II
Difficult Problems

87. BLACKSMITH KHECHO'S INGENUITY

Last summer, as we traveled through the Georgian Republic, we would make up all sorts of unusual stories. Seeing a relic of old times often inspired us.

One day we came across an old and isolated tower. One of us, a student mathematician, invented an amusing puzzle story:

"Some three hundred years ago a prince lived here, a man of ill heart and much pride. His daughter, who was ripe for marriage, was named Daridjan. He had promised her to a rich neighbor, but she had a different plan: she fell in love with a plain lad, the blacksmith Khecho. The lovers tried to run off to the mountains, but were caught.

"Angered, the prince decided to execute them both the next day. He had them locked up in this tower—a somber structure, unfinished and abandoned. A young girl, a servant who had helped the lovers in their unsuccessful flight, was locked up with them.

"Khecho, calmly looking around, climbed the steps to the tower's upper part and glanced out the window. He realized it would be impossible to jump out and survive. But he saw a rope, forgotten by the masons, hanging near the window. The rope was thrown over a rusty tackle fastened to the tower wall above the window. Empty baskets were tied to each end of the rope. These baskets had been used by the masons to lift bricks and lower rubble. Khecho knew that if one load was 10 pounds more than the other, the heavier basket would descend smoothly to the ground while the other rose to the window.

"Looking at the two girls, Khecho guessed Daridjan's weight at 100 pounds and the servant's at 80. He himself weighed nearly 180. In the tower he found 13 separated pieces of chain, each weighing 10 pounds. Now all three prisoners succeeded in reaching the ground. At no time did a descending basket weigh over 10 pounds more than the ascending basket.

"How did they escape?"

88. CAT AND MICE

Purrer has decided to take a nap. He dreams he is encircled by 13 mice: 12 gray and 1 white. He hears his owner saying: "Purrer, you are to eat each thirteenth mouse,

keeping the same direction. The last mouse you eat must be the white one."
Which mouse should he start from?

89. SISKIN AND THRUSH

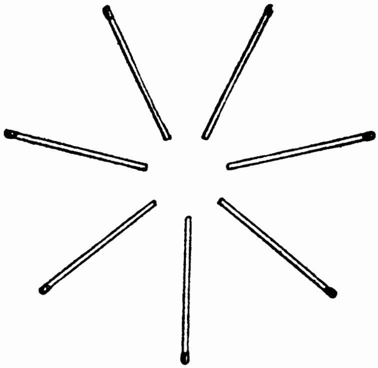
At the end of summer camp, the children decided to free the 20 birds they had caught. The Counselor suggested:

"Line up the cages in a row. Counting from left to right, open each fifth cage with a bird in it. When you reach the end of the row, start over. You can take the last 2 birds to the city."

Most of the children did not care which birds would be taken to the city, but Tanya and Alik set their hearts on a siskin and a thrush. As they helped line up the cages, they remembered about the cat and the mice (Problem 88). Which cages did they put the 2 birds in?

90. MATCHES AND COINS

Get 7 matches and 6 coins. Place on a table to form a star, as shown. Count clockwise starting with any match and place a coin at the head of the third match.



Continue clockwise, beginning with any match that has no coin at its head and placing a coin at the head of the third match. Do not skip matches that already have coins at their heads.

Can you place the 6 coins without placing 2 at the head of any match?

91. LET THE PASSENGER TRAIN THROUGH!

A work train, made up of a locomotive and 5 cars, stops at a small station. The station has a small siding that can hold an engine and 2 cars.

A passenger train is due. How do they let it through?

92. THE WHIM OF THREE GIRLS

The theme of this problem goes back many centuries. Three girls, each with her father, go for a stroll. They come to a small river. One boat, able to carry two persons at a time, is at their disposal. Crossing would be simple, except for the girls' whim: none is willing to be in the boat or ashore with one or two strange fathers unless her own father is present too. The girls, of course, can row.

How do they all get across?

