5

Birthday Gift

He must have been born on February 29th and was 28 years old.

6

Foot Race

In Race 1 you overtake the person in second place. You are now in second place (not first). In Race 2 you overtake the person in last place, but this is impossible, since no one can be behind the runner in last place.

7

Deuce Power

Each player must take a trick with a deuce. This is because if one player has no deuce he must be out of the led suit for the tricks taking the first three deuces so must have the suit of the fourth deuce when it is led. The defense must take at least 4 tricks. Each defender must take a trick to get the lead and another one to cash a deuce. Thus the contract must be 3 no trump. One way for all four deuces to take tricks is for each player to have 7222 distribution as shown in the diagram.

	♠ A10 ♥ AKQJ542 ♦ 43 ♣ 63	
♠ KQJ9872 ♥ 87 ♦ 65 ♣ 109		♠ 65 ♥ 109 ♦ KQJ9872 ♣ 87
	♠ 43 ♥ 63 ♦ A10 ♣ AKQJ542	

Trick	W	N	E	S
1	♠J	♠A	♠5	♠3
2	♠K	♠10	♠6	♠4
3	♠2	♥4	♦7	♣4
4	♥7	♥A	♥9	♥3
5	♥8	♥K	♥10	♥6
6	♠7	♥2	♦8	♣5
7	♦5	♦3	♦9	♦A
8	♦6	♦4	♦K	♦10
9	♠8	♥5	♦2	♣J
10	♣9	♣3	♣7	♣A
11	♣10	♣6	♣8	♣K
12	♠9	♥J	♦J	♣Q
13	♠Q	♥Q	♦Q	♣2

8

Expansion Problem

Because of the implied (x-x) term the expression calculates to E=0.

115

Four-digit Squares

2116
1225
1296
6561

116

Number Square

5	4	1
1	4	9
2	1	6

117

Curious License Plate

The license number is 741; the next number is 7425741.

36

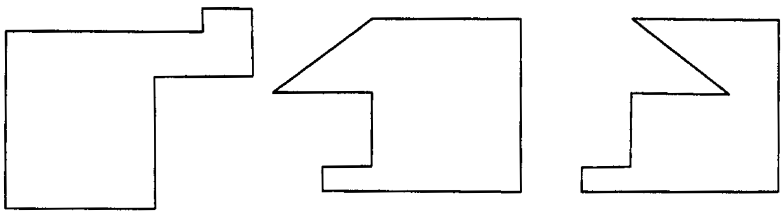
Grid Point Polygon

The first 4 legs (lengths 1, 2, 3, 4) must be connected by right angle bends. Characterize the legs as vectors and define leg 1 as (1,0). Then leg 2 will be (0,2), which can be added to or subtracted from the first leg to get the vertex at the end of leg 2. The next two legs will be (3,0) and (0,4) to be added or subtracted at will. There can be no four-sided solution since the x-components (1 and 3) cannot add to zero.

Because five can be the side of a Pythagorean triangle (3-4-5) leg 5 can take 10 possible directions from the end of leg 4. These come from the vectors (5,0), (3,4) and (4,3) with arbitrary signs on the components. A solution with five sides would require the first four sides to add to one of these ten vectors. A 5-sided solution is impossible because any sum of the first four legs gives even numbers for both the x and y coordinates. A 6-sided solution is impossible for the same reason since the sum of all legs other than leg 5 gives a vector with even coordinates for both components.

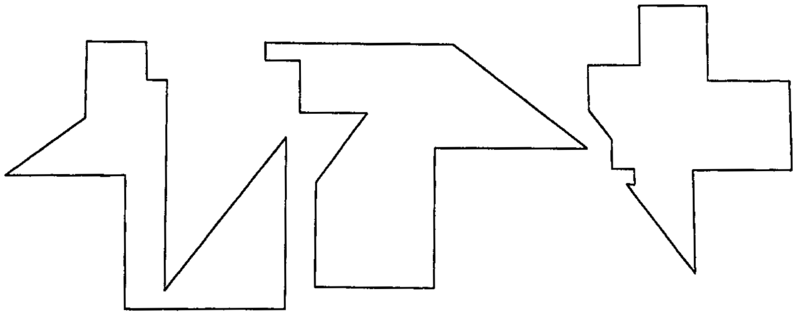
If there is a 7-sided solution leg 7 must be (0,7) or (0,-7) so as to meet leg 1 properly and leg 6 must be (6,0) or (-6,0). The sum of all legs but leg 5 will add or subtract 1, 3 and 6 in the x direction and add or subtract 2, 4 and 7 in the y direction. We cannot get any of the 10 vectors for leg 5 with these numbers so a 7-sided solution is impossible.

