PREFACE TO THE REVISED 2010 EDITION

This book is intended to provide the reader with a complete foundation in modern set theory: how the subject is axiomatized, what ordinal and cardinal numbers are, what the role of the axiom of choice is, what constructible sets are, what forcing is. All the details are present, from initial axiomatic development to the relative consistency and independence of the continuum hypothesis. We have made use of our rather broad experience as logicians to organize things in what we hope will be seen as an elegant and insightful manner. The Preface to the first edition discusses the novelties of our approach in some detail—we do not repeat this here. Our coverage begins with Zermelo (and Cantor, of course), and ends with the work of Cohen. There are no new results, but there are new presentations of important results. After this book, the reader will have a full background in the subject and will be ready to move on to further and more current study. And whether or not further investigation of contemporary work in set theory is desired, the reader will have seen a thorough presentation of one of the great achievements of twentieth century mathematics.

The only changes made to this work, from the first edition, involve the correction of errors. While there were quite a few, most were minor. The most significant error concerned the proof that constructibility is absolute, in Chapter 14, and correcting this required the replacement of several theorems and proofs. Except for this, theorem and definition numbering are the same as in the original edition.

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PREFACE

In the late nineteenth century Georg Cantor startled the mathematical world by proving that there are different orders of infinity. In particular he showed that for any set S of elements, the size of the power set $\mathcal{P}(S)$, of all *subsets* of S, is greater than the size of S. (His proof is given in Chapter 1 of this volume.) He then raised the following question: Given an infinite set S, does there exist a set X larger than S but smaller than the set $\mathcal{P}(S)$? He *conjectured* that there is no such intermediate set X, and this conjecture is known as the *generalized continuum hypothesis*. To this day, no one knows whether it is true or not (Gödel conjectured that it is false). But in 1938 Kurt Gödel made the amazing discovery that whether true or not, it is at least not *disprovable* from the accepted axioms of set theory (the Zermelo-Fraenkel axiom system, explained in this book) assuming these axioms themselves are consistent, and in 1963 Paul Cohen proved that the continuum hypothesis, even if true, is not *provable* either. So the continuum hypothesis is thus *independent* of these axioms (again assuming that the axioms are consistent, which we will henceforth do).

Gödel and Cohen proved similar results about the so-called *axiom of choice*—one form of which states that given any collection of non-empty sets, it is possible to simultaneously assign to each of the sets one of its members. The axiom of choice is certainly useful in modern mathematics, and most mathematicians regard it as self-evident, unlike the continuum hypothesis, which does not appear to be either self-evidently true or self-evidently false. Gödel showed that the axiom of choice is not disprovable from the Zermelo-Fraenkel axioms, and Cohen showed that the axiom is not provable either.

Later Gödel turned his attention to the more elegant system of *class-set* theory known as *NBG*, in place of the Zermelo-Fraenkel axioms, and this is the system we use here. We provide a general introduction to *NBG* and to the Gödel and Cohen proofs of relative consistency and independence of the axiom of choice and the continuum hypothesis. Our book is intended as a text for advanced undergraduates and graduates in mathematics and philosophy. It is close to self-contained, involving only a basic familiarity with the logical connectives and quantifiers. A semester course in symbolic logic is more than enough background.

About two-thirds of the book originated as class notes, used in teaching set theory over the years. We revised these chapters to make them more suitable as a text, and we added the remaining chapters. Now for some special features of the book.

Part I (Chapters 1–9) on basic set theory deals mainly with standard topics, but from a novel viewpoint. Standard topics include the axiomatic development of set theory, ordinal and cardinal numbers, and the investigation of various equivalent forms of the axiom of choice. (We do not cover the construction of the real number system—see

(Vaught 1995) or (Moschovakis 1994) for this.) Our main novelty lies in the use of Smullyan's double induction and double superinduction principles (in Chapters 3 and 4), which have hitherto appeared only in research papers, and which provide a unified approach to several of these topics, including Zermelo's well ordering theorem, Zorn's lemma, the transfinite recursion theorem, and the entire development of the theory of ordinals. It is high time that these should appear in a textbook. Closely connected with this is Cowen's theorem which also has not yet appeared in any textbook. Together they provide a particularly smooth and intuitive development of the ordinals.

We do as much as we can without using the axiom of substitution, which is not introduced until Chapter 6. In Chapter 4 we present the double superinduction principle and its applications to well ordering and various maximal principles such as Zorn's lemma. We present these principles in sharper forms than those usually given—forms which might be termed *local*, rather than global. For example, instead of proving the global statement that the axiom of choice implies that every set can be well ordered (Zermelo's theorem), we prefer the sharper local form that for any set S, if there is a choice function for S, then S can be well ordered. Thus the only choice function needed to well order S is one for S itself. In general, in dealing with maximal principles, we are interested in just what choice functions are needed to do a given job. Some of the results of these maximal principles may be new (for example, Theorem 6.2 of Chapter 4). The chapter concludes with the statement and proof of Cowen's theorem, which deserves to be better known.

Our treatment of ordinals (Chapter 5) is non-standard, in that our very definition of *ordinal number* involves quantifying over all *classes*, instead of all sets (x is an ordinal if and only if x belongs to all classes having certain closure properties) and it is therefore not immediate in the class-set theory that we are using that the class of all ordinals exists! Yet the definition leads to such a beautifully natural and elegant treatment of ordinals, if only it would work in *NBG*! Well, fortunately it *does* work by virtue of Cowen's theorem, from which the existence of the class of ordinals is immediate, and so this is the approach we happily take. This natural definition of *ordinal* was first published by Sion & Willmott (1962), and was independently thought of by Hao Wang (written communication) and by Smullyan. It is high time that this neat approach should be known!

