}_m^H \bar{J} \bar{X}_m[k]$$

in which the column vectors $\bar{a}_m$ and $\bar{X}_m[n]$ are defined as the $(m+1)$ dimensional vectors

$$\bar{a}_m = \begin{bmatrix} 1 \\ a_m[1] \\ \vdots \\ a_m[m] \end{bmatrix} = \text{linear prediction vector}, \quad \bar{x}_m[k] = \begin{bmatrix} x[k] \\ x[k-1] \\ \vdots \\ x[k-m] \end{bmatrix} = \text{data vector}$$

and $\bar{J}$ is an $(m+1)\times(m+1)$ dimensional reflection matrix. Now consider all orders $n \le m$ and the associated backward linear prediction error $e_n^b[k]$ which can be expressed as

$$e_n^b[k] = \sum_{i=0}^{n} a_n^*[i]\, x[k-m+i] = \bar{a}_n^H \bar{J} \bar{X}_m[k]$$

where the $(m+1)$ dimensional vector $\bar{a}_n$ is defined as

$$\bar{a}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ a_n[1] \\ \vdots \\ a_n[n] \end{bmatrix} \quad \text{so that} \quad e_n^b[k] = \left(0 \cdots 0\, 1\, a_n^*[1] \cdots a_n^*[n]\right) \begin{pmatrix} x[k-m] \\ \vdots \\ x[k-n-1] \\ x[k-n] \\ x[k-n+1] \\ \vdots \\ x[k] \end{pmatrix}$$

which has $(m-n)$ zeros at the top of the vector.
Defining $e_n^b[k]$ in this manner allows us to represents the expectation of the product

$$\mathcal{E}\left\{e_m^b[k]\, e_n^{b*}[k]\right\} = \mathcal{E}\left\{\bar{a}_m^H \bar{J} \bar{X}_m[k] \cdot \bar{x}_m^H[k] \bar{J} \bar{a}_n\right\} = \bar{a}_m^H J \cdot \mathcal{E}\left\{\bar{x}_m[k] \bar{X}_m^H[k]\right\} \cdot \bar{J} \bar{a}_n$$

in a vector/matrix representation. Note that $\mathcal{E}\left\{\bar{x}_m[k] \bar{x}_m^H[k]\right\} = \bar{R}_m^*$ is the $(m+1)\times(m+1)$ dimensional autocorrelation matrix of order m. Because $\bar{R}_m$ is hermitian Toeplitz, then $\bar{J}\bar{R}_m^*\bar{J} = \bar{R}_m$, with the result

$$\mathcal{E}\left\{e_m^b[k]\, e_n^{b*}[k]\right\} = \bar{a}_m^H \bar{J} \bar{R}_m^* \bar{J} \bar{a}_n = \bar{a}_m^H \bar{R}_m \bar{a}_n$$

However, based on equation (3.164), $\bar{A}_m^H \bar{R}_m \bar{A}_m = \bar{P}_m$ where $\bar{A}_m = \begin{bmatrix} 1 & & 0 \\ a_m[1] & \ddots & \\ \vdots & & \\ a_m[m] & \cdots & a_1[1] \end{bmatrix}$, $\bar{P}_m = \begin{bmatrix} \rho_m & & 0 \\ & \ddots & \\ 0 & & \rho_0 \end{bmatrix}$

Noting that $\bar{A}_m = \left[\bar{a}_m\ \bar{a}_{m-1} \cdots \bar{a}_n \cdots \bar{a}_0\right]$ (assuming $n \le m$), one may immediately infer from (3.164) that

$$\mathcal{E}\left\{e_m^b[k]\, e_n^{b*}[k]\right\} = \begin{Bmatrix} \rho_m & \text{if } m=n \\ 0 & \text{if } m\neq n \end{Bmatrix} = \rho_m\, \delta[m-n]$$

Once the Burg algorithm has been run with data samples to a desired maximum order M, the descending order Levinson recursion derived in Problem 7 of Chapter 7. This has the form (given $a_m[i]$ for $i=1$ to m),

$$a_{m-1}[i] = \frac{1}{(1-|a_m[m]|^2)} \left[a_m[i] - a_m[m]\left(a_m[m-i]\right)^* \right] \quad \text{for } i=1 \text{ to } m-1$$

with the associated descending-order recursion for variance of the driving white noise

$$\rho_m = \rho_{m-1}\left(1-|a_m[m]|^2\right)$$

or

$$\rho_{m-1} = \frac{1}{(1-|a_m[m]|^2)}\,\rho_m$$

Chapter 8, # 6 Derive from Levinson recursion (use 3.129 or 7.5, or class notes)

$$\bar{R}_p = \begin{bmatrix} \bar{R}_{p-1} & J r_p^* \\ \bar{r}_p^T J & r_{xx}[0] \end{bmatrix} = \begin{bmatrix} r_{xx}[0] & \bar{r}_p^H \\ \bar{r}_p & \bar{R}_{p-1} \end{bmatrix} \quad \text{where } \bar{r}_p = \begin{bmatrix} r_{xx}[1] \\ \vdots \\ r_{xx}[p] \end{bmatrix}$$

by definition

Yule-Walker equations and Toeplitz property

$$\begin{bmatrix} \bar{R}_p \end{bmatrix} \begin{bmatrix} 1 \\ \bar{a}_p \end{bmatrix} = \begin{bmatrix} \rho_p \\ \bar{0} \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{R}_{p-1} & \bar{J}\bar{r}_p^* \\ \bar{r}_p^T \bar{J} & r_{xx}[0] \end{bmatrix} \begin{bmatrix} 1 \\ \bar{a}_{p-1} \\ 0 \end{bmatrix} = \begin{bmatrix} \rho_{p-1} \\ \bar{0} \\ \Delta \end{bmatrix} \quad \text{where } \Delta = \bar{r}_p^T \bar{J} \begin{bmatrix} 1 \\ \bar{a}_{p-1} \end{bmatrix}$$

or $\Delta = r_{xx}[p] + \sum_{i=1}^{p-1} a_{p-1}[i]\, r_{xx}[p-i] = \sum_{i=0}^{p-1} a_{p-1}[i]\, r_{xx}[p-i]$ where $a_{p-1}[0] = 1$ by definition

and similarly:

$$\begin{bmatrix} r_{xx}[0] & \bar{r}_p^H \\ \bar{r}_p & \bar{R}_{p-1} \end{bmatrix} \begin{bmatrix} 0 \\ J\bar{a}_{p-1}^* \\ 1 \end{bmatrix} = \begin{bmatrix} \Delta^* \\ \bar{0} \\ \rho_{p-1} \end{bmatrix}$$

Now

$$\begin{bmatrix} 1 \\ \bar{a}_p \end{bmatrix} = \begin{bmatrix} 1 \\ \bar{a}_{p-1} \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ J\bar{a}_{p-1}^* \\ 1 \end{bmatrix} \Rightarrow c = a_p[p]$$

and

$$\begin{bmatrix} \bar{R}_p \end{bmatrix} \begin{bmatrix} 1 \\ \bar{a}_p \end{bmatrix} = \begin{bmatrix} \bar{R}_p \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ \bar{a}_{p-1} \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ J\bar{a}_{p-1}^* \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} \rho_{p-1} \\ \bar{0} \\ \Delta \end{bmatrix} + c \begin{bmatrix} \Delta^* \\ \bar{0} \\ \rho_{p-1} \end{bmatrix} = \begin{bmatrix} \rho_p \\ \bar{0} \end{bmatrix}$$

which implies $\Delta + c\rho_{p-1} = 0$ or $c = a_p[p] = -\Delta/\rho_{p-1} = -\sum_{i=0}^{p-1} a_{p-1}[i]\, r_{xx}[p-i] \Big/ \rho_{p-1}$

Chapter 8, #7 Assume k_i for $i = 1$ to p and ρ_0 are given.

