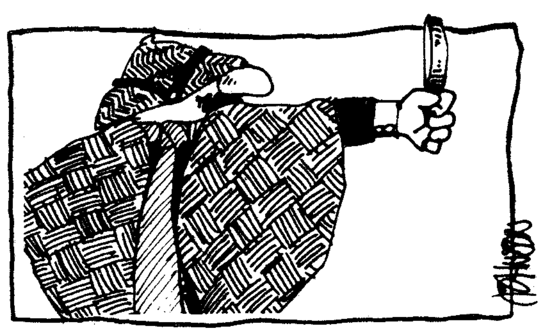
Solutions Manual

PROBLEM SOLVING THROUGH RECREATIONAL MATHEMATICS

Bonnie Averbach
Orin Chein

Temple University



Solutions Manual

PROBLEM SOLVING THROUGH RECREATIONAL MATHEMATICS

Bonnie Averbach
Orin Chein

Temple University

Printed in the United States of America

TABLE OF CONTENTS

CHAPTER 1

COMMENTS AND SUGGESTIONS

1. The purpose of this chapter is to familiarize the students with some techniques of attacking problems. It is also our intention to begin by considering problems which most students are able to handle, so that the students can begin to acquire the feeling of accomplishment associated with the successful solution of a problem.

2. Solutions of double starred problems are difficult to write up and class discussions of these problems may require a great deal of time. We therefore caution against assigning too many of them and getting bogged down as a result. On the other hand, we do like to assign at least one of them as an "overnight" because they are fun to solve and very challenging.

CROSS-REFERENCES

1. Tree diagrams are introduced here to help enumerate cases. They appear also in Chapter 6 as examples of graphs, and are useful in analyzing some games in Chapter 7.

2. The Multiplication Principle is introduced here to help count cases. It is useful for this same purpose in some of the exercises of Chapter 2. It is also referred to in Chapters 7 & 8.

3. The matchstick game illustrates the use of a tree diagram and is an example of the method of simplification. We find this a convenient spot to introduce the definitions of position, winning strategy, etc., from Chapter 7. This enables us to assign game problems throughout the semester.

SOLUTIONS TO EXERCISES

Most of the problems in this chapter can be handled by setting up an appropriate chart.

1.1: After we fill in as much as of the given information as possible, our chart looks like

woman	pet	pet's name
Toni	hog	Jo
Belle	frog	Sue
Janet	crow	
Jo	snake	
Sue		
	pony	x
x		Toni

1. By the process of elimination Sue owns the pony.
2. By the process of elimination, x must be Janet, and as a result
3. Jo's pet is Belle and
4. Jo is Sue's mother and Sue's mother's pet is a snake.

1.2:

name	handed	height
Adam	right	
Robert		under 6
Clifton	right	
Stephen		
Brent		under 6

1. Since Adam, Clifton and either Stephen or Brent are of the same handedness, they must be right handed.
2. Similarly Robert and Brent must be under 6 ft.
3. The only person who could be both left handed and over 6 ft is Stephen.
4. Therefore <u>Stephen is the center</u>.

<u>1.3</u>: For reference purposes, we label the rows of our chart.

row	sender	plant	recipient
i	Rose	holly	Forsythia
ii	Azalea	y	
iii	Fors.		x
iv	x	rose	
v	z	fors.	y
vi		z	Azalea

1. z ≠ azalea (row vi), forsythia (row v), holly (rows i and vi), or rose (rows i and v). ∴ z = iris.
2. y ≠ azalea (row ii), forsythia (row v), holly (rows i and ii), or iris (row v, since z = iris). ∴ y = rose.
3. By ii, azalea sends a rose, and, by iv, x sends a rose, so x = azalea.
4. The completed chart follows.

sender	plant	recipient
Rose	holly	Forsythia
Azalea	rose	Holly
Forsythia	iris	Azalea
Iris	forsythia	Rose
Holly	azalea	Iris

<u>1.5</u>:

1st	2nd	3rd	4th	5th
C-P	C-SJ	SJ-LB	P-S	
			SJ-?	

1. Consider SJ's opponent on the 4th day. (This choice is made since we can eliminate all but one possibility. Consideration of any other day or player leaves more than one possibility to consider.) This opponent must be F (P,S play each other and SJ has already played C and LB).
2. The 3rd match on the 4th day must therefore be C-LB.
3. Consider SJ's opponent on the 1st day. The chart can then be completed easily.

1st	2nd	3rd	4th	5th
C-P	C-SJ	SJ-LB	P-S	LB-S
SJ-S	P-LB	P-F	SJ-F	SJ-P
LB-F	S-F	C-S	LB-C	C-F

<u>1.6</u>:

	Sa	Su	M	Tu	W	Th	F	Sa	Su
milk	√		√		√				
dog		√			√				√
garb			√					√	

1. Since the milkman and dog both made noise on Wednesday, the milkman did not make noise on either Sunday and the dog did not make noise on the first Saturday or Monday.
2. Since the milkman and garbagemen both made noise on Monday, the milkman did not make noise on the second Saturday and the garbagemen did not make noise on Wednesday or the first Saturday.
3. Since the milkman did not make noise on the second Saturday and Sunday he must have made noise on Friday.
4. But then the dog and garbagemen must have been quiet on Friday.

5. Since the garbagemen were quiet on Wednesday and Friday, they must have made noise on Thursday.

6. Therefore the only day on which Joe was able to sleep late was Tuesday. (The completed chart is shown below.)

	Sa	Su	M	Tu	W	Th	F	Sa	Su
milk	√	X	√	X	√	X	√	X	X
dog	X	√	X	X	√		X		√
garb	X	X	√	X	X	√	X	√	

1.7:

Events

		h	s	d	p	r	total
schools	C						24
	F						
	L						
	M						
	O	5			3		
totals		15	15	15	15	15	75

1. Central can only get 24 by four 5's and a 4. Since the 5 in h is already accounted for, C gets 4 for h, and 5 for everything else.

2. Look at the total number of points for each school. Olney has at least 11. If O had 12 or more, then M would have at least 13, L at least 14, F at least 15 giving a total of at least 78 points. Since 15 points are awarded for each event, the total must be 75. Hence O has 11. By similar reasoning M has 12, L has 13, F has 15.

3. L either has four 2's and a 5 or four 3's and a 1. But all the 5's are accounted for.

4. The chart at this point is

	h	s	d	p	r	total
C	4	5	5	5	5	24
F						15
L	3	3	3	1	3	13
M						12
O	5	1	1	3	1	11

5. The numbers remaining are 4, 4, 4, 4, 2, 2, 2, 2, 1. Since F's total is odd, F must receive a 1.

This 1 must occur in h.

1.8:

1. The possible scores are 10-0, 9-1, 8-2, 7-3, 6-4, 5-5.

2. The total number of points in the games played by any one player is 30.

3. Since Alice outscored her opponents by 22 points, she scored 26 and her opponents scored 4. So she won her three games by scores of 10-0, 9-1, 7-3.

4. Which one game did Carol win? Either 8-2 or 6-4. If she won by 6-4, since she lost by more than 2 points to Alice, she would not have outscored her opponents. Therefore Carol won by 8-2 and lost to Alice by 7-3 and the third game must have been a tie.

5. Bob beat Ted. The score must have been 6-4.

6. Finally, since Bob scored 6 points against Ted, Ted scored 6 points in all. Four of these against Bob, and the remaining 2 must have come in an 8-2 loss to Carol.

A–B	A–C	A–T	B–C	B–T	C–T
9–1	7–3	10–0	5–5	6–4	8–2

1.9:

	sp	tc	gc	gf
H	√			
C	√	√		
S		√	√	
P			√	√

1. The possible pairs of requirements are (sp,tc), (sp,gc), (sp,gf), (tc,gc), (tc,gf), (gc,gf).
2. Since C has (sp,tc), H cannot have tc, S cannot have sp.
3. Similarly C cannot have gc, P cannot have tc, S cannot have gf.
4. The chart so far is

	sp	tc	gc	gf
H	√	X		
C	√	√	X	
S	X	√	√	X
P		X	√	√

5. The only way (tc,gf) can occur is for C to have gf. So <u>C must be the chosen resort</u>. (It is not hard to check that this is consistent.)

<u>1.10</u>:
1. The winner of a game scores 5, so that no more than 5 points is scored against a player in any one game.
2. Therefore M lost all three games and T lost two games and won one game by 5–4. The 5–4 game must be against M.
3. Since N beat T and M, she scored 10 against them and hence scored 0 against P.
4. A partial table is

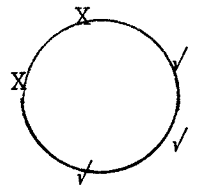
P–N	P–M	P–T	N–M	N–T	M–T
5–0	5–	5–	5–	5–	4–5

5. Since P's opponents scored a total of 2 and since T scored more against P than M did, T scored 2 against P and M scored 0 against P.
6. This gives

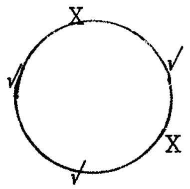
P–N	P–M	P–T	N–M	N–T	M–T
5–0	5–0	5–2	5–4	5–1	4–5

<u>1.11</u>:
1. The possible seating arrangements for each of the characteristics are

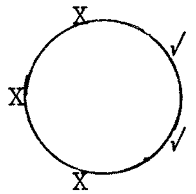
<u>know odds (o)</u>



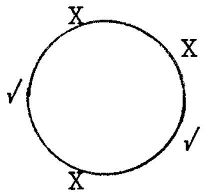
<u>wise bluffers (b)</u>



<u>poker face (p)</u>

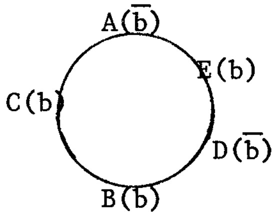


<u>lucky (*L*)</u>

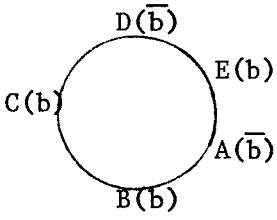


2. Consider the situation with regard to bluffing. Edie is not sitting next to anyone who knows how to bluff; the person on Cleo's left does not know how to bluff; Babs knows how to bluff. The only possible arrangements are as follows (b = bluffers, $\overline{b}$ = non-bluffer).

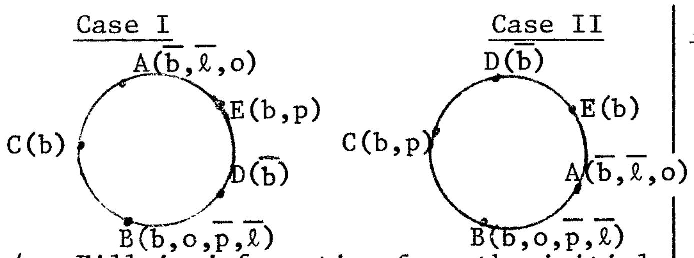
<u>Case I</u>



<u>Case II</u>



3. Fill in the other information to get



4. Fill in information from the initial diagrams in step one. For example, in Case I, C must know the odds, while D and E do not. When this is done, we determine that, in Case I, no one can possibly have all four traits and, in Case II, Cleo is the only one who can have all four traits.

1.12:

	li	t	e	g	p	le
Lou	X					X
Ted		X				X
Eli			X	X		X
Gary			X	X		√
Pete					X	√
Leo				X		X

1. Since Ted and Peter could not have both photographed the gazelle, Lou must have and Eli must have photographed the lion; and Gary did not photograph the lion.

2. Since Gary and Ted have photographs of the same animal and Gary doesn't have the lion, the only possibility is for Gary and Ted to have the panther.

3. The rest of the chart can now be filled in.

	li	t	e	g	p	le
Lou	X	√	X	√	X	X
Ted	X	X	X	√	√	X
Eli	√	√	X	X	X	X
Gary	X	X	X	X	√	√
Pete	X	X	√	X	X	√
Leo	√	X	√	X	X	X

1.13: 1. First find out who is who:

	b	l	d	p	g
Mr. B	X	X	X	X	√
Mr. L	X	X	X	√	X
Mr. D	√	X	X	X	X
Mr. P	X	X	√	X	X
Mr. G	X	√	X	X	X

2. Make a second chart as follows. (Each person is represented by his occupation).

intended callee

caller	b	d	g	l	p
b	b	g			l
d	p	d	l	b	g
g			g		
l		p		l	
p					p

The entries in the table represent the person actually reached. The diagonal entries indicate that each person knows his own number.

3. Each letter must appear once in each row and once in each column. In the third column, the p must be entered in the first row. Continue to fill in the table.

	b	d	g	l	p
b	b	g	p	d	l
d	p	d	l	b	g
g			g		
l		p		l	
p					p

4. At this point we can answer the question: Mr. Butcher (gardner) reached Mr. Lawyer (plumber) when he called Mr. Gardner (lawyer).

5. There are two ways to fill in the rest of the table consistently.

1.15: Bently reasoned as follows: If my handkerchief said Professor Moriarty, then the gentleman behind me would know that, if he had Moriarty, the man behind him would immediately know that he had Sherlock Holmes. Since this third man is quiet, the man behind me would realize that he can't have Moriarity and would therefore announce that he has Sherlock Holmes. Since he hasn't made this announcement, I cannot have professor Moriarty. Therefore, I have Sherlock Holmes.

1.16: 1. In order for Ann to be able to make her statement, she must have either ordered 3S or 2S, 1B. In order for Bob to make his statement, the situation must be one of the following (for any other order, Bob would not be able to deduce what he received):

name	ordered	received
Ann	3S	2S, 1B
Bob	1S, 2B	3S

or

name	ordered	received
Ann	2S, 1B	3S
Bob	1S, 2B	2S, 1B

2. Since Carla claims to have received 3B, she did not order them, which means that John did, and he must have received 1S, 2B. That is there are two possibilities:

name	ordered	received
Ann	3S	2S, 1B
Bob	1S, 2B	3S
Carla	2S, 1B	3B
John	3B	1S, 2B

and

name	ordered	received
Ann	2S, 1B	3S
Bob	1S, 2B	2S, 1B
Carla	3S	3B
John	3B	1S, 2B

Both lead to the same answer regarding John. (Note: In each case, Carla is able to deduce what she received from Ann's and Bob's statements and the fact that she did not receive what she ordered.)

1.18: The following charts follow immediately from the given clues.

claim to fame

people	prs	chr	vp	bp	val	pc
B						
F		X			X	
G	X	X				
J	X		X		X	
R						
S						X

whom they like

people	prs	chr	vp	bp	val	pc
B	X	X	X			
F	X		X	X	X	X
G		X	X	X	X	X
J	X	X	X			
R	X	X	X	X		
S	X	X		X	X	X

2. From the above charts, the last clue, and the clues that refer to J and S, only the following two seating arrangements are possible:

wall	pres	J	vp
	S		pc

wall	pres S	J	vp
			pc

3. Since B is sitting opposite the person he or she loves and since B does not like the volleyball player (S does), B is not the principal's child.

4. Since F is sitting opposite to the valedictorion, F cannot be the volleyball player or the principal's child.

The following seating arrangements are now possible:

I

wall	pres F	J	vp
	val S		pc

II

wall	pres	val J	vp
	S	F	pc

III

wall	pres S	val J	vp
		F	pc

5. Since F is not the cheerleader and G is sitting next to the cheerleader, charts II and III are impossible.

6. Chart I leads to two cases (note that F is president and S is the valedictorian):

Ia

wall	pres F	cheer J	vp G
	val S		pc

Ib

wall	pres F	J	vp
	val S	cheer	pc G

7. Since B is sitting opposite the person he or she loves and B doesn't like the volleyball player (S does) or cheerleader (F does), chart Ia is impossible. This leaves two possibilities.

wall	pres F	bp J	vp R
	val S	cheer B	pc G

wall	pres F	bp J	vp B
	val S	cheer R	pc G

In both cases J is the basketball player.

8. At this point we can fill in the original two charts as follows:

claim to fame

people	prs	chr	vp	bp	val	pc
B	X			X	X	X
F	✓	X	X	X	X	X
G	X	X		X	X	
J	X	X	X	✓	X	X
R	X			X	X	
S	X	X	X	X	✓	X

whom they like

	prs	chr	vp	bp	val	pc
B	X	X	X			
F	X	√	X	X	X	X
G	√	X	X	X	X	X
J	X	X	X	X		
R	X	X	X	X		
S	X	X	√	X	X	X

9. We now see that B likes the basketball player. Since B is sitting opposite the person he or she likes, the solution is

wall

pres F	bp J	vp R
val S	cheer B	pc G

1.19: We make the following chart and enter the information immediately available from the schedule and the clues.

day	1st	2nd	3rd
Mon	E–W	N–S Emily	C–
Tues	S–	– Emily	–
Wed	S–	– Irv	– Paul Sylvia
Thur	C– Vicki	– Max	– Ted

(It is also helpful to make a second chart showing which players are on which teams. We do not exhibit this chart here.)

1. Since W won on Monday, it had to win the first game on Monday (M_1). It therefore played in M_3.

2. Since W only won once, it had to lose M_3 and play and lose in T_3 and W_3. (W was eliminated as a result of losing W_3.)