Using the axiom of substitution, in Chapter 6 we establish various forms of the transfinite recursion theorem, but by quite a novel method: Our first form of this theorem (Theorem 5.1) appears to be new, and the proof is almost immediate from the double superinduction principle. From this theorem, the various other and more standard forms of the transfinite recursion theorem follow without further use of the Axiom of Substitution.

Chapters 7, 8, and 9 contain more or less standard material on rank, well foundedness, and cardinals. One general remark about style: Our approach is rather leisurely (particularly at the beginning) and free-wheeling, and is definitely *semantic* rather than syntactic. But we do indicate how it can all be formalized.

Part II (Chapters 10–15) is devoted to two of Gödel's proofs of the relative consistency of the continuum hypothesis. The first—and central—one is a modification of Gödel's 1938 proof, using later results such as the Mostowski-Shepherdson mapping theorem, reflection principles, and various contributions of Tarski, Vaught, Scott, Levy, Karp, and others. The second proof (given largely as a sequence of exercises) comes closer to Gödel's original proof, and is more involved, but reveals additional information of interest in its own right, information which was *implicit* in Gödel's proof, but which we bring out quite explicitly. These proofs combine four major themes—Mostowski-Shepherdson maps (Chapter 10), Reflection Principles (Chapter 11), Constructible sets (Chapter 12) (which is the central notion) and the notion of *absoluteness*. Our treatment of Mostowski-Shepherdson maps (also called *collapsing maps*) is considerably more thorough than usual, and brings to light interesting but less widely known results about rank, induction, and transfinite recursion, all generalized to well founded relational systems. We learned of this approach from Dana Scott. We also do more in Chapter 11 on Reflection Principles than is strictly necessary for Gödel's proof.

The four themes mentioned above unite in a remarkable way in the last chapters of Part II, leading to the basic result that the class of constructible sets satisfies all the axioms of class-set theory, as well as the axiom of choice and the generalized continuum hypothesis. Then, as explained in Chapter 15, all this can be formalized to show that if the formal axioms of class-set theory are consistent, then neither the axiom of choice nor the generalized continuum hypothesis can be disproved in the system.

Part III (Chapters 16–22) presents Paul Cohen's proofs of independence results using forcing. The approach it takes is novel and is, we believe, pedagogically an improvement on the usual versions. It is not hard to show that methods like those of Gödel that show the consistency of the continuum hypothesis will not allow one to construct a model of set theory that establishes its independence. But this fact applies only if a classical first-order model is desired. We introduce the notion of a *modal* model for set theory, and show there is one in which a version of the continuum hypothesis fails. We then show, by a simple argument, that this implies the *classical* independence.

Currently there are in the literature two main approaches to forcing. One version uses partially ordered sets of "forcing conditions," and follows Cohen's approach generally. Mathematically, such an approach can be hard for students to follow. In addition, certain countability assumptions are commonly made in order to present the proofs most naturally, then one shows how to avoid these assumptions. This is pedagogically confusing, at the least. The other standard version in the literature uses Boolean-valued models—in effect a kind of non-classical logic. This is mathematically easier to follow, but it is hard to see motivation for the specific models that are constructed. In fact, for motivation one generally returns to the Cohen-style approach.

We use the well-known modal logic $S4$. Partial orderings yield Kripke models for it, so we are in the Cohen tradition rather than in the Boolean-valued one. Even so, countability assumptions play no role. It is not that we can avoid them—they simply do not come up. Since we are following Cohen, details of the construction of particular models can be motivated fairly naturally. On the other hand, mathematical complexity is minimized in a way that is familiar to logicians generally—informal becomes formal; meta-language becomes language. Let us be more precise about this point, since it is a crucial one.

In most Cohen-style presentations one finds talk of *denseness*: a set S of forcing conditions is dense below p if every condition that strengthens p in turn has a strengthening that is in S. Then one finds assertions like: the set of conditions at which X is true is dense below p. There is much that must be established about which sets are dense, and all of this takes place at the meta-level, via arguments in English. But if one introduces a modal language, the assertion that X is true in a set that is dense below p becomes simply: $\Box\Diamond X$ is true at p. What had been meta-level arguments about sets of conditions become formal manipulations of the standard modal operators $\Box$ and $\Diamond$. Simple rules for these operators can be given once and for all. In this way the reader can concentrate on the key ideas of the independence proofs, without the details needing constant attention.

In a nutshell, we show various independence results using a natural $S4$ generalization of sets-with-rank (unramified forcing, in effect). Classical models are not constructed and countability assumptions are not made, or needed. Afterwards, countability assumptions are added, and it is shown how classical models can be produced. At this point, connections with other treatments are apparent, and the reader is prepared to consult the standard literature.

Our treatment is self-contained, and the ideas we need from modal logic, which are both standard and fairly minimal, are fully presented.

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CHAPTER

GENERAL BACKGROUND

Infinity and the continuum problem

§1 What is infinity?