Use equations (8.2), (8.3), (8.4) to solve for $r_{xx}[p]$.

At $p = 0$, $\rho_0 = r_{xx}[0]$

At $p = 1$,

$$\begin{bmatrix} r_{xx}[0] & r^*_{xx}[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} 1 \\ \underbrace{a_1[1]}_{k_1} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ 0 \end{bmatrix} \Rightarrow r_{xx}[1] + r_{xx}[0]k_1 = 0$$

$a_1[1] = k_1$ or $r_{xx}[1] = -k_1 r_{xx}[0]$

$\rho_1 = \rho_0 [1 - |k_1|^2)$

For $p > 1$, restructuring (8.3) yields recursively

$$r_{xx}[p] = k_p \rho_{p-1} + \sum_{i=1}^{p-1} a_{p-1}[i] r_{xx}[p-i]$$

KEY RECURSIVE RESULT

and

$$a_p[n] = a_{p-1}[n] + k_p a^*_{p-1}[p-n]$$

$$\rho_p = \rho_{p-1}[1 - |k_p|^2]$$

increment to next order

Note: real-valued data case version of this suggested by N. Anderson, Proc. IEEE, vol. 66, pp 1581-1582, Nov

According to equation (8.15), the definition

$$DEN_p = \sum_{n=p+1}^{N} \left(|e^f_{p-1}[n]|^2 + |e^b_{p-1}[n-1]|^2 \right)$$

reduces, at order $p=1$, to

$$DEN_1 = \sum_{n=2}^{N} \left(|e^f_0[n]|^2 + |e^b_0[n-1]|^2 \right) = \sum_{n=2}^{N} |x[n]|^2 + \sum_{n=1}^{N-1} |x[n]|^2$$

or

$$DEN_1 = 2\sum_{n=1}^{N} |x[n]|^2 - |x[1]|^2 - |x[N]|^2 \qquad \boxed{A}$$

The line of code DEN= P*2 actually computes the value of $2\sum_{n=1}^{N} |x[n]|^2$. The line of code DEN = TEMP * DEN − REAL (EF(K))**2 − AIMAG (EF(K))**2 − REAL (EB(N))**2 − AIMAG (EB(N))**2 then generates the value of $\boxed{A}$ above during the first pass through the main loop.

From equation (8.15), the definition of DEN_p is

$$DEN_p = \sum_{n=p+1}^{N} \left(|e^f_{p-1}[n]|^2 + |e^b_{p-1}[n-1]|^2 \right)$$

and similarly for DEN_{p-1}.

The following modification is possible by changing the summation range

$$DEN_p = \sum_{n=p}^{N} \left(|e^f_{p-1}[n]|^2 + |e^b_{p-1}[n]|^2 \right) \quad - |e^f_{p-1}[p]|^2 \quad - |e^b_{p-1}[N]|^2$$

The recursive relationships of equation (8.7) [assuming the estimate $\hat{k}_p$ has been computed]

$$e^f_{p-1}[n] = e^f_{p-2}[n] + \hat{k}_{p-1} e^b_{p-2}[n-1]$$
$$e^b_{p-1}[n] = e^b_{p-2}[n-1] + \hat{k}^*_{p-1} e^f_{p-2}[n]$$

may be substituted for $e^f_{p-1}[n]$ and $e^b_{p-1}[n]$ to yield

$$DEN_p = \sum_{n=p}^{N} \left(|e^f_{p-2}[n] + \hat{k}_{p-1} e^b_{p-2}[n-1]|^2 + |e^b_{p-2}[n-1] + \hat{k}^*_{p-1} e^f_{p-2}[n]|^2 \right) - |e^f_{p-1}[p]|^2 - |e^b_{p-1}[N]|$$

$$= (1 + |\hat{k}_{p-1}|^2)\left(\sum_{n=p}^{N} |e^f_{p-2}[n]|^2 + |e^b_{p-2}[n-1]|^2 \right) + 2\hat{k}_p \left(\sum_{n=p}^{N} e^{f*}_{p-2}[n] e^b_{p-2}[n-1] \right)$$
$$+ 2\hat{k}^*_p \left(\sum_{n=p}^{N} e^f_{p-2}[n] e^{b*}_{p-2}[n-1] \right) \quad - |e^f_{p-1}[p]|^2 \quad - |e^b_{p-1}[N]|^2$$
$$= (1 + |\hat{k}_{p-1}|^2) DEN_{p-1} + 2\,\mathrm{Re}\left\{ 2\hat{k}^*_p \left(\sum_{n=p}^{N} e^f_{p-2}[n] e^{b*}_{p-2}[n-1] \right) - |e^f_{p-1}[p]|^2 - |e^b_{p-1}[N]|^2 \right.$$

Equation (8.14) for order $(p-1)$ can be expressed as

$$\hat{k}_{p-1} = \frac{-2 \sum_{n=p}^{N} e^f_{p-2}[n] e^{b*}_{p-2}[n-1]}{DEN_{p-1}}$$

Multiply both sides by $\hat{k}^*_{p-1}$ to yield

$$\hat{k}^*_p \sum_{n=p}^{N} e^f_{p-2}[n] e^{b*}_{p-2}[n-1] = -2 \cdot |\hat{k}_{p-1}|^2 \cdot DEN_{p-1}$$

Substituting,

$$DEN_p = (1 + |\hat{k}_{p-1}|^2) DEN_{p-1} - 2|\hat{k}_{p-1}|^2 DEN_{p-1} - |e^f_{p-1}[p]|^2 - |e^b_{p-1}[N]|^2$$

$$= (1 - |\hat{k}_{p-1}|^2) DEN_{p-1} - |e^f_{p-1}[p]|^2 - |e^b_{p-1}[N]|^2$$

Chapter 9, Problem 1

Eqn (9.24) $$\bar{P}_N^{-1} = \omega \bar{P}_{N-1}^{-1} + \bar{x}^*_{p-1}[N]\,\bar{x}^T_{p-1}[N]$$

