I

What Is Topology?

Topology is a fairly new branch of mathematics, and it may seem odd to talk of *experiments* in mathematics unless one is, so to speak, at the front line—so advanced that one can hope to make a new contribution—while we are assuming that the reader knows nothing of the subject. But perhaps because it is so new, additions can be made at the side, like branches, if not at the top. Also certain experiments can be made that, while adding nothing, still help one to understand this rather elusive subject.

Topology is curiously hard to define, whereas the following are much less so. *Arithmetic:* "The science of positive real numbers" (*Webster's New Collegiate Dictionary*), or: "The art of dealing with numerical quantities in their numerical relations" (*Encyclopaedia Britannica,* 11th ed.). *Al-*

gebra: "The generalization and extension of arithmetic" (*Enc. Brit.,* 11th). Mark Barr defined mathematics as being "devised to keep facts in abeyance while we dispassionately examine their relations," but this definition applies especially to algebra. *Geometry:* "The study of the [mathematical] properties of space" (*Enc. Brit.,* 11th). Topology started as a kind of geometry, but it has reached into many other mathematical fields. One might almost say it is a state of mind—and is its own goal. (Later we shall see that this last phrase has a topological sound to it.)

In one sense it is the study of continuity: beginning with the continuity of space, or shapes, it generalizes, and then by analogy leads into other kinds of continuity—and space as we usually understand it is left far behind. Really high-bouncing topologists not only avoid anything like pictures of these things, they mistrust them. This is partly because it is not only impossible to make a visually recognizable picture of some of their "spaces," but meaningless. We can, however, get to an understanding of their goal by easy stages, and by looking at certain shapes (or "spaces") from *the topologists' point of view,* if we start with ones that we can see and feel.

A topologist is interested in those properties of a thing that, while they are in a sense geometrical, are the most permanent—the ones that will survive distortion and stretching.

The roundness of a circle obviously will not: one can tie or glue the ends of a bit of string together and make it into a circle, and, without cutting or disconnecting it, make it into a square. But the fact that *it has no ends* remains unchanged, and if we had strung numbered beads on it they would retain their *order* even if we tied it in knots, provided we count along the string, like a crawling bug (Fig. 1). This would also be true if we used elastic instead of string, because we could only alter the distance between the beads—not their order.

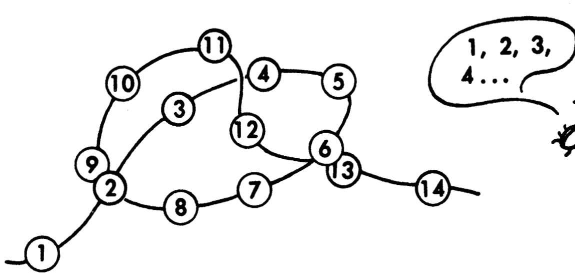


Fig. 1

In projective geometry we get somewhat the same state of affairs: a straight line casts a straight shadow, and a triangle will give a triangular shadow at any angle, even when its own angles change. In topology, though, the straight line doesn't have to remain straight: but it retains the quality of being continuously connected along itself, and with its ends disconnected—or not, as the case may be. (The latter could be so if the line were drawn on a globe, and regarded as straight by the crawling bug, who would report that it did not deviate to either side: like the equator.) It is this connectedness, this continuity, that topology holds on to, and for this reason distortions are only allowed if one does not *disconnect* what was connected (like making a cut or a hole), nor *connect* what was not (like joining the ends of the previously unjoined string, or filling in the hole).

According to this rule, we can take a lump—say round—ot clay and make a cup, but we cannot give it a handle because of the hole in the handle. However we could make both cup and handle from a doughnut-shaped piece (Fig. 2).