3. Since S played in T_1, N beat S in M_2; since S played in W_1, S lost in T_1.

We now have the following (winners are circled)

day	1st	2nd	3rd
Mon	E–(W)	(N)–S	(C)–W
Tues	S–(E)	N–C	(E)–W
Wed	S–	E–	W–( )
Thur	C–( )	–	–

4. By clue 1, Emily is on N. By clue 5, Helen is on E. By clue 7 and clue 10, Vicki is not on C. She is also not on W (W was eliminated on Wednesday). Therefore Vicki is on S and S won in Th_1. This implies that S won in W_2 or W_3, but S couldn't have played in W_2. Hence S won in W_3 and also won in W_1. This in turn implies C won in W_2 and hence in T_2. We now have the following.

day	1st	2nd	3rd
Mon	E–(W)	(N)–S	(C)–W
Tues	S–(E)	N–(C)	(E)–W
Wed	(S)–N	E–(C)	W–(S)
Thur	C–(S)	N–E	S–

5. Since N played only four times through Th_2, it follows from clue 3 that Max played on E and N won in Th_2.

6. From clue 6, Irv is on C.

7. Since the only possibilities left for Sylvia are W or C, it follows from clue 8 that Sylvia is on W

and Paul is on S. This leaves Becky on C.

8. Since the only possibilities remaining for Ted were W and N, and W was eliminated on Wednesday, it follows from clue 9 that Ted is on N and that S beat N in Th_3. This eliminates N from the tournament. It also implies that Lee is on W. We now have

day	1st	2nd	3rd
Mon	E-W	N-S	C-W
Tues	S-E	N-C	E-W
Wed	S-N	E-C	W-S
Thur	C-S	N-E	S-N

9. At this point, only S and C remain. From clue 11, S played in two games on Friday, but did not win the championship. Hence S beat C in F_1 and C beat S in F_2 to win the championship.

1.20: 1. Make the chart in Figure 1.1, entering the given information.

2. The only three letters which might be the same for player and school are H, J, M. Since the competitors from Jupiter and Holbrook were in positions 1 and 5 after the first five events, Manners must have been in second place. His school was either Marmaduke or Mamaraneck. Therefore x = 2.

3. Consider v next: from columns III and IV, v ≠ 1, 2, 3, or 5 (compare rows 3 and 7 for example). Therefore v = 4, v + 1 = 5, and Twofeathers is from Holbrook.

4. Consider w: w ≠ 1, (Jimenez does not attend Jupiter), w ≠ 2 or 3 (cols. III and V, x = 2) w ≠ 5 (cols. I, III, v = 4). Hence w = 4. Jimenez is from Mamaraneck leaving Manners from Marmaduke. And u = 3.

5. The chart becomes Figure 1.2 (next page).

I Name	II School	III Standing at end of five events	IV Jersey No.	V Position in high jump
	Jupiter	1		
	Holbrook	5		
		3	1	
		4	5	
Boone		y+1		y
Manners		x		x+1
Twofeathers		v+1	v	z
Harris		z	z+1	
Jimenez		w		w
	Maraduke		u	u
	Mamaraneck	v		
*	*	2		

* The two entries so marked begin with the same initial

Figure 1.15

I Name	II School	III Standing at end of five events	IV Jersey No.	V Position in high jump
Manners	Marmaduke	2	3	3
Twofeathers	Holbrook	5	4	z
Jimenez	Mamaraneck	4	5	4
Boone		y+1		y
Harris		z	z+1	
	Jupiter	1		
		3	1	

Figure 1.2

6. z must = 1, and y + 1 = 3. Therefore, Harris goes to Jupiter and Boone to West Rochedale, etc.

1.21: 1. We fill in the chart in Figure 1.3 according to a first reading of the clues. ("Negative" information such as that coffee cannot be drunk by E--clues 1, 3--is included.)
2. Also consider the order of the houses by color. By clues 1,5,9,14, there are two possible arrangements--both giving that N lives in the yellow house and doesn't drink coffee (clue 3) or milk (clue 8).

Case I

y	b	i	g	r
N				E

Case II

y	b	r	i	g
N		E		

3. N lives in the yellow house (clues 1,9) and therefore smokes Kools (clue 7) and does not drink orange juice (clue 12). By elimination, N drinks water.
4. Furthermore the horse is in the blue house (clue 11) and, therefore, not in the yellow or red. This also implies that S does not live in the blue house.
5. N does not own the snail (clues 6, 7).
Filling in this new information on the chart we obtain Figure 1.4 (next page).
6. At this point we consider the alternatives of step 2, above. Consider first Case I: The ivory house is in the middle. Then E does not drink milk (clues 1 & 8) and therefore must drink orange juice

	House color					Pets					Drink					Cigarettes				
	r	g	y	b	i	d	s	f	h	z	c	t	m	o	w	og	k	c	ls	p
E	√	x	x	x	x	x					x	x					x			x
S	x					√	x	x	x	x		x				x				x
U	x	x				x					x	√	x	x	x				x	x
N	x			x		x						x								x
J	x		x			x	x					x		x		x	x	x	x	√

Figure 1.3

	House color					Pets					Drink					Cigarettes				
	r	g	y	b	i	d	s	f	h	z	c	t	m	o	w	og	k	c	ls	p
E	√	x	x	x	x	x			x		x	x			x		x			x
S	x		x	x		√	x	x	x	x		x			x	x	x			x
U	x	x	x	√		x					x	√	x	x	x		x		x	x
N	x	x	√	x	x	x	x		x		x	x	x	x	√	x	√	x	x	x
J	x		x			x	x					x		x	x	x	x	x	x	√

Figure 1.4

(by elimination) and therefore smokes Lucky Strikes (clue 12). This implies that S smokes Chesterfields and U smokes Old Golds and owns snails (clue 6). In addition U does not live in the ivory house (clues 4, 8 and our assumption) and hence must live in the blue. But then, as we saw above, he must own a horse. This is a contradiction.
This leaves only Case II.

7. E is in the middle house which is red. Therefore E drinks milk (clue 8), J must drink coffee (by elimination) and S drinks orange juice. This implies J lives in the green house (clue 3). Also S smokes Lucky Strikes (clue 12). Therefore, S must live in the ivory house and U lives in the blue house.

8. Since S lives in the ivory house and owns a dog, and since the horse is in the blue house, E is flanked by a horse and a dog and hence cannot smoke Chesterfields (clue 10). Therefore, E smokes Old Golds and owns a snail (clue 6); U smokes Chesterfields (elimination), and N owns the fox (clue 10). Since U owns the horse, J must have the zebra.

1.22: This is a very lengthy problem and requires reading through the clues several times. Part of the difficulty is that no matter what few charts we select to use, there will be many clues which cannot immediately be entered.

We selected the charts suggested in the hint. After running through the clues once, we easily determine which show is at which theater:

	P	Q	R	S	T
K	x	x	x	√	x
L	x	x	x	x	√
M	x	x	√	x	x
N	√	x	x	x	x
O	x	√	x	x	x

After running through the clues again, substituting show names for the appropriate theater names, we reach the following position (if we are careful to extract as much information as possible from each clue):

	K	L	M	N	O	W	T	F	Sm	Sn
AF	x		x	x						x
AG	x		x			x	x			
AH	x	x	√	x	x					
AI	√	x	x	x	x	x				
AJ	x		x		x	x	x			x
BF	√	x	x	x	x	x	x			x
BG	x	x	√	x	x	x	√	x	x	x
BH	x	x	x			x	x			
BI	x		x			x	x			
BJ	x		x		x	√	x	x	x	x
CF	x	x	x	√	x	x				x
CG	√	x	x	x	x	√	x	x	x	x
CH	x	√	x	x	x	x				
CI	x	x	√	x	x	x				
CJ	x	x	x	x	√	x				
DF	x		x	x						x
DG	x	x	x			x	x			
DH	√	x	x	x	x	x	x			
DI	x		x		x					x
DJ	x	x	√	x	x	x				
EF	x	x	√	x	x	x	x	x	x	√
EG	x		x			x	x			x
EH	x	x	x							x
EI	x		x							x
EJ	√	x	x	x	x	x				x

(For example, in entering clue 1, cross out not only the L, M, T, F, S mat, and S night boxes for CG, but also all the other W boxes for C and for G. In entering clue 2, GD did not see L or M and did not date each other on W or T. Clue 5 implies that C did not see "Ottoman" on W, so he did not see it with G. Clue 7 tells us that F saw K and M on Saturday. Since she was with E on S night and did not see K with him, they must have seen M together, etc. In addition to entering the clues, when four boxes in a category were eliminated we checked the fifth box in that category.) At this point, we have not yet made full use of all of the clues. In particular, note that clue 8 tells us that N had no intermission; so by combining with clue 2, we see that DG did not see N. Also, clue 10 tells us that A saw N with G (since CG saw K). These two additional pieces of information enable us to complete the who saw what with whom part of the chart. Running through the clues one more time, we are able to complete the rest of the chart: From clue 2 and the fact that G saw K on Wednesday, the earliest she could have dated D was on Saturday. From clue 3 and the fact that DG saw O, DH must have dated S night and DG S mat; and, from clue 8, DI saw N on T. The rest of the chart is now easily filled in.

1.23: Solution

a) Considering AB as one car and LCD as the locomotive, switch AB and LCD as in the text. This results in LCDAB. Leaving AB sitting at the right, switch L and CD. This gives the desired result.

b) One way to proceed is to switch A with BLCD; then switch B with LCD; then D with LC and finally C with L.

CHAPTER 2

COMMENTS AND SUGGESTIONS

1. The purpose of this chapter is to help the students become more aware of the precise use of language, and to teach them how to make elementary deductions from compound statements. In particular, the students should be able to determine what conclusions logically follow from given information.

Symbolism is introduced to help with this learning and also because it can be used to get to the heart of complicated verbal problems.

2. The amount of time we spend on this chapter varies from semester to semester. The determining factors include the amount of time we've had to spend on Chapter 1, the percentage of the students who have previously been exposed to formal logic, and the reaction of the class to the symbolic aspects of the material. If we decide to dwell on the material (which we almost certainly would do if we were teaching a two semester course), we devote additional time to truth tables, and to the meanings of important tautologies, and we include a discussion of syllogisms, for which relevant exercises may be found in Chapter 9 (Exercises 9.17-9.23). (This latter topic is useful for students who intend to take the LSAT Examination or similar exams.)

3. Exercises 2.1 and 2.5 present the background for most of the truthful-liar problems, so we always assign these exercises when we start the chapter.

4. Exercises 2.18, 2.20 and 2.21 require lengthy arguments and it is probably best not to assign these to be written up and handed in.

5. Exercise 2.23 can be delayed until Chapter 3, and tied in with Exercises 3.55 and 3.72.

CROSS REFERENCES

1. To help students determine what cases need be considered in analyzing some of the problems and exercises, we often refer back to tree diagrams and the Multiplication Principle which were discussed in Chapter 1.

PRACTICE PROBLEM ANSWERS

2.A:

2. b), d) and e) are statements.

2.B:

2. a) T, b) F, c) T, d) F, e) T.

4. a) $g \rightarrow e$. If you want to grow, then you must eat.

b) $d \to a$. If you are to be permitted to drive, then you must acquire a driver's license.

c) $r \to s$. If a rectangle is a square, then all four sides are the same length.

d) $m \to e$. If a number is a multiple of four, then it is even.

6. a) $r \vee (\sim s)$. The statement is true if Alice and Carol both passed history, if they both failed it, or if Alice passed and Carol failed.

b) $\sim r$. This statement is true if Alice and Carol both failed history or if Carol passed and Alice failed.

c) $r \to s$. This statement is true if Alice and Carol both passed history, or if they both failed, or if Alice failed and Carol passed.

8. a) If Macbeth is the Thane of Cawdor, then Ruth is the Sultan of Swat.

b) If Ruth is the Sultan of Swat, then Macbeth is the Thane of Cawdor.

c) Macbeth is not the Thane of Cawdor.

d) Either Macbeth is the Thane of Cawdor or Ruth is the Sultan of Swat.

e) Macbeth is the Thane of Cawdor if and only if Ruth is the Sultan of Swat.

2.C:

2. a) Carrie is not sleeping.

b) Bonnie is not tired.

c) Bonnie is not tired and Carrie is not sleeping.

2.D:

2. a) $(p \wedge q) \to (r \wedge s)$

b) $(r \wedge s) \vee [(\sim p) \vee (\sim q)]$ or, more simply, $(p \wedge q) \to (r \wedge s)$

c) $(p \to q) \wedge [r \to (\sim s)]$

4. a) Jack and Jill did not go up the hill if and only if Little Bo Peep has not lost her sheep; or, more simply, Jack and Jill went up the hill if and only if Little Bo Peep has lost her sheep.

b) It's not true that either Jack and Jill went up the hill or Little Bo Peep has lost her sheep; or, more simply, Jack and Jill did not go up the hill and Little Bo Peep has not lost her sheep.

c) If it is true that if Little Bo Peep has lost her sheep then Jack and Jill went up the hill, then it is also true that Little Tommy Tittlemouse lived in a little house.

6. a) Either today is not Thursday or the sun is not shining.

b) We can come to no conclusion.

c) Either the sun is not shining or it is not summer.

2.E:

2. a) If you don't get a tan then you don't sit in the sun.

b) If Juanita doesn't know the news of the day, then she doesn't read the newspaper.

c) If an angle is not acute, then its measure is not 30^0.

d) If someone answers the phone, then my wife is home.

4. a) is equivalent to c); b) is equivalent to d).

2.F:

2. a) The premises are

P_1: Either Lucretia is forceful or she is creative.

P_2: Lucretia is not forceful.

The conclusion is

C: Lucretia is creative.

Symbolically,

$$\begin{array}{l} f \vee c \\ \underline{\sim f} \\ \therefore c \end{array}$$

b) The premises are

P_1: If Lucretia is not efficient, then either she is forceful or she will be a good executive.

P_2: Lucretia is not forceful.

P_3: Lucretia will not be a good executive.

The conclusion is

C: Lucretia is efficient.

Symbolically,

$$\begin{array}{l} (\sim e) \rightarrow (f \vee g) \\ \sim f \\ \underline{\sim g} \\ \therefore e \end{array}$$

c) The premises are

P_1: If Bessie is a cow then she moos.

P_2: Bessie moos.

The conclusion is

C: Bessie is a cow.

$$\begin{array}{l} c \rightarrow m \\ \underline{m} \\ \therefore c \end{array}$$

d) The premises are

P_1: If Bessie is a cow, then she moos.

P_2: Bessie doesn't moo.

The conclusion is

C: Bessie is not a cow.

$$\begin{array}{l} c \rightarrow m \\ \underline{\sim m} \\ \therefore \sim c \end{array}$$

e) The premises are

P_1: If Bessie is a cow, then she moos.

P_2: Bessie is not a cow.

The conclusion is

C: Bessie doesn't moo.

$$\begin{array}{l} c \rightarrow m \\ \underline{\sim c} \\ \therefore \sim m \end{array}$$

4. b) and d) are valid; the others are not.

6. a), b) and d) are valid; the others are not.

8. c) is valid; a) and b) are not.

9. If the conclusion $(p \rightarrow r)$ is false, then p must be true and r false. But then, either $p \rightarrow q$ is false (if q is false) or $q \rightarrow r$ is false (if q is true). In other words, it is not possible for the

conclusion to be false if both premises are true. Therefore, the argument is valid.

SOLUTIONS TO EXERCISES

Comment: Natives in each problem will be referred to as N_1, N_2, ..., unless their names are given.

2.1: Clearly N_1 and N_2 belong to different tribes since they disagree about the name of the island. On the other hand, N_2 and N_3 belong to the same tribe, since their final statements agree with each other. Thus N_1 and N_3 belong to different tribes, and so N_3's statement is a lie. That is, N_2 and N_3 are liars while N_1 is a truthteller. The island is therefore Hamlock.

2.2: If N_2 is truthful, then N_1 is a liar (by N_2's statement) and they belong to different tribes. But then N_1's statement is true--a contradiction. Therefore N_2 is a liar, and N_1 is truthful (since N_2's statement is a lie).

2.3: Since all three disagree, at most one is telling the truth and at least two are Liars. It is not possible that all three are Liars, for then N_1 is telling the truth--a contradiction. Thus two are Liars, and N_2 is a Truthful.

2.5: Observe that, for any native making three statements, the first and third statements must have the same truth value.

Since N_1's and N_2's first statements agree with each other, they are either both true or both false. If true, then their third statements are true and so their second statements are both false (N_1's third statement contradicts N_2's second and vice versa). Thus neither is a Truthful. Similarly, if their first statements are false, neither is a Truthful. Thus, in either case N_3 (the tall native) must be the Truthful.

Thus N_1 is Waldar, N_2 is Gaut and N_3 is Lowax. This implies that N_2 is a Liar and N_1 is an Alternator, lying first.

2.6: Since neither a Truthful nor a Liar can claim to be a Liar, the spokesman must be an Alternator, lying first. His second statement is true, so the lady is a Truthful. This leaves the young gentleman as a Liar.

2.7: Since a Truthful cannot claim to be an Alternator, Winken is not a a Truthful. If he were an Alternator, then Blinken and Finken are both lying, so none of the three is a Truthful. This means that two of them belong to the same tribe--a contradiction.