Using the matrix inversion lemma, eqn (3.54),

$$P_N = \left(P_N^{-1}\right)^{-1} = \left(\omega \bar{P}_{N-1}^{-1}\right)^{-1} - \frac{\left[\left(\omega \bar{P}_{N-1}^{-1}\right)^{-1} \bar{x}^*_{p-1}[N]\right]\left[\bar{x}^T_{p-1}[N]\left(\omega \bar{P}_{N-1}^{-1}\right)^{-1}\right]}{1 + \bar{x}^T_{p-1}[N]\left(\omega \bar{P}_{N-1}^{-1}\right)^{-1}\bar{x}^*_{p-1}[N]}$$

or

$$P_N = \omega^{-1}\bar{P}_{N-1} - \omega^{-1}\frac{\left[\bar{P}_{N-1}\,\bar{x}^*_{p-1}[N]\,\bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\right]}{\left[\omega + \bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\,\bar{x}^*_{p}[N]\right]}$$

Then,

$$\bar{c}_{p-1,N} = \bar{P}_N\,\bar{x}^*_{p-1}[N] = \omega^{-1}\bar{P}_{N-1}\bar{x}^*_{p-1}[N] - \omega^{-1}\frac{\left[\bar{P}_{N-1}\bar{x}^*_{p-1}[N]\,\bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\,\bar{x}^*_{p-1}[N]\right]}{\left[\omega + \bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\,\bar{x}^*_{p-1}[N]\right]}$$

$$= \omega^{-1}\bar{P}_{N-1}\bar{x}^*_{p-1}[N]\,\frac{\left[\omega + \bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\,\bar{x}^*_{p-1}[N]\right]}{\left[\omega + \bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\,\bar{x}^*_{p-1}[N]\right]} - \omega^{-1}\frac{\left[\bar{P}_{N-1}\bar{x}^*_{p-1}[N]\,\bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\bar{x}^*_{p-1}[N]\right]}{\left[\omega + \bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\,\bar{x}^*_{p-1}[N]\right]}$$

$$= \frac{1}{\left[\omega + \bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\bar{x}^*_{p-1}[N]\right]}\left[\bar{P}_{N-1}\bar{x}^*_{p-1}[N] + \cancel{\omega^{-1}\bar{P}_{N-1}\bar{x}^*_{p-1}[N]\,\bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\bar{x}^*_{p}[N]} - \cancel{\omega^{-1}\bar{P}_{N-1}\bar{x}^*_{p-1}[N]\,\bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\bar{x}^*_{p-1}[N]}\right]$$

$$= \frac{\bar{P}_{N-1}\bar{x}^*_{p-1}[N]}{\omega + \bar{x}^T_{p-1}[N]\,\bar{P}_{N-1}\,\bar{x}^*_{p-1}[N]}$$

$\boxed{\text{r 9, Problem 4}}$

$$\bar{R}_{p,N} = \sum_{n=1}^{N} \omega^{N-n}\, \bar{x}_p^{*}[n]\, \bar{x}_p^{T}[n]$$

Eqn (9.C.6) definition variation

$$\bar{R}_{p,N+1} = \sum_{n=1}^{N+1} \omega^{N+1-n}\, \bar{x}_p^{*}[n]\, \bar{x}_p^{T}[n] = \omega \bar{R}_{p,N} + \bar{x}_p^{*}[N+1]\, \underline{\bar{x}_p^{T}[N+1]}$$

Eqn. (9.C.12) definition variation

$$\bar{x}_p^{T}[N+1] = \omega\, \bar{c}^{H}_{p,N+1}\, \bar{R}^{H}_{p,N} = \omega\, \bar{c}^{H}_{p,N+1}\, \bar{R}_{p,N} \quad \text{since } \bar{R}^{H}_{p,N} = \bar{R}_{p,N}$$

Substituting,

$$\bar{R}_{p,N+1} = \omega \bar{R}_{p,N} + \bar{x}_p^{*}[N+1]\left(\omega\, \bar{c}^{H}_{p,N+1}\, \bar{R}_{p,N} \right) = \omega \left(\bar{I} + \bar{x}_p^{*}[N+1]\, \bar{c}^{H}_{p,N+1} \right) \bar{R}_{p,N}$$

Eqn (3.52) on page 68 : $\det \bar{A}\bar{B} = \det \bar{A} \cdot \det \bar{B}$, so above case

$$\det \bar{R}_{p,N+1} = \omega \det\left[\left(\bar{I} + \bar{x}_p^{*}[N+1]\, \bar{c}^{H}_{p,N+1} \right) \bar{R}_{p,N} \right]$$

$$= \omega \det\left(\bar{I} + \bar{x}_p^{*}[N+1]\, \bar{c}^{H}_{p,N+1} \right) \cdot \det \bar{R}_{p,N}$$

It can be found in linear/matrix algebra books that the following property holds true (proof not provided here):

$$\det\left(\bar{I} + \bar{x}_p^*[N+1]\,\bar{c}^H_{p,N+1}\right) = 1 + \bar{c}^H_{p,N+1}\,\bar{x}_p^*[N+1] = \gamma_{p,N+1}$$

see definition (9.C.17)

Substituting,

$$\det \bar{R}_{p,N+1} = \omega\,\gamma_{p,N+1} \det \bar{R}_{p,N}$$

or

$$\boxed{\gamma_{p,N+1} = \det \bar{R}_{p,N+1} \big/ \omega \det \bar{R}_{p,N}}$$

(sorry: typo in book forgot ω !)

On page 87 of text, it is shown that a Hermitian Toeplitz matrix, like the ACS matrices $\bar{R}_{p,N}$ and $\bar{R}_{p,N+1}$, have positive determinants (eqn 3.159). We conclude that

$$\gamma_{p,N+1} > 0$$

Using a process similar to eqn (3.146), plus the partitions shown in eqn. (9.C.7), we can write:

pxp identity matrix

$$\bar{R}_{p+1,N+1}\begin{bmatrix} \bar{I}_p & \bar{a}^b_{p+1,N+1} \\ \bar{0}_p^T & 1 \end{bmatrix} = \begin{bmatrix} \bar{R}_{p,N+1} & \bar{s}_{p+1,N+1} \\ \bar{s}^H_{p+1,N+1} & \bar{r}_{p+1,N+1}[p+1,p+1] \end{bmatrix}\begin{bmatrix} \bar{I}_p & \bar{a}^b_{p+1,N+1} \\ \bar{0}_p^T & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \bar{R}_{p,N+1} & \bar{0}_p \\ \bar{s}^H_{p+1,N+1} & \rho^b_{p+1,N+1} \end{bmatrix}$$

