Chapter 2

Representations of numbers

2.1 How were numbers written in former times?

The need to record numbers is ancient. It probably originated in the practical requirement to represent a constant amount or its increase or decrease over time. If a goatherd wanted to check whether his herd contained the same number of animals on two consecutive days, then he made one notch for each animal and checked the next day whether the number of notches matched the number of goats.

Much later, potentates wished to be remembered after death. Numbers representing the year, amount of accumulated gold, etc., were thus carved in stone.

Other requirements gained importance later still: A number system should allow for a representation of large numbers, or better yet, of arbitrarily large numbers. Finally, a number system should allow for efficient computation.

The most elementary way of representing numbers is to make a corresponding number of tallies or notches. This method is ancient. Archeologists have found bones as old as 30,000 years with many notches. The Ishango bone, which was introduced in Chapter 1, shows the prime numbers 11, 13, 17, and 19 in the form of tallies:
Soon early humans noticed that such lists of markings become confusing when recording even moderately large numbers. Therefore, in practically all cultures, the idea of grouping certain numbers became accepted, most often in sets of five and ten. This may mean leaving a small gap after each set of five notches, or it may mean indicating five-ness by a horizontal bar, as we do today. Or it might mean introducing new symbols for five and ten.

The Egyptians recorded numbers systematically as early as 2000 B.C.: They had symbols for one, ten, one hundred, one thousand, ten thousand, one hundred thousand, and for one million. The Egyptians represented a number by writing down the appropriate number of copies of each individual symbol. They began on the left starting with the symbol of the highest value. This is an example of an additive system.

The Romans used the familiar system consisting of I (one), V (five), X (ten), L (fifty), C (one hundred), D (five hundred), and M (one thousand). Initially, this was also an additive system, so that for example the number 44 was written like this: XXXXIII. Later the number 4 was no longer written as IIII but as IV, for brevity.

As compelling as the number systems of the Egyptians and the Romans may be at first glance, from a mathematical point of view they are both problematic.
2.1. NUMBERS WRITTEN IN FORMER TIMES?

The first problem is quite obvious: How did the Romans write 10,000? Perhaps by writing \( M \) ten times? And how did they represent one million? By writing \( M \) one thousand times? Of course the Romans did come up with solutions to these problems. For example, an \( M \) in a box meant 100,000. However, this does not truly solve the problem, it only delays it. The Romans (and the Egyptians) had to introduce new symbols for each new order of magnitude.

The second problem becomes apparent when one attempts to calculate with Roman numerals. Addition is quite all right: What is \( XVII + LXI \)? That’s easy: Write the number symbols consecutively, and then group them together. Thus \( XVII + LXI = XVII \cdot LXI = LXXVIII \). Of course this only works if one uses a strictly additive system; as soon as abbreviations such as \( IV \) are used, even addition becomes challenging and prone to error.

Multiplication is even harder. What is \( XVII \times LXI \) supposed to be? This calculation does not work at all in the Roman system; Roman numerals are not made for computing. The Romans carried out all calculations using an abacus, which is essentially a decimal place value system (see Section 2.2).

A completely different system was used by the Babylonians around 2000 B.C. in Mesopotamia, a region in today’s Iraq. It is considered the most powerful number system of antiquity, far superior to any other system. It is what we now call a “place value system.” Such systems are characterized by a limited supply of digits, which have different values depending on their “place.” We are quite familiar with this: The one in the number 1,302 has a different value than the one in the number 2,310: In the first example, it is worth one thousand, in the second only ten.

The Babylonians did not use the decimal system we are used to, that is, a base 10 system. Instead they used base 60. The digits were the numbers 1, 2, ..., 59. The last digit of a number was the ones digit, the second to last was the 60s digit, and in the third to last place, a one had a value of 3,600. The “sexagesimal number” 234 would be 7,384 in the modern base 10 system because \( 2 \times 3,600 + 3 \times 60 + 4 = 7,384 \).

The Babylonians used an additive system to represent the digits, that is, the numbers 1, ..., 59. A vertical bar meant one, and a wedge (\( \langle \) ) represented ten. The digit 14 was therefore written as \( \langle ||| \) \).