Hence, Winken is a Liar. Since Finken is lying but cannot be a Liar, he must be the Alternator. This leaves Blinken as the Truthful.

2.8: If either were a Truthful, then the other would be a Liar. But this is impossible, since they agree about Cloy.

If Hocus were an Alternator, his first and third statements would have to be false and his second and fourth statements true. Thus Pocus would be a Liar and Cloy would be a Truthful--a contradiction since Pocus says that Cloy is a Truthful.

Thus Hocus is a Liar.

Since Pocus is not a Truthful but his first statement is true, he must be an Alternator (telling truth first). Thus Hocus is a Liar, Pocus and Crocus are Alternators and all we can say about Cloy is that he's not a Truthful.

2.9: One of the three was a Truthful.

If Benge is the Truthful, then the order of finish was C-B-A. But then Ange's and Conge's statements are all false, so there is no Alternator--a contradiction.

Similarly, if Conge is the Truthful, then the order of finish was A-B-C. This makes both A and B Liars.

Thus Ange must be the Truthful and the order of finish was C-A-B. (B is the Alternator and C the Liar.)

2.10: The first robot cannot be a Mendible because the only way a Mendible could claim to be a Mendible is if a flacus coin is used--but then the robot would truthfully identify the coin as being flacus--not grenjo.

Thus the first robot is a Lawbake and the coin is made of telic (since the robot is lying).

Since the second robot does not give the same answer as the first, it must be a Mendible.

2.11: He should insert the coin, point to one of the roads and ask: "If I were to insert a coin made of the same material I just inserted and I were to ask if this is the correct road to Logos, would you say yes?"

If the robot answers "yes," then the captain should take the road to which he is pointing: Otherwise, he should take the other road.

2.12: Since Werner and Virna contradict each other, one of them is lying. Since Werner and Myrna contradict each other, one of them is lying. Since there is only one liar, Werner must be lying and Virna and Myrna are telling the truth. Therefore, Werner is the oldest.

2.13: Since Fagin makes three statements implying his own innocence, and since at least two of these must be true, Fagin didn't do it.

The same argument applies to Tolliver.

Thus Sykes is the culprit.

This implies that Fagin's third statement is false, so his second statement is true. He therefore was not present when the picture was made.

Tolliver's first and third statements are true, so the picture was not on the board when he arrived. He was therefore present when Sykes did his artwork.

2.14: Since one of Queen's statements must be true, Queen didn't win. Similarly, King's statements imply that he didn't win.

If Ace won, then both Jack's statements are false--a contradiction.

Therefore, Jack won.

King's true statement must be that he himself came in second.

Ace's true statement must then be that Queen's house was taller than his own.

The order of finish is thus: Jack, King, Queen, Ace.

2.16: There are six possibilities as to which two of Gerald's statements were true. (Gerald was chosen since his statements leave no doubt as to the order of finish.)

If Gerald's first two statements are true, then the order of finish was HGCL; etc. We obtain the following table:

Gerald's true statements	Order of finish	contradiction
1st & 2nd	HGCL	Leo-no 2 true
1st & 3rd	HCLG	Har-no 2 true
1st & 4th	HLGC	Har-no 2 true
2nd & 3rd	CGLH	Car-none true
2nd & 4th	LGHC	Har-none true
3rd & 4th	GHLC	o.k.

2.17: Consider which of Sir Hilary's statements is true:

Case 1: If the Snowman was over 7' tall, then by also considering Matty

Horne's statements the Snowman does not have long white fur nor brown fur and does not have 6 toes nor 5 toes on each foot. Thus he has no fur at all and hooves. But then Torkay made two true statements--contradiction.

Case 2: If the Snowman had long white fur, then by also considering Monte Everesto's statements, the Snowman is not over 7' tall nor 6' tall and does not have 6 toes on each foot nor hooves. Thus he is under 5' tall and has 5 toes on each foot. This case is possible.

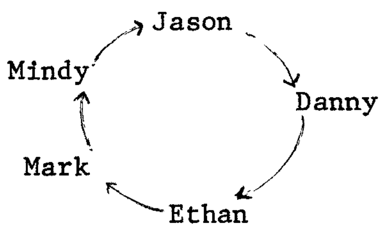
Case 3: If the Snowman has 6 toes on each foot, then, in a manner similar to that above, by considering Manjaro's statement we find that the Snowman must be 6' tall and have no fur. But then Matte Horne made no true statement--contradiction.

Therefore, Case 2 is correct.

2.18: Since rumor 1 implies that the cardiologist and the gastroenterologist are both women, and rumor 2 implies they are of the opposite sex, one of these rumors must be false and rumors 3 and 4 must be true. From rumor 3, the hemotologist must be male (Marvin). Therefore the cardiologist and the gastroenterologist are both women, rumor 2 is false, and rumor 1 is true. Thus Dr. Coates is the endocronologist. Marvin's girlfriend (Dr. Mavis) is not the gastroenterologist (rumor 4), and so must be the cardiologist. This leaves Rosalinde to be the gastroenterologist.

2.19: Note that Jason and Mindy contradict each other, Jason and Mark contradict each other and so do Ethan and Danny. If Jason's statement is true then Mark's, Mindy's and either Ethan's or Danny's statements would be false--leaving only two correct statements. Therefore Jason's is incorrect and Mark and Mindy are telling the truth (if either is incorrect there would again be three false statements).

Since Mark received Ethan's present, Ethan could not have received Mark's present (by the rules of giving) so Ethan is incorrect and Danny is correct. The correct order is



2.21: We can immediately conclude that Abby is not the comic, Linda is not the writer, and neither Janice, Martin, nor Roberto is the singer. Thus the singer is either Abby or

Linda. Since they contradict each other in their statements regarding the singer and the agent, neither can be the agent. Therefore, either Janice, Martin, or Roberto is the agent. Since the statements made by these three about the agent contradict each other, at most one of these statements is true. Therefore neither Janice nor Martin could be the comic (otherwise two of the three statements would have to be true). This leaves Linda and Roberto as possibilities for the comic. However their statements regarding the comic and the dancer contradict each other; thus neither can be the dancer.

At this point we have the following chart:

Occupations

	s	d	c	w	a
A			x		x
J	x		x		
L		x		x	x
M	x		x		
R	x	x			

Since Abby, Janice, and Martin contradict each other about the writer and the dancer no two of them can between them be both the writer and the dancer. Thus either Linda or Roberto must be the writer or the dancer. From the chart, the only possibility is that Roberto is the writer. By elimination the comic is Linda. This leaves the singer to be Abby. Since Abby's statement about the singer and the agent implies that they are of opposite sex, Martin must be the agent, leaving Janice to be the dancer.

2.22: Consider the statements about quarters:

A had one if and only if B did; if C had one, so did A and B; if D had one, so did everyone. Since one girl had only a dime, D could not have had a quarter, and since at least one girl had a quarter, A and B must both have had quarters.

Consider the statements about dimes:

B had one; if A had one, so did D; if C had one, so did everyone. Since one girl had only a quarter, C did not have a dime.

Now consider the statements about nickels:

B had one if and only if C did; if they had one, so did D. But one girl had only a dime and another had only a quarter so that neither B nor C could have had a nickel.

We thus have the following chart:

	q	d	n
A	√		
B	√	√	x
C		x	x
D	x		

The only person left who could have had only a dime is D. Also C must be one who has exactly one quarter. This leaves B with .35 and A with .40 (either one quarter, one dime and one nickel or one quarter and three nickels).

2.23: Since there are the same number of red cards as there are black cards in the deck, the number of red cards in the top half will always equal the number of black cards in the bottom half. (If r is this number, then there are 26-r black cards in the top half, leaving r black cards in the bottom half.)

Since the antecedent of the given statement is always false, the statement is always true.

2.25: Let s, p, d represent the statements

s: the spark plugs are o.k.

p: the points are o.k.

d: the distributor cap is o.k.

and let Q and R be the compound statements.

$Q = (s \wedge p) \rightarrow (\sim d)$

$R = [p \wedge (\sim s)] \rightarrow d$

The mechanic's statement is that either Q or R is true but not both.

Suppose that d were true. Then R would be true and so Q would have to be false. This only happens if s and p are both true. But then all three items are o.k.--contrary to assumption.

Therefore, d must be false. In this case Q is true, and so R must be false. This only happens if p is true and s is false.

Thus the spark plugs and distributor cap need replacement.

2.26: Let S_g, W_g, B_g, S_b, W_b, B_b represent the statements "Stanley can play the guitar," etc.

If Bing cannot play the guitar, then Stanley can (at least one of them can), and Walter can ($[\sim B_g] \rightarrow W_g$). But this contradicts the first statement ($[S_g \wedge W_g] \rightarrow B_g$).

Therefore Bing can play the guitar. This means that Stanley can't (not both of them can). But then, Stanley must play the banjo. This in turn implies that Walter does not play the banjo ($W_b \rightarrow [\sim S_b]$). This leaves Bing as the only one who is able to play both instruments.

2.27: Suppose Nat is wearing a blue suit. Then he is either wearing a blue shirt or white socks. If he is wearing white socks then he is wearing a blue shirt anyway, so in either case he is wearing a blue shirt. Since he is also wearing a blue suit he must be wearing a blue tie--contradiction.

Therefore he is not wearing a blue suit. But then he must be wearing white socks and, hence, a blue shirt, in addition to his gold tie. (If he is also wearing a suit, it must be the brown one.)

2.28: If Archie does not own the Chevy, then the Chevy is green and Archie owns the Chrysler which must be brown. This means that the Ford must be blue and therefore the Chrysler green--contradiction.

Therefore Archie owns the Chevy, and the Chevy is green or brown. This implies that Brian does not own the Ford and therefore owns the Chrysler. If the Chrysler were blue, then Brian's car would be blue and the Chrysler would have to be green--contradiction.

Therefore the Ford must be blue and the Chrysler green. This means that Archie owns the brown Chevy; Brian, the green Chrysler; and Joaquim, the blue Ford.

2.29: Let f, m, p represent the statements

f: the plane is able to fly more than 25 ft. high

m: the plane is model #25

p: the plane requires four penlight batteries

The given facts can be represented as follows:

P_1: $(f \wedge m) \rightarrow p$

P_2: $m \rightarrow [(\sim f) \vee (\sim p)]$

P_3: $f \rightarrow (p \rightarrow m)$

P_4: $m \vee p$

Does the conclusion f follow? (If not, does the conclusion $\sim f$ follow?)

Suppose m is true. Then from P_2 either f is false or p is false. If f is true, then, from P_1, p is true--contradiction. Therefore f must be false.

On the other hand, if m is false, then by P_4, p must be true. But then $p \rightarrow m$ is false so, by P_3, f must be false. Since f is seen to be false in both cases the plane cannot fly higher than 25 ft. and we see that the conclusion $\sim f$ is the valid one.

2.30: Let p, q, and r represent the following statements:

p: Pamela's pease porridge is putrid

q: Pablo painted potted palms

r: Peter picked a peck of pickled peppers

We now have the following statements:

P_1: $q \rightarrow p$

P_2: $q \vee (\sim r)$

P_3: $r \vee \sim[(\sim q) \wedge (\sim p)]$

Does the conclusion p follow? (If not, what about $\sim p$?) Suppose p is false. Then, by P_1, q must be false. By P_2, r is false. But this contradicts P_3. Therefore, p must be true. That is, Pamela's porridge is putrid.

2.31: Let t, m, e, f, and w represent the following statements:

t: Manfred is a telepath

m: Manfred is a marvelous magician

e: electronic devices are used in his act

f: mind reading is a form of magic

w: Manfred is wealthy

We now have

P_1: $t \to m$

P_2: $\sim e \vee (\sim f \to \sim m)$

P_3: $\sim w \to (\sim m \vee e)$

P_4: $\sim f \vee \sim t$

P_5: t

Does the conclusion w follow? (If not, does the conclusion $\sim w$ follow?)

By P_5, t is true. Therefore, by P_4, f is false, and, by P_1, m is true. Therefore, by P_2, e is false. This implies by P_3 that w is true. That is Manfred is wealthy.

2.32: Let b, d, v, r, e, and f represent the following statements:

b: life begins at 80

d: Attila died at 79

v: Attila lived

r: Attila was reincarnated as a snake

e: the emotional development of primates parallels that of reptiles

f: the female viper is deadlier than the male

The given information can be represented as follows:

P_1: $(b \wedge d) \to (\sim v \wedge \sim r)$

P_2: $(\sim b \wedge e) \to (\sim v \vee \sim f)$

P_3: $e \to (\sim v \vee d)$

P_4: $f \to e$

P_5: $\sim f \to \sim (b \vee d)$

P_6: v

By P_6, v is true. Therefore, by P_1, either b or d is false.

If d is false, then, by P_3, e is false; and by P_4, f is false. But then, by P_5, b is also false. Therefore b is false (regardless of whether d is or not).

If f is true, then e is true by P_4. But then P_2 leads to a contradiction. Hence f is false.

But then, by P_5, d is false.

Finally, if e were true, then P_3 would not be true. Therefore e is false.

Note that nothing may be concluded concerning r.

CHAPTER 3

COMMENTS AND SUGGESTIONS

1. The purpose of this chapter is to help students learn how to translate word problems into algebraic equations or inequalities. (Not only will it be necessary for the students to be able to do so in order to handle some of the exercises in Chapter 4, but, more importantly, the ability to translate verbal problems into symbols is required for problem solving in all fields. In particular, problems of the type considered in this chapter frequently appear in exams such as the GRE's or the LSAT's.)

Although solving verbal problems algebraically involves two procedures--setting up the relevant equations or inequalities and then solving them, we devote most of our attention to the former. Our discussion of algebraic techniques of solving an equation or system of equations is usually limited to those needed to solve the problems which we happen to go over in class. We do, of course, point out to the class that Appendix A deals with techniques of solving algebraic equations, and we answer any questions the student may have.

If we were to teach a two semester course, or if our course were required of all students, then we would devote more time to discussing algebraic techniques and to the exercises of the chapter than we now do.

2. If you intend to assign Exercise 4.30 when you discuss Chapter 4, then it is interesting to contrast it with Exercise 3.4, and so the latter should be assigned.

3. Exercises 3.7 and 3.54 are helpful in understanding the set up of Sample Problem 4.2, so we recommend including at least one of them in the exercises you assign and discuss.

4. Exercises 3.10 and 3.11 have been included since they are typical of problems which often appear on aptitude tests.

5. Exercises 3.55 and 3.72 can be related to Exercise 2.23.

6. Exercises 3.62-3.68 have been included because many students find it interesting to discover that problems of this type were considered centuries ago. These may be assigned almost immediately.

7. The tricks found in Exercises 3.69-3.79 constitute our and our students' favorite segment of this chapter. We usually spend an

entire class period performing some of the tricks, without explanation. The class is then sent home to figure out how they work. The following class is then spent discussing the mathematics of the tricks.

CROSS-REFERENCES

1. Being able to set up problems algebraically is helpful in attacking some of the problems and exercises of Chapter 4.

PRACTICE PROBLEM ANSWERS

3.A:

2. a) s; m; $s + m$
   b) s; m; $(s + 5) - (m + 5)$
   c) s; m; $\frac{1}{2}(s-2)$
   d) a; g; $\frac{1}{a} + \frac{1}{g}$

3.B:

2. a) $n^3 - n = 10n-6$
   b) $(m-2) = \frac{1}{2}(s-2)$
   c) $6a + 5p > 7p + 5a$
   d) $\frac{1}{2}s + \frac{1}{3}h = \frac{1}{2}h - \frac{1}{2}s$

SOLUTIONS TO EXERCISES

3.1: Let v be the total number of votes cast.

$$\frac{2}{5}v + \frac{5}{12}v + 33 = v$$

$v = 180$.

3.3: Let n be the amount still needed.

$$\frac{1}{3}(70000-n) = \frac{3}{5}n$$

$n = \$25,000$.

3.4: We work backward. (Note that if there are C candies, and if $1 + \frac{C-1}{3}$ are taken, then $\frac{2(C-1)}{3}$ remain. Thus, the number taken is one more than half the number remaining.)

Let L be the number of candies Leonard received.

Ethel received $\frac{L}{2} + 1$

Mildred received

$$\frac{1}{2}(\frac{L}{2} + 1 + L) + 1 = \frac{3L + 6}{4}$$

Samuel received

$$\frac{1}{2}(\frac{3L + 6}{4} + \frac{L}{2} + 1 + L) + 1 = \frac{9L + 18}{8}$$

$$\frac{3L + 6}{4} + \frac{L}{2} + 1 = \frac{9L + 18}{8} + 7$$

$L = 54$. (There were 187 candies in the bag.)

3.6: Observe the similarity with Exercise 3.5.

Each person's share is four vans. Thus Stan is giving Ann three vans and Dan is giving Ann one. Stan should therefore receive three times as much money as Dan.

$3x + x = 120,000$

Stan receives \$90,000; Dan, \$30,000.

3.7: Let s represent the number of sacks of wheat seed he bought.

$s^2 + (3s)^2 + (6s)^2 = 184.$

$s = 2.$

He bought 20 sacks.

3.8: Let w represent the width of the first field.

$(w + 700)w = (w + 250)400$

$w = 200.$

The fields are 900 yd x 200 yd and 450 yd x 400 yd.