Again, we apply $\det \bar{A}\bar{B} = \det \bar{A} \cdot \det \bar{B}$ property

$$\det\begin{bmatrix} \bar{R}_{p,N+1} & \bar{0}_p \\ \bar{s}^H_{p+1,N+1} & \rho^b_{p+1,N+1} \end{bmatrix} = \det\left[\bar{R}_{p+1,N+1}\right] \cdot \det\begin{bmatrix} I_p & \bar{a}^b_{p+1,N+1} \\ \bar{0}_p^T & 1 \end{bmatrix}$$

or

$$\rho^b_{p+1,N+1} \cdot \det \bar{R}_{p,N+1} = \det \bar{R}_{p+1,N+1} \cdot 1$$

Rearranging,

$$\boxed{\rho^b_{p+1,N+1} = \det \bar{R}_{p+1,N+1} \big/ \det \bar{R}_{p,N+1}}$$

(Sorry, there was another typo on the ρ^b subscript)

Using a similar process as above,

$$\bar{R}_{p+1,N+1}\begin{bmatrix} 1 & \bar{0}_p^T \\ \bar{a}^f_{p+1,N+1} & \bar{I}_p \end{bmatrix} = \begin{bmatrix} \bar{r}_{p+1,N+1}[0,0] & \bar{r}^H_{p+1,N+1} \\ \bar{r}_{p+1,N+1} & \bar{R}_{p,N} \end{bmatrix}\begin{bmatrix} 1 & \bar{0}_p^T \\ \bar{a}^f_{p+1,N+1} & \bar{I}_p \end{bmatrix} = \begin{bmatrix} \rho^f_{p+1,N+1} & \bar{r}^H_{p+1,N+1} \\ \bar{0}_p & \bar{R}_{p,N} \end{bmatrix}$$

$$\det\begin{bmatrix} \rho^f_{p+1,N+1} & \bar{r}^H_{p+1,N+1} \\ \bar{0}_p & \bar{R}_{p,N} \end{bmatrix} = \det\left[\bar{R}_{p+1,N+1}\right] \cdot \det\begin{bmatrix} 1 & \bar{0}_p^T \\ \bar{a}^f_{p+1,N+1} & \bar{I}_p \end{bmatrix}$$

or $\rho^f_{p+1,N+1} \det \bar{R}_{p,N} = \det \bar{R}_{p+1,N+1} \cdot 1 \quad \Longrightarrow \quad \boxed{\rho^f_{p+1,N+1} = \det \bar{R}_{p+1,N+1} \big/ \det \bar{R}_{p,N}}$ (another typo)

Prove $a[m] + \sum_{n=1}^{q} b[n]\, a[m-n] = \delta[m] = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases}$

The system function of a MA(q) process is:

$$B(z) = 1 + \sum_{k=1}^{q} b[k]\, z^{-k}$$

The system function of an AR(∞) process is $1/A(z)$, where

$$A(z) = 1 + \sum_{k=1}^{\infty} a[k]\, z^{-k}$$

So,

$$B(z) = \frac{1}{A(z)}$$

$$A(z)\, B(z) = 1$$

By taking the inverse z-transform of both sides,

$$a[m] \circledast b[m] = \delta[m]$$

Calculating the convolution yields:

$$a[m] + \sum_{n=1}^{q} b[n]\, a[m-n] = \delta[m] = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

since by definition, $a[0]=1$ and $a[k]=0$ for $k<0$.

Chapter 11, Problem 5

Let $x[n] = \exp(-.2nT)\sin(2\pi .05nT)$ for :

(1) $T=1$, $N=20$

(2) $T=.5$, $N=40$

We make the following MATLAB calls :

```
[h1,z1] = ls_prony (1, 2, x)
[amp1, damp1, freq1, phase1] = exp_par (1, h1, z1)

[h2, z2] = ls_prony (1, 2, x)
[amp2, damp2, freq2, phase2] = exp_par (.5, h2, z2)
```

which yield respectively :

$$\text{amp1} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \text{damp1} = \begin{pmatrix} -0.2 \\ -0.2 \end{pmatrix}, \text{freq1} = \begin{pmatrix} 0.05 \\ -0.05 \end{pmatrix}, \quad \text{amp2} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \text{damp1} = \begin{pmatrix} -.2 \\ -.2 \end{pmatrix}, \text{freq2} = \begin{pmatrix} .05 \\ -.05 \end{pmatrix},$$

$$\text{phase1} = \begin{pmatrix} -\pi/2 \\ +\pi/2 \end{pmatrix} \qquad \text{phase2} = \begin{pmatrix} -\pi/2 \\ \pi/2 \end{pmatrix}$$

Both yield the same result, irregardless of the sample rate :

$$\hat{x}[n] = \tfrac{1}{2}\exp(-.2nT)\left[\exp(j2\pi .05T - \pi/2) + \exp(-j\,2\pi .05T + \pi/2)\right]$$

or

$$\hat{x}[n] = \cancel{2}\,\tfrac{1}{\cancel{2}}\exp(-.2nT)\cos(2\pi .05nT - \pi/2) = \exp(-.2nT)\sin(2\pi .05nT)$$

It was intended that some additive noise be added to the signal, which affects the quality of the estimates. In that case, it would be discovered that the data sequence that is sampled at a higher sample rate ($T=.5$) will yield a slightly better estimate than the data sequence sampled slower ($T=1$).

Chapter 11, Problem 7 To prove

$$\bar{R}_{2p}\begin{bmatrix} J\bar{g}_p \\ 1 \\ \bar{g}_p^* \end{bmatrix} = \begin{bmatrix} \bar{0}_p \\ 2e_p^s \\ \bar{0}_p \end{bmatrix}$$

all zero column vector of p zero elements

, we shall use the least squares approach pages

Eqn. (11.44)

in which

$$\bar{R}_{2p} = \begin{bmatrix} \bar{X}_{2p} \\ \bar{X}_{2p}^* \bar{J} \end{bmatrix}^H \begin{bmatrix} \bar{X}_{2p} \\ \bar{X}_{2p}^* \bar{J} \end{bmatrix} = \bar{X}_{2p}^H \bar{X}_{2p} + \bar{J}\bar{X}_{2p}^T \bar{X}_{2p}^* \bar{J} \qquad \text{Eqn. (8.48)}$$

and

$$\bar{X}_{2p} = \begin{bmatrix} x[2p+1] & \cdots & x[1] \\ \vdots & \ddots & \vdots \\ x[N-2p] & & x[2p+1] \\ \vdots & \ddots & \vdots \\ x[N] & \cdots & x[N-2p] \end{bmatrix} \qquad \text{Eqn. (8.22)} \qquad \bar{g}_p = \begin{bmatrix} g_p[1] \\ \vdots \\ g_p[p] \end{bmatrix}$$

The linear smoothing error is given as:

$$e_p^s[n] = x[n] + \sum_{k=1}^{p} \left(g_p[k]\, x[n+k] + g_p^*[k]\, x[n-k] \right) \qquad \text{for } p+1 \le n \le N-p$$

as squared error:

$$\rho_p^s = \sum_{n=p+1}^{N-p} \left| e_p^s[n] \right|^2 = \sum_{n=p+1}^{N-p} \left(e_p^s[n] \right)^* e_p^s[n]$$