The sexagesimal system performed outstandingly well when it came to calculations, so it was always used when many difficult
calculations had to be performed, especially in astronomy. Even today, this system has left obvious traces: We divide one hour into 60 minutes, and one minute into 60 seconds. This means that the sexagesimal number 234 is the number of seconds in 2 hours, 3 minutes, and 4 seconds. The way we measure angles in degrees also has its origins in Babylonia.

By the way, the Mayans in Central America also used a place value system during the peak of their culture (until around A.D. 900), a base 20 system. The Mayans even had a concept of zero, and their symbol for it was a small seashell, arguably the most beautiful zero that has ever existed. The Mayan culture experienced a dramatic decline in the 10th century, and the culture did not continue. Thus the advanced mathematics developed by the Mayans had no impact on the further development of number systems.

2.2 The abacus and counting board

How did the Romans calculate? Undoubtedly, they did calculate. Without the elementary operations, one can neither build the Colosseum nor maintain an effective military, let alone establish a workable tax system. The Romans must have calculated! However, Roman numerals are terribly unsuitable for calculations. In principle, one can add, but multiplying is inconceivable. Roman numerals were almost exclusively used to record dates.

The Romans calculated using an abacus. The abacus was invented in China as early as 1000 B.C. It is the oldest calculating tool, and has been adopted worldwide: In addition to the “classic” Roman abacus, there are the Chinese Suanpan, the Japanese Soroban, and the Russian Stschoty. Some of these were used well into the 20th century.

The abacus is a decimal system in disguise. Its most elementary function is to represent a natural number. An abacus consists of several horizontal rods. The lowest rod represents the ones, the next one up the tens, followed by the hundreds, the thousands, and so on. The rods hold beads or pebbles, which are used to represent the value of the respective digit. Every rod has two halves. The beads on the left side represent one, the beads on the right, five. The value of a digit is determined by how many beads are moved to the center. The simplest version of the abacus has four beads to the left and one bead
to the right of the center. This allows for the unique representation of the digits 1 through 9.

Two numbers are added by placing the first number on the abacus, followed by the second one. To compute the sum of 21 and 13, we place 21 on the abacus and add on 13. This means sliding 3 beads on the lowest rod to the middle, and one bead on the second lowest rod. Then we may read the result directly off the abacus: 34.

An abacus representing the number 1954

The calculation $21 + 34$ is more challenging. In addition to the one bead on the bottom, we have to slide another four beads over—but there are only three beads there! Thus we slide these three beads to the middle and make a mental note that technically we have to slide another bead over. We know the bead on the right has a value that is one greater than the total value of the beads on the left. Therefore, we slide the bead on the right into the middle and the beads on the left back to the outside.

An abacus with carried numbers

Not only does this sound complicated, it really is complicated and also terribly prone to error. The abacus was thus developed further,
with five beads to the left and two to the right of the middle on each rod. This allows for much better control of the carried numbers. In our example of 21 + 34, we add four beads to the one bead on the left, then replace these five beads with one bead on the right: We slide the five beads on the left to the outside, and at the same time move one bead on the right to the middle. Similarly, we replace two beads in the middle on the right by one bead on the left on the next rod up.

But even with this type of abacus it is difficult to calculate. One must learn how to do it, practice it, and remain focused. There is no way to retrieve a previous result, because the abacus does not provide a written record. For an observer, the pattern of the procedure is always the same: A bunch of numbers are read and represented on the abacus. Then a knowledgeable arithmetician spends some time calculating, and finally a result appears, which is read and converted to a Roman numeral.

The idea of the abacus was adopted in the Middle Ages in Europe, but instead of the abacus the Europeans used a counting board or a counting cloth. Sometimes the necessary lines were simply drawn onto a table top. This was known as “Reckoning on the Lines.”