3.9: Let t represent the number of white tiles on one side of the small square.

$t^2 + (t + 8)^2 = 1000;$

$r = 2t(t + 8)$

936 red tiles are required.

3.11: $x = 2 + \frac{15}{x};$

$x = 5.$

3.12: $t + w = 2.20; \quad t - w = 1.20.$

The towel costs \$1.70; the washcloth, \$.50.

3.13: Let a represent the number of animals and let r represent the number of riders.

$4a + 2r = 4248; \quad a + r = 1078$

$a = 1046; \quad r = 32.$

He collected \$109.40.

3.14: Let n represent the number of 3 cent screws Calvin bought.

Let y represent the total number of cents he spent.

$3n + 4n + 5n = y$

$\frac{y}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = 3n + 6$

$n = 45.$

He spent \$5.40.

3.16: $A + B = 180; B + C = 430;$ $C + D = 400; D + E = 380;$ $E + F = 310; A + E = 280.$

Ada was fined \$.25; Brendan, \$1.55; Corinne, \$2.75; Darryl, \$1.25; Eva, \$2.55; and Floyd, \$.55.

3.17: $M + M_w = P + P_w = D + D_w = 51;$ $M + P + D_w = 43; \quad M + D + P_w = 93;$ $P + D + M_w = 81.$

Minski spent \$17; his wife, \$34. Pinski spent \$11; his wife, \$40. Dubinski spent \$36; his wife, \$15.

3.18: Let the value (in bonors) of each note be represented by the first initial of the monarch whose picture appears on that note.

$B + C = 102; G + W + R = 73;$

$H + B = 22; H + C = 120;$

$H + G + R = 43; C + W + R = 168.$

$B = 2, G = 5, C = 100, H = 20,$

$R = 18, W = 50.$

Waldo's picture appears on the fifty bonor note.

3.19: Let g be the value of one glomek (in morms), and let G be the number of glomeks I handed the teller. Define n and N analogously.

$4g = 7n$; $g + n = 33$; $2gG + nN = 120$; $gG + 2nN = 114$.

$n = 12$, $g = 21$, $G = 2$ and $N = 3$.

He gave me 78 morms.

As a second, shorter, approach, which we only noticed as an afterthought, if g and n respectively denote the value in morms of the number of glomeks, nindars, which I handed the teller, then

$2g + n = 120$; $g + 2n = 114$; so

$3g + 3n = 234$, and $g + n = 78$.

3.21: Let n be the number of people that attended.

$9n = 8(n + 8)$

$n = 64$.

64 people attended.

3.22: Let w represent the number of wagons and n the number of people per wagon that left Fort McConnell.

$$nw - 4 = (w - 8)(n + 2);$$

$$nw - 11 = (w - 11)(n + 3).$$

$$n = 4, \ w = 22.$$

Therefore (in addition to the baby), 76 people who left Fort McConnell arrived in California.

3.23: $a = 2(a + 4) - 2(a - 4)$.

$a = 16$.

That is, I am 16 years old.

3.24: $(a - 2) = 3[\frac{1}{2}(a + 6) - 8]$

$a = 26$.

Thus, I am 26 years old.

3.26:

	Present	Past	Future
Mutt	m	$\frac{1}{2}j$	2x
Jeff	j	x	2m

$$m - j = \frac{j}{2} - x = 2x - 2m$$

$$(m+5) + (j+5) = 100$$

Mutt is 40; Jeff is 50. ($x = 35$.)

3.27:

	t_1	t_2	t_3
Erica	2	y	e
Leroy	4x	6y	2m
Miriam	x	2y	m

$$3x = 4y = m$$

$$x - 2 = y = m - e$$

$x = 8$, $y = 6$, and $m = 24$, and Erica was 18.

3.28:

	Present	t_1	t_2
butcher	x	$\frac{1}{2}x$	$\frac{1}{2}z$
baker	y	u	$\frac{1}{2}x$
can.mak.	z	2u	

$$x - y = \frac{x}{2} - u = \frac{z}{2} - \frac{x}{2}$$

$$z - y = u$$

$$y - 12 = \frac{1}{2}(x + 12)$$

The butcher is 48; the baker, 42; and the candlestick maker, 60.

($u = 18$.)

3.29: $n = 3j$;

$i+3 = (n+3) + (s+3) + (j+3)$;

$i+5 = 3(s+5)$;

$i+x = 2(n+x)$;

$s+x = \frac{3}{2}(j+x)$

Izzy is 52 months; Norma, 24; Susie, 14; and John, 8. ($x = 4$.)

3.30: $L = B+2$; $LB = 575$.
Brenda is 23, and Lee is 25.

3.31: The hare can't win; just as he is finishing the first half of the course, the tortoise is crossing the finish line.

3.33: $\frac{d}{5}t + \frac{d}{7}t = d$
The trains meet at 10:55 A.M.
($t = \frac{35}{12}$)

3.34: $\frac{d_1}{r_1} = \frac{d_2}{r_2}$; $d_2 = d_1 + 24$
$\frac{d_1}{r_2} = 9$; $\frac{d_2}{r_1} = 16$
(d_1 = distance to Deadwood);
Deadwood is 168 miles from Tombstone.

3.35: The train from Miami takes 15 hours for its trip and 9 hours to the meeting point. This point is therefore $\frac{3}{5}$ of the way from Miami to Washington, and $\frac{2}{5}$ of the way from Washington to Miami. The trip for the train from Washington will therefore take $\frac{5}{2}\cdot 9 = 22\frac{1}{2}$ hours. Therefore, the train from Washington will arrive in Miami at 6:30 A.M.

3.36: In an hour, the boats travel a total of sixteen nautical miles. Hence they are sixteen nautical miles apart an hour before they meet.

3.37: The trains are closing at the rate of 130 miles per hour. Therefore it will take half an hour until they meet. In half an hour the fly flies 55 miles.

3.39: Let n represent the number of chairs on their way up the slope at the time she starts (including the one at the top). Let 10t be the number of seconds she was skiing.
$n + t = 97$; $n - t = 61$
$n = 79$, $t = 18$.
Her average speed was 40 miles per hour.

3.40: $j + c = \frac{5}{7}$; $j - c = \frac{5}{9}$.
He could go $\frac{40}{63}$ miles per minute, or 5 miles in $7\frac{7}{8}$ minutes.

3.41: Since the current has the same effect on the driftwood that it does on the boat, the rate at which the distance between the two is changing depends only on the rate of rowing. Since Rosalind rows away from the wood for half an hour, it must take her half an hour to catch up with it again. Thus the drift-

wood had been floating for an hour when it was retrieved.
Therefore, the current was moving at 2 miles per hour.

3.43: Let n be the number of steps visible on a motionless escalator.

Let r be the number of steps which appear (disappear) each second.

$n - 18r = 18; \quad n + 30r = 90$

At any time, 45 steps are visible. ($r = 3/2$.)

3.45: Define n, m, R_F and R_{EE} as suggested in the hint.

$$\frac{n}{R_F} + 2 = \frac{n-1}{R_{EE}}\ ; \quad \frac{m}{R_F} = \frac{10}{R_{EE}}\ ;$$

$$\frac{m}{R_F} + 2 = \frac{m}{R_{EE}}\ ;$$

$$\frac{n-m}{R_{EE}} = \frac{n-m}{R_F} + 2 - 1$$

There are 18 stations between Continental Avenue and 34th Street; Harriet's trip will take 16 minutes.
($R_{EE} = 1$; $R_F = \frac{6}{5}$; $m = 12$.)

3.46: The only time saved was the time it would have taken the wife to drive from the meeting point to the station (3 miles) and back to the meeting point. Thus the wife drives 6 miles in 12 minutes, or 3 miles in 6 minutes. The Muters therefore met at 5:54.
Therefore, Comm. Muter walks 3 miles in 54 minutes or 3 1/3 miles per hour.

3.47: $\frac{1}{8n} = \frac{1}{6(n+3)}$

$n = 9$.

Therefore, it would take one person working alone 72 hours.

3.48:

$$\frac{1}{h} + \frac{1}{w} = \frac{1}{36}$$
$$\frac{1}{a} + \frac{1}{w} = \frac{1}{60}$$
$$\frac{1}{a} + \frac{1}{h} = \frac{1}{30}$$

a) It would take all three together $25\frac{5}{7}$ minutes

b) It would take Hank alone 45 minutes.

3.49: $2\left(\frac{24-m}{24}\right) = \frac{28-m}{28}$

Therefore, the water in the circular trough will be twice as deep as the water in the rectangular trough after 21 minutes.

3.50: One chicken lays one egg every $1\frac{1}{2}$ days, or $\frac{2}{3}$ eggs every day.
Seven chickens lay $32\frac{2}{3}$ eggs in seven days.

3.51: $\frac{1}{P_g} + \frac{1}{W_g} = \frac{1}{3}$; $\frac{1}{W_g} = \frac{1}{12}$;

$\frac{1}{P_o} + \frac{1}{W_o} = \frac{1}{2}$; $\frac{1}{P_o} = \frac{1}{10}$.

It takes Pepper 2½ hours to finish the orchard. It then takes the two of them together $1\frac{1}{8}$ hours to finish the garden. They therefore save one hour, 22½ minutes.

3.52: $\frac{1}{C} + \frac{1}{H} + \frac{1}{S} - \frac{1}{G} = \frac{1}{20}$

$\frac{1}{C} + \frac{1}{H} - \frac{1}{G} = \frac{1}{25}$

$\frac{1}{C} + \frac{1}{S} - \frac{1}{G} = \frac{3}{100}$

$\frac{1}{H} + \frac{1}{S} - \frac{1}{G} = \frac{1}{50}$

The grass existing and growing in the field would sustain the cow for 50 days, the horse for 100 days, and the sheep forever.

3.53: $m_\ell + f_d + m_d + f_\ell = 15$ (16);

$m_\ell + f_\ell < f_d + m_d$

$m_\ell < m_d < f_d < f_\ell$;

Since $m_\ell + f_\ell \leq 7$, $f_\ell \leq 6$.

If $f_\ell \leq 5$, then it is not possible for $f_d + m_d$ to be eight or more. Therefore $f_\ell = 6$, and $m_\ell = 1$ (regardless of whether or not I am included).

Without me, $f_d + m_d = 8$, with $1 < m_d < f_d < 6$; so $f_d = 5$ and $m_d = 3$.

With me, $f_d + m_d = 9$ and $1 < m_d < f_d < 6$; so $f_d = 5$ and $m_d = 4$, and I am a male doctor.

3.54: $P^2 - S^2 = 39$; $3S = P + 7$.

Leroy bought eight pairs of slacks (P) and five shirts (S).

3.56: a) $96 = .99(100-c)$

b) $99 = .96(100+m)$

a) 3 1/33 gallons of cream must be removed.

b) 103 1/8 gallons are needed.

3.57: $940+80p+700 = 130(8+p)$

She should buy 12 lb of peanuts.

3.58: $w + w-20 + w-40 + w-60 = 1160$

Therefore, Whelan weighed 320; Wally, 300; Bubba, 280; and Bobby, 260.

Together, the women weigh 620. Belinda's husband's weight is $\frac{3}{2}$ times Belinda's weight, so his weight is divisible by 3. Therefore, Belinda is married to Wally, and weighs 200 lb. The other two wives weigh 420. The only combination which works is Beulah (married to Bobby) weighs 260, and Barbra (married to Whelan) weighs 160.

3.59: $w + s = 3c$ (w = tweezer)

$s = 3t+c$

$3w = c+t$

Therefore,

$$3t+c = s = 3c-w = 3c-\frac{1}{3}(c+t)$$

$$10t=5c$$

$$2t=c$$

so $s=3t+c=5t$.

Five toothbrushes balance a bar of soap.

3.60: $p > j > h > c > b$ (h = Chuck)

$b + c = 7$

$j + p = 24$

$4b + 4c + 4h + 4j + 4p =$

$$7+10+11+14+15+16+17+18+20+24$$

$b + h = 10$

$p + h = 20$

Bruce spent \$3; Cookie, \$4; Chuck, \$7; Jeff, \$11; and Pearl, \$13.

3.61: $(T + B)L = 25R$

$$24L = TR$$

$$12L = BR$$

giving --

a) The ratio of the length of the left arm to that of the right is 5 to 6;

b) The tomatoes weigh 20 oz.

c) The bag of beans weighs 10 oz.

3.62: $x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 33$; *
$x = \frac{1386}{97}$.

3.63: $r = \frac{1}{2}$; $\ell = \frac{1}{3}$; $f = \frac{1}{4}$; $m = 2$;
$\frac{12}{37}$ twelve hour days $= 3\frac{33}{37}$ hours.

3.64: $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x$
He was 84 years old.

3.65: $9c + 7w = 107$; $7c + 9w = 101$;
A citron costs 8 and a wood-apple 5.

3.66: $\frac{h}{4} + 2\sqrt{h} + 15 = h$

36 camels were in the herd.

3.67: $4[3(2x-30) - 54] - 72 = 48$
He started with \$29.

3.68: $m = 7b + 24$; $m + 32 = 9b$
There are 28 beggars, and I have 220 cents.

3.69: $5\{2[5(2n + 8) - 3] - 31\}=915$
$100n + 215 = 915$
The number is 7.

3.70: Let x = the face value of the card and s = the suit value.
$5(2x+3) + s = 79$; $10x + s = 64$
The card is the six of spades.

3.71: $(x-y)(x+y) = x^2-y^2= 39$
$x^2= 39+y^2$
39 + 1 is not a square; 39 + 4 is not a square; etc.
$y = 5$, $x = 8$.
Note: It might be worthwhile returning to this problem after solving Sample Problem 4.2.

* An alternative interpretation (counting mass twice) yields $x = \frac{1386}{139}$

3.72: Before shuffling: the number of cards face up is n; the number of cards face down is irrelevant.
After shuffling: n are face up; the number face down is irrelevant.
After removing the packet of n cards:

in the packet: the number of cards face up is f; face down, n-f;

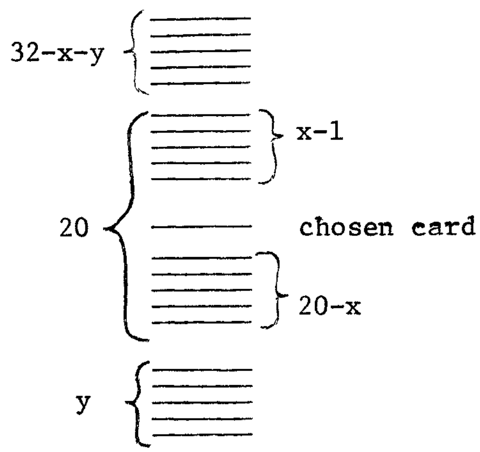
in the rest of the deck: the number of cards face up is n-f; face down, irrelevant.

The magician turned the entire packet upside down behind his back. Note: It might be worthwhile to point out the connection with Exercise 3.55.

3.74: If the three cards turned face up have values V_1, V_2, and V_3, then the number of cards on the table when the counts are completed to ten will be $(11-V_1) + (11-V_2) + (11-V_3) = 33 - (V_1+V_2+V_3)$. When $V_1+V_2+V_3$ are dealt on the table, the total number of cards on the table will be 33, regardless of what three cards have been turned face up. Thus, the selected card must be the (52-33) = 19th card from the bottom of the deck. To insure this, when the magician divides his half of the deck, he makes sure that one part contains seven cards. The subject's card is then placed on top of the other part (which will contain 18 cards).

3.75: By an analysis similar to that of the previous trick, the procedure described will result in there being 44 cards dealt onto the table, with the "predicted" card being the 45th card of the deck. But, considering the seven cards which were added to the bottom of the deck, the 45th card of this deck was the bottom card of the deck prior to the start of the trick. The magician therefore peeked at the bottom card prior to the start of the trick and predicted it.

3.77: Let x be the number of cards in the selected packet, and let y be the chosen number. Then, when the deck is finally placed on the table, the situation is as shown in the following figure:



Note that the chosen card is the

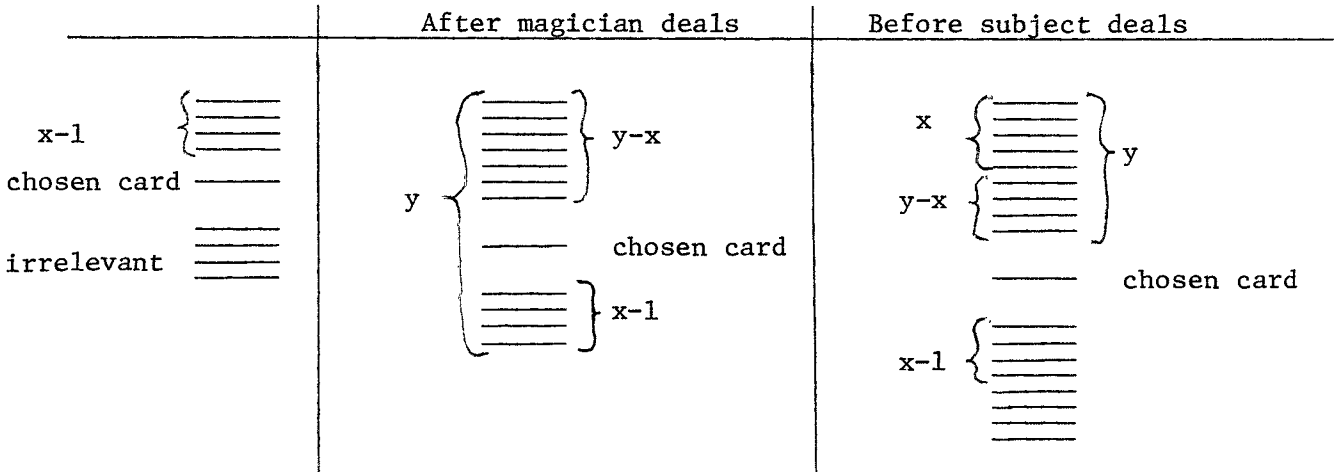


Figure 3.1

32nd - yth card from the top of the deck (regardless of the value of x). Since y is known, it is a simple matter to count to the chosen card.