All the smoothing errors can be expressed in matrix-vector form as: $\boxed{2}$

$$\bar e_p^s = \bar x + \bar X_1 \bar J \bar g_p + \bar X_2 \bar g_p^* = \left[\bar X_1 \ \ \bar x \ \ \bar X_2\right]\begin{bmatrix}\bar J \bar g_p \\ 1 \\ \bar g_p^*\end{bmatrix} = \bar X_{2p}\begin{bmatrix}\bar J \bar g_p \\ 1 \\ \bar g_p^*\end{bmatrix} = \bar X_{2p}\bar J\begin{bmatrix}\bar J \bar g_p^* \\ 1 \\ \bar g_p\end{bmatrix}$$

in which

$$\bar x = \begin{bmatrix} x[p+1] \\ \vdots \\ x[N-p]\end{bmatrix}, \quad \bar X_1 = \begin{bmatrix} x[2p+1] & \cdots & x[p+2] \\ \vdots & & \vdots \\ x[N] & \cdots & x[N-p+1]\end{bmatrix}, \quad \bar X_2 = \begin{bmatrix} x[p] & \cdots & x[1] \\ \vdots & & \vdots \\ x[N-p-1] & \cdots & x[N-2p]\end{bmatrix}$$

The smoothing squared error is then expressed as

$$\rho_p^s = (\bar e_p^s)^H(\bar e_p^s) = \left[\bar J \bar g_p^H \ 1 \ \bar g_p^T\right]\left(\bar X_{2p}^H \bar X_{2p}\right)\begin{bmatrix}\bar J \bar g_p \\ 1 \\ \bar g_p^*\end{bmatrix}$$

and also

$$2\rho_p^s = (\bar e_p^s)^H(\bar e_p^s) + (\bar e_p^s)^T(\bar e_p^s)^*$$

$$= \left[\bar J \bar g_p^H \ 1 \ \bar g_p^T\right]\underbrace{\left(\bar X_{2p}^H \bar X_{2p} + \bar J \bar X_{2p}^T \bar X_{2p}^*\right)}_{\bar R_{2p}\,!}\begin{bmatrix}\bar J \bar g_p \\ 1 \\ \bar g_p^*\end{bmatrix} = \left[\bar J \bar g_p^H \ 1 \ \bar g_p^T\right]\bar R_{2p}\begin{bmatrix}\bar J \bar g_p \\ 1 \\ \bar g_p^*\end{bmatrix}$$

[illegible], we could "complete the square" to find the $\bar g_p$ that minimized $2\rho_p^s$, Based on eqns (3.64) and (3.69), we can see that the minimum must satisfy

$$\bar R_{2p}\begin{bmatrix}\bar J \bar g_p \\ 1 \\ \bar g_p^*\end{bmatrix} = \begin{bmatrix}\bar 0_p \\ 2\rho_p^s \\ \bar 0_p\end{bmatrix}$$

since

$$\left[\bar J \bar g_p^H \ 1 \ \bar g_p^T\right]\bar R_{2p}\begin{bmatrix}\bar J \bar g_p \\ 1 \\ \bar g_p^*\end{bmatrix} = \left[\bar J \bar g_p^H \ 1 \ \bar g_p^T\right]\begin{bmatrix}\bar 0_p \\ 2\rho_p^s \\ \bar 0_p\end{bmatrix} = \cancelto{0}{\bar J \bar g_p^H \bar 0_p} + 1 \cdot 2\rho_p^s + \cancelto{0}{\bar g_p^T \bar 0_p}$$

$$= 2\rho_p^s$$, the desired result

Chapter 12, Problem 1 First, prove eqn (12.20).

$$\frac{1}{\rho_p}\left|\sum_{i=0}^{p} a_p[i]\exp(-j2\pi f i T)\right|^2 = \frac{1}{\rho_p}\sum_{l=0}^{p}\sum_{i=0}^{p} a_p[l]\, a_p^*[i]\exp(-j2\pi f[l-i]T)$$

defining $a_p[0] = 1$

Make change of variables: $l \to k+i$

$$\frac{1}{\rho_p}\sum_{i=0}^{p}\sum_{k=-i}^{p-i} a_p[k+i]\, a_p^*[i]\exp(-j2\pi f k T) = \frac{1}{\rho_p}\sum_{k=-p}^{p}\sum_{i=0}^{p-k} a_p[k+i]\, a_p^*[i]\exp(-j2\pi f k T)$$

$$= \frac{1}{\rho_p}\sum_{k=-p}^{p}\psi_{AR}[k]\exp(-j2\pi f k T)$$

by defining ψ_{AR} as shown in eqn (12.20)

To prove (12.25), we need to use the analytic form of the inverse ACS matrix, given in eqn. (12.21)

$$\bar{e}^H(f)\,\bar{R}_p^{-1}\,e(f) = \frac{1}{\rho_p}\,\bar{e}^H(f)\left[\bar{T}_p\bar{T}_p^H - \bar{S}_p\bar{S}_p^H\right]\bar{e}(f)$$

$$= \frac{1}{\rho_p}\left[\bar{e}^H(f)\bar{T}_p\right]\left[\bar{e}^H(f)\bar{T}_p\right]^H - \frac{1}{\rho_p}\left[\bar{S}_p^H\bar{e}(f)\right]^H\left[\bar{S}_p^H\bar{e}(f)\right]$$

$$= \frac{1}{\rho_p}\Bigg[\left|\sum_{i=0}^{p} a[i]\exp(-j2\pi f i T)\right|^2 + \left|\sum_{i=0}^{p-1} a[i]\exp(-j2\pi f[i+1]T)\right|^2 + \cdots$$

$$\cdots + \left|\sum_{i=0}^{0} a[i]\exp(-j2\pi f[i+p]T)\right|^2 - \left|\sum_{i=1}^{p} a[i]\exp(+j2\pi[p+1-i]T)\right|^2$$

$$- \left|\sum_{i=1}^{p-1} a[i]\exp(+j2\pi f[p+2-i]T)\right|^2 - \cdots - \left|\sum_{i=1}^{1} a[i]\exp(+j2\pi f[2p-i]T)\right|^2\Bigg]$$

Each $|x|^2$ term above is like the (12.20) expression. Combining like terms

$$\bar{e}^H(f)\,\bar{R}_p^{-1}\,\bar{e}(f) = \frac{1}{\rho_p}\sum_{k=-p}^{p}\sum_{i=0}^{p-k}(p+1-k-2i)\,a_p[k+i]\,a_p^*[i]\,\exp(-j2\pi fkT)$$

and the definition for $\Psi_{MV}[k]$ as given in (12.25) applies.