If you ask the average person what the word "infinite" means, the answer is likely to be: "endless," or "without end." This answer, though it somewhat captures the true picture, is not altogether satisfactory, since, for example, the circumference of a circle of radius 1 inch is certainly endless, in the sense of having no beginning or end, but one would not call it infinite (in the sense that a straight line extending infinitely far in at least one direction would be called infinite).

Now, the word "infinite" is used in mathematics in a very precise sense, and this is the sense we shall adopt. The first question is what are the sorts of things that can properly be said to be infinite or finite. The answer is that it is *sets of objects* to which these adjectives are applicable. And so, just what do we mean when we say that a set of objects is infinite or that it is finite? Before answering this question, some preliminaries are in order.

Let us first ask what it means for two sets to have the *same number of elements*, or to be of *the same size*. The clue here lies in the notion of a 1-1 correspondence, read "one to one correspondence." We say that a set A can be put into a 1-1 correspondence with a set B if it is possible to match each element of A with one and only one element of B in such a way that no element of B is left out and no two distinct elements of A are matched with the same element of B. For example, suppose you look into a theater and see that every seat is taken (and no one is sitting on anyone's lap) and that no one is standing. Then, without having to count either the number of seats or the number of people, you will know that the numbers are the same. The reason is that the set of people is in a 1-1 correspondence with the set of seats—each person corresponds to the seat on which he or she sits.

At this point we rely on the reader's familiarity with the natural numbers 0, 1, 2, $\ldots, n, \ldots$, that is, 0 and the positive whole numbers. (Later in this study we will see how the natural numbers can be explained in terms of the more basic notion of *set*, but for now, the reader's intuitive notion of "number" is sufficiently reliable.) Now, what does it mean to say that a set—say the set of fingers on my left hand—has exactly 5 members? It means that it can be put into a 1-1 correspondence with the set of positive whole numbers from 1 to 5. More generally, for any positive whole number n, to say that a set x has n elements is to say that x can be put into a 1-1 correspondence with the set of positive whole numbers from 1 to n. (Incidentally, the process of putting the elements of x into a 1-1 correspondence with the numbers from 1 to n is commonly called *counting*.) And so we have defined what it means for a set to have n elements, where n is a *positive* whole number. (The word "number" in this volume will always mean natural number, unless specified to the contrary.) As for the case $n = 0$, we say that a set has 0 elements if it has no elements at all (as, for example, if at a given time there is no one in a theater, we say that the set of people in the theater has *zero* elements, or that the set is *empty*).

Now that we have defined what it means for a set to have n elements, where n is any natural number, we can define what it means for a set to be finite or infinite.

Definition 1.1. A set S is said to be *finite* if there is a natural number n such that S has n elements. If there is no such natural number n, then S is said to be *infinite*.

This is the definition we shall adopt. Examples of finite sets abound. An obvious example of an *infinite* set is the set of all natural numbers themselves. Also the set of all *even* natural numbers is clearly infinite. As another example, consider a line segment one inch long. Although the *length* of the line is finite, the set of points on the line is infinite, since between any two distinct points there is a point (for example, the one halfway between them).

§2 Countable or uncountable?

The true father of the field known as *set theory* is the late nineteenth century mathematician Georg Cantor. A most fundamental problem about infinite sets that engaged his attention is this: are all infinite sets of the same size—that is, can any two infinite sets be put into a 1-1 correspondence with each other—or do they come in different sizes? What is your guess? (I have posed this question to non-mathematicians, and about half guess *yes* and half guess *no*.) The answer (as discovered by Cantor) will be given in this chapter.

An infinite set is called *denumerable* or *countably infinite* if it can be put into a 1-1 correspondence with the set $1, 2, 3, \ldots, n, \ldots$ of positive whole numbers. A set is called *uncountable* or *non-denumerable* if it is infinite but not denumerable. And so our question can be rephrased thus: is every infinite set denumerable or do there exist non-denumerable sets?

As I understand the history of the situation, Cantor spent 12 years trying to prove that every infinite set is denumerable and then in the thirteenth year discovered that the opposite is the case. What he did was to consider various sets which at first sight *seemed* too large to be denumerable, but then in each case he thought of an ingenious way of enumerating them (i.e., of putting them into a 1-1 correspondence with the set of positive integers).

I like to illustrate Cantor's constructions in the following manner. Imagine that you and I are immortal. I write down a positive whole number on a slip of paper and tell you: "Each day you have one and only one guess as to what the number is. If and when you guess it, you win a grand prize." Now, if you guess numbers at random, then it is perfectly possible that you will never get the prize. But if you guess systematically in the order $1, 2, 3, 4, \ldots, n, \ldots$, then obviously you will get the prize sooner or later.

Now, for another test a wee bit more difficult. This time I say: "I have now written down either a positive whole number or written down a negative whole number. Each day you have one and only one guess as to what it is. Again, if you ever guess it, you win a prize." Now do you have a strategy that will guarantee success sooner or later? Of course you do; you guess in the order 1, −1, 2, −2, 3, −3, ... and sooner or later you are bound to come to my number. (This shows that the set of all positive and negative whole numbers taken together, which at first sight appears to be "twice as large" as the set of positive whole numbers alone, is really denumerable after all!)