93. AN EXPANSION OF PROBLEM 92

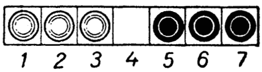
(A) After the six get across, they wonder if, given the conditions, four pairs can cross. Well, they can—if the boat will hold three.

(B) What's more, a boat holding only two can take four girls and their fathers from one shore to the other—if there is an island in the middle which can be used for intermediate loading and unloading.

Show how, for both.

94. JUMPING CHECKERS

Place 3 white checkers in squares 1, 2, and 3 of the figure, and 3 black ones on squares 5, 6, and 7. Shift the white checkers to the squares occupied by the black



ones, and vice versa. You may move a checker forward to the adjacent unoccupied square, if any. You may jump a checker forward over an adjacent checker into the vacant square. The solution requires 15 moves.

95. WHITE AND BLACK

Take 4 black and 4 white checkers (or 4 pennies and 4 other coins) and put them on a table in a row, white, black, white, black, and so on. Leave a vacant place at one end which can hold 2 checkers. After 4 moves, all the black checkers should be on one side and the white ones on the other.

A move consists of shifting 2 adjacent checkers, keeping their order, into any vacant space.

96. COMPLICATING THE PROBLEM

In the last problem, 8 checkers took 4 moves. Show how 10 checkers take 5 moves, 12 take 6, and 14 take 7.

97. THE GENERAL PROBLEM

From Problems 95–96, can you derive a general procedure for arranging $2n$ checkers in n moves?

98. SMALL CARDS PLACED IN ORDER

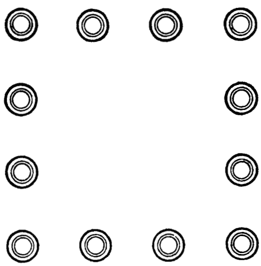
Take the ace through 10 from a deck of cards. Deal the ace face down on the table, put the 2 at the bottom of the pile you are holding, deal the 3, put the 4 on the bottom, and so on until all the cards are dealt.

Naturally the cards on the table are not in numerical order.

What order should you start with, top to bottom, to end with cards on the table from ace to 10 with the 10 on top?

99. TWO ARRANGEMENT PUZZLES

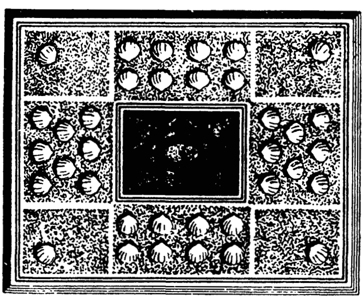
(A) Twelve checkers (coins, pieces of paper, etc.) are in a square frame with 4 checkers on each side. Try placing them so there are 5 on each side.



(B) Arrange 12 checkers to form three horizontal and three vertical rows, with 4 checkers in each row.

100. A MYSTERIOUS BOX

Misha brought a pretty little box for his sister Irochka from his Crimean summer camp. She was not of school age yet, but could count to 10. She liked the box because she could count 10 sea shells along each side, as shown.



One day Irochka's mother, while cleaning the box, accidentally broke 4 shells.

"No great trouble," Misha said.

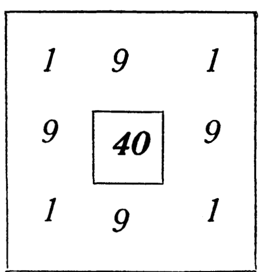
He unstuck some of the remaining 32 shells, then pasted them on so that there were again 10 shells along each side of the cover.

A few days later, when the box fell on the floor and 6 more shells were crushed, Misha again redistributed the shells—though not quite so symmetrically—so Irochka could count 10 on each side, as before.

Find both arrangements.