Chapter 12, Problem 2 From equation (12.18)

- $$P_{MV}^{-1}(p,f) = \sum_{k=0}^{p} P_{AR}^{-1}(k,f)$$ This problem shows that
- $$\bar{P}_{MV}^{-1}(p-1,f) = \sum_{k=0}^{p-1} P_{AR}^{-1}(k,f)$$

Thus, $$\bar{P}_{MV}^{-1}(p,f) - \bar{P}_{MV}^{-1}(p-1,f) = \sum_{k=0}^{p} P_{AR}^{-1}(k,f) - \sum_{k=0}^{p-1} P_{AR}^{-1}(k,f) = P_{AR}^{-1}(p,f)$$

From this, we can infer that

$$P_{AR}(p,f) = \frac{1}{1/P_{MV}(p,f) - 1/P_{MV}(p-1,f)} = \frac{P_{MV}(p,f)\,P_{MV}(p-1,f)}{P_{MV}(p-1,f) - P_{MV}(p,f)}$$

which shows that a pth-order autoregressive is a special mapping of a pth-order and a (p-1)th-order minimum variance spectral estimator.

$\boxed{\text{Chapter 13, Problem 2}}$ From eqn (7.5),

$$\begin{bmatrix} r_{xx}[0] & \bar{r}_p^H \\ \bar{r}_p & \bar{R}_{p-1} \end{bmatrix} \begin{bmatrix} 1 \\ \bar{a}_p \end{bmatrix} = \begin{bmatrix} \rho_p \\ \bar{0}_p \end{bmatrix} \Rightarrow \begin{matrix} r_{xx}[0] + \bar{r}_p^H \bar{a}_p = \rho_p \\ \bar{r}_p + \bar{R}_{p-1}\bar{a}_p = \bar{0}_p \end{matrix}$$

from which we obtain: $\bar{a}_p = -(\bar{R}_{p-1})^{-1} \bar{r}_p$

From eqn (3.88), we can eigendecompose $\bar{R}_{p-1}^{-1} = \sum_{k=1}^{p} \frac{1}{\lambda_k} \bar{v}_k \bar{v}_k^H$ into p eigenvectors and p eigenvalues, where λ_k is an eigenvector of $\bar{R}_{p-1}$. Substituting,

← Note: could have indexed from 0 to $p-1$

$$\bar{a}_p = -\left(\sum_{k=1}^{p} \frac{1}{\lambda_k} \bar{v}_k \bar{v}_k^H\right)\bar{r}_p = -\sum_{k=1}^{p}\left(\frac{\bar{v}_k^H \bar{r}_p}{\lambda_k}\right)\bar{v}_k$$

Defining the scalar from the vector inner product $\alpha_k = \bar{v}_k^H \bar{r}_p$, and assuming that $\lambda_1 \geq \cdots \geq \lambda_p$ and there are M signals, a rank-reduced estimate of the autoregressive parameters based on the signal subspace eigenvectors would be

$$(\bar{a}_p)_{sig\ sub} = -\sum_{k=1}^{M}\left(\frac{\alpha_k}{\lambda_k}\right)\bar{v}_k \qquad M \leq p$$

← Note: could also have indexed from 0 to $M-1$

Chapter 13, Problem 4

$$P_{music}(f) = \left[\bar{e}^H(f) \left(\sum_{k=M}^{p} \bar{V}_k \bar{V}_k^H \right) \bar{e}(f) \right]^{-1}$$

Since $\bar{R}_p = \sum_{k=0}^{p} \lambda_k \bar{V}_k \bar{V}_k^H$, and $\bar{I} = \sum_{k=0}^{p} \bar{V}_k \bar{V}_k^H$ since eigenvectors are orthonormal

$\leftarrow$ dimension $(p+1) \times (p+1)$ identity matrix

$$\bar{P}_{music}(f) = \left[\bar{e}^H(f) \left(\bar{I} - \sum_{k=0}^{M-1} \bar{V}_k \bar{V}_k^H \right) \bar{e}(f) \right]^{-1}$$

or

$$\bar{P}_{music}(f) = \left[\bar{e}^H(f) \bar{e}(f) - \bar{e}^H(f) \left(\sum_{k=0}^{M-1} \bar{V}_k \bar{V}_k^H \right) \bar{e}(f) \right]^{-1}$$

Since $\bar{e}^H(f)\bar{e}(f) = (p+1)$ is a vector inner product with scalar result $(p+1)$, $\boxed{5}$

$$\bar{P}_{music}(f) = \left[(p+1) - \sum_{k=0}^{M-1} \left(\bar{v}_k^H \bar{e}(f)\right)^H \left(\bar{v}_k^H \bar{e}(f)\right) \right]^{-1}$$

$$\mathcal{E}\left\{ \bar{\bar{e}}^f_p[n] \left(\bar{\bar{e}}^b_p[n-1]\right)^H \right\} = \mathcal{E}\left\{ \bar{\bar{a}}_p \bar{x}_p[n] \bar{x}^H_p[n-1] \bar{\bar{b}}^H_p \right\} = \bar{\bar{a}}_p \, \mathcal{E}\left\{ \bar{x}_p[n] \bar{x}^H_p[n-1] \right\} \bar{\bar{b}}^H_p$$

$$= [\bar{\bar{a}}_p \ \bar{\emptyset}] \underbrace{\mathcal{E}\left\{ \bar{x}_{p+1}[n] \bar{x}_{p+1}[n] \right\}}_{\text{ACS matrix of order } p+1} \begin{bmatrix} \bar{\emptyset} \\ \bar{\bar{b}}^H_p \end{bmatrix}$$

$\longrightarrow$ in which $\bar{\emptyset}$ is a $m \times m$ all-zeros matrix (m channels) and the dimension of the $\bar{x}$ vector has increased by 1 and the $\bar{a}$ and $\bar{b}$ vectors are augmented by $\bar{\emptyset}$

$$= [\bar{\bar{a}}_p \ \bar{\emptyset}] \, \bar{\bar{R}}_{p+1} \begin{bmatrix} \bar{\emptyset} \\ \bar{\bar{b}}^H_p \end{bmatrix}$$

$\longrightarrow$ $\bar{\bar{R}}_{p+1}$ is block ACS matrix

$$= [\bar{\bar{a}}_p \ \bar{\emptyset}] \begin{bmatrix} \bar{\bar{R}}_p & \bar{\bar{s}}^H_{p+1} \\ \bar{\bar{s}}_{p+1} & \bar{R}_{xx}[0] \end{bmatrix} \begin{bmatrix} \bar{\emptyset} \\ \bar{\bar{b}}^H_p \end{bmatrix}$$

$\longrightarrow$ using the matrix partition of equation (15.63) in which $\bar{\bar{s}}_{p+1}$ is a block row vector

$$= \left[\bar{P}^f_p \ \bar{\emptyset} \cdots \bar{\emptyset} \ \bar{\Delta}_{p+1} \right] \begin{bmatrix} \bar{\emptyset} \\ \bar{\bar{b}}^H_p \end{bmatrix}$$