With eight legs there are finally enough possibilities to come up with three solutions shown here.



For a 9-sided solution we must have 1, 3, 6 and 8 contributing in the x direction and 2, 4, 7 and 9 contributing in the y direction in an attempt to add to one of the ten vectors possible for leg 5. A 9-sided solution is impossible because any sum of these eight legs gives even numbers for both the x and y coordinates.

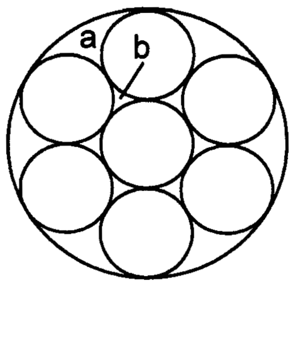
The 11-sided swan and woodpecker shaped polygons shown are two examples of a polygon with the least odd number of sides possible. For interest the 15-sided Texas-shaped polygon is shown as well.



37

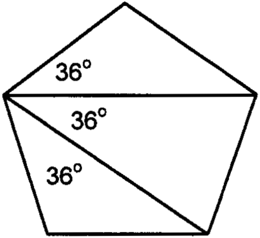
Seven Cookies

The large circle has 9 times the area of each small circle so that 6(a+b) = 2 cookies and a+b = one third of a cookie.



Grid Point Pentagon

Assume that three vertices do fall on grid points. The triangle they form will always include a vertex with an angle of 36° as shown. Place that vertex at the origin and suppose the other two vertices of the triangle are at grid points (a, b) and (c, d), where a, b, c and d are integers. Then $\cos^2 36^\circ = (ac+bd)^2/((a^2+b^2) \times (c^2+d^2))$. But $\cos^2 36^\circ$ is irrational and therefore cannot be the ratio of two integers. This contradiction proves that three vertices of the regular pentagon cannot lie on grid points.



60

Logical Hats 1

Because A doesn't know, he cannot have seen two 11's. Because B doesn't know, and he knows A doesn't know, he cannot have seen any 11's. Thus C has a 7 on his hat.

61

Elves and a Troll

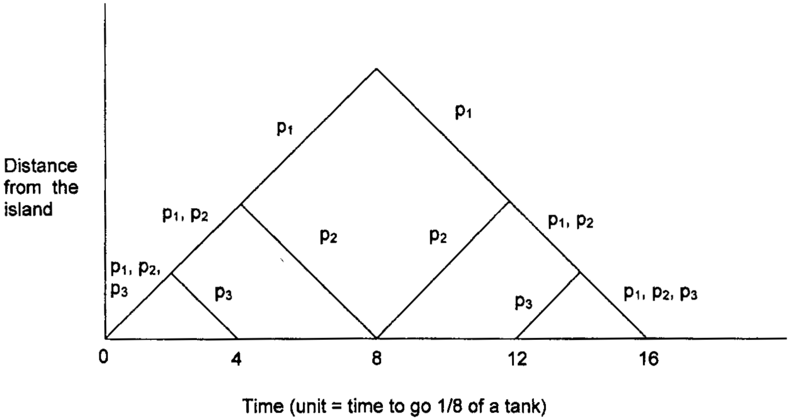
With the proper strategy all but the first elf can go free. What is needed is agreement from all that whoever goes first will be a martyr. All elves agree on a natural ordering of the cards that labels them from 0 to 51. Whoever goes first adds the numbers of the cards he can see and takes the result mod 52 (that is he divides the sum by 52 and takes the remainder). Call this result

R. He guesses the card corresponding to R. Each other elf then knows that the sum of cards they can see, leaving out the card from the first elf, will equal R if their card is added in. Thus each elf other than the first deduces his card and is set free.