Now I give you a more difficult test. I write down *two* positive whole numbers on a slip of paper (maybe two different numbers, or maybe the same number written twice). Again you have one and only one guess each day as to what I wrote, and if you guess correctly you win a grand prize. However, you must guess *both* the numbers on the same day—you are not allowed to guess one of the numbers on one day and the other on another day, as this would be hardly any different from the first test. Is there *now* a strategy by which you can certainly win? At first thought, this may seem hopeless, since there are infinitely many possibilities for the first number that I wrote, and with each of the possibilities, there are infinitely many possibilities for the second number. But I can assure you that there is a strategy (a very simple one in fact) in which you can be sure to win the prize sooner or later. What strategy will work? I will leave this as a problem (call it *Problem 1*) which the reader is invited to try to solve. The solution will be given a bit later.

As a slight variant of this problem, suppose I require that you must not only name the numbers I have written, but also the order in which I wrote them (say, the first is written to the left of the second one). Now what strategy will work?

Next problem (Problem 2). I write down a positive fraction (a quotient of two positive whole numbers). On each day you have one and only one guess as to what the fraction is. What strategy will enable you to certainly name my fraction on some day or other? Or is there no such strategy? That is, is the set of positive fractions denumerable or not?

Now for a slightly harder problem (Problem 3). Is the set of all *finite* sets of positive whole numbers denumerable or not? That is, suppose I write down some *finite* set of positive integers (whole numbers). I don't tell you how many numbers are in the set, nor what is the highest number of the set. All you know is that the set is finite. Now do you feel that there is any chance of your certainly guessing my set?

At this point, let us discuss the solutions to these problems.

Solutions—For Problem 1, there is only one possibility in which the highest of the two numbers is 1—namely $\langle 1,1\rangle$; there are two possibilities in which the highest number is 2—namely, $\langle 1,2\rangle$ and $\langle 2,2\rangle$, and for each positive integer n, there are only n possibilities for the highest number being n—namely $\langle 1,n\rangle, \langle 2,n\rangle, \ldots, \langle n-1,n\rangle, \langle n,n\rangle$. And so you successively guess $\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \ldots, \langle 1,n\rangle, \langle 2,n\rangle, \ldots, \langle n,n\rangle, \ldots$. Sooner or later you are bound to guess my pair.

As for the variant of the problem in which you must guess also the order, you simply mention each pair of *distinct* numbers in both orders before proceeding to the next—that is you name the ordered pairs in the order: $\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 1,3\rangle, \langle 3,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \ldots, \langle 1,n\rangle, \langle n,1\rangle, \langle 2,n\rangle, \langle n,2\rangle, \ldots$.

Problem 2 is essentially the same as the last problem and differs only in that in writing a fraction $\frac{m}{n}$, where m, n are positive *whole* numbers, the numerator m is written *above* the denominator n, instead of to the left of it. Thus we can enumerate all the positive fractions in the order $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{2}{3}, \frac{3}{2}, \frac{3}{3}, \ldots, \frac{1}{n}, \frac{n}{1}, \frac{2}{n}, \frac{n}{2}, \ldots$. This result of Cantor—that the set of fractions (and consequently the set of rational numbers) is denumerable—was quite a shock to the mathematical world, when first announced.

For Problem 3, the key is that for each positive integer n, there are only finitely many sets whose highest member is n—namely, there are 2^{n-1} such sets. (Reason: each such set consists of n together with some subset of the numbers from 1 to $n-1$, and there are 2^{n-1} possible sets of numbers from 1 to $n-1$.) And so you first name the one and only set whose highest member is 1, then the two sets whose highest member is 2, then the four sets whose highest member is 3, then the eight sets whose highest member is 4, and so forth.

§3 A non-denumerable set

We have just seen that the collection of all *finite* sets of positive integers is denumerable. Now, what about the collection of *all* sets of positive integers—infinite sets as well as finite ones? Is this set denumerable or not? Well, Cantor showed that it is not! His argument is ingenious in its very simplicity and I would like to first illustrate it as follows.

Imagine that we have a book with denumerably many pages—the pages being consecutively numbered: page 1, page 2, ..., page n, ... (Thus the book has infinitely many pages, but these pages are in a 1-1 correspondence with the set of positive integers.) On each page is listed a set of positive integers. If *every* set of positive integers is listed in the book, then the book wins a blue ribbon. But the book *cannot* win the ribbon; there must be at least one set of positive integers that is *not* listed in the book! Why? Let us see.

We let S_1 be the set listed on page 1, S_2 the set listed on page 2, ..., S_n the set listed on page n, ... We wish to define a set S of positive integers which is different from every one of the sets $S_1, S_2, \ldots, S_n, \ldots$. Well, we first consider the number 1—whether it should go into our set S or not. We do this by considering the set S_1 listed on page 1 of the book. Either 1 belongs to the set S_1 or it doesn't. If it doesn't, then we shall include it in our set S, but if 1 *does* belong to S_1, then we *exclude* 1 from S. Thus whatever future decisions we make concerning the numbers 2, 3, ..., n, ..., we have secured the fact that $S \neq S_1$ because, of the two sets S and S_1, one of them contains 1 and the other doesn't. Next we consider the number 2. We put it into our set S just in case 2 does *not* belong to the set S_2, and this guarantees that $S \neq S_2$ (since one of them contains 2 and the other doesn't). And so on with every positive integer n. We thus take S to be the set of all positive integers n such that n does *not* belong to the set S_n. Then for every n, $S \neq S_n$ because, of the two sets S and S_n, one of them contains n and the other doesn't.