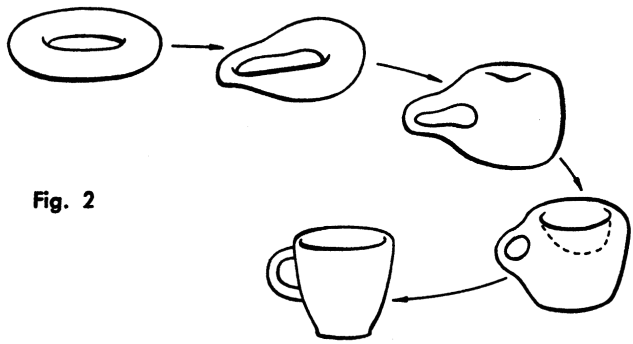


Fig. 2

To be more explicit: we are allowed to make a break, provided we rejoin it afterward. For example, some topologists have said that one can change or distort the first arrangement of a loop of string in Fig. 3 into the second, without alter-

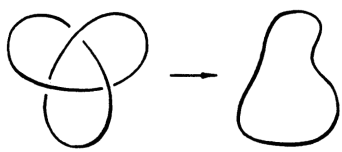


Fig. 3

ing its connectedness. It is true that both are connected the same way, but we obviously cannot do it with string without cutting it and rejoining: but that is allowed. Some say it is possible to do in a 4-dimensional space, but perhaps this modification of the no-cutting-or-joining rule is clearer at this time: Any distortion is allowed provided the end result is connected in the same way as the original.

Another example of this is that one cannot make a flat plate without a hole in it from the doughnut-shaped piece. The latter, incidentally, is called a *torus*. These characteristics—like having or not having a hole—are called *topological invariants*. Sometimes one finds one that turns out to be merely the result of another, but we need not insist on this fact right now.

The lump of clay without a hole is called *simply* *connected,* and as a result of being so, we find that, if we draw a circle—or any closed curve—on it (Fig. 4), it divides the whole surface into two: the part inside and the part outside, just as it

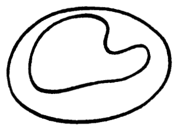


Fig. 4

would on paper. The equator does this for the globe, except that it would be hard to say which was the "inside" and which the "outside," but at least it does divide the surface in two.

Now, if we draw another circle, it will either not cut or intersect the first one at all, or it will do so in two places. This means "cut" in the sense of going right through and not merely touching, like the two circles in Fig. 5. This is because if we

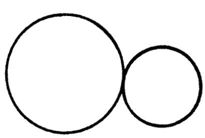


Fig. 5

start drawing the second circle at a point *outside* the first, and then cross over into the *inside,* we cannot get back to the outside to finish the new circle—to join the new line to the point we started at—unless we cross over again. The same applies when we start inside.

Now take the case of the torus (doughnut, Fig. 6). First draw the line L. We can see that it has *not* divided the whole surface into two, and so, if we start a second circle at any point, say P, this point is neither inside nor outside the circle L. Therefore if we cross L, the dotted line we are making is not necessarily barred by the line L from returning to P. As the drawing shows, we *can* have two circles that intersect at *one point only*.

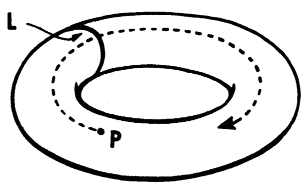


Fig. 6

This fact—not true of a simply connected surface with no holes—is true of anything with a hole, and is a topological invariant.

As was pointed out before, a torus can be distorted into anything with one hole; and a circle into any closed curve that does not join itself anywhere, except for being joined into an endless line. The latter kind are called Jordan curves, after the mathematician who proved that they divide the surface into two distinct regions, which have no points in common but which have the curve as a common boundary—provided the curve is drawn on a simply connected surface: e.g., a plane or a sphere. This may seem to be obvious, but it is unexpectedly difficult to prove. A Jordan curve that divides the surface in two *can* be drawn on a torus, so long as it does not circle the hole, or go through it, as the ones in Fig. 6 do. But on a plane or a sphere *all* Jordan curves divide the surface in two: while on a torus they do not do so necessarily. When one shape or curve can be distorted into another, following our rule, they are said to be *homeomorphic* to one another.