3.78: Let x be the number of cards removed, and let y be the selected number. The situation at various times during the trick is shown in Figure 3.1 at the top of the page.

3.79: Let n be the first number selected, and let m be the second number, m > n. The situation at various times during the trick is shown in Figure 3.2 at the bottom of the page.
The four of diamonds was originally the top card of the deck.

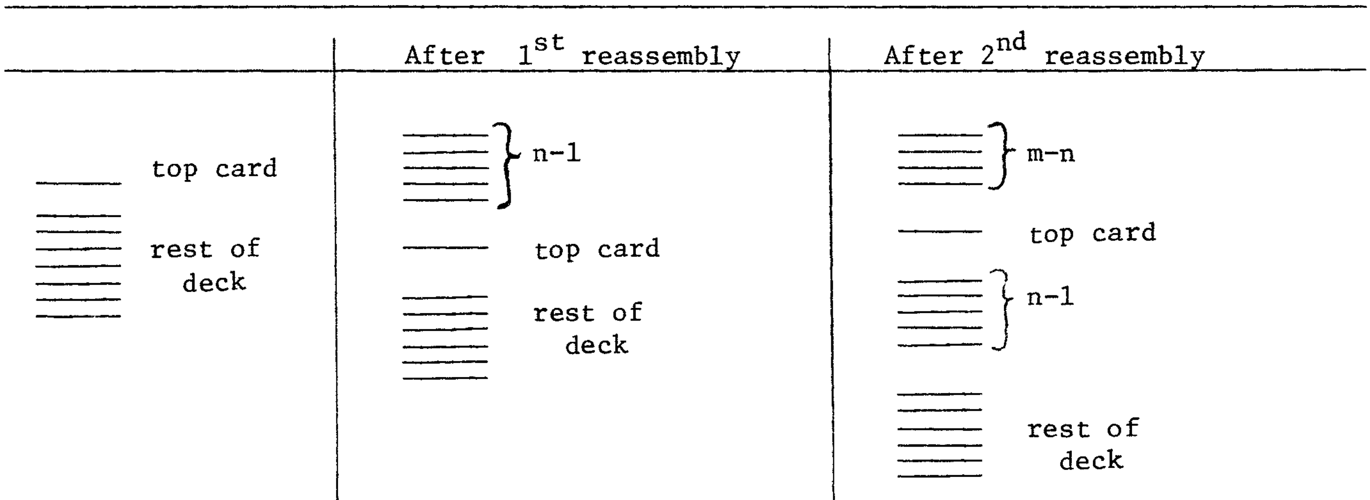


Figure 3.2

CHAPTER 4

COMMENTS AND SUGGESTIONS

1. The purpose of this chapter is several fold. On the one hand, the techniques of elementary number theory are both interesting and useful, and the concepts discussed are important to a full understanding of our number system.

A second consideration is that we wish to introduce our students to some of the mathematical recreations for which a knowledge of elementary number theory is needed. (Our favorites in this chapter include the census taker and the calendar problems-Exercises 4.12 and 4.36 respectively. We also think it important that students see that one equation in two unknowns may suffice, under certain circumstances, to determine the values of these unknowns.) In addition, congruence is a useful concept in attacking some of the exercises in Chapter 5 and in analyzing some of the games in Chapter 7.

Finally, we view this chapter as an opportunity to introduce the student to the nature of mathematics. The notions of definition and theorem are ones which every student should encounter at some time in his or her life.

2. In general, we do not dwell (in this chapter) on the proofs of theorems, although we usually do prove some of the first ones which follow almost immediately from the definition of divisibility. In a two semester course, we would devote time to proving some of the others.

3. We have found that our method of solving linear Diophantine equations is easier for the students to handle than is the arithmetic of the Euclidean algorithm. With our approach, however, it is advisable to postpone assignment of the exercises on linear Diophantine equations (4.14-4.20) until after the class has learned to solve linear congruences.

If we were to teach a full year course, we would probably also discuss the Euclidean algorithm and related topics.

4. In a two semester course, we would supplement our text with a discussion of the Chinese Remainder Theorem. Suitable recreational problems may be found in many of the puzzle books, such as those of Loyd or Dudeney, or may easily be made up.

5. We have found that it is especially important in this chapter to devote time to the practice problem sets, to help the students master the basic mathematical principles which many are seeing here for the first time.

6. We use Sample Problems 4.1 and 4.2 to point out that the algebraic techniques of Chapter 3 are not adequate here and that other techniques are needed. Although students are able to find solutions to Sample Problem 4.1, the question "How do we know that we have all solutions?" leads naturally to a discussion of divisibility and a development of the related material.

7. By the time we get to this chapter, we have usually already discussed the material from Chapter 7 dealing the definitions of winning, losing and drawing positions in games. We now present the material on working backward as a technique for analyzing games, and we apply the language of congruence to the analysis of some of the matchstick games in Chapter 7.

8. We usually include the material on digital roots, and then devote a class to related tricks (Exercises 4.24-4.28). Occasionally, we delay these tricks until we are ready to also discuss those tricks found in Chapter 5 (Exercises 5.18-5.24).

9. The solution of Exercise 4.9 is closely related to that of Sample Problem 4.2 in the text. (You might also relate Exercise 3.71.)

10. It is interesting to contrast Exercise 4.30 with Exercise 3.4, and also to contrast Exercise 4.11 with Exercise 3.30.

11. If Exercises 4.32 and 4.33 are to be assigned, Problem 5 of Practice Problem Set 4.M should be considered first.

12. Exercise 4.36 is an interesting problem, but some of parts c-h are time consuming. As a result, we usually do not solve all parts of the problem, but we nevertheless find it worthwhile to comment on the results.

13. The number theoretic approach to the solution of the colored cubes problem (Chapter 8) may be presented in this chapter, although we prefer to wait until we consider Chapter 8 later in the semester.

CROSS-REFERENCES

1. The ability to translate word problems into equations is needed for some of the problems and exercises in this chapter. This topic is presented in Chapter 3.

2. To help the students understand the algebraic set-up of Sample Problem 4.2, we usually refer back to Exercises 3.7 and 3.54 which have been considered previously.

3. There is a natural tie-in between congruence and some of the matchstick games of Chapter 7. (For example, Exercises 7.1-7.4.)

4. The results on divisibility and congruences are helpful in attacking some of the exercises in Chapter 5. (These exercises are denoted by #.)

5. In the discussion of possible divisors of a number, there is a brief reference to tree diagrams and the Multiplication Principle, which are discussed in Chapter 1.

PRACTICE PROBLEM ANSWERS

4.A:

2. Since $a|b$, there exists an integer d such that $ad = b$. But then, $bc = adc = a(dc)$, and so $a|bc$ since dc is an integer.

4.B:

1. 2,3,5,7,11,13,17,19,23,29,31,37,41, 43,47,53,59,61,67,71,73,79,83,89,97, 101,103,107,109,113,127,131,137,139, 149,151,157,163,167,173,179,181,191, 193,197,199.

4.C:

2. a) $3816 = 2^3 \cdot 3^2 \cdot 53$;
   b) $353 = 353$;
   c) $64680 = 2^3 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11$;
   d) $6006 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13$;
   e) $44296 = 2^3 \cdot 7^2 \cdot 113$.

4.D:

2. a) $784 = 2^4 \cdot 7^2$ is a perfect square but not a perfect cube.
   b) $353 = 353$ is neither a square nor a cube.
   c) $8000 = 2^6 \cdot 5^3$ is a perfect cube not a perfect square.
4. a) $\sqrt{784} = 2^2 \cdot 7 = 28$;
   b) $\sqrt[3]{8000} = 2^2 \cdot 5 = 20$.
6. $2^5 = 32$.

4.E:

2. $44296 = 2^3 \cdot 7^2 \cdot 113$ has 24 divisors.
4. a) 1,2,3,4,6,8,12,16,24,32,48,96.
   b) 1,3,5,7,9,15,21,35,45,63,105, 315.
   c) 1,2,3,4,6,8,12,24,43,86,129, 172,258,344,516,1032.
   d) 1,199.

4.F:

2. No. $4x + 6y$ is divisible by 2, but 7 is not.

4.G:

2. a) i) 17; ii) 1; iii) 2; iv) 1.

b) 12 and 35 are relatively prime; so are 113 and 197.

4. a) 1; b) $5^2 = 25$; c) 72.

4.H:

2. quotient = 18; remainder = 3.
4. $q = 8$; $r = 3$.
6. $q = -6$; $r = 7$.

4.I:

2. $\overline{0} = \{\ldots,-10,-5,0,5,10,15,\ldots\}$
   $\overline{1} = \{\ldots,-9,-4,1,6,11,16,\ldots\}$
   $\overline{2} = \{\ldots,-8,-3,2,7,12,17,\ldots\}$
   $\overline{3} = \{\ldots,-7,-2,3,8,13,18,\ldots\}$
   $\overline{4} = \{\ldots,-6,-1,4,9,14,19,\ldots\}$

4.J:

2. a), c), d) and e) are true.
4. 3.
6. 56.
8. $x = 5k + 3$.
9. The proof in one direction is given in the text. In the other direction, if $d \mid (b-a)$, then $b-a = kd$ and so $b = kd + a$ for some k. Suppose that a has remainder r when divided by d. Then $a = qd + r$, and so $b = kd + qd + r = (k + q)d + r$, so b leaves remainder r when divided by d. Therefore, $b \equiv a \pmod{d}$.

4.K:

2. $x + y \equiv 1 \pmod{7}$;
   $xy \equiv -6 \equiv 1 \pmod{7}$.
4. a) $4x \equiv 0 \pmod{7}$;
   b) $-x \equiv 1 \pmod{3}$.
6. a) $3x + y \equiv 0 \pmod{13}$;
   b) $11x \equiv 4 \pmod{13}$.
8. 1.

4.L:

2. a) 4; b) 9; c) 9.

4.M:

2. 3.
4. The digital roots of the three subproducts are 4, 1, and 5, whereas the main product has digital root 3, so there must be a mistake in the addition.
6. 1; 1; 3; 1; 3; 7; 4.

4.N:

2. a) $x \equiv 9 \pmod{14}$;
   b) $x \equiv 21 \pmod{23}$;
   c) $x \equiv 6 \pmod{15}$.

4.O:

2. a) 4; b) 1; c) 3; d) 3.
4. a) $x \equiv 9 \pmod{12}$;
   b) $x \equiv 3 \pmod{4}$;
   c) no solutions.

4.P:

2. a) $x \equiv 13 \pmod{16}$;
   b) $x \equiv 4 \pmod{11}$;
   c) $x \equiv 11 \pmod{41}$.

4.Q:

2. $x = 8 + 19k$, $y = 7 + 17k$.

4. $x = 5 + 23k$, $y = -9 - 44k$.

SOLUTIONS TO EXERCISES

4.1: Let x = the number of girls
y = the price per coke in cents.
then $xy = 187 = 11 \cdot 17$.
Therefore $x = 11$, $y = 17$ cents.

4.2: Let x = the number of people
y = the fare in cents.
Then $xy = 2183 = 37 \cdot 59$. Since 59¢ cannot be paid with 5 coins, $x = 59$, $y = 37$. Since $37 = 1+1+5+5+25$, the driver received $59 \cdot 2 = 118$ pennies.

4.3: $24{,}949{,}501 = 499 \cdot 49{,}999$. Since no prime less than 499 divided 24,949,501, no such prime divides 49,999. But $499 > \sqrt{49999}$. Therefore 49,999 is a prime.

4.4: Let b = the amount of the budget in cents. Then b is a square, a cube, and a fifth power. Therefore b is a thirtieth power. The smallest such number greater than 1 is $b = 2^{2 \cdot 3 \cdot 5} = 2^{30} =$ 1073741824.

4.5: Let g = the number of guests. $g = 2x^2 = 3y^3 = 2^a 3^b$ (any other prime factor just results in a larger number).
$\frac{g}{2} = 2^{a-1} \cdot 3^b$, so a is odd and b is even.
$\frac{g}{3} = 2^a \cdot 3^{b-1}$, so a and b-1 are divisible by 3.
Since a and b are to be as small as possible, $a = 3$ and $b = 4$. Thus $g = 2^3 \cdot 3^4 = 648$.

4.6: The number of tens which divide 50! is determined by the number of fives which divide the integers from 1 to 50. (There are clearly at least as many 2 factors as there are 5 factors.) There are ten integers from 1 to 50 which are divisible by 5 and two integers from 1 to 50 which are divisible by 25. Therefore 5^{12} divides 50! and therefore there are 12 zeros at the end of 50!.
(Note, for any prime p, the power of p which divides n! is given by $[\frac{n}{p}] + [\frac{n}{p^2}] + [\frac{n}{p^3}] + \ldots$, where [ ] denotes the greatest integer function.)

4.8: The nth light is switched a number of times equal to the number of divisors of n. Since perfect squares have an odd number of divisors and non-squares have an even number of divisors, the lights numbered 1, 4, 9, 16, ..., k^2, ... will be on; the others will be off.

4.9: Let X = the number of chickens owned by the person with initial X.

X^2 = the number of eggs laid by X's chickens.

X_s = the number of chickens owned by the sister of X.

Then: 1) $Q = 3 \cdot Q_s$

2) $Q = 8 + R_s$

3) $Q^2 = 56 + P^2$

4) $R^2 = 52 + S^2$

5) $P^2 = S^2 + T^2$.

By (3), $Q^2 - P^2 = 56 = 2^3 \cdot 7$

$(Q - P)(Q + P) = 1 \cdot 56$

$= 2 \cdot 28 = 4 \cdot 14 = 7 \cdot 8$

$Q - P = 2$	$Q - P = 4$
$Q + P = 28$	$Q + P = 14$
$Q = 15,\ P = 13$	$Q = 9,\ P = 5$

(other cases do not give integer solutions).

By (4), $R^2 - S^2 = 52$

$(R - S)(R + S) = 1 \cdot 52 = 2 \cdot 26 = 4 \cdot 13$

$R - S = 2$

$R + S = 26$

$R = 14,\ S = 12$

(other cases do not give integer solutions).

By (5), $P^2 = S^2 + T^2$

$169 = 144 + T^2$

($25 = 144 + T^2$ does not work)

Therefore $P = 13$, $Q = 15$, $S = 12$, $R = 14$, $Q_s = 5$, $R_s = 7$, $T = 5$.

Therefore, T is Q's sister, V is R's sister, and S is P's sister.

4.11: Let x, y, z be the integers.

1) $x + y + z = 25$

2) $xyz = 540 = 2^2 \cdot 3^3 \cdot 5$

1) implies that $x, y, z \leq 25$;

2) implies that exactly one of the integers is divisible by 5.

Let x be this integer.

x = 5,10,15, or 20.

case 1	case 2
$x = 5$	$x = 10$
$y + z = 20$	$y + z = 15$
$yz = 108$	$yz = 54$

case 3	case 4
$x = 15$	$x = 20$
$y + z = 10$	$y + z = 5$
$yz = 36$	$yz = 27$

Cases 1,3 and 4 are easily eliminated by considering the product restriction. (E.g., in case 4, $yz = 1 \cdot 27$ or $3 \cdot 9$; neither case gives the correct sum.)

This leaves case 2:

$yz = 1 \cdot 54 = 2 \cdot 27 = 3 \cdot 18 = 6 \cdot 9$;

only $6 + 9 = 15$. Hence the three numbers are 6, 9, and 10.

4.12: Let x,y,z be the ages of the three children ($x \leq y \leq z$).

n = the number on the door (known to the census taker).

$x + y + z = n$

$xyz = 36 = 2^2 \cdot 3^2$

Possibilities for x,y,z, x + y + z are

x	y	z	x+y+z = n
1	1	36	38
1	2	18	21
1	3	12	16
1	4	9	14
1	6	6	13
2	2	9	13
2	3	6	11
3	3	4	10

Since the census taker knows n, he would know the ages in any case except if n = 13 (i.e., 1,6,6 and 2, 2,9). Since he had to ask an additional question, n = 13; and, since the youngest is not a twin, the ages are 1, 6 and 6.

4.13: Let n = the desired number and let r = the remainder.

$$\begin{aligned} \text{Then } 887 &= nq_1 + r \\ 959 &= nq_2 + r \\ 1007 &= nq_3 + r \\ 1187 &= nq_4 + r\,. \end{aligned}$$

Taking differences to eliminate r,

$$\begin{aligned} 72 &= n(q_2 - q_1) \\ 120 &= n(q_3 - q_1) \\ 300 &= n(q_4 - q_1) \end{aligned}$$

so n is a common divisor of 72, 120, and 300. Since n is to be as large as possible, $n = \gcd(72,120,300) = 12$.