101. THE COURAGEOUS GARRISON

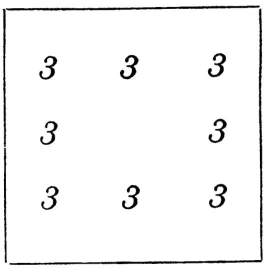
A courageous garrison was defending a snow fort. The commander arranged his forces as shown in the square frame (the inner square showing the garrison's total strength of 40 boys): 11 boys defending each side of the fort.



The garrison "lost" 4 boys during each of the first, second, third, and fourth assaults and 2 during the fifth and last. But after each charge 11 boys defended each side of the snow fort. How?

102. DAYLIGHT LAMPS

A technician was lighting a room for a TV broadcast with tubular neon lamps. At first he put 3 lamps in each corner and 3 lamps along each of the room's four sides, a total of 24 lamps, as shown. He added 4 lamps and again 4 lamps. Then he tried



20 lamps, and 18. Always each wall had 9 lamps. How? Could he do it with other numbers of lamps?

103. ARRANGEMENT OF EXPERIMENTAL RABBITS

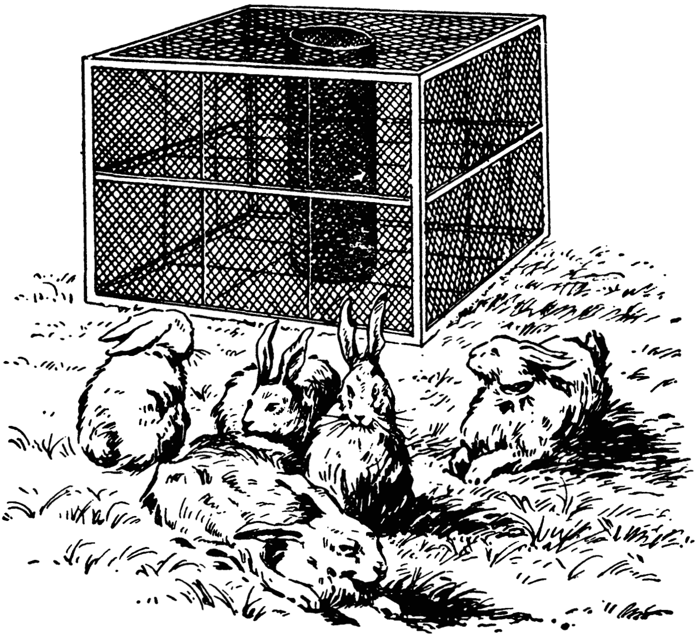
A special two-level cage has been prepared for experiments with rabbits at a research institute, each level having 9 sections. The rabbits are to occupy 16 sections, 8 on the upper level and 8 on the lower. (The 2 central sections are set aside for equipment.)

There are four conditions for the experiments:

1. All 16 sections must be occupied.
2. No section can hold more than 3 rabbits.
3. Each of the four outer sides (total of both levels) must hold 11 rabbits.
4. The whole upper level must hold twice as many rabbits as the whole lower level.

Although the institute received 3 fewer rabbits than expected, it housed the rabbits in conformity with the four conditions.

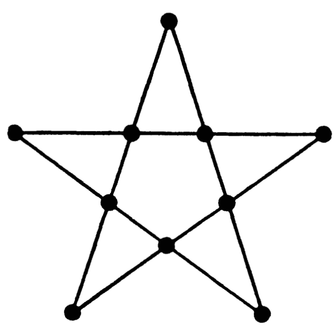
How many rabbits were expected, and how many arrived? How were they housed?



104. PREPARING FOR A FESTIVAL

The preceding five problems involved arranging objects along the sides of a rectangle or a square so that their number along each side remained the same when their total number changed. Besides regarding an object in a corner as belonging to two sides, we can regard the intersection of two lines, in general, as belonging to both lines.

For example, in preparing a festive illumination, can you arrange 10 light bulbs in 5 rows with 4 bulbs in each row? The answer is the 5-pointed star shown.