Reckoning on the Lines
(From Karl Menninger: *Number Words and Number Symbols,* Göttingen, 1987)

Arithmeticians drew four horizontal lines. The lowest line represented the ones, the second lowest the tens, the third the hundreds,
and the one on top the thousands. “Counting pennies,” or simply pebbles, were placed on or between the lines. The number of pebbles determined the value of a digit; one pebble in the space between two lines (in a “spatum”) counted five times as much as one pebble on the line below it. Often a vertical line divided the board in two halves so that two numbers could be represented at the same time. In the following picture we see the number 328 on the left, and 2,763 on the right.

![Two numbers represented on a counting board](image)

How was this used for calculations? Adding numbers was, in principle, easy: Arithmeticians represented the summands on the left and on the right as shown above, then combined them as shown below on the left, and finally cleaned things up. In the following picture, we literally push the numbers from the above example together, then convert the interim result to the standard representation in the next two steps. The final result is the number 3,091.

![Addition and converting to the standard representation](image)

The operations of doubling and halving were of particular importance. Doubling is easy, since it just means adding a number to itself. Halving is quite a mechanical procedure when using the “counting pennies”: From a line with an even number of pebbles, remove half. If a lines starts with an odd number of pebbles, slide one into the
space below, then remove half of those remaining. A pebble that starts in a space is replaced by two pebbles on the line below and one pebble in the space below. Odd numbers are trickier; in this case, a pebble slides into the space below the lowest line. Such a pebble is then omitted. This procedure is thus “halving with rounding down”; so 13 is halved to 6.

Doubling and halving are important because these operations can be used to multiply efficiently by only halving and doubling. To compute 83 times 56, we divide 83 into halves until we arrive at the number 1; and we double the number 56 as often. Now we add those numbers in the right column that correspond to an odd number in the left column. We obtain: 56 + 112 + 896 + 3584 = 4648, the result of the multiplication.

<table>
<thead>
<tr>
<th>Left number odd?</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>83</td>
<td>✓</td>
</tr>
<tr>
<td>41</td>
<td>✓</td>
</tr>
<tr>
<td>20</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>✓</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>✓</td>
</tr>
</tbody>
</table>

This procedure probably emerged in several places at different times. For example, the Egyptians and the Romans may have used it to multiply; it is also known as “Russian farmer’s multiplication.” In the end, this procedure is based on the binary system (see Section 2.5). One might deduce this when considering what happens when we multiply a power of two, for example $16 = 2^4$, by an arbitrary number $x$ in this way:

<table>
<thead>
<tr>
<th>Left number odd?</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$x$</td>
</tr>
<tr>
<td>8</td>
<td>$2x$</td>
</tr>
<tr>
<td>4</td>
<td>$2 \cdot 2x = 4x$</td>
</tr>
<tr>
<td>2</td>
<td>$2 \cdot 4x = 8x$</td>
</tr>
<tr>
<td>1</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>$2 \cdot 8x = 16x$</td>
</tr>
</tbody>
</table>
2.3. **THE DECIMAL SYSTEM**

Since only the last row has an odd number in the left column, the result of the calculation is the entry in the last row.

2.3 The decimal system

There are few things on Earth that are perfect, and perfect customs which man has successfully created are fewer still. But he can pride himself on this: these new Indian numerals are indeed perfect.

Karl Menninger

Our numbers come from India. In particular, our zero was invented in India. Together with the other nine digits, it forms the basis for the decimal system. Place value systems were invented in several locations at different times, but the unrivaled decimal system used worldwide today has its origins in India.

The first documented Indian zero occurred in the year A.D. 786. On a stone tablet from Gwalior, a town about 500 km south of New Delhi, zero was used to represent the numbers 270 and 50. It appears so nonchalantly that one may safely assume that zero had previously been in use.

As nondescript as it may seem, zero is indispensable for the representation of numbers and computing. The Babylonians had a place value system over 2,000 years before the Indians, but they did not have a zero! If we did not have a zero, then we would not be able to write the number 205 as we do. Instead we would have to write 2, leave a gap for the “empty” tens digit, and then write 5. Clearly, this notation inevitably leads to misunderstandings: One person might interpret “2 5” as 25, assuming that the spacing is just a bit off. Someone else might think that there is a gap at the end and call it 250.