4.14: Let s = the number of senior citizens.

r = the number of non-senior citizens.

Then, $23r + 17s = 1500$.

This gives

$$\begin{aligned} 6r &\equiv 4 \pmod{17} \\ 3r &\equiv 2 \pmod{17} \\ r &\equiv 12 \pmod{17} \\ r &= 12 + 17k \\ 23(12 + 17k) + 17s &= 1500 \\ 276 + 17(23k + s) &= 1500 \\ 23k + s &= 72 \\ s &= 72 - 23k. \end{aligned}$$

The minimal possible positive value for s is s = 3, so there are 3 senior citizens.

4.15: Let x = the number of dollars in Judy's check.

y = the number of cents in the check.

100x + y = the amount of the check in cents.

Then

$$\frac{2}{3}(100x + y + 3000 - 240) = 100y + x$$

$$298y - 197x = 5520$$

giving

$$\begin{aligned} 101y &\equiv 4 \pmod{197} \\ -96y &\equiv 4 \pmod{197} \\ 24y &\equiv -1 \equiv 196 \pmod{197} \\ 6y &\equiv 49 \equiv -148 \pmod{197} \\ 3y &\equiv -74 \equiv 123 \pmod{197} \\ y &\equiv 41 \pmod{197} \\ y &= 41 \\ x &= 34. \end{aligned}$$

The check was for $34.41.

4.16: Let x = the number of cars sold.

y = the number of cars originally purchased.

Then $225x - 89y = 2327$

$$47x \equiv 13 \pmod{89}$$

$$-42x \equiv -76 \pmod{89}$$

$$21x \equiv 38 \equiv -51 \pmod{89}$$

$$7x \equiv -17 \equiv 161 \pmod{89}$$

$$x \equiv 23 \pmod{89}$$

$$x = 23,\ y = 32.$$

Therefore, there are 9 cars remaining.

4.17: Let H = the number of half dollars,

F = the number of five dollar bills,

T = the number of ten dollar bills.

Then $H + F + T = 100$

$$\frac{1}{2} H + 5F + 10T = 100.$$

Eliminate H to get

$$9F + 19T = 100.$$

Therefore, T = 1, F = 9 and H = 90.

4.18: Let P = the number of pennies,

N = the number of nickels,

D = the number of dimes,

Q = the number of quarters,

H = the number of half dollars.

Then $P + N + D + Q + H = 50$,

$$P + 5N + 10D + 25Q + 50H = 100.$$

Eliminating P,

$$4N + 9D + 24Q + 49H = 50.$$

Therefore H = 0 and

$$4N + 9D + 24Q = 50.$$

Q = 2 is clearly impossible.

Therefore Q = 0 and $4N + 9D = 50$; or Q = 1 and $4N + 9D = 26$.

We get two solutions:

H = 0, Q = 1, D = 2, N = 2, P = 45,

and

H = 0, Q = 0, D = 2, N = 8, P = 40.

Therefore, there are at most 2 people at the meeting.

4.20: Let k = the number of kitchen clocks sold,

c = the number of cuckoo clocks sold,

g = the number of grandfather clocks sold.

Then

$$17k + 31c + 61g = 300.$$

Since she sold at least one of each, $1 \le k$; $1 \le c$; $1 \le g$.

Let $k' = k-1$, etc.

Then

$$17k' + 31c' + 61g' = 191.$$

Only four values of g' are possible: $g' = 0,1,2$ or 3. Trying these respectively we obtain four cases

$g' = 0$, $17k' + 31c' = 191$;

$g' = 1$, $17k' + 31c' = 130$;

$g' = 2$, $17k' + 31c' = 69$;

$g' = 3$, $17k' + 31c' = 8$.

Only the second case has positive integral solutions: $k' = 4$, $c' = 2$.

Hence g = 2, k = 5, c = 3.

4.21: Since the first tomb is given the number 29 when it is reached the second time, we can consider the counting modulo 28. Since $221 \equiv 25 \pmod{28}$, the jewels are buried under the fifth tombstone.

4.22: Let r = the remainder. Then

$$r \equiv 3^{666,666} \pmod 7$$
$$\equiv (27)^{222,222} \pmod 7$$
$$\equiv 1 \pmod 7$$

Therefore r = 1.

4.23:

$$3^{3n+1} + 2^{n+1} = 3\cdot(27)^n + 2\cdot 2^n$$
$$\equiv 3\cdot 2^n + 2\cdot 2^n = 5\cdot 2^n \equiv 0 \pmod 5.$$

4.24: Let x = the first number selected, $1 \leq x \leq 12$;

y = the number announced, $1 \leq y \leq 12$;

N = the number at which the subject is to say stop.

When the magician taps position y, the subject counts x. When the magician taps y-1, the subject counts x + 1; when the magician taps y − k (modulo 12), the subject counts x + k. When x + k = N, k = N − x, so the subject is pointing at y − (N − x) = x + y − N when he is told to stop. If N is chosen to be y + 12, $y - (N - x) \equiv x \pmod{12}$.

4.25: Let n be the number selected. Then, modulo 9, the number obtained will be $\frac{6}{2}(3n + 7) + 66$
$= 9n + 21 + 66 \equiv 6 \pmod 9$.

4.26: Suppose the card which was ultimately chosen was in the deck at the xth position before the trick was started.

Let n = the number selected, $11 < n < 20$; n = 10 + y, $2 < y < 9$.

at the start	after cards are dealt	on the clock
—— 1	—— n	n → 1
—— 2	—— n-1	n-1 → 2
—— 3	—— .	. .
—— .	—— .	. .
—— .	—— .	. .
—— .	—— x	n-k → k+1
—— x	—— .	
—— .	—— .	
—— .	—— .	
—— .	—— 3	
—— .	—— 2	
	—— 1	
Face down	Face down	

The sum of the digits of n is y + 1 and the card in the (y + 1)st position on the clock is the card which was originally in the $(n - y)^{th}$ position. Hence x = n − y = 10; and so the desired card is placed in the 10th position prior to the start of the trick.

4.27: Let n = the number chosen.

$2n$

$2n + 7$

$5(2n + 7) = 10n + 35$

$9n + 35$

Consider this modulo 9.

$9n + 35 \equiv 8$ modulo 9.

So the sum of the digits of the answer must $\equiv 8 \pmod 9$.

4.28: The two numbers have the same digital root. Therefore, their difference is congruent to zero modulo 9.

4.29: Note that T must have the same digital root, d, as 77^{77}.

$d \equiv (77)^{77} \equiv 5^{77} = (5^3)^{25} \cdot 5^2 \equiv$

$(-1)^{25} \cdot 7 \equiv 2 \pmod 9$.

Therefore d = 2.

$77^{77} < 100^{77}$,

so $S \leq 9 \cdot 154 = 1386$.

Therefore $T \leq 1 + 9 + 9 + 9 = 28$.

Since T has digital root 2, T = 2, 11, or 20.

In all cases, the sum of the digits of T is 2.

4.30: Let G = the number of pieces of gum in the machine.

$G \equiv 1 \pmod 4$; i.e., $G = 4K + 1$ for some positive integer K.

Kevin took K + 1 balls, leaving 3K balls.

$3K \equiv 1 \pmod 4$ or $K \equiv 3 \pmod 4$ or $K = 4J + 3$.

Esteban took $1 + \frac{3K-1}{4} = 3J + 3$ gumballs, leaving $\frac{3}{4}(3K-1) = 9J+6$.

$9J + 6 \equiv 1 \pmod 4$ or $J \equiv 3 \pmod 4$ or $J = 4L + 3$.

Tony took 9L + 9 gumballs leaving 27L + 24 balls.

Sean got 27L + 24; the smallest such number is with L = 0.

Therefore Sean got 24,

Tony got 9,

Esteban got 12,

Kevin got 16,

and the total number of gumballs is 61. Sean got the best of the deal; the gumball vendor got the worst.

4.32: The rules for finding the remainder when a number is divided by 3,4,5,8 or 11 are presented in the solution of Problem 5 of Practice Problems 4.M.

The remainder upon division by 9 is just the digital root of the number (or 0 if the digital root is 9).

The remainder upon division by 10 is just the last digit of the number.

4.33: The rules for divisibility by 2,3,4,5,7,8,11 are presented in the solution of Problem 5 of Practice Problems 4.M.

The hint to Exercise 4.33 discusses divisibility by 6,12,13,14,15 and 16.

Divisibility by 9 and 10 is discussed in 4.32 above.

Using these rules, to find the smallest (largest) number divisible by a given n, we arrange the digits in increasing (decreasing) order and try to satisfy the necessary properties by making changes in as few of the final digits as possible.

For example, to find the smallest ten digit number divisible by 4, we start with 1023456789 (note 0123456789 would not be permissible). Clearly the last digit must be even. Permuting only the 9 and the 8 does not give a number divisible by 4. The same is true if we permute only the 7,8 and 9. If we permute the 6,7,8 and 9, we can get several different numbers divisible by four; we want the smallest possible of these. Since the 6 cannot be left where it is, the smallest number which can occupy its place is 7. With 6,8 and 9, the 6 cannot come first, so the smallest number of these would have the 8 first. This leads to 1023457896. The argument in most of the remaining cases is done similarly.

The case of divisibility by eleven is worthy of further comment. We know that we must divide the ten digits into two sets of five each so that the difference between the sums of the digits in each set is divisible by 11.

Let x be the sum of the digits in one set, and y the sum of the digits in the other. There is no loss of generality in assuming that $x \leq y$.

$10 = 0 + 1 + 2 + 3 + 4 \leq x \leq y$
$\leq 5 + 6 + 7 + 8 + 9 = 35.$

$0 \leq y - x \leq 25$, so $y - x = 0, 11$, or 22.

$x + y = 45$, so x and y are of opposite parity, so $y - x = 11$.

$y = 28$; $x = 17$.

This may occur in several ways. Since we wish to stay as close to 1023456789 as possible, we see if it is possible for one set to contain 1,2,4 and 6 and the other to contain 0,3 and 5. This isn't possible. Continuing in this manner we try

x or y	y or x	possible or not
1,2,4	0,3,5	no
1,2,4	0,3	no
1,2	0,3	no
1,2	0	yes

$x = \{1,2,3,5,6\}$ or $x = \{1,2,3,4,7\}$. The first of these enables us to build a smaller number: 1024375869.

The case for the largest number is done similarly, as are the cases for nine digit numbers (although this is a little more complicated since one set contains five digits and the other four).

The answers for nine digit numbers are given in the text.

The ten-digit answers are:

divisor	smallest	largest
2	1023456798	9876543210
3	1023456789	9876543210
4	1023457896	9876543120
5	1023467895	9876543210
6	1023456798	9876543210
7	1023456798	9876543201
8	1023457896	9876543120
9	1023456789	9876543210
10	1234567890	9876543210
11	1024375869	9876524130
12	1023457896	9876543120
13	1023456798	9876542130
14	1023456798	9876543012
15	1023467895	9876543210
16	1023457968	9876543120

4.34: Let x,y be the missing digits of the number. Since $99 = 9\cdot 11$,

1) $25x37y411 \equiv 0 \pmod 9$

$x + y + 5 \equiv 0 \pmod 9$

$x + y \equiv 4 \pmod 9$

$x + y = 4$ or $x + y = 13$;

2) $25x37y411 \equiv 0 \pmod{11}$

$1-1+4-y+7-3+x-5+2 \equiv 0 \pmod{11}$

$x - y \equiv 6 \pmod{11}$

$x - y = -5$ or $x - y = 6$.

Solving simultaneously,

$x + y = 4$, $x - y = 6$: impossible

$x + y = 4$, $x - y = -5$: impossible

$x + y = 13$, $x - y = 6$: impossible since y must be nonnegative

$x + y = 13$, $x - y = -5$: $x = 4,\ y = 9$.

4.35: Let x,y,z be the missing digits of the number. Since $792 = 9\cdot 8\cdot 11$,

$5x3y1z672 \equiv 0 \pmod{8,9,11}$

$5x3y1z672 \equiv 672 \equiv 0 \pmod 8$, for any x,y,z.

$5x3y1z672 \equiv x + y + z + 6 \pmod 9$

$5x3y1z672 \equiv 2-7+6-z+1-y+3-x+5$

$= 10 - x - y - z \pmod{11}$.

Therefore, $x + y + z \equiv 3 \pmod 9$;

$x + y + z \equiv 10 \pmod{11}$.

This is true for $\{x,y,z\} = \{4,8,9\}$ regardless of order.

Thus the answer is: The blanks may be filled in in six ways.

4.36: b) The number of days in a 28 year period which does not include the turn of a century is

$28\cdot 365 + 7 \equiv 0 \pmod 7$,

where 7 is the number of leap years in 28 such years.

In the following, d through h, label Sunday-0, Monday-1, Tuesday-2, Wednesday-3, Thursday-4, Friday-5, Saturday-6.

d) For February to have five Sundays, the year must be a leap year with February 1 a Sunday. Suppose this occurs in the kth leap year of the twentieth century. We will solve for k.

By part a), January 1, 1901 is a Tuesday. Therefore February 1, 1901 is Friday:

$$\underset{\text{Feb.1, 1901}}{x} \equiv \underset{\text{Tues. Jan.1, 1901}}{2} + 31 \equiv \underset{\text{Fri.}}{5} \pmod 7$$

Therefore, modulo 7,

$$0 \equiv 5 + 3\cdot 365 + 4(k-1)\cdot 365 + k-1$$

0	5	3·365	4(k−1)·365	k−1
Sun. Feb.1 kth leap year	Fri. Feb.1 1901	till Feb.1 1904	# of days excluding leap days	# of leap days

$0 \equiv 5 + 3\cdot 1 + 4(k-1)\cdot 1 + (k-1) \pmod 7$

$0 \equiv 3 + 5k \pmod 7$, $1 \le k \le 20$

$4 \equiv -3 \equiv 5k \pmod 7$

$k \equiv 5 \pmod 7$

k = 5,12,19 giving the years <u>1920, 1948, 1976</u>.

e) As in d, we consider the kth leap year of the twentieth century. In a leap year, February 29th is the same day as February 1.
Therefore, modulo 7,

$$x \equiv 5 + 3\cdot 365 + 4(k-1)\cdot 365 + k-1$$

5	3·365	4(k−1)·365	k−1
Fri. Feb.1 1901	till Feb.1 1904	# of days excluding leap days	# of leap days

$x \equiv 5 + 3\cdot 1 + 4(k-1)\cdot 1 + (k-1) \pmod 7$

$x \equiv 3 + 5k \pmod 7$, $1 \le k \le 20$.

As k runs from 1 through 7, x runs through all the days of the week. Therefore February 29 can be any day of the week.

f) Consider a four hundred year period starting on January 1, 1901.

Let x represent the first day of the kth subsequent century, $0 \le k \le 3$. Modulo 7,

$$x \equiv 2 + 100k\cdot 365 + 24k + \left[\frac{k+3}{4}\right]$$

2	100k·365	24k	[(k+3)/4]
Tues. Jan.1 1901	# of days excluding leap days	# of leap years excl. 2000	the leap year 2000

where [x] = the integer part of x.

$$x \equiv 2 + 2k + 3k + \left[\frac{k+3}{4}\right] \pmod 7$$
$$= 2 + 5k + \left[\frac{k+3}{4}\right] \pmod 7$$

k = 0: $x \equiv 2 \pmod 7$ (<u>Tuesday</u>)

k = 1: $x \equiv 1 \pmod 7$ (<u>Monday</u>)

k = 2: $x \equiv 6 \pmod 7$ (<u>Saturday</u>)

k = 3: $x \equiv 4 \pmod 7$ (<u>Thursday</u>)

Therefore the first day of a century cannot be on a Wednesday, Friday or Sunday.

g) For a year to contain 53 Sundays, either January 1 must be a Sunday, or the year must be a leap year and January 2 must be a Sunday.

So we seek

1) years of the 20th century in which January 1 is a Sunday,

2) leap years of the 20th century in which January 1 is a Saturday.

By part b, we only need consider a 28 year period.

k = the kth year of the century, $1 \le k \le 28$.

$$0 \equiv 2 + (k-1)\cdot 365 + \left[\frac{k-1}{4}\right]$$

0	2	(k−1)·365	[(k−1)/4]
Sun. Jan.1 kth year	Tues. Jan.1 1901	# of days in k−1 yr excluding leap days	# of leap days

$0 \equiv 2 + k-1 + [\frac{k-1}{4}] \pmod 7$

$6 \equiv k + [\frac{k-1}{4}] \pmod 7$

Trying values of k = 1, ..., 28, we get k = 5, 11, 22, 28.

So Jan. 1 is a Sunday in 1905, 1911, 1922, 1928, 1933, 1939, 1950, 1956, 1961, 1967, 1978, 1984, 1989, 1995.

2) Jan. 1 is a Saturday:

Let k = kth leap year, $1 \leq k \leq 25$.