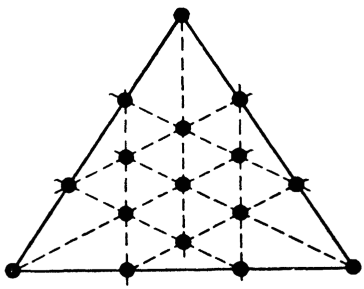
Here are some similar problems. Try to make your solutions symmetrical.

(A) Place 12 light bulbs in 6 rows with 4 bulbs in each row. (There is more than one solution.)

(B) Plant 13 bushes in 12 rows with 3 bushes per row.

(C) On the triangular terrace shown, a gardener raises 16 roses in 12 straight-line rows with 4 roses in each row. Then he prepares a flower bed and transplants to it the 16 roses in 15 rows with 4 roses in each.

How?



(D) Now arrange 25 trees in 12 rows with 5 trees in each row.

105. PLANTING OAKS

A pretty sight, these 27 oaks in a six-pointed star—9 rows with 6 oaks in each—but a true forester might object to the three isolated trees. An oak loves sunshine from above, but on its sides it prefers greenery. As the saying goes, it likes to wear a coat but no hat.

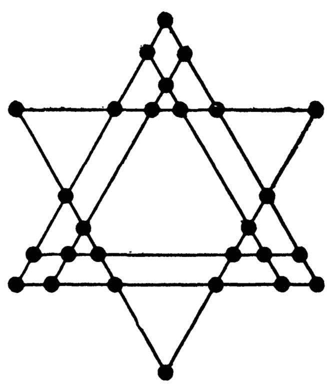


Diagram the 27 oaks in 9 rows with 6 oaks per row, preserving symmetry, but with all the oaks in three clustered groups.

II. Difficult Problems

87. BLACKSMITH KHECHO'S INGENUITY

The prisoners placed 1 piece of chain (10 pounds) in a basket and sent it down. Into the empty basket that came up they put 2 pieces of chain (20 pounds). They kept adding 2 pieces to each basket that came up until they sent a 70-pound load down, getting back a 60-pound load.

Khecho replaced the 6 pieces of chain (60 pounds) with the servant (80 pounds). The girl descended as 7 pieces of chain came up. He unloaded 6 pieces and signaled the girl below to climb out. He lowered the remaining piece of chain, bringing the empty basket up.

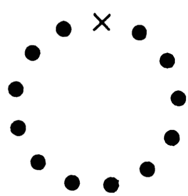
The servant got in the basket again (total weight 80 + 10 = 90 pounds) and Daridjan (100 pounds) descended. They both got out, Daridjan on the ground, the servant in the tower. Down went the basket still with 1 piece of chain in it, and up came the other basket, now empty.

Khecho repeated his first set of actions and soon lowered the servant to the ground again. He signaled Daridjan and the servant (100 + 80 = 180 pounds) to get in, allowing Khecho (180 pounds) to descend with 1 piece of chain. Now the two women were in the tower and Khecho on the ground.

The servant was brought down as before, then Daridjan replaced her on the ground. In due time, the servant made her fourth and last trip down, bringing up 7 pieces of chain. As she stepped out, Khecho fastened the basket to keep the chain in the top basket from falling.

88. CAT AND MICE

Start from the cross in the diagram (position 13) and go clockwise through positions 1, 2, 3, . . . , crossing out each thirteenth dot: 13, 1, 3, 6, 10, 5, 2, 4, 9, 11, 12, 7, and 8. Call position 8 the white mouse, and Purrer starts clockwise from the fifth mouse clockwise from the white mouse (i.e., position 13 relative to position 8). Or he starts counterclockwise from the fifth mouse counterclockwise from the white mouse.