The Indians didn’t just use zero, represented by a small circle, to write numbers. Instead, they considered it a number in its own right. For example, they noticed that a number minus itself equals zero, and that one may add zero to any number without changing its value.

Starting in India, zero, and with it the decimal system, began its victory lap around the world. The new number system gained
speed with the rapid spread of Islam. During its expansion after Mohammed’s death (A.D. 632), Islam eagerly picked up, processed, and developed the cultures of conquered countries, and thus preserved and harnessed them for modern times.

Let us highlight two mile-markers along the way. The most distinguished mathematician at the time, Abu Dscha’far Muhammad ibn Mūsā al-Khwārizmī, lived in Baghdad from around A.D. 780 to 850. His “Book of Addition and Subtraction According to the Hindu Calculation” is the first book outside of India to describe the decimal system, including the number zero. It was the first step toward worldwide recognition of the Indian system. The book itself was only passed on to us in a Latin translation. The author’s name al-Khwārizmī was Latinized as “al-gorismi.” It developed into the word “algorithm,” which generally means a computational procedure following a set of fixed rules. Moreover, the title al-dschabr of another one of al-Khwārizmī’s books developed into the word “algebra.”

Fibonacci (actually, Leonardo of Pisa, c. 1180–1250) is considered the first “modern” European mathematician. In his 1202 Book of Calculation (liber abaci), he introduces the new Indian system and its advantages. In particular, Fibonacci writes: “The nine Indian digits are 9 8 7 6 5 4 3 2 1. With these nine digits, and with the symbol 0, which the Arabs call zephirum, any number may be represented.” This mathematical theorem proves to be of immense value in the plethora of exercises that follows in his book.

However, the number zero and the Indian decimal system were not immediately accepted; on the contrary, they had to prevail in a fight against established traditions over hundreds of years. One argument against them involved employment: With the new decimal system, pretty much anyone could calculate, whereas with the old system, special skills were required. Another argument against the new system was the claim that the new digits were easy to forge: One brush stroke sufficed to change a 0 into a 6 or 9. Not until the 16th century was there a general demand for learning and using the new method. This is documented by a large number of books on arithmetic, among them the bestseller Rechenung auff der linihen und federn (1522) by Adam Ries. In addition to computing with a counting board (the “linihen” being the lines), Ries describes how to perform written calculations (with “federn” being feather pens) using the Hindu-Arabic digits. Subsequently there has been no stopping the triumph of the decimal system.
2.4 Divisibility rules

“A number is even when its ones digit is even.” This theorem is often regurgitated without much thinking. Some even think this is the definition of an even number—but this is wrong. The definition of an even number is that it is divisible by 2 without leaving a remainder.

The fact that we can read this property off the last digit of a number is a mathematical theorem, and like every mathematical theorem is something of a miracle. Suppose we want to know whether the number 782,573,728,764,104 is even. By definition, we have to divide this monster by 2 and check whether the remainder is 0. The theorem shows us a much easier method: It suffices to take a closer look at the last digit, 4, and we know at first glance whether this huge number is even. This is a “David and Goliath theorem”: Based on a tiny observation (the last digit), we knocked out a gigantic number!

We can convince ourselves of the truth of this theorem in several ways. The most illustrative method uses the figurate numbers from Chapter 1. We write the number as a rectangle of height 10. Of course in general, this will be an imperfect rectangular number; it probably will not actually make a rectangle, and there is a remainder that is less than ten. This remainder is the ones digit. Let us consider the number 54. It consists of 5 tens and 4 ones. Written as a figurate number, it looks like this:

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

54 as a figurate number

The first 5 columns consist of ten dots each. Since 10 is an even number, the number containing all but the last column must be even. Whether the overall number is even therefore only depends on the last column. If that number is even, then the overall number is even, and otherwise it is odd.