If we draw a triangle on a lump of clay, it is conceivably possible to distort it *homeomorphically* so as to get rid of the three angles and make it into a circle, but if we mark or otherwise identify the apexes as points on the line, they will remain *on it,* and *in the same order* (counting clockwise). Also, if we draw Fig. 7, which is one closed curve joined at two distinct points by another, no distortion that follows our rule can change that description of the

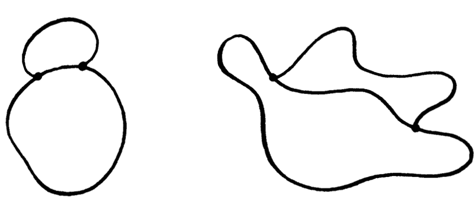


Fig. 7

figure. Not only will the two joints remain as joints, but no new ones will appear, as that would mean making a *new connection*. Thus a sphere with its equator, and another line connecting with the equator at two points, p and p' (Fig. 8), cannot be distorted so that the arrangement of these lines is altered *topologically* (Figs. 9–10).

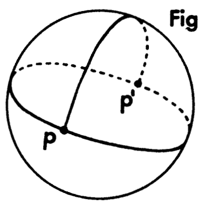


Fig. 8

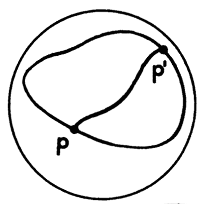


Fig. 9

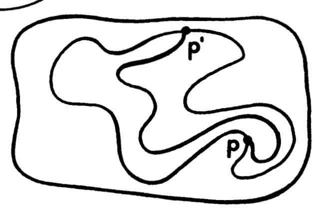


Fig. 10

Fig. 9 shows the whole arrangement of lines pulled around onto this side, and bent into arbitrary shapes. (One may distort a drawing on a surface, if we follow the rule.) We see that it still divides the surface into three areas; it still consists of three segments of line, which still meet at two distinct points. These basic facts have survived distortion, while nothing else has. These facts are the kind that topologists are concerned with.

Euler's Theorem

A prime example of topological invariants comes from a theorem the Swiss mathematician Leonhard Euler stated in 1752. It has to do with *polyhedra:* solid geometrical figures, like the cube, or the tetrahedron (Fig. 11), i.e., solids which are bounded by flat planes (*faces*) which have straight *edges* and the edges meet at points, or corners, called *vertices.* You can have more complicated

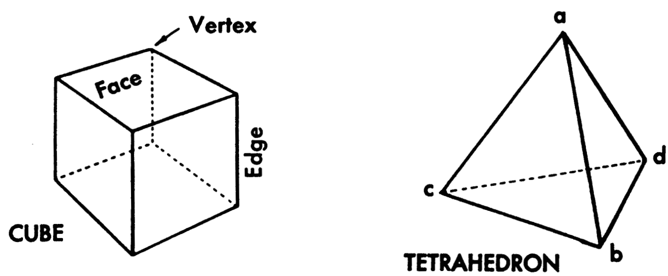


Fig. 11

polyhedra with as many faces as you wish: but never less than 4, as in the tetrahedron. Euler proved that if you add the number of faces to the number of vertices, and subtract the number of edges, you *always* get 2 for an answer. no matter how complex the polyhedron.

Instead of giving the proof, we shall generalize this rule still further in a topological way. The proof will then include Euler's, and be, unexpectedly, easier to follow. First, remembering that in topology we can bend lines, let us draw the tetrahedron on a sphere (Fig. 12). We still have (compare it to Fig. 11) 4 faces (no longer flat but bulging), 6 edges (now curved), and 4 vertices. With

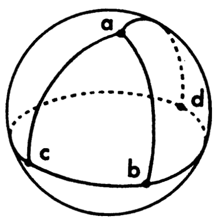


Fig. 12

Euler's rule: 4 faces plus 4 vertices minus 6 edges equals 2. F—E+V=2 is the way most books give the equation. Now, as we saw in Fig. 9, page 9, we can pull this whole arrangement of lines around to the front (if we make no breaks or new joints) and get Fig. 13. We still have the 4 vertices, *a*, *b*, *c*, and *d*, and the 6 edges joining them. Three of the original 4 faces are the triangles 1, 2, and 3,

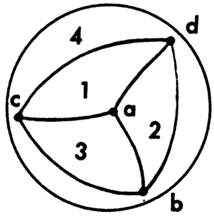
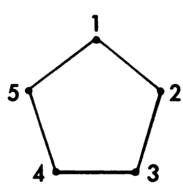


Fig. 13

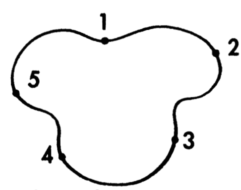
and the fourth is the space outside the new figure. It is still, topologically speaking, a triangle, as it is bounded by the same 3 edges. This can be drawn on flat paper—all polyhedra can, though in some cases they are hard to recognize—if we remember that the blank space around the figure represents the missing face.