Circumnavigation

(a) **Three planes.** The best that can be done is a circumference C = 2. This is analyzed generally by setting up a coordinate system where x = 0 at the home island. Start by sending p_1, p_2 and p_3 out to point x_1. p_3 tops off p_1 and p_2 with fuel at x_1 and returns home. p_1 and p_2 continue to x_2, where p_2 tops off p_1. p_1 continues around the planet. p_2 returns home, refuels and flies to x_3 in the reverse direction to meet p_1.They share fuel and fly to x_4, where p_3 meets them for a safe return home. The fuel constraint at x_1 requires $x_1 \le 1/4$ to allow a safe return of p_3. The fuel constraint at x_2 requires $2-2(x_2-x_1) \ge 1+x_2$. Getting p_1 from x_2 to x_3 requires $1 \ge C-x_2-x_3$. The fuel constraint at x_3 requires $1-x_3 \ge 2(x_3-x_4)$. The fuel constraint at x_4 requires $x_4 \le 1/4$. There is also a timing constraint for p_1 and p_2 traveling from x_2 to x_3. p_2 travels x_2+x_3 while p_1 travels $C-x_2-x_3$, so we must have $C-x_2-x_3 \ge x_2+x_3$. This constraint assures that p_2 can get from x_2 back to the home island, refueled there and out to x_3 in time to meet p_1 before it runs out of fuel. We pick x_1 to x_4 to obey all constraints and maximize C. There are multiple ways to do this but a symmetric one occurs for $x_1 = 1/4$, $x_2 =$

$x_3 = ½$ and $x_4 = ¼$, giving $C = 2$. The figure shows a time vs. distance graph for each plane.



(b) Four planes. The best that I could do is a circumference $C = {}^{106}/_{45}$. Start by sending p_1, p_2, p_3 and p_4 out to point x_1. p_4 tops off p_1, p_2 and p_3 with fuel at x_1 and p_4 returns home to refuel and go back out later to meet p_2. p_1, p_2 and p_3 continue to x_2, where p_3 tops off p_1 and p_2 and returns home to refuel and go back out later to meet p_1. p_1 and p_2 continue to x_3, where p_2 tops off p_1. p_1 continues around the planet while p_2 returns to x_4, where p_4 shares fuel with p_2 to get them both safely home. From x_2, p_3 returns home, refuels and flies to x_5 toward the incoming p_1, share fuel with it and get both planes to x_6. Meanwhile p_2 and p_4 return home, refuel and fly toward p_1 to x_7. At x_7 p_4 tops off p_2 and returns home safely. p_2 continues on to meet p_1 and p_3 at x_6. p_1, p_2 and p_3 continue toward home to x_8 where they are met by p_4 to share fuel and return all planes safely home. There are fuel constraints at each of the transfer points.

At x_1: $x_1 \leq {}^1/_5$

At x_2: $3-3(x_2-x_1) \geq 2+x_2$

At x_3: $2-2(x_3-x_2) \geq 1+(x_3-x_4)$
At x_4: $1-x_4 \geq 2x_4$
At x_5: $1-x_5 \geq 2(x_5-x_6)$
At x_6: $1-(x_6-x_7) \geq 3(x_6-x_8)$
At x_7: $2-2x_7 \geq 1+x_7$
At x_8: $x_8 \leq 1/5$

Getting p_1 from x_3 to x_5 requires $1 \geq C-x_3-x_5$. There are also timing constraints.