This proof may also be expressed in a symbolic way. To check whether a natural number \( n \) is even, we write it as \( n = 10a + b \).
In the case of \( n = 54 \), we write \( n = 10 \times 5 + 4 \). Here, the number \( b \) represents the ones digit. We proceed in a similar manner: \( 10a \) is always an even number, since 10 is even, and so is every multiple of 10, in particular \( 10a \). Now we apply the old rule of the Pythagoreans that “even plus even is even.” In our example this means: If we take the even number \( 10a \) and add an even number \( b \) to it, then we obtain another even number. Therefore \( n = 10a + b \) is even. On the other hand, “odd plus even is odd.” So if we add an odd number \( b \) to the even number \( 10a \), then the sum \( n = 10a + b \) is odd. In summary, this means: \( n = 10a + b \) is even if and only if the last digit \( b \) is even.

Even more exciting than divisibility by 2 is divisibility by 9. The theorem in question is another example of a “David and Goliath theorem.” To decide whether the number 123,456,789 is divisible by 9, all we have to do is add its digits, or in other words, find its “checksum”; the checksum of 123,456,789 is \( 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 \). Since 45 is divisible by 9, the monster number above is also divisible by 9.

Every number is either even or odd. In other words, with divisibility by 2, there are only two cases. When divided by 2, a natural number leaves a remainder of either 0 or 1.

But divisibility by 9 has even greater potential! Of course one might say: Either a number is divisible by 9, or not. Alternatively we might take a closer look and determine not only “remainder 0” and “other remainder,” but distinguish between all possible remainders: When divided by 9, a natural number leaves a remainder of 0, 1, 2, 3, 4, 5, 6, 7, or 8. This is called the “nines remainder” of a number. For example, the number 1,024 has a nines remainder of 7, because \( 1,024 = 9 \times 113 + 7 \).

It is easier to come to this conclusion by considering the checksum. The checksum of 1,024 is \( 1 + 0 + 2 + 4 = 7 \). The nines remainder of 7 is, of course, 7—and this is also the nines remainder of our original number.

This surprising mathematical fact holds true in general: A natural number leaves the same remainder as its checksum when divided by 9. This leads us to a generalization of the nines rule. Since a number is divisible by 9 if and only if its nines remainder is 0, one may rewrite the nines rule as:

The nines remainder of a number is 0 if and only if the nines remainder of its checksum is 0.
2.4. DIVISIBILITY RULES

This mathematical fact provides a framework for a quite intriguing magic trick. Ask someone to think of an arbitrary five-digit number and write it on a piece of paper, so that you are unable to see the number. Then instruct them to write the digits of the number down in a different order. Then they should subtract the smaller of the two numbers from the larger one. Finally, ask them to choose and circle one digit of the result. If the result contains a zero as a digit, then it may not be circled. You can always say that the number zero is already a circle itself. Now ask them to read the other digits to you—and you will be able to name the circled digit immediately.

The calculation may for example look like this: The original number was 54,831; the permutation was 38,415. Then the difference is 16,416. The digit in the middle was circled, and they read the digits 1, 6, 1, 6. All you have to do is add the digits that were read back to you: $1 + 6 + 1 + 6 = 14$, and find the difference to the next nines number. Here, we have to add 4, and that turns out to be the circled number.

But wait, there is more! The nines remainder has an application of historical importance, the “nines check,” which Adam Ries held in high regard. The nines remainder allows us to check whether a complicated calculation was carried out correctly. It is an indicator of the inner harmony of a calculation.

Roughly, the rule may be expressed like this: If a calculation was carried out correctly, then it must also be correct when carried out with the respective nines remainders. This is because the nines remainder gets along famously with the elementary operations: If we want to calculate the nines remainder of a sum $a + b$, then we may find the nines remainders of the summands $a$ and $b$ and then add them. Notice that our magic trick above depends on the fact that the nines remainder cooperates with subtraction.

In order to check whether the sum $247 + 354 = 601$ was calculated correctly, we add the nines remainders 4 and 3 of the numbers 247 and 354 and compare this to the nines remainder of the sum 601. If these numbers are not the same, then we know for sure that a mistake was made in the calculation; if the two results are the same, then the calculation was probably done correctly.

The same thing goes for multiplication: The nines remainder of a product is the product of the nines remainders. For example, the nines remainder of 19 times 31 is the same as the nines remainder of 19 (i.e., 1) times the nines remainder of 31 (i.e., 4); the result is 4.
The product of 19 and 31 is 589; since this also has a nines remainder of 4, the calculation was probably carried out correctly.