To make matters a bit more concrete, suppose, for example, that the first eight sets S_1–S_8 are the following:

$* S_1$ — The set of all numbers greater than 5
S_2 — The set of all even numbers
S_3 — The set of all prime numbers
$* S_4$ — The set of all odd numbers
S_5 — The set of all (positive whole) numbers
$* S_6$ — The empty set (the set with no numbers at all)
$* S_7$ — The set of all numbers divisible by 3
S_8 — The set of all numbers divisible by 4

I have starred those lines n such that n does *not* belong to S_n. (For example, 1 is not greater than 5, so 1 does not belong to S_1, so line 1 is starred. On the other hand, 2 is even, so 2 does belong to S_2, so line 2 is unstarred.) Then the numbers of the starred lines belong to S and those of the unstarred lines don't. Thus, of the first eight numbers, those that belong to S are 1, 4, 6 and 7. And so we see that $S \neq S_1$, because 1 belongs to S but not to S_1. As for S_2, $S \neq S_2$ because 2 doesn't belong to S, but does belong to S_2. As for 3, 3 belongs to S_3 but not to S, so $S \neq S_3$. And so forth.

What we have shown is that given any countably infinite (denumerable) sequence $S_1, S_2, \ldots, S_n, \ldots$ of sets of positive integers, there exists a set S of positive integers (namely, the set of all n such that n doesn't belong to S_n) such that S is different from each of the sets $S_1, S_2, \ldots, S_n, \ldots$. This means that *no* denumerable set of sets of positive integers contains every set of positive integers—in other words *the set of all sets of positive integers is non-denumerable*!

This is a special case of Cantor's theorem.

The more general case—For any set A, by the *power set of* A—symbolized $\mathcal{P}(A)$—is meant the set of all *subsets* of A. (A set B is called a subset of A if every member of B is also a member of A—as for example, the set of all even whole numbers is a subset of the set of all whole numbers.) We have just proved that if A is the set of positive integers, then $\mathcal{P}(A)$ cannot be put into a 1-1 correspondence with A. More generally, Cantor's theorem is that for *any* set A, it is impossible for $\mathcal{P}(A)$ to be put into a 1-1 correspondence with A. The proof is not much different from the proof for the special case that we have considered and runs as follows. Suppose we have a 1-1 correspondence that matches each element x of A with a subset S_x of A. Let S be the set of all elements x of A such that x does not belong to the set S_x. Then for every element x of A, $S \neq S_x$ because, of the sets S and S_x, one of them contains x and the other doesn't. And so there is *no* x in A that corresponds to S, and so in the purported 1-1 correspondence between A and the set of *all* subsets of A—the set S has been left out.

§4 Larger and smaller

Given two sets A and B, what does it mean to say that A is (numerically) *smaller* than B, or *larger* than B? It would be tempting to define A as being smaller than B, or B as being larger than A, if A can be put into a 1-1 correspondence with a *proper* subset of B—that is, with a subset of B which is not all of B (it leaves out at least one element of B). This definition would work fine for finite sets, but not for infinite sets, since, for example, the set of natural numbers can be put into a 1-1 correspondence with the set of natural numbers starting at 2, but we certainly wouldn't want to say that the set of natural numbers is smaller than itself! No, the correct definition is the following: we say that A is *smaller* than B—in symbols $A \prec B$—if A can be put into a 1-1 correspondence with some subset of B *and also* A *cannot* be put into a 1-1 correspondence with the whole of B (in other words, every 1-1 correspondence from A to part of B leaves out at least one element of B). We note that if A can be put into a 1-1 correspondence with a subset of B, then either A is smaller than B or A is the same size as B—in symbols, $A \preceq B$.

Two interesting problems immediately arise. Suppose that A can be put into a 1-1 correspondence with a subset of B and B can be put into a 1-1 correspondence with a subset of A; does it necessarily follow that (all of) A can be put into a 1-1 correspondence with (all of) B? The answer is *yes*, and this has been proved in several ways, two of which we will study in this volume. (The result is known as the Schröder-Bernstein theorem.) A second and deeper problem is this. Two sets A and B are called *comparable* (comparable in size, that is), if either A can be put into a 1-1 correspondence with a subset of B, or B can be put into a 1-1 correspondence with a subset of A. Equivalently, A is *comparable* with B if and only if A is either smaller than B, larger than B, or the same size as B. If A and B are *finite* sets, then of course they are comparable. But is this necessarily true for infinite sets? Isn't it possible that A and B are such that there is no 1-1 correspondence between A and any subset of B, and also there is no 1-1 correspondence between B and any subset of A? Well, the statement that any two sets A and B *are* comparable is certainly accepted by most mathematicians, but it is equivalent to a somewhat controversial (though highly important) principle known as the *axiom of choice* (which in one form roughly says that, given any non-empty collection of non-empty sets, it is possible to simultaneously choose exactly one member from each of the sets). This equivalence is one of the topics we will study.

Cantor's theorem in full—We have shown that no set A can be put into a 1-1 correspondence with $\mathcal{P}(A)$, and so A is not of the same size as $\mathcal{P}(A)$. But A can be easily put into a 1-1 correspondence with a subset of $\mathcal{P}(A)$—namely, let any element x of A correspond to the set whose only element is x (this set is denoted $\{x\}$). And so we have:

Theorem 4.1 (Cantor's theorem). *$A \prec \mathcal{P}(A)$ for any set A. That is, A is smaller than $\mathcal{P}(A)$—i.e., A can be put into a 1-1 correspondence with a subset of $\mathcal{P}(A)$, but not with $\mathcal{P}(A)$.*

It of course follows from Cantor's theorem that for any infinite set A, the set $\mathcal{P}(A)$ is larger than A, and then $\mathcal{P}(\mathcal{P}(A))$ is in turn larger than $\mathcal{P}(A)$, then $\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))$ is in turn larger than $\mathcal{P}(\mathcal{P}(A))$, and so we can generate an *infinite* sequence A, $\mathcal{P}(A)$, $\mathcal{P}(\mathcal{P}(A))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))$, ... of sets—each one of a larger size than the preceding one. And so there is at least a denumerable infinity of different infinities (and, as we will later see, there are in fact many more than that!).