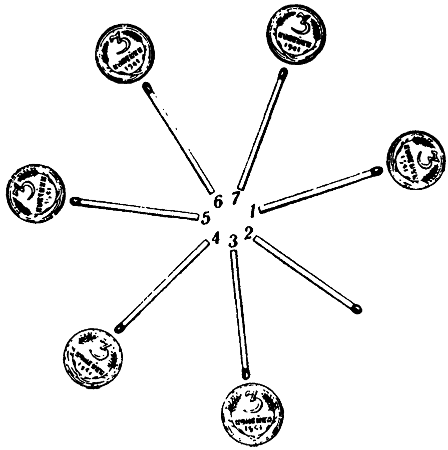


89. SISKIN AND THRUSH

From left to right, the seventh and fourteenth cages.

90. MATCHES AND COINS

A good method is to aim at the match you just started from. Say you start with the fifth match and put a coin by the seventh. Now start from the third so you can put a coin by the fifth, start from the first so you can put a coin by the third, and so on (see diagram).



91. LET THE PASSENGER TRAIN THROUGH!

The work train backs into the siding, which can hold its rear 3 cars.

Uncoupling them in the siding, the rest of the work train goes forward a sufficient distance.

The passenger train comes up and couples on the 3 cars left by the work train. It backs up on the main track.

The work train backs up into the siding, which will now hold its engine and the remaining 2 cars.

The passenger train uncouples the 3 cars it took from the siding and goes through.

92. THE WHIM OF THREE GIRLS

Call the three fathers *A, B, C,* and their daughters correspondingly *a, b, c.*

First shore	Second shore
A B C	. . .
a b c	. . .

1. First two girls go:

First shore	Second shore
A B C	. . .
a . .	. *b c*

2. A girl returns and rows the third girl across:

First shore	Second shore
A B C	. . .
. . .	*a b c*

3. One of the three girls returns and remains ashore with her father. The other two fathers row across:

A . .	. *B C*
a . .	. *b c*

4. A father returns to the first shore with his daughter; the girl remains there, and two fathers go:

. . .	*A B C*
a b .	. . *c*

5. The last girl returns to the first shore and crosses with the second girl:

. . .	*A B C*
a . .	. *b c*

6. The girl on the first shore is fetched by her father (or one of the other two girls):

. . .	*A B C*
. . .	*a b c*

93. AN EXPANSION OF PROBLEM 92

(A) Crossing in a boat that holds three: Call the fathers *A*, *B*, *C*, *D*, and their daughters correspondingly *a*, *b*, *c*, *d*.

	First shore	In the boat	Second shore
	A B C D		. . . .
	a b c d		. . . .
1. Three girls go:			
	A B C D		. . . .
	a . . .	*b c d* →	. *b c d*
Two return:			
	A B C D		. . . .
	a b c .	← *b c*	. . . *d*
2. A father goes with his daughter and the father whose daughter is on the other shore:			
	A B . .	*C D* } →	. . *C D*
	a b . .	*c*	. . *c d*
A father and his daughter return:			
	A B C .	← { *C*	. . . *D*
	a b c .	*c*	. . . *D*
3. Three fathers go:			
	. . . .	*A B C* →	*A B C D*
	a b c .		. . . *d*
A girl returns:			
	. . . .		*A B C D*
	a b c d	← *d*	. . . .
4. The girl who has just returned takes two with her:			
	. . . .		*A B C D*
	a . . .	*b c d* →	. *b c d*

Father A returns for his daughter (or one of the other 3 girls does):

A . . .	$\leftarrow A$	. B C D
a . . .		. b c d

5. The last pair go:

. . . .	$\left.\begin{matrix} A \\ a \end{matrix}\right\} \rightarrow$	A B C D
. . . .		a b c d

(B) Crossing in a boat that holds two persons (solution by Y. V. Morozova):

	First shore	Island	Second shore
	A B C D	. . . .	. . . .
	a b c d	. . . .	. . . .
1.	A B C D	. . . .	. . . .
	a b . .	. . . .	. . c d
2.	A B C D	. . . .	. . . .
	a . . .	. . . .	. b c d
3.	A B . .	. . . .	. . C D
	a b . .	. . . .	. . c d
4.	A B C .	. . . .	. . . D
	a b . .	. . c .	. . . d