Adam Ries describes the nines check as a schematic procedure. To verify the correctness of the calculation $7,869 + 8,796 = 16,665$, we begin by drawing a cross in the form of a big X. We write the nines remainder of the first number, that is, of 7,869, on the left; it is 3. The nines remainder of the second number, that is, of 8,796, goes on the right; it is also 3. We add the two nines remainders and put the result on top, i.e., $3 + 3 = 6$. If the result is greater than or equal to nine, then we subtract 9 until we obtain a number between 0 and 8. Finally, we put on the bottom the nines remainder of the sum 7,869 + 8,796, that is, the nines remainder of 16,665: It is 6. If the same number appears on top and on the bottom, then everything checks out; if the numbers are different, there is a mistake in the calculation.

### 2.5 Binary numbers

The Pythagoreans discovered the distinction between even and odd numbers. From there originated the description of the world through polar opposites: above and below, left and right, before and after, day and night, old and young, life and death. Mathematicians express this, very matter-of-factly, with plus and minus, or simply with 0 and 1. Not only is it possible to identify opposites this way, but moreover, all numbers may be written in terms of 0s and 1s, something that the great Gottfried Wilhelm Leibniz published in 1703.

His idea was to develop a place value system using only 0 and 1. This is known as the binary system. In principle, things work just like they do in the decimal system: The value of a digit depends on its place. In the decimal system, a 1 as the last digit just means one, in the second to last place it has a value of ten, in the third to last place one hundred, and so on.

In the binary system, a 1 in the second to last place has a value of two; the number two is thus written as 10. The binary number 11 means three; since the 1 in the last place has a value of one, and the 1 in the second to last place has a value of two, and these add up to three.

Does that make sense? Let’s take a step beyond this. A 1 in the third to last place has a value of four (since 2 times 2 is 4). This
means that the number four is written as 100, the number five 101, the number six 110, and finally the number seven as 111. Clearly, since 111 means: 1 one, 1 two, and 1 four, together this adds up to seven. The first few numbers are therefore written in binary as follows: 1, 10, 11, 100, 101, 110, 111, 1000, ....

<table>
<thead>
<tr>
<th>Decimal number</th>
<th>Binary number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
</tr>
<tr>
<td>9</td>
<td>1001</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
</tr>
<tr>
<td>11</td>
<td>1011</td>
</tr>
<tr>
<td>12</td>
<td>1100</td>
</tr>
<tr>
<td>13</td>
<td>1101</td>
</tr>
<tr>
<td>14</td>
<td>1110</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
</tr>
</tbody>
</table>

The binary numbers from 0 to 15

This is the representation of numbers used by modern computers. All data, regardless of whether they are text, audio, or video data, are represented by numbers, and numbers are represented by zeros and ones, i.e., bits. What we take for granted today is the original vision of the American mathematician Claude E. Shannon (1916–2001), who published an article in 1948, in which he described his insights: Any kind of information may be represented in the form of “bits” (“binary digits”). Without Shannon’s idea, the development of information technology would most likely have taken a different path.

Before Shannon, Leibniz had envisioned a binary computer. However, he did not pursue his idea of a “Machina Arithmeticae Dyadicae,” because his computing machine based on the decimal system had already confronted him with almost unsolvable problems.
Leibniz had predecessors: Representations of numbers using only two symbols were found in India and China; whether the symbols in question represent numbers or whether they are only a systematic sequence of combinatoric patterns remains unclear. In Europe, too, binary number systems had been published a few decades before Leibniz. However, it would be fair to say that Leibniz was the first to recognize the enormous potential of binary numbers. This potential lies in the fact that it is extremely easy to calculate with them. For example, the entire “binary multiplication table” reduces to the one trivial equation, $1 \times 1 = 1$.

Leibniz writes: “Addition of numbers by this method is so simple that they can be added as quickly as they can be written.” Today, we say that the “cost of addition is linear.” This means that the time required for adding two numbers is just a constant multiple of the time required for writing the numbers.