As we said, in topology you can distort if you don't alter the way a figure is connected, and in the case of a polygon, although you may smooth out the angles, you must retain the vertices as *points marked on it.* The pentagon on the left of Fig. 14 becomes the figure on the right, but still has its 5 vertices, and edges: There are certain rules about the way faces, edges, and vertices can be connected in polyhedra—quite complicated—one being that 4 faces is the minimum, another: a vertex is the meeting place of at least 3 edges, and so on, but I am going to generalize Euler's rule to apply to *any* figure we can draw, provided it follows these rules:

Fig. 14



Pentagon



Topological Pentagon

It must be completely connected: no unattached parts. Every line has a vertex at its free end if there are any free ends, and where it touches or crosses another line—which might be at a previously made vertex. Any enclosure counts as a face, including the outside space. It must be drawn on a simply connected surface—no doughnuts allowed, because then the rule—the formula—changes. We now find that the Euler theorem is, rather surprisingly, easier to prove—or at least to follow, and if we prove the foregoing, we shall have proved it for polyhedra, too. We start with a single line (Fig. 15), and since it has 2 free ends and encloses nothing, it gives 1 face (the space

Fig. 15

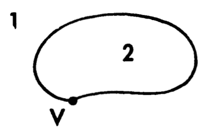


Fig. 16

around it), 1 edge (itself), and 2 vertices. In the somewhat unorthodox notation used in the following equations, where a number is followed by a space and then a letter, it indicates what the number is *of;* 2 V means 2 vertices, thus identifying the 2. 1 F—1 E+2 V=2. If we now join the ends (Fig. 16) it is regarded as making a vertex, which can be put anywhere on the line arbitrarily. This has enclosed a space, giving 2 faces, 1 edge, and 1 vertex (2 F—1 E+1 V=2).

Now, instead, we cross the first line with another, still enclosing nothing, and we get 1 face, 4 edges, and 5 vertices (1 F—4 E+5 V=2). If they merely *met* we would get 1 face, 3 edges, and 4 vertices: (again getting 1 F—3 E+4 V=2). Also we can put any number of arbitrary vertices on an edge: and each would divide the line into new edges, giving, in Fig. 17, 1 F—4 E+5 V=2.

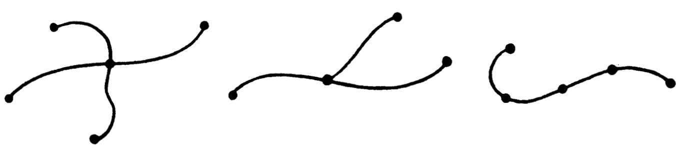


Fig. 17

When a new line, or edge, meets a loop (a self-connected edge) at its *vertex,* we get 2 F—2 E+2 V=2. If not at the vertex we would have 2 F—3 E+3 V=2. Likewise a line meeting a loop at 2 points gives 3 F—3 E+2 V=2 (Fig. 18).

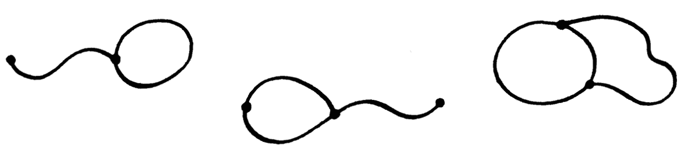


Fig. 18

It is obvious that the only way to get a new face is by adding at least 1 edge, and this edge must either connect with both its ends or be itself a loop: otherwise it would enclose nothing. Keep in mind that, although in topology we distort things, in the following proof we cannot change anything *after* it has been drawn. The following apply in all cases (or figures).

1. If we add a vertex to an edge between vertices, it divides it: making 1 edge into 2, thus it adds 1 E, canceling the new V, in the expression F—E+V.