$\longrightarrow$ using equation (15.65)

$$= \left[\bar{P}^f_p \ \bar{\emptyset} \cdots \bar{\emptyset} \ \bar{\Delta}_{p+1} \right] \begin{bmatrix} \bar{\emptyset} \\ \bar{B}^H_p[p] \\ \vdots \\ \bar{B}^H_p[1] \\ I \end{bmatrix}$$

$$= \bar{\Delta}_{p+1}$$

CHAPTER 15, PROBLEM 4 Want to show that $\hat{\Lambda}_{p+1} = (\hat{P}_p^{f\ 1/2})^{-1}(\hat{P}_p^{fb})(\hat{P}_p^{b\ 1/2})^{-H}$

Minimization of the trace of the matrix $\sum_{n=p+1}^{N} \tilde{e}_p^f[n](\tilde{e}_p^f[n])^H + \tilde{e}_p^b[n](\tilde{e}_p^b[n])^H$ with respect to Λ_p

is equivalent minimizing the scalar sum of channel squared errors

$$\rho = \sum_{n=p+1}^{N}\left[(\tilde{e}_p^f[n])^H\tilde{e}_p^f[n] + (\tilde{e}_p^b[n])^H\tilde{e}_p^b[n]\right] \qquad \boxed{A}$$

with respect to Λ_p.

From equations (15.84), (15.85), (15.73) and (15.74)

$$e_p^f[n] = e_{p-1}^f[n] + A_p[p]e_{p-1}^b[n-1] = e_{p-1}^f[n] - \Delta_p(P_{p-1}^b)^{-1}e_{p-1}^b[n-1]$$
$$e_p^b[n] = e_{p-1}^b[n-1] + B_p[p]e_{p-1}^f[n] = e_{p-1}^b[n-1] - \Delta_p^H(P_{p-1}^f)^{-1}e_{p-1}^f[n]$$

Using equations (15.79) and (15.13), these can be also be expressed as

$$\Lambda_p = (P_{p-1}^{f\ 1/2})^{-1}(P_{p-1}^{fb})(P_{p-1}^{b\ 1/2})^{-1} \leftarrow \text{note error in text eqn. (15.79)?}$$

$$e_p^f[n] = e_{p-1}^f[n] - (P_{p-1}^{f\ 1/2})\Lambda_p(P_{p-1}^{b\ 1/2})^{-1}e_{p-1}^b[n-1] \qquad \boxed{B}$$
$$e_p^b[n] = e_{p-1}^b[n-1] - (P_{p-1}^{b\ 1/2})\Lambda_p^H(P_{p-1}^{f\ 1/2})^{-1}e_{p-1}^f[n] \qquad \boxed{C}$$

If the normalized residuals $\tilde{e}_p^f[n] = (P_{p-1}^{f\ 1/2})^{-1}e_p^f[n]$ and $\tilde{e}_p^b[n] = (P_{p-1}^{b\ 1/2})^{-1}e_p^b[n]$ are used, then multiplying $\boxed{B}$ through by $(P_{p-1}^{f\ 1/2})^{-1}$ and $\boxed{C}$ through by $(P_{p-1}^{b\ 1/2})^{-1}$ yields

$$\tilde{e}_p^f[n] = \tilde{e}_{p-1}^f[n] - \Lambda_p\tilde{e}_{p-1}^b[n-1] \qquad \boxed{D}$$
$$\tilde{e}_p^b[n] = \tilde{e}_{p-1}^b[n-1] - \Lambda_p^H\tilde{e}_{p-1}^f[n]. \qquad \boxed{E}$$

Results $\boxed{D}$ and $\boxed{E}$ motivate the definition for $\tilde{e}_p^f[n]$ and $\tilde{e}_p^b[n]$ using estimates $\hat{P}_{p-1}^{f\ 1/2}$ and $\hat{P}_{p-1}^{b\ 1/2}$ rather than ideally known matrices $(P_{p-1}^f)^{1/2}$ and $(P_{p-1}^b)^{1/2}$.

Using the normalized residual definitions $\tilde{e}_p^f[n] = (\hat{P}_{p-1}^{f\ 1/2})^{-1}e_p^f[n]$ and $\tilde{e}_p^b[n] = (\hat{P}_{p-1}^{b\ 1/2})^{-1}e_p^b[n]$ and substituting the recursions $\boxed{D}$ and $\boxed{E}$ into $\boxed{A}$

$$\rho = \sum_{n=p+1}^{N}\left[\tilde{e}_{p-1}^{fH}[n] - \tilde{e}_{p-1}^{bH}[n-1]\Lambda_p^H\right]\left[\tilde{e}_{p-1}^f[n] - \Lambda_p\tilde{e}_{p-1}^b[n-1]\right] + \left[\tilde{e}_{p-1}^{bH}[n-1] - \tilde{e}_{p-1}^{fH}[n]\Lambda_p\right]\left[\tilde{e}_{p-1}^b[n-1] - \Lambda_p^H\tilde{e}_{p-1}^f[n]\right]$$

$$= \sum_{n=p+1}^{N}\Big[\tilde{e}_{p-1}^{fH}[n]\tilde{e}_{p-1}^f[n] - \tilde{e}_{p-1}^{bH}[n-1]\Lambda_p^H\tilde{e}_{p-1}^f[n] - \tilde{e}_{p-1}^{fH}[n]\Lambda_p\tilde{e}_{p-1}^b[n-1] + \tilde{e}_{p-1}^{bH}[n-1]\Lambda_p^H\Lambda_p\tilde{e}_{p-1}^b[n-1]$$

$$+\tilde{e}_{p-1}^{bH}[n-1]\tilde{e}_{p-1}^b[n-1] - \tilde{e}_{p-1}^{fH}[n]\Lambda_p\tilde{e}_{p-1}^b[n-1] - \tilde{e}_{p-1}^{bH}[n-1]\Lambda_p^H\tilde{e}_{p-1}^f[n] + \tilde{e}_{p-1}^{fH}[n]\Lambda_p\Lambda_p^H\tilde{e}_{p-1}^f[n]\Big] \qquad \boxed{F}$$

To minimize ρ of $\boxed{F}$, set the derivative of ρ with respect to Λ_p to zero, yielding

$$\hat{\Lambda}_{p} = \sum_n \tilde{e}_{p-1}^f[n]\tilde{e}_{p-1}^{bH}[n] = \sum(\hat{P}_{p-1}^{f\ 1/2})^{-1}e_{p-1}^f[n]\,e_{p-1}^{*b}[n]\,(\hat{P}_{p-1}^{b\ 1/2})^{-1} = (\hat{P}_{p-1}^{f\ 1/2})^{-1}\left(\sum_n e_{p-1}^f[n]e_{p-1}^{*b}[n]\right)(\hat{P}_{p-1}^{b\ 1/2})^{-1} =$$

This problem, in retrospect, appears to be ill-defined. Nuttall originally proved that the inverse weightings $\underline{V}_p = (\underline{\underline{P}}^f_p)^{-1}$ and $\underline{\underline{W}}_p = (\underline{\underline{P}}^b_p)^{-1}$ (eqs. 15.96 and 15.97) lead to positive definite $\hat{\underline{\underline{P}}}^b$ and $\hat{\underline{\underline{P}}}^f$ matrices. However, it also appears to be true in the case $\underline{\underline{V}}_p = \underline{\underline{W}}_p = \underline{\underline{I}}$.