§5 The continuum problem

Two sets of the same size are also said to be of the same *cardinality*, or the same *power*. A set A that is larger than a set B is also said to be of higher cardinality or higher power than B (and then B is said to be of lower cardinality or lower power than A).

We let ω be the set of natural numbers (starting with 0). It of course has the same cardinality as the set of positive whole numbers (starting with 1)—since we can correspond to each natural number x the positive whole number $x + 1$, and this correspondence is clearly 1-1—so ω is a denumerable set. As we have seen, $\mathcal{P}(\omega)$ has higher power than ω and it is said to be of the power of the continuum, because it can be proved to be of the same size as the set of all real numbers, or as the set of all points on a straight line. Now, the continuum problem is this: does there exist a set x of higher cardinality than ω but of lower cardinality than $\mathcal{P}(\omega)$, or is the size of $\mathcal{P}(\omega)$ the "next" size after the size of ω? Cantor's continuum hypothesis (and it is only a *hypothesis*!) is that there is no such intermediate set x—in other words, any set that is larger than ω must be at least as large as $\mathcal{P}(\omega)$. And Cantor's *generalized* continuum hypothesis is that for *every* infinite set A, there can never be a set of higher cardinality than A but of lower cardinality than $\mathcal{P}(A)$. These hypotheses are, of course, only *conjectures* of Cantor. So far, no attempts have been the slightest bit successful in determining whether the continuum hypothesis is true or false! Another question *not to be confused with the truth or falsity of the continuum hypothesis* is whether it can be formally proved or disproved from the present day axioms of set theory. *This* question is completely settled.

We use the expression "present day axioms of set theory" to mean the system *ZF*—Zermelo-Fraenkel set theory—or the closely related system *NBG* of *class-set theory* (the system studied in this book), which are the systems currently in widespread use. Kurt Gödel (Gödel 1938, Gödel 1939) proved the celebrated result that the generalized continuum hypothesis is formally consistent with the axioms of *ZF* (assuming *ZF* is itself consistent, which we do throughout this book). And Paul Cohen (Cohen 1963, Cohen 1964, Cohen 1966) settled the matter in the other direction—he showed that the negation of the generalized continuum hypothesis (in fact even the negation of the special continuum hypothesis) is consistent in *ZF*. Thus the continuum hypothesis is *independent* of the axioms of *ZF*—i.e., it can neither be proved nor disproved in *ZF*. So the axioms of *ZF* are *not strong enough* to settle the continuum problem.

Another important independence result concerns the *axiom of choice*. Gödel has also shown that the axiom of choice is not disprovable in *ZF*, and Cohen has shown that it is not provable in *ZF*. So the axiom of choice is also independent of the axioms of *ZF*. The most remarkable result of Cohen is that even if we add the axiom of choice to the other axioms of set theory (i.e., to *ZF*) it is still not possible to prove the continuum hypothesis.

§6 Significance of the results

There has been a remarkable diversity of opinions concerning the significance of the independence results.

First of all we must realize that these results have been proved for the particular axiom systems *ZF* and *NBG* (though the arguments go through for many related systems). There are, however, very different *formal axiom systems* of set theory in which the arguments of Gödel and Cohen do *not* go through. But it is highly questionable whether these other systems really describe the notion of "set" as used by the working mathematician. For example, in Quine's systems *NF* (New Foundations) and *ML* (Mathematical Logic) (Quine 1937, Quine 1940) the axiom of choice is provably false. For one who regards all these alternative systems to be on an equal footing (and this includes many so-called *formalists*) the independence results may well seem insignificant; they could be construed as merely saying that the continuum hypothesis comes out positive in one system and negative in some other (equally good) system. On the other hand there are those who concede that *ZF* is the more "natural" set theory, but who cannot understand what could be meant by mathematical *truth* other than provability in *ZF*. They would therefore construe the independence results as saying that the continuum hypothesis is neither true or false. A slightly modified viewpoint is to the effect that the only propositions we can ever *know* to be true are those provable in *ZF*, hence that we can never *know* whether the continuum hypothesis is true or false.

The so-called *mathematical realist* or *Platonist* (and this seems to include a large number of working mathematicians) looks upon the matter very differently. We can describe the realist viewpoint as follows. There is a well defined mathematical reality of sets, and in this reality, the continuum hypothesis is definitely true or false. The axioms of *ZF* give a true but incomplete description of this reality. The independence results cast no light on the truth or falsity of the continuum hypothesis, nor do they in any way indicate that it is neither true nor false. Rather they highlight the inadequacy of our present day axiom system *ZF*. But it is perfectly possible that new principles of set theory may one day be found which, though not derivable from the present axioms, are nevertheless self-evident (as the axiom of choice is to most mathematicians) and which might settle the continuum hypothesis one way or the other. Indeed Gödel (1947)—despite his proof of the formal *consistency* of the continuum hypothesis—has conjectured that when such a principle is found, the continuum hypothesis will then be seen to be *false*. In this volume we adopt a realist viewpoint.