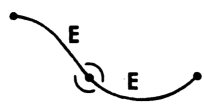


Fig. 19

2. Add an edge that meets a vertex—its own vertex on the free end cancels the new edge (in F—E+V).

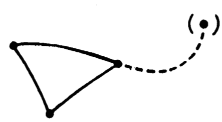


Fig. 20

3. Add an edge that meets an edge between vertices: it adds 2 E and 2 V (having divided the old edge). These cancel as before.

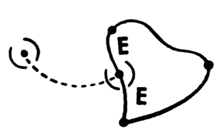


Fig. 21

4. Add an edge with each end meeting a vertex: it adds 1 F and 1 E (but no V) and they cancel.

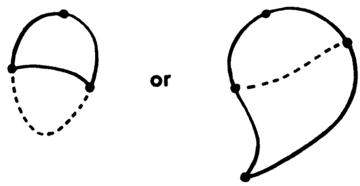


Fig. 22

5. Add an edge with both ends meeting the same V: it adds 1 F and 1 E, which cancel.

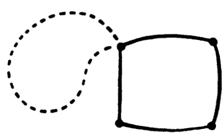


Fig. 23

6. Add an edge that meets 1 V and 1 E: it adds 1 F, 2 E, and 1 V, which cancel (1 F—2 E+1 V=0).

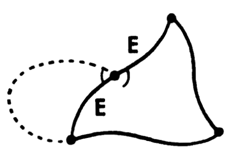


Fig. 24

7. Add an edge that meets 2 edges: it adds 1 F, 3 E, and 2 V, which cancel (1 F—3 E+2 V=0).

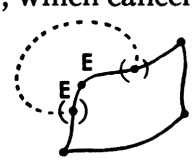


Fig. 25

8. Add an edge with both ends meeting at one V in one edge: it adds 1 F, 2 E, and 1 V, which cancel.

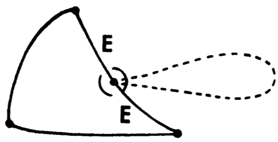


Fig. 26

That exhausts all the ways of adding lines and vertices, and therefore one can draw any figure made of them, and if it is connected, and on a simply connected surface, F—E+V=2. Thus it must be true of polyhedra, also. Try it with a complicated figure drawn at random. We have been stressing the rule that these figures must be on a simply connected surface: what happens to Euler's theorem when they are on a torus? Remembering Fig. 6 (page 7), we can see that it breaks down at once: redrawn in Fig. 27, it shows that 1 F—2 E+1 V=0. And as we said (page 8), a Jordan curve *can* be drawn on the side of the torus and still divide it into two, but not if it circles, or goes

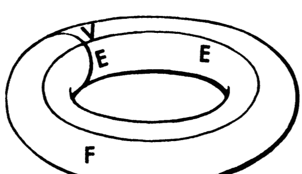


Fig. 27

through the hole. In the same way, any of the connected figures we have just discussed can also be drawn on a torus and the Euler law applies if they do not connect either *around* or *through* the hole. If these lines represent a polyhedron with a hole, they will do both, and polyhedra were what Euler had in mind. The simplest polyhedron with a hole is shown in Fig. 28—made transparent so as to show all the edges. It has 9 faces, 18 edges, and 9 vertices, giving 9 F—18 E+9 V=0.

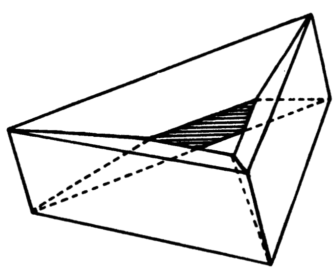


Fig. 28

Without going into the proof, the above is the new Euler law for doubly connected surfaces, and it will work for all figures drawn on them provided we have at least *one line going around the hole,* and *one going through it.* NOTE: Euler's law can be generalized to include any drawing at all that is in lines and dots: starting with one dot on a sheet of paper, 1 F—0 E+1 V=2, we can also include disconnected parts if we change the formula to $F - E + V - n = 2$, where n is the number of disconnected parts (dots, lines, or figures) minus 1. The reader can prove this by experimentation, which will disclose the underlying reason for the formula. The proof turns out to be really quite simple—after we have it. (It applies, of course, only to simply connected surfaces.)