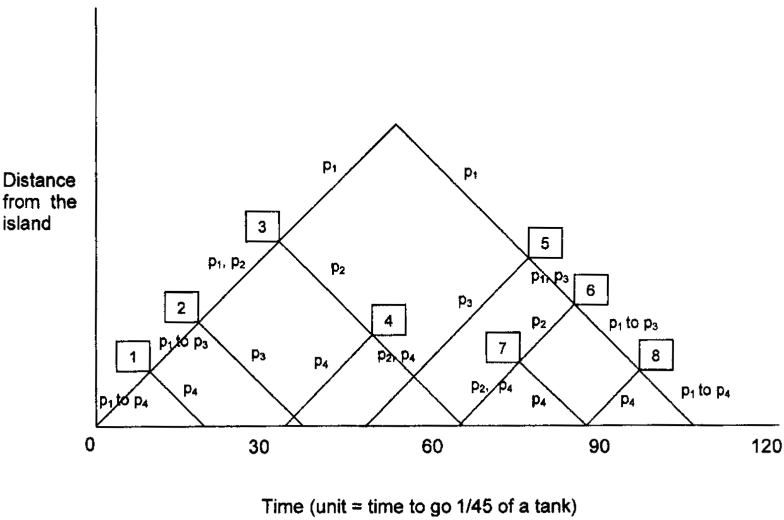
p_2 and p_4 travel from x_1 to x_4 so we must have $(x_3-x_1) + (x_3-x_4) \geq x_1+x_4$.

p_1 and p_3 travel from x_2 to x_5 so that $C-x_2-x_5 \geq x_2+x_5$.

p_1 and p_2 travel from x_3 to x_6 so that $C-x_3-x_6 \geq x_3+x_6$.

p_2 and p_4 travel from x_7 to x_8 so that $(x_6-x_7) + (x_6-x_8) \geq x_7+x_8$.

We pick x_1 to x_8 to obey all constraints and maximize C. There are multiple ways to do this and one occurs for $x_1 = 1/5$, $x_2 = 2/5$, $x_3 = 32/45$, $x_4 = 1/3$, $x_5 = 29/45$, $x_6 = 7/15$, $x_7 = 4/15$, $x_8 = 1/5$ and $C = 106/45 = 2.355555\ldots$ The figure shows a time vs. distance graph for each plane.



90

Two Triangles

A quick numerical check gives (x, y, u, v) = (15,8,15,13).
A computer search showed the next primitive solution to be (x, y, u, v) = (8109409, 10130640, 12976609, 9286489).

91

Three Integer Triangles

a, b, and c are integers only if $\cos\theta = p/q$, where p and q are relatively prime integers. In (a) it follows that $15\theta < 180^\circ$ or $\theta < 12^\circ$. Thus $\cos\theta > .98714...$and we get $q > 45$. Though this problem can be solved with algebra it is simpler to do it numerically. Pick $\cos\theta = 45/46$ and solve the four triangles independently. a = 529k, b = 1035k, c = 1496k;
d = 418642136m, e = 529447005m, f = 709644761m;
g = 172726942962199n, h = 189783039512880n,
i = 149252376812351n; c = 910507711420985554417765r,
f = 911289037060712700994862r,
i = 7494149547368268741303r.

k, m, n and r must be chosen so that both expressions for c, f and i agree. This gives i=1187452389139022621502501244 348687897422398794968 from which the remaining lengths can be determined. Different values of $q > 45$ must be tried to see if common factors can be cancelled out producing even smaller i. In general even values of q allow a common factor of 64 to be removed; other factors must be searched for numerically. A search up to q=400 found no improvement to the above result.

In (b) it follows that $15\theta < 90^\circ$ or $\theta < 6^\circ$. Thus $\cos\theta > .99452...$ and we get $q > 182$. Though q=184 looks like a natural approach it turns out that q = 192 produces a common removable factor of 71 and gives the best result known.
$\cos\theta = 191/192$ gives a = 9216k, b = 18336k, c = 27265k;
d = 2315743395840m, e = 3050002612224m,
f = 5033291921281m; g = 326365726223997773576521n,
h = 384063664654108411590258n,
i = 602100418462702407496537n;
c = 5948971974648842680073134907535O9r,

f = 66167250529090574733887764689 7817r,
i = 90695404291939029856635558826 6574r. Thus
i = 150784063433011623421741956789701086887309120
593342113502985192202682110
from which the remaining lengths can be computed.