This book is divided into three parts. Part I contains the basics of axiomatic set theory. In Part II we give Gödel's proofs of the consistency of the axiom of choice and the continuum hypothesis. (Gödel proved these in two somewhat different ways (Gödel 1938, Gödel 1939) and (Gödel 1940), and both proofs are of considerable interest.) Part III is devoted to Paul Cohen's proofs of the consistency of the negation of the continuum hypothesis (even with the addition of the axiom of choice) and the consistency of the negation of the axiom of choice, as well as some related results.

Introduction to sets and classes

Before embarking on a rigorous development of class-set theory—which we begin in the next chapter—some brief historical background might be helpful.

One purpose of the subject known as *axiomatic set theory* is to develop mathematics out of the notions of logic (the logical connectives and quantifiers) together with the notion of an element being a *member* of a collection of elements. We presume the reader to be familiar with the basic symbols and concepts of first-order logic with identity: $\neg$ (not), $\wedge$ (and), $\vee$ (or, in the sense of at least one), $\supset$ (implies), $\equiv$ (if and only if), $\forall x$ (for every element x), $\exists x$ (there exists at least one x such that), and $=$ (equality or identity). We also use the symbol "$\in$" for "is a member of"—thus "$x \in A$" is read "x is a member of the collection A." For the moment, we will use the words *set, class, collection* synonymously; later, however, we will have to make certain technical distinctions between them.

§7 Frege set theory

One of the principal pioneers of formal set theory was Gottlob Frege. His system had, in addition to the axioms of logic, just one axiom of set theory—namely that given any property P, there exists a (unique) set A consisting of those and only those things that have the property P. Such a set is written "$\{x \mid P(x)\}$" and is read: "the set of all x's having the property P," or "the set of all x such that P of x." This principle of Frege is sometimes referred to as the *abstraction principle*, or the *unlimited* abstraction principle. It has the marvelous advantage of allowing us to obtain just about all the sets necessary for mathematics. For example, we can take some property $P(x)$ which doesn't hold for any x at all—such as $\neg(x = x)$—and form the set of all x having this property. We thus have the *empty* set whose usual name is "$\emptyset$." Thus $\emptyset = \{x \mid \neg(x = x)\}$.

Next, given any two sets a and b, we can form the set of all x such that $x = a$ or $x = b$. This set has a and b as its only elements and is denoted $\{a, b\}$. Thus $\{a, b\} = \{x \mid x = a \vee x = b\}$ and its existence is guaranteed by Frege's abstraction principle. The set $\{a, b\}$ is sometimes called the "unordered pair" of a and b.

We have already defined the *power set* $\mathcal{P}(a)$ of any set a as the set of all subsets of a. Then $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ ($x \subseteq a$ means that x is a subset of a) and its existence is again guaranteed by Frege's abstraction principle.

For any set a, by $\cup a$ (read "the union of all the elements of a," or more briefly "union a") is meant the set of all elements of all elements of a. Thus $\cup a = \{x \mid (\exists y)(y \in a \wedge x \in y)\}$, and so its existence is guaranteed by Frege's principle.

As we shall see, the natural numbers can be defined as certain special sets, and the existence of the set ω of all natural numbers is a consequence of Frege's principle. But now we must pause for an important consideration.

§8 Russell's paradox

Despite the marvelous advantage of Frege's system of freely giving us all the sets we need, it has one serious drawback: It is logically inconsistent!

This startling observation was made by Bertrand Russell (and independently by Zermelo) who derived the following contradiction from Frege's unlimited abstraction principle. Given any set x, either x is a member of itself or it isn't. Call a set *ordinary* if it is not a member of itself and *extraordinary* if it is a member of itself. For example, the set of all chairs is not itself a chair, so it is an ordinary set. On the other hand, the set of all sets (which exists under Frege's principle) is itself a set, hence is one of its own members, hence is extraordinary. Whether extraordinary sets really exist or not, ordinary sets certainly do, and so we let O be the set of all ordinary sets. By Frege's principle, the set O exists since $O = \{x \mid \neg(x \in x)\}$. Now, is the set O ordinary or not? Either way leads to a contradiction. Suppose on the one hand that O is ordinary. Then, since *every* ordinary set belongs to O, then O belongs to O, making O extraordinary. This is a contradiction. On the other hand, suppose O is extraordinary. This means that O is a member of itself, but only ordinary sets are members of O, so we again have a contradiction.

More briefly, the argument is this. By definition of the set O, for every set x we have $x \in O \equiv \neg(x \in x)$. Since this is true for *every* x, then we can take O for x, and we thus have: $O \in O \equiv \neg(O \in O)$, which is a clear contradiction.

Thus, by Russell's famous argument, there cannot be such a thing as the set of all ordinary sets, yet the existence of such a set is implied by Frege's principle, and so Frege's principle leads to an inconsistency.