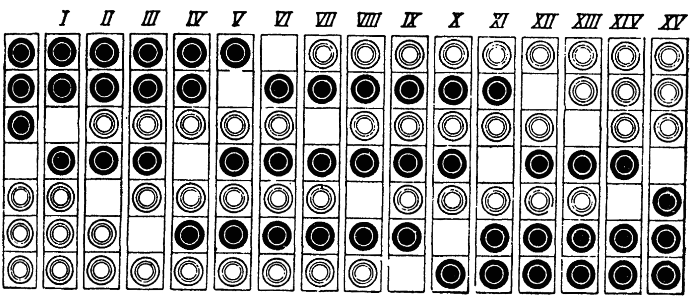
(C took c to the island, returned to the first shore, and gave the boat to two girls.)

5.	A B C .	. . . .	. . . D
	. . . .	a b c .	. . . d
6.	A . . .	. . . .	. B C D
	a . . .	. b c .	. . . d
7.	A . . .	. . . .	. B C D
	a . . .	. b . .	. . c d
8.	. . . .	. . . .	A B C D
	a . . .	. b . .	. . c d

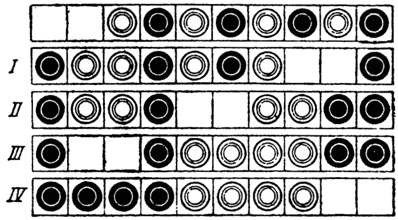
(B went to fetch A and took him directly to the second shore.)

9.	. . . .	. . . .	A B C D
	a . . .	. . . .	. b c d
10.	. . . .	. . . .	A B C D
	. . . .	. . . .	a b c d

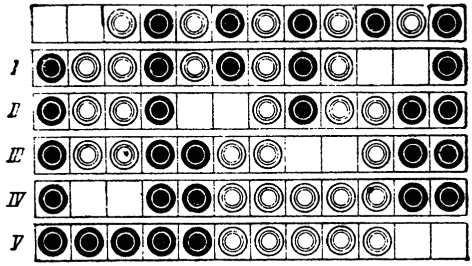
94. JUMPING CHECKERS

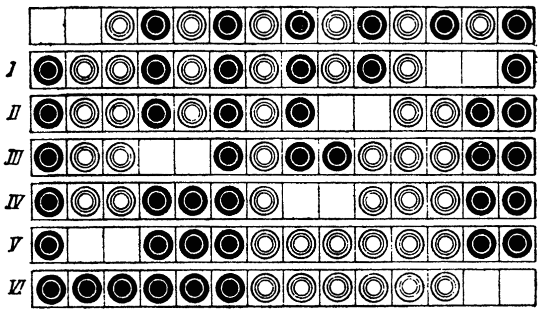


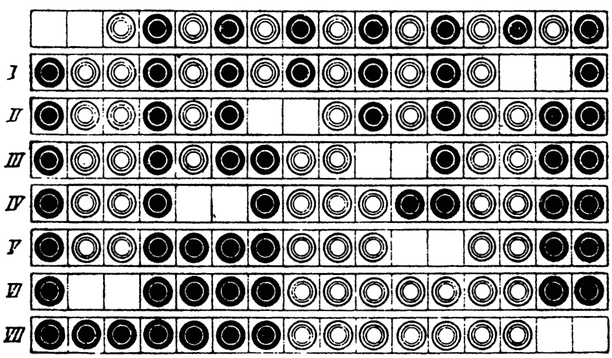
95. WHITE AND BLACK



96. COMPLICATING THE PROBLEM

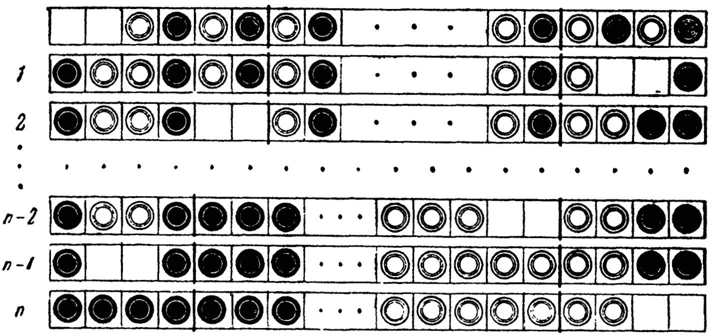






97. THE GENERAL PROBLEM

In the diagram, vertical strokes help us concentrate on the 2 pairs at left and 2 pairs at right.