6	≡ 2	+ 3·365	+ 4(k-1)·365	+ k-1
Sat. Jan.1 kth leap year	Tues. Jan.1 1901	# of days until Jan.1 1904	# of days in 4(k-1) years excl. leap days	# of leap days

$6 \equiv 2 + 3 + 4k-4 + k-1 \pmod 7$

$6 \equiv 5k \pmod 7$

$k \equiv 4 \pmod 7$

k = 4, 11, 18, 25, giving 1916, 1944, 1972, 2000.

h) For the sake of simplifying our computations, we will consider year periods beginning on March 13 and ending on the following March 12. For each such period, we determine the day on which the thirteenth of each month falls as a function of the day on which March 13th falls.

That is, if x represents the day of the week on which March 13th falls, then April 13th will fall on x + 3 (since there are 31 days in March and $31 \equiv 3 \pmod 7$), May 13 will fall on x + 5, etc. Continuing in this manner, we arrive at the following chart:

Days on which March 13th falls	Months during which the thirteenth of the month is a Friday
Friday	March, November
Saturday	August
Sunday	May, January(following yr)
Monday	October
Tuesday	April, July
Wednesday	September, December
Thursday	June, February(following yr)

Now we need the years with March 13th falling on various days of the week.

Let z = the day of the week of March 13 in the kth year, $1 \leq k \leq 28$.

Modulo 7,

z ≡	3	+ 365(k-1)	+ $[\frac{k}{4}]$
	Mar.13 1901	# of days excl. leap days	# of leap days

$z \equiv 2 + k + [\frac{k}{4}] \pmod 7$.

We determine z for k = 1 to k = 28 (28 year repetition of calendar). This gives the chart on the following page.

Therefore, Friday falls on the thirteenth day of the month 171 times in the twentieth century.

year	day of March 13	Months with Friday 13
1901,1929,1957,1985	Wednesday	September, December
1902,1930,1958,1986	Thursday	June, February*
1903,1931,1959,1987	Friday	March, November
1904,1932,1960,1988	Sunday	May, January*
1905,1933,1961,1989	Monday	October
1906,1934,1962,1990	Tuesday	April, July
1907,1935,1963,1991	Wednesday	September, December
1908,1936,1964,1992	Friday	March, November
1909,1937,1965,1993	Saturday	August
1910,1938,1966,1994	Sunday	May, January*
1911,1939,1967,1995	Monday	October
1912,1940,1968,1996	Wednesday	September, December
1913,1941,1969,1997	Thursday	June, February*
1914,1942,1970,1998	Friday	March, November
1915,1943,1971,1999	Saturday	August
1916,1944,1972,2000	Monday	October
1917,1945,1973	Tuesday	April, July
1918,1946,1974	Wednesday	September, December
1919,1947,1975	Thursday	June, February*
1920,1948,1976	Saturday	August
1921,1949,1977	Sunday	May, January*
1922,1950,1978	Monday	October
1923,1951,1979	Tuesday	April, July
1924,1952,1980	Thursday	June, February*
1925,1953,1981	Friday	March, November
1926,1954,1982	Saturday	August
1927,1955,1983	Sunday	May, January*
1928,1956,1984	Tuesday	April, July

* of the following year.

CHAPTER 5

COMMENTS AND SUGGESTIONS

1. This chapter has been included for several reasons. In the first place, arithmetic in other bases is a topic which is discussed in many "liberal arts" courses. It is also one which is easily handled by most students. As a second reason, cryptarithmetic problems are among our favorites; and an understanding of the elementary operations in our positional number system is necessary if we are to successfully attack problems of this type.

2. If you would like to discuss mathematical induction (something we usually do very briefly but to which we would devote much more time in a two semester course), the discussion of Bachet's weight problem presents a convenient place for doing so. Appendix B contains material on this topic and may be introduced at any time. Some of the exercises of the chapter, for example 5.16, also lend themselves to inductive consideration.

We generally prefer to tie our discussion of induction to the Tower of Brahma problem (Sample Problem 8.1) in Chapter 8. At that time we return to Bachet's weight problem and discuss it more rigorously.

3. We usually discuss the game of Nim (Sample Problem 7.3) immediately after we finish discussing the binary system.

4. The cryptarithmetic problems in other bases (Exercises 5.57-5.64) frequently prove to be difficult for many of the students. In recent semesters, we have usually not considered more than one or two of them.

CROSS REFERENCES

1. The exercises marked # are more easily handled if Chapter 4 has been discussed.

2. The binary system is needed in Chapter 7 for the discussion of the game of Nim.

3. Appendix B on mathematical induction is referred to in the discussion of Bachet's weight problem.

PRACTICE PROBLEM ANSWERS

5.A:

2. a) 157; b) 1227; c) 20457; d) $22\frac{2}{7}$.

5.B:

2. a) $(100010110111110)_{two}$;

b) $(220111021)_{three}$;

c) $(1032404)_{five}$; d) $(T3ET)_{twelve}$;

e) $(45EF)_{sixteen}$;

where T represents ten; E, eleven; and F, fourteen.

5.C:

2. a)

+	0	1
0	0	1
1	1	0

•	0	1
0	0	0
1	0	1

b)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

c)

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

•	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

4. a) $(12030)_{seven}$;

b) $(10452)_{seven}$.

6. a) $(745)_{eight}$;

b) $(1712)_{eight}$.

8. a) $(24435356)_{seven}$;

b) $(1033362)_{seven}$.

9. $1 + 2 + 2^2 + \cdots + 2^k =$

$(1)_{two} + (10)_{two} + (100)_{two} + \cdots + (1\underset{k}{0\ldots0})_{two} = (\underset{k}{11\ldots1})_{two}$

$(\underset{k}{11\ldots1})_{two} + (1)_{two} = (1\underset{k+1}{0\ldots0})_{two}$

$= 2^{k+1}$

Therefore

$(1 + 2 + 2^2 + \cdots + 2^k) + 1 = 2^{k+1}$

and so

$(1 + 2 + 2^2 + \cdots + 2^k) = 2^{k+1}-1$

5.D:

2. a) Since C•C ends in C, C = 0,1,5, or 6. The first two can be discarded since C times ABC = EDC.

b) Since B•C ends in B, and B is clearly non-zero, there is no possibility for B if C = 5; and B = 2,4, or 8 if C = 6.

c) C = 6.

d) F = 5 (since at most 1 is carried from the third column to the fourth).

e) Since A = 1, and F = 5, B $\leq$ 5. Therefore B = 2 or B = 4. If B = 2, we would need A = 2 to get B times ABC = FEB. Since this is impossible, B = 4 and A = 1.

f)
```
 146
  46
 ---
 876
584
----
6716
```

SOLUTIONS TO EXERCISES

5.1: There are 9 one digit numbers, 90 two digit numbers, 900 three

k	2	3	4	5	6	7	8	9	10
10T+U	18	27	12,24,36,48	45	54	21,42,63,84	72	81	10,20,30,40,50,60,70,80,90

Figure 5.1

digit numbers, etc. Together, these require 9 + 180 + 2700 = 2889 digits. 31676 - 2889 = 28787 = 4·7196 + 3 The final digit is 9 (the third digit of 8196).

5.2: a) $(10k+r)^2 \equiv r^2 \pmod{10}$. Since $0 \le r \le 9$, there are only ten cases to consider. These give the stated possible final digits.

b) $(10k+r)^3 \equiv r^3 \equiv 0,1,8,7,4,5,6,3,2,9 \pmod{10}$

Thus all digits are possible.

c) $(10k+r)^4 \equiv r^4 = (r^2)^2 \equiv 0^2,1^2,4^2,9^2,6^2,5^2,6^2,9^2,4^2,1^2$

$\equiv$ 0,1,5,6 (mod 10).

5.4: From Exercise 5.2 a), L. Cohn can be eliminated, since a perfect square cannot end in 8. By Exercise 5.3, M. Addison can also be ruled out.

In order for r^2 to end in 5, r must also end in 5. But $(10k+5)^2 = 100k^2+100k+25 \equiv 25 \pmod{100}$. That is, in order for r^2 to end in 5, it must end in 25. Therefore, Jax can be eliminated too.

Since any number is congruent to its digital root modulo 9, $r^2 \equiv [d(r)]^2 \pmod 9$. Therefore, the digital root of a perfect square must be congruent to $0^2,1^2,2^2,3^2,4^2,5^2,6^2,7^2$ or $8^2 \equiv 0,1,4,0,7,7,0,4,1 = 0,1,4,7 \pmod 9$.

But 174,329 ≡ 8 (mod 9), so Wills was not the winner. This leaves R. Velt as the next president of the company.

5.5: Suppose the original sign read x0y, then x0y - yx = yx - xy. Since the latter difference is less than 100, x must be 1.

In two hours, the automobile traveled 10y - 1y = (100+y)-(10+y) = 90 miles. It therefore traveled 45 mph.

y1 - 1y = 45; y = 6.

At 45 mph, it takes 16/45 hr = 21 1/3 min to travel 16 miles.

5.7: $(T+U) \mid (10T+U)$.

$(10T+U) = k(T+U)$.

$(10-k)T = (k-1)U$

For each value of k, $1 \le k \le 10$, we obtain values for T and U. These are shown in Figure 5.1, above.

5.8: $(T+U) \mid (10T+U-10U-T)$

$(T+U) \mid 9(T-U)$

$k(T+U) = 9(T-U), \quad 1 \le k \le 9$

$(9-k)T = (9+k)U$

We check each value of k (see Figure 5.2 at the top of the next page.)

k	1	2	3	4	5	6	7	8	9
10T+U	54	none	21,42,63,84	none	72	51	81	none	none

The reversals of these numbers also work.

Figure 5.2

5.9: a) $10T+U + 10U+T = n^2$

$11(T+U) = n^2$

$T+U = 11$

10T+U = 29,38,47,56,65,74,83,92.

c) $(10U+T) - (10T+U) = n^3$

$9(U-T) = n^3$

$U-T = 3,0$

10T+U = 14,25,36,47,58,69, 11,22,33, 44,55,66,77,88,99.

5.10: $66 | (10T+U+10U+T)$

$6 | (T+U)$

10T+U = 15,24,33,42,51,39,48,57,66, 75,84,93,99.

5.11: $(10T+U)^2 - (10U+T)^2 = n^2$

$99(T^2-U^2) = n^2$

$T^2-U^2 = 11k^2$

$(T+U)(T-U) = 11 \text{ or } 44$

10T+U = 65(or 56) .

5.12: $(100H+10T+U)-(100U+10T+H)=99(H-U)$

5.14: Let m = the number representing the month of the subject's birth, and let d represent the day of the month.

$2\{5[2(5m+17)-13]-8\}+9+d=100m+d+203$

$100m+d+203 = 1332$

The subject's birthday is November 29; today is December 4.

5.16: In n guesses, he can determine a number which does not exceed 2^n-1 (induction). The largest number Tweedledum should be allowed to pick is $2^8-1 = 255$.

5.17: He says "Is the number greater than 1." "Is it greater than 2?," and so on. Once he receives a "no" answer he has pinpointed the selected number.

5.18: Card A contains all numbers which contain a 1 in the units digit of their binary representations; Card B contains all numbers which contain a 1 in the "twos digit"; Card C contains numbers with a 1 in their "fours digit"; and so on. To guess the selected number, the magician just recaptures the number from its binary representation--that is, he or she adds up the powers of two which correspond to the cards which are indicated by the subject.

5.19: b) Dealing cards face down reverses their order. After the first deal and reassembly, the order of the cards will be: all numbers with zero in the units digit of their binary representation, largest to smallest, followed by all numbers with 1 in the units digit, largest to smallest. After the second deal and reassembly, the order will be: numbers whose last two binary digits are 11; numbers ending in 10; numbers ending in 01; numbers ending in 00. Within each group, the numbers are arranged from smallest to largest. After the third deal and reassembly, the numbers, arranged largest to smallest within each group, will appear in groups in the following order. Those (with binary representation) ending in 000; those ending in 001; those ending in 010; those ending in 011; those ending in 100; those ending in 101; those ending in 110; and those ending in 111.

The fourth deal sorts according to the fourth from last digit of the binary representation. After reassembly, the order is from 1111 down to 0000 - i.e., from 15 to 1.

The same approach is applicable to parts c and d. For d), the answer is: First method--five; Second method--five; Third method--ten.

5.20: Let A = the number of sticks received by the person who removed the apple; let B (banana) and P (peach) be defined similarly.

Then, of the twenty-two sticks remaining on the table at the time that the magician was blindfolded, A+3B+9P were removed by the subjects. Since this left seven,

$$(PBA)_{three} = 22 - 7 = 15.$$

But $15 = (120)_{three}$; hence P=1; B=2; A=0. That is, Alvin (who received one stick) took the peach; Julia (who received two sticks) took the banana; and Melonie (who received no sticks) took the apple.

In general, expressing twenty-two minus the number of sticks remaining in base three reveals who took which fruit.

5.21: The explanation is similar to that of Exercise 5.19a, except that the use of three piles means that the magician is working in base three rather than base two. That is, the first deal sorts the cards according to their units digit in base three; etc.

5.22: a) In accordance with the solution of part b (see the text), the magician would have expressed 22 in base three: $22 = (211)_{three}$. He would therefore have placed the indicated piles in the middle after each of the first two deals and on the bottom after the final deal.

5.23: Card A contains all numbers (up to 26) which have 1 as the units digit of their base three representation; Card B - numbers with 2 as their units digit; Card C - numbers with 1 as their middle digit; etc.

If the subject announced A, C and F, the student would add 1, 3 and 18, and announce "your number is twenty-two."

5.25: As is shown in the solution of Exercise 5.24, every integer may be expressed as a sum of powers of three, with coefficients 0,1,-1.

To balance an object with integral weight w, it is only necessary to have weights corresponding to each of the powers of three: Express w as suggested above, place the object in one pan together with all powers of three which have coefficients -1 in the representation of w. In the other pan place all powers of three which have coefficients +1. The two pans will then balance.

5.26: Express 1,013,652 in base seven:

$1,013,652 = 1\cdot 823543 + 1\cdot 117649 + 4\cdot 16807 + 2\cdot 2401 + 1\cdot 343 + 1\cdot 49 + 5\cdot 7 + 3\cdot 1.$

Eighteen heirs received money.

5.27: $6\cdot 4 = (30)_b$; hence b = eight. $\frac{1}{4}$ of $(10)_{\text{eight}} = 2$.

5.28: For b>4,

$$(b^2+\frac{b}{2})^2 < b^4+b^3+b^2+b+1 < (b^2+\frac{b+1}{2})^2$$

(since $\frac{b^2}{4} < b^2+b+1$ and $b+1 < (\frac{b+1}{2})^2$).

Hence $b^4+b^3+b^2+b+1$ could only be a perfect square for b=2, 3 or 4. But $2^4+2^3+2^2+2+1 = 31$ and $4^4+4^3+4^2+4+1 = 341$ are not perfect squares leaving only b=3 as a possibility.

$3^4+3^3+3^2+3+1 = 121=(11)^2$, so

$(11111)_{\text{three}} = (102)^2_{\text{three}}$

5.29: In any base b>9, $4\cdot 297>4\cdot 200>792$ and $2\cdot 297<2\cdot 300<792$. Hence 792 must equal $3\cdot 297$. In order for $3\cdot 7$ to end in 2, the base must be 19.

5.30: $(ABCDE)_b \equiv E \pmod b$. Hence, if b is even, ABCDE is odd if and only if E is; If b is odd,

$$(ABCDE)_b = Ab^4+Bb^3+Cb^2+Db+E \equiv A+B+C+D+E \pmod 2.$$

So, in this case, ABCDE is odd if and only if A+B+C+D+E is.

5.31: $(121)_b = b^2+2b+1 = (b+1)^2 = (11)^2{}_b$;
$(1331)_b = b^3+3b^2+3b+1 = (b+1)^3 = (11)^3_b$.

5.32: b) $3(b-1) = (xy)_b$ where $x = 2$, $y = b-3$.
Let $z = b-1$.
Then $(yxz)_b = (b-3)b^2+2b+b-1 = b^3-3b^2+3b-1 = (b-1)^3$

5.33: If I^2 has two distinct digits which are distinct from I, then I = 4, 7, 8 or 9. Of these, only $8^3 = 512$ has three digits distinct from I and I^2. Hence ME = 64, and ME-I = 64-8 = 56 = YM. The reversal is MY.

5.34: If O^3 = DAD, then O = 7 and DAD = 343; if $(IM)^2$=MOM, IM = 26 and MOM = 676. Therefore, MAID = 6423.

5.35: Since NOT and TO are squares, TO = 16 and NOT = 361 or 961. TWO = 196; so NOT = 361. AGE is a perfect square with three distinct digits none of which are 1,3,6 or 9. The only possibility is 784.
TWO + TO + TOO = 196 + 16 + 166 = 378 = NAG.

5.36: X = 2 or 3, C = 4 or 9. But if X=2, then C=4, so L=2. This is impossible. Hence X=3, C=9, L=4.
V = I + 3; I + V = 13. I = 5, V=8.