Remark Bertrand Russell later gave a popular version of his paradox in terms of a male *barber* of a certain town who shaved all and only those male inhabitants who didn't shave themselves. Did the barber shave himself or didn't he? If he didn't shave himself, then he failed to shave some man (namely himself) who didn't shave himself, thus violating the given conditions. On the other hand, if he shaved himself, then he again violated the conditions by shaving someone (namely himself) who shaves himself. The solution to this paradox is that there cannot exist any such barber—just as there cannot be any set that contains all and only those sets that are not members of themselves. More generally (and this takes care of both sets and barbers) given any relation $R(x, y)$ there cannot exist any element x that bears the relation R to all and only those y which do not bear the relation R to y—symbolically $\neg(\exists x)(\forall y)[R(x, y) \equiv \neg R(y, y)]$. (Indeed, this formula is a theorem of first-order logic.)

§9 Zermelo set theory

Frege was extremely upset by Russell's paradox, but more than need be, since his system, despite its inconsistency, has been subsequently modified to provide a most useful system indeed. Many of the main principles of modern day logic and set theory derive ultimately from the work of Frege.

It was Zermelo who replaced Frege's unsound abstraction principle by a less liberal, but presumably consistent, principle known as the *limited* abstraction principle, or *separation* principle—also known as *Aussonderungs*—which is this: Given any property P *and given any set* a there exists the set of all elements *of the set* a that have property P. Thus we cannot speak of the set of *all* x's having property P (as Frege did), but we can speak of the set of all x's *in* a that have property P. Thus we cannot (in general) talk of $\{x \mid P(x)\}$, but we can talk of $\{x \mid x \in a \wedge P(x)\}$. Now, this principle has never been known to lead to any contradiction and is indeed a principle in common use by the everyday mathematician (who speaks, e.g., of the set of all *numbers* having a given property P, or the set of all *points on a plane* having a given property P). Russell's paradox then disappears. We can no longer form the set of *all* ordinary sets, but given a set a in advance, we can form the set b of all ordinary elements *of the set* a. This leads to no paradox, but merely to the conclusion that b, though a subset of a, cannot be a member of a (for if it were, it couldn't be either ordinary or extraordinary without contradiction). It then follows that for any set a there is at least one subset of a that is not a member of a. (This result also follows from Cantor's theorem, since this theorem is to the effect that there are more subsets of a than elements of a.)

As a price for having given up Frege's *unlimited* abstraction principle, Zermelo had to take the existence of the sets $\emptyset$, $\{a, b\}$, $\cup a$, $\mathcal{P}(a)$ as separate axioms. He also took an axiom of *infinity*, which provided for the existence of the set ω of natural numbers. This constitutes *Zermelo* set theory. Later Fraenkel, and independently Skolem, added a powerful axiom known as the *axiom of substitution*, or the *axiom of replacement*, which roughly says that given any set x, one can form a new set by simply replacing each element of x by some element. (A more precise formulation of this axiom will be given later in this book.) The resulting system is known as *Zermelo-Fraenkel* set theory—abbreviated *ZF*.

One might also wish to state axioms providing for the existence of various *properties* of sets. Zermelo did *not* do this, and to that extent his system was not a completely *formal* axiom system in the modern sense of the term. It was Thoralf Skolem who proposed to identify *properties* with *first-order properties*, by which is meant conditions defined by *first-order formulas*—i.e., well formed expressions built from the set-membership symbol "$\in$," the variables $x, y, \ldots$ ranging over all sets, the connectives of propositional logic, and the quantifiers for the set variables $x, y, \ldots$. In Skolem's formulation, it was necessary to express Zermelo's separation principle as an *infinite* number of axioms, one for every first-order formula. (The system *ZF* when formulated along these lines of Skolem is sometimes referred to as *ZFS*.)

Now, Zermelo protested vigorously against this interpretation by Skolem. For Zermelo, properties were to be thought of as *all* meaningful conditions, not just those conditions given by first-order formulas (or higher order formulas, for that matter!). To which Skolem replied that Zermelo's notion of "property" was too vague to be satisfactory.

There are surely many realists who feel that Zermelo's system, despite its non-formal character, comes far closer to the true Cantorian set theory. Of course there is no harm in laying down formal axioms which force at least the first-order properties into the picture, but these are certainly not *all* the properties that there are. A still stronger system of set theory called *Morse-Kelley* set theory forces more properties into the picture—properties defined by quantifying over properties as well as sets, but this still does not provide for *all* the properties of sets that there are. In fact (by incompleteness results of Gödel) there is *no* formal axiomatization of properties that captures the entire picture, and this (we believe) is a key factor in the difficulty of deciding whether the continuum hypothesis is true or false.

§10 Sets and classes

Conceptually, Zermelo-Fraenkel set theory is a simple one, but technically it is in many ways quite awkward and inelegant. A far more attractive system was developed by Von Neumann, later revised by Robinson, Bernays, and Gödel and is now known as *NBG* (sometimes *VNB*). This is the main system that we study in this book. The basic idea here is that certain collections of things are called *classes* and certain collections are called *sets*. The term "class" is the more comprehensive one, since every set is also a class, but not every class is a set. Which classes are sets? Rather than attempt an absolute answer to this (which some authors have done with dubious success), we regard it as philosophically more honest to take these notions as only *relative* to any given *model* of the axioms of class-set theory. That is, a collection V is called a *model* of class-set theory if it satisfies the axioms of *NBG*, which will be given in the next several chapters. The elements of V are called the *sets* of the model and the subcollections of V are called the *classes* of the model. When the model V is fixed for the discussion, then the sets of the model are more briefly called "sets" and the classes of the model are simply called "classes." This is the procedure that we will adopt. And now we turn to a more formal development.

MATHEMATICS

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