In the first 2 moves leave the inner $(n - 4)$ pairs alone and arrive at the position shown for the outer 4 pairs, leaving the vacancy to the right of the 2 left pairs.

In the next $(n - 4)$ moves you will be able to put the inner pairs in order, black left, white right. At move $(n - 2)$ leave the vacancy to the left of the two right pairs.

The last 2 moves put the outer pairs in order, completing the solution.

98. SMALL CARDS PLACED IN ORDER

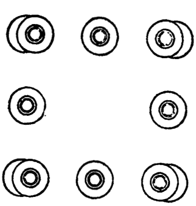
The first deal described produces this sequence on the table, with the 4 on top:

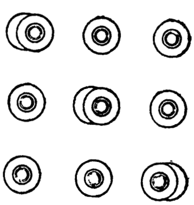
1, 3, 5, 7, 9, 2, 6, 10, 8, 4.

Since the 4 is tenth, put the 10 fourth from the top as you form a new pile of cards. The 8 is in ninth, so put the 9 eighth, and so on. The new pile, from top down, will now be in the desired order:

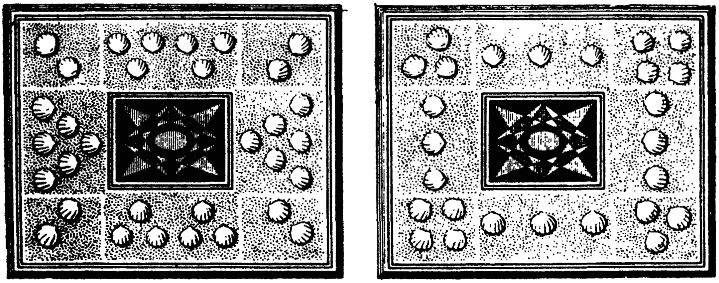
1, 6, 2, 10, 3, 7, 4, 9, 5, 8.

99. TWO ARRANGEMENT PUZZLES





100. A MYSTERIOUS BOX



101. THE COURAGEOUS GARRISON

2	7	2
7	36	7
2	7	2

3	5	3
5	32	5
3	5	3

4	3	4
3	28	3
4	3	4

5	1	5
1	24	1
5	1	5

6	—	5
—	22	—
5	—	6

102. DAYLIGHT LAMPS

He can use any number of lamps from 18 through 36, in some cases with some loss of symmetry. The maximum case is shown in the last diagram.

5		4
	18	
4		5

4	1	4
1	20	1
4	1	4

2	5	2
5	28	5
2	5	2

1	7	1
7	32	7
1	7	1

	9	
9	36	9
	9	

103. ARRANGEMENT OF EXPERIMENTAL RABBITS

It follows from condition 3 that from 22 through 44 rabbits can be housed (see solution of Problem 102).

But the number of rabbits must be a multiple of 3 (condition 4). Thus the number can be 24, 27, 30, 33, 36, 39, or 42. Further, trial shows 24 rabbits cannot be housed 11 to a side (condition 3) without leaving empty sections (condition 1), and that 33, 36, 39, or 42 rabbits cannot be housed 11 to a side without placing more than 3 rabbits in some sections (condition 2).

By elimination, 30 rabbits were expected, and 27 arrived. The diagrams show how they were housed. (In both pairs of diagrams, the second floor is on the left, and the first floor on the right.)