5.37: A=1; G=0; E=8 (since 1 must be carried from the third column).
F = 4 or 9; C = 3 or 7,
H+B(+1)=10; B+F=H(+10);
2B+F(+1) = 10(+10).
F=9, B=5, H=4 or F=4, B=3, H=7.
The latter case leaves no value for C.

```
  14579
  85919
 100498
```

5.38: N=1. Since T ≠ 0, T = 9 (column 5); E=8; P+S =11;
2·O+1 = S(+10); R+O(+1) = W.

S	P	O	R	W
3	8	contradiction (E=8)		
5	6	2	impossible (R+2=W)	
5	6	7	impossible (R+8=W)	
7	4	3	2	5
7	4	8	contradiction (E=8)	

5.40: C = 5 or 6 (C^2 ends in C);
E ≠ 0; Therefore C=6.
E > D (since ExABC > DxABC).
ExC ends in E, so E is even;
E-D = 5 (since DxC ends in E).
Therefore E = 8, D = 3.
From ExABC, we now find B = 4 or 9.
In either case, G = 7 (from CxABC);
I = 5; H = 9 (H=G+F+1).
So B = 4; F = 1; A = 2:

```
   246
   386
  1476
 1968
 738
 94956
```

5.41: ExC ends in C; therefore C = 0 or 5 or E = 1 or 6. Of these, only E = 6 is viable. C must be even. C^2 ends in H; H≠6; hence C = 2 or 8 and H = 4.
Since ExABC has only three digits, A = 1 and B ≤ 5.
I + A + carryover = 10 + A; and F + A + E = carryover•10 + G.
Since A + E = 7, the carryover is 1 and I = 9.
Since ExABC ≥ 900, B ≥ 5. Therefore B = 5, G+C = C, so G = 0; Hence, from CxABC, C = 2.
Finally, since DxABC = 1216, D = 8.

```
    152          F = 3.
    862
    304
   912
 1216
 131024
```

5.42: C^2 ends in C, so C = 5 or 6; AxC ends in C and A≠1, so C = 5.
H = 0; B is even; F = 2.
D+A+A(+ ?) = I, with nothing carried; so A ≤ 4.

AxABC has four digits, the first of which is D, and the second of which is H = 0; so A = 3, D = 1, and B = 4 or 6.

Since 6 x 360 > 2000, B = 4:

```
    345
    345
   1725        E = 7
  1380         G = 8
 1035
 119025        I = 9
```

5.43: Label the missing digits:

```
     ab3c
      def
    ghi0j
   kmn7p
   q23r
  s4tuv5j
```

s = 1 and p = 5. e x ab3c ends in 75 only if e = c = 5, so this must be the case. Since at most 2 can be carried from the addition of g, m and 2, k is at least 3. Hence (from e x ab3c), a ≥ 6. But then d = 1 and b = 2, r = c = 5, q = a. Since k + q (+1 or 2) = 14, and (5 x a) + 1 = km, a = q = 8 or 9 and k = 4. If a = q = 8, then m = 1 and at most one is carried from g + m + 2 meaning k + q (+1) will not give 14. Therefore a = q = 9. Also, for f x ab3c to end in 0j, f = 3:

```
     9235
      153
    27705
   46175
   9235
  1412955
```

5.44: Label as follows:

```
  aXb
   cX
 dXXe
  fgh
 ijkXm
```

h is clearly zero so either c = 5 or b = 0 or 5.

Since c times aXb has only three digits, c < X. Since X times aXb has four digits, X≠1. Therefore, since X times aXb ends in XXe, only a few possibilities exist.

X	5	6	8	9
b	0 or 1	0 or 1	5 or 6	9
a	none	none	4 or 9	9

The last case can be ruled out since **b x c ends in 0. Therefore X = 8.**

c cannot be 5 in this case, because c times aXb has only three digits. Therefore b = 5 and, c is even. Furthermore, c times aXb has only three digits, so c = 2 and a = 4:

```
  485
   28
-----
 3880
 970
-----
13580
```

5.45:

```
 ab
 cd
---
XXX
 ef
----
g0hi
```

XXX = X(111) = X(37)(3) = d(ab)

By the Unique Factorization Theorem, ab = 37 and d = 3(X) or ab = 74 and d = 3(X/2).

If ab = 74, then ef = 74 and X = 3 (to give X+e=10). But this is impossible, since X must be even in this case. Hence ab = 37 and X≤3. In order for X+e = 10 to hold, c = 2, ef = 74 and X = 3:

```
  37
  29
----
 333
 74
----
1073
```

5.46: ABxD = EEE = E(37)(3). As in 5.45, AB=37 or 74. Since G = 0, AB = 74 and C = 5. E = 6, so D = 9.

```
  74
  59
----
 666
370
----
4366
```

5.47: Since CHEIN < $\frac{99999999}{1978}$ <50557, CH ≤ 50

Also, since 7 x EIN ends in 44—, we have only the following possibilities:

N	0	1	2	3	4	5	6	7	8	9
I	2	2	9	6	6	3	0	0	7	4
E	9	9	4	0	0	6	2	2	7	3

The next to last case can be ruled out since E and I can't both be 7.

In each case, by adding the last two columns of the problem, we can find CH.

H	0	8	6	4	2	0	8	6		2
C	6	3	7	1	9	3	6	4		2

Ruling out the cases in which we get duplication of values or in which CH>50, three possibilities remain.

CHEIN = 46207, 38921, or 14063

Of these, only 38921 fits the rest of the pattern (e.g. the duplicated digit in the bottom row):

```
      38921
       1978
     311368
    272447
   350289
   38921
   76985738
```

5.48: The quotient is clearly 90809, since 8 times the divisor has two digits and the first and last products have three digits each. The divisor must be 12.

5.49: Label the rows of the divisor as in the solution of Sample Problem 5.4 in the text.

Since row four has a nonzero first digit, row three is less than 900. On the other hand, since row four minus row five has only two digits, row five = 9 _ _. Since rows 1 and 7 have four digits, the quotient must be 97809.

Furthermore, since 8 x divisor $\leq$ 999, 9 x divisor $\leq \frac{9}{8} \cdot 999 < 1125$; hence the first digits of rows six and seven must be 10 or 11, and so row five $\geq$ 989. Therefore $989 < 8 \cdot$ divisor ≤ 999. This leaves only one possibility: The divisor is 124.

```
         97809               97809
124)--------        124)12128316
    1116                1116
    ---                  968
     868                 868
     ----                1003
      992                 992
      ----                1116
      1116                1116
```

5.50: Since A x ABC = ABC, A=1; also, the E in the quotient must be zero, so E=0.

Since G x ABC and B x ABC both end in D, C is even and G-B=5.

Since HHF - IHD = GG, it follows that H=I+1, G=9, and D=F+1. Therefore, since G-B=5, B=4.

F x 14C = 994. Therefore, F=7, C=2. We now know the divisor (142) and the quotient (90147), so it is easy to reconstruct the entire problem.

012456789 = EACBIHFDG; No 3 appears in the problem.

5.51: Since F x ABC and A x ABC both end in C, and since neither F nor A is one, either C=0 or C=5. The first of these cases is easily eliminated (F-C=D), so C=5 and F and A are odd; also (from DxABC), I = 0 and D and G are even. (G>D.)

But D x ABC has only three digits, and neither D nor A is 1; hence A=3 and D=2.

From A x ABC, K=1; from

D x ABC = FBI, 2xB + 1 = 10+B, so B=9 and F=7, (giving E=8); from G x 395 = 1580, G=4. We now have the divisor and the quotient and can reconstruct the entire problem: 0123456789 = IKDAGCJFEB.

5.52: Label as in the solution of **Sample Problem 5.4 in the text.**

Since all the zeros and ones are shown, the first digit in row one must be 8. (Otherwise row one >911 and row two would only have three digits).

The first two digits of the divisor must divide 81, so the divisor is 271.

From here, it is a simple matter to fill in the rest of the missing digits. The quotient turns out to be 372731.

5.53: Label as in Sample Problem 5.4, and let abc be the divisor and defgh be the quotient, where f=0.

Since rows five and seven both end in 4, c is even and g=h+5.

Since all the fours are shown, c≠4, and h≠0,1, or 4. Therefore, h=2 or 3 and g=7 or 8. In either case, a is at most 3.

In each case, it is now possible to recapture abc from g x abc = _4_4, and e from e x abc = _ _4_.

h	g	c	a	b	e
2	7	2	2	0	impossible
2	7	2	2	1	impossible (e≠4)
2	7	2	3	5	impossible
3	8	8	1	7	7
3	8	8	3	0	5 or 6

Working from the bottom up, we find that the last two cases are not compatible with the placement of the 4 in row four. Therefore, the divisor is 178.

Working from the bottom up, we are able to complete the entire pattern. The quotient is 17083.

5.54: Since D x DON < DIV, D=1. Row eight is therefore 1_ _, so the last digit in row six is 5. Since this digit is just the I brought down from the dividend, I=5.

Clearly E=0, so O is at least 2. If O were 3 or more, then row three would be at least 300, so that row one could be at most 129. As this is impossible, O=2.

Row three = 24_ or 25_ and the same is true for row five. So the first entry in row four is 2 or 3, the first entry in row two is 2, and the second entry in row two is 6, 7, 8 or 9. Therefore N>V and 10+V-N = 6, 7, 8 or 9.

N - V = 1, 2, 3, 4. Make a chart: (0,1,2, and 5 are already used)

N	4	6	7	8	9
M	6	4	impossible	3	6
V	3	3		4,6 or 7	3,4,7,8

In each case, work backward to find S.

S	2	2		4	5

The only case possible is N = 8, M = 3, S = 4. Continuing to work backward, we find V=6. From here, it is easy to complete the problem.

5.55: Consider the first division: Since row two clearly starts with 1, row one is 999 and the first four digits of the dividend are 1000. The divisor could be 111, 333, or 999. But since row seven has four digits, the divisor cannot be 111; and since row five cannot be 999, the divisor must be 333.

The first four digits of the quotient are 3003, and the fifth digit is either one or two.

Turning to the second division and filling in the first four digits of the dividend, we see that the divisor must be 29 (since the first digit in row two must be 1).

Row two is 103, and row three is 87. The first two digits of row four are 16, meaning that row five is 145. The last digit of row four is either 1 or 2 (the fifth digit of the quotient of the first division), so the first two digits of row six are 16 or 17. But row seven equals row six and must be a multiple of 29. Therefore, row six equals 174, row four is 162, and the quotient of the first division is 300324:

```
       300324          10356
 333)100007892      29)300324
      999              29
       1078             103
        999              87
         799             162
         666             145
         1332             174
         1332             174
```

5.56: Since all digits of the dividend after the decimal point are zeros, row nine is an even multiple of $1000=2^3\cdot 5^3$, and the divisor is divisible by 25. Since the last digit of row six is 0, row seven cannot end in 0, so the divisor must be an odd multiple of 25. The last digit of the quotient must therefore be 8, and so the divisor must be an odd multiple of 125.

The last digit of row seven must be 5, so row eight=5000 and the divisor is 625. The quotient is 1011.1008.

5.57: Since $(GOO)^2$ and $(OOG)^2$ both have only five digits, G^2 and O^2 both have only one digit. Hence G and O, in some order, are 1 and 2. Since G^2 ends in G, G=1 and O=2. The rest of the problem follows easily: ALONG=53241.

5.59: ABCABC = ABC(1001)
Converting to base ten, we get (25A+5B+C)(126) is a perfect square. Since $126 = 9\cdot 14$, $25A+5B+C = 14n^2$. Since $A\neq 0$, $25 < 25A+5B+C \leq 124$. Therefore, $n^2=4$ and $25A+5B+C = 56$. But this gives A=2, B=C=1 -- a contradiction.

5.60: Regardless of the base, N=1, I=0, T=b−2, E=b−1, and O=2. Therefore, there is a unique solution as long as b−2>2--i.e., b>4.

5.61: Since there are six letters, $b\geq 6$. Regardless of the base, N=1.

$$T\ (+b) = E + O$$
$$O + T\ (+1) = E + b$$
$$2T(+b)(+1) = 2E+b$$

Consider two cases:

Case 1: b is even.

Then the right side of the equation is even and so the (+1) cannot occur on the left, and so neither can the (+b), since $T\neq E$. Hence $T = E + \frac{b}{2}$, so $O = \frac{b}{2}$
Also X=W+1.
If b=6, then O=3, E=2 and T=5. But this leaves no room for X and W

If b=8, then O=4, E=3, T=7, W=5 and X=6 is the unique solution.

If $b\geq 10$, then there exists more than one solution. E.g., $O = \frac{b}{2}$, E = 2, $T = \frac{b}{2}+2$ with $W = \frac{b}{2}+3$ and $X = \frac{b}{2}+4$ or with W=3 and X=4.

Case 2: b is odd.

To avoid an obvious contradiction, we must have 2T+1 = 2E+b

$$T = E + \frac{b-1}{2};\ O = \frac{b-1}{2}$$

X = 0 and W = b−1;

If b=7, T=5, E=2 gives the unique solution.

If $b\geq 9$, there exist at least two solutions:

$T = \frac{b+3}{2}$, E = 2, and $T = \frac{b+5}{2}$, E = 3.

5.62: N=1

```
 BE     BE
+OR    +TO
 TO    NOT
```

If E+R = 0, then B+O = T. But E+O = T or 9+T both of which are impossible since $E\neq B$ and $E\neq B+9$.

Therefore, E+R = 9+O; B+O+1 = T.

If E+O = T+9, then E = B+10 which is impossible. Therefore E+O = T (so E = B+1), and B+T = 9+O

B+O+1 = T

B+T = 9+O

2B+1 = 9, so B = 4, E = 5

R = O+4

T = O+5

Only two sets of values are possible: O=2, R=6, T=7; or O=3, R=7, T=8.

5.63: Since the last entry in row two is 0 and the first entry in row four is 3, the base must be four. Since row three is odd, the divisor is odd. It cannot end in 1 (otherwise row three would have only two digits), so the divisor ends in 3. Since 2(23)=112, the divisor must be 13 or 33 in order for row five to begin with 3.

The multiples of 13 are 13,32,111. If the divisor were 13, then row one would have to be 32 and the first digit in row two would be at least 2. But row 3 would have to be 111, which when subtracted from 2 _ _ would leave more than a one digit difference. As this is not what happens, the divisor must be 33.

It is now easy to complete the problem

```
     1031
33 |102003
    33
     300
     231
       33
       33
```

<u>5.64</u>: $(ONE)_{b+1} - (ONE)_b$
$= O(b+1)^2+N(b+1)+E - [O(b^2)+N(b)+E]=$
$O(2b+1)+N = (ON)_{2b+1}$.

CHAPTER 6

COMMENTS AND SUGGESTIONS

1. Again we have included this chapter for a number of reasons. Firstly, graph theory is an important branch of mathematics in today's world, and it is a topic which is often covered in "liberal arts" math courses. The fact that the student needs no mathematical background to understand the material of this chapter and that the proofs of the theorems we present are, for the most part, intuitive enough so that the students can be led to develop the theory by themselves is another reason why we have included it.

2. As mentioned in the paragraph above, we try to lead the students in the class to develop the theory as a group-to pick out which concepts need be defined, to conjecture appropriate theorems, and to develop proofs of these theorems.

3. If Chapters 7 and 8 have been handled as we've indicated that we often handle them--introduced during the discussion of other chapters, as the need and/or opportunity arises--then this might be a good place to introduce the graphical solution of the colored cubes problem (from Chapter 8) and the game of Sim (Chapter 7). You might also mention some of the other graph-related games from Chapter 7, such as Dots, Bridgit, Slither, and Sprouts.

4. As is mentioned in the book, although all of the exercises of the chapter are graph-related, many may be solved more efficiently by other methods. Nevertheless, the graph theoretic view of the problem is usually of interest.

5. Because we teach a one semester course and we wish to spend more time on Chapter 7, we often omit the starred sections in this chapter or cover them cursorily.

CROSS REFERENCES

1. Tree diagrams (Chapter 1) and state diagrams (Chapter 7) are examples of graphs.

2. Some of the games from Chapters 7 and 8 are graph-related.

PRACTICE PROBLEM ANSWERS

6.A:

2. a)

vertex	A	B	C	D
degree	2	4	3	3

b)

vertex	A	B	C	D	E	F	G	H	I	J	K	L
degree	2	4	2	4	4	2	4	2	4	4	4	4

6.B:

2. a) none exists; b) BACFGEBCDFEDB; c) BCGLKJFABDFHKIGEIHDEB.

6.C:

2. a) ACFABCEBDEFDA; b) none exists;

   c) FHGEBACFADBGDFG.

6.D:

1. ABDCDBCA.

6.E:

2. a) ABEGFCADB; EDG; CDF;

   b) GFEACHEIGBEDC; ADH; BFI.

4. a) ABEGDEGFCDFCADB;

   b) IGBEIFBGFEDCHEACADH.

6. a) ABEGDEGFCDFCADBA;

   b) ACHEBGIEFGBFIEDCADHEA.

6.F:

1. An Eulerian dipath is an Eulerian path which obeys directional signs.

3. a) dicircuit:

   ABIJFADFGJHIEBCEHGDCA;

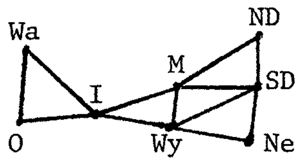
   b) dipath: EBACBDEFCDF.

6.G:

2. a) 4; b) 