### 2.6 Applications: Barcodes

Errare humanum est.

We read and write numbers, and we make mistakes. How often do we overlook a digit? How easy is it to read 8 instead of 3? How often do we say “twenty-three” but write 32?

There are a number of traditional methods for dealing with such errors. If we did not understand a word, then we ask for it to be repeated. Sometimes numbers are written as words, and difficult words may be spelled out. All of these methods have something in common: The actual information is augmented by additional elements. A word is repeated, or in addition to a digit, a word is transmitted; a word is verified by the sequence of its letters. In short, we add redundancy. Often, a lot of redundancy is added, and the number of symbols to be transmitted at least doubles. Mathematical procedures for error detection are based on the same idea, but they make do with minimal redundancy, that is with only one additional symbol.

This idea is best illustrated by a small example. Suppose we send the message 123456 and would like to protect it from erroneous reading or writing. We may add a check digit to the original message 123456 in such a way that the total sum of the digits is a multiple of ten. Since $1 + 2 + 3 + 4 + 5 + 6 = 21$, the check digit is 9, and the complete message reads 1234569. We call such a procedure, or the set of messages created in this way, a “code.”
The receiver of the coded message reads some sequence of digits. In order to check whether the message was transmitted correctly or not, they compute the check digit. Or even easier: They determine whether the checksum is correct. If an error occurred, for example if instead of 3 an 8 was transmitted, then the received message is 1284569. Since the checksum of this number is 35, the message will not be accepted.

In general, these types of codes will detect an incorrect digit, in the sense that the checksum is no longer a multiple of ten. However, if errors occur in two or more places, then the effects may cancel out. We say that such a code “detects one error.”

For many purposes these types of codes are sufficient. However, they do not catch transposition errors: The message 1324569 is accepted as much as the original message 1234569.

In order to detect the latter type of error, one has to use another version of the checksum, which distinguishes the neighboring digits. This may be realized by “weighting” the digits differently. Often the first, third, fifth, etc., digits are added, while the second, fourth, and sixth digits are multiplied by 3 before they are added to the other digits. In our example, we would compute $1 + 3 \times 2 + 3 \times 4 + 5 + 3 \times 6$, which is 45. Again, we complete this number to the next multiple of ten. The complete message is then 1234565.

This code will detect single errors and most transposition errors. For example, the sequence 1324565 leads to a checksum of $1 + 3 \times 2 + 3 \times 4 + 5 + 3 \times 6 + 5 = 52$. Since this is not a multiple of ten, the message will be rejected.

The weighting of 1, 3, 1, 3, 1, …, 3 is the basis for the “European Article Number (EAN),” which is known worldwide as a “barcode.” The bars are simply the translation of the number below into a machine-readable form. The barcode is directly useful to us as consumers: When the cashier scans an item, then the check digit is computed automatically. The beep means that everything is alright; and we can be sure that the correct item was detected and we only pay for this particular item.

These types of error correcting codes are but the beginning of an extensive theory, which has numerous practical applications. In practically every transmission of data, error correcting codes are used. No cell phone or CD can work properly without them. In practically every transmission of data, error correcting codes are working behind the scenes.
Numbers
Histories, Mysteries, Theories
Albrecht Beutelspacher

Posing the question “What exactly is a number?” a distinguished German mathematician presents this intriguing and accessible survey. Albrecht Beutelspacher—founder of the renowned interactive mathematics museum, Mathematikum—characterizes the wealth of experiences that numbers have to offer. In addition, he considers the many things that can be described by numbers and discusses which numbers possess special fascinations and pose lasting mysteries.

Starting with natural numbers, the book examines representations of numbers, rational and irrational numbers, transcendental numbers, and imaginary and complex numbers. Readers will explore the history of numbers from Pythagoras to Fermat and discover such practical applications as cryptography and barcodes. A thoughtful and enlightening introduction to the past, present, and future of numbers, this volume will captivate mathematicians and nonmathematicians alike.


Translated by
Andrea Bruder, Andrea Easterday & John J. Watkins