2	3	3
3	20	2
3	2	2

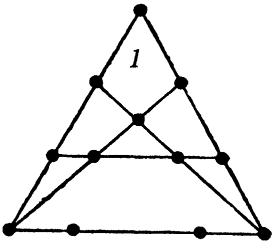
1	1	1
1	10	2
1	2	1

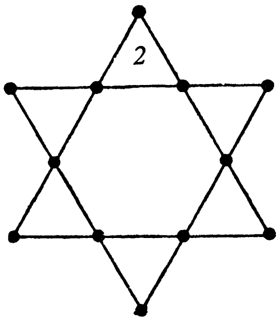
3	1	3
1	18	2
3	2	3

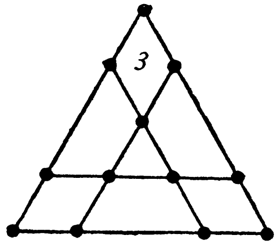
2	1	1
1	9	1
1	1	1

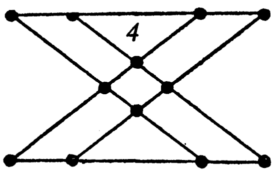
104. PREPARING FOR A FESTIVAL

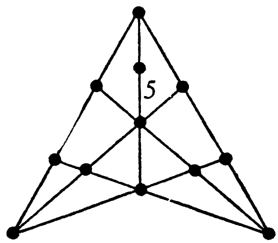
(A) Four solutions are shown below. The third and fourth were found by fourth-grader Batyr Erdniyev of Stavropol. He also arranged 12 lamps in 7 rows as shown in the fifth diagram, the one that looks like a dunce cap.



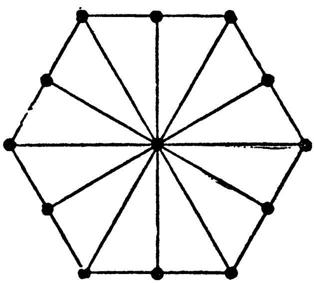




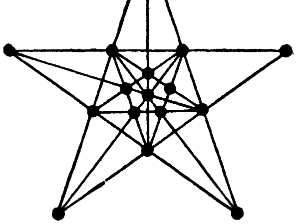


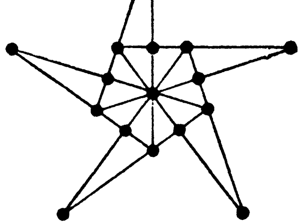


(B)

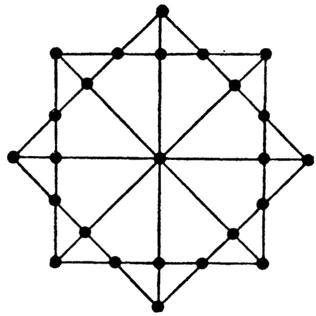


(C) The problem's basic condition is satisfied by the five-pointed star (left) but it is better not to have intersections where there are no objects. The gardener, therefore, chose the irregular star at right (found by V. I. Lebedev, a Moscow engineer).

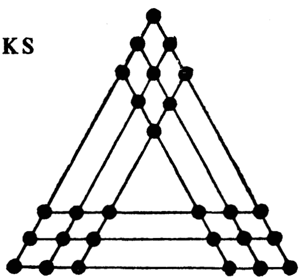




(D) A square superimposed on a square. A simpler solution: form a 5-by-5 square array!



105. PLANTING OAKS



106. GEOMETRICAL GAMES

(A) All the possible solutions can be quickly and easily obtained by simple geometrical constructions. Represent the checkers by dots on a piece of paper. Strike out any 3 dots and 1 bottom dot. Connect one of the remaining 2 top dots to any 2 remaining dots, then connect the second top dot with the other 2 remaining bottom dots (diagram below). Discard combinations that result in parallel lines. Place 4 checkers corresponding to the crossed-out dots at the 4 intersection points of the lines drawn.