Nuttall then shows that the M-channel correlation sequence recursion

$$\underline{\underline{R}}_{xx}[m] = \sum_{k=1}^{M} \underline{\underline{A}}_p[k]\,\underline{\underline{R}}_{xx}[m-k] \qquad \text{for } m>0 \text{ and order } p$$

is stable, i.e., the elements of $\underline{\underline{R}}_{xx}[m]$ tend to finite values (non-infinite) as m tends to infinity (for fixed order p). The essence of the proof for stability is based on the observation that for arbitrary column vector $\underline{V} = \begin{bmatrix} v_0 \\ \vdots \\ v_m \end{bmatrix}$

$$\underline{V}^H \underline{\underline{R}}_{xx}[m] = \underline{v}_0\, \underline{\underline{P}}^f_p\, v_0 + [\underline{v}_1 \cdots \underline{v}_m]\, \underline{\underline{R}}_{xx}[m-1] \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

is positive definite. Since it has been shown (regardless of weightings $\underline{V}_p$ and $\underline{\underline{W}}_p$) that $\underline{\underline{P}}^f$ and $\underline{\underline{P}}^b$ are positive definite, which (assuming $\underline{\underline{R}}_{xx}[m-1]$ was previously shown to be positive definite) that $\underline{\underline{R}}_{xx}[m]$ remains finite ($\underline{\underline{R}}_{xx}[0]$ fixed) $\Longrightarrow$ stable recursion.

$$A(1) = \begin{bmatrix} -.85 & +.75 \\ -.65 & -.55 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Using equation (15.49),

$$P_{AR}(z) = \frac{1}{D(z)D(1/z^*)} \begin{bmatrix} -.55z + 1.865 - .55z^{-1} & .65z + 0.280 - .75z^{-1} \\ -.75z + 0.280 + .65z^{-1} & -.85z + 2.145 - .85z^{-1} \end{bmatrix}, \quad D(z) = 1 - 1.4z^{-1} + 0.955z^{-2}$$

Factoring the $D(z)$ polynomial provides the poles and factoring the four matrix entries provides the zero locations. For a two-channel process,

$$P_{AR}(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix}$$

then

$$P_{11}(z) = (-.55z + 1.865 - .55z^{-1}) / D(z)D(1/z^*)$$
$$P_{12}(z) = P_{21}^*(1/z^*) = (.65z + 0.280 - .75z^{-1}) / D(z)D(1/z^*)$$
$$P_{22}(z) = (-.85z + 2.145 - .85z^{-1}) / D(z)D(1/z^*)$$

We see that the auto spectra $P_{11}(z)$ and $P_{22}(z)$ and cross spectra $P_{12}(z)$, $P_{21}(z)$ have two zero and four poles each. From equation (15.16), the MSC is

$$MSC(f) = \left. \frac{P_{12}(z)P_{21}(z)}{P_{11}(z)P_{22}(z)} \right|_{z=\exp(j2\pi fT)}$$

$$= \left. \frac{(.65z + 0.280 - .75z^{-1})(-.75z + 0.280 + .65z^{-1})}{(-.55z + 1.865 - .55)(-.85z + 2.145 - .85z^{-1})} \right|_{z=\exp(j2\pi fT)} \qquad \boxed{A}$$

which has 4 zeros and 4 poles. Evaluation of $MSC(f)$ over a fine evaluation range shows that the maximum value is 0.999013, which may be found by setting the derivative of $\boxed{A}$ with respect to f to zero and finding the f that maximizes the derivative (approximately 0.12).

To prove $|P_{xy}(f)|^2 \le P_{xx}(f)P_{yy}(f)$, we can actually prove this for the multichannel case of m channels. Define the complex scalar random process $y[n]$ which is a linear combination of the m individual complex channel random processes,

$$y[n] = a_1^* x_1[n] + a_2^* x_2[n] + \cdots + a_m^* x_m[n] = \underline{a}^H \underline{x}[n]. \qquad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \ \underline{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_m[n] \end{bmatrix}$$

The ACS of random process $y[n]$ is

$$r_{yy}[m] = \mathcal{E}\{y[n+m]y^*[n]\} = \underline{a}^H \mathcal{E}\{\underline{x}[n+m]\,\underline{x}^H[n]\}\underline{a} = \underline{a}^H \underline{\underline{R}}_{xx}[m]\underline{a} \qquad \boxed{A}$$

in which $\underline{\underline{R}}_{xx}$ is the multichannel correlation sequence defined by eq. (15.18). The discrete-time Fourier transform of $\boxed{A}$ can be expressed in terms of the power spectral density of process $y[n]$ and the multichannel power spectral density matrix of the multichannel process $\underline{x}[n]$.

$$P_{yy}(f) = \text{DTFT}\{r_{yy}[m]\} = \underline{a}^H \text{DTFT}\{\underline{\underline{R}}_{xx}[m]\}\underline{a} = \underline{a}^H \underline{\underline{P}}_{xx}(f)\underline{a} \qquad \boxed{B}$$

in which $\boxed{B}$ is a quadratic form of multichannel power spectral density matrix $\underline{\underline{P}}_{xx}(f)$. Because $P_{yy}(f) \ge 0$, and real-valued, because it is a power spectral density function, then we may infer that $\underline{\underline{P}}_{xx}(f) = \underline{\underline{P}}_{xx}^H(f)$ [has a Hermitian property] and it is a positive semi-definite matrix. A positive semi-definite Hermitian matrix has the property that every principal minor of $\underline{\underline{P}}_{xx}(f)$ is non-negative. For example, if $m=2$, then

$$\begin{vmatrix} P_{11}(f) & P_{12}(f) \\ P_{21}(f) & P_{22}(f) \end{vmatrix} \ge 0 \quad \Longrightarrow \quad P_{11}(f)P_{22}(f) - |P_{12}(f)|^2 \ge 0 \qquad \text{since } P_{21}^* = P_{12}$$

or

$$|P_{12}(f)|^2 \le P_{11}(f)P_{22}(f)$$

For $m=3$,

$$\begin{vmatrix} P_{11}(f) & P_{12}(f) & P_{13}(f) \\ P_{21}(f) & P_{22}(f) & P_{23}(f) \\ P_{31}(f) & P_{32}(f) & P_{33}(f) \end{vmatrix} \ge 0$$

with related conditions among the 3 auto and 3 cross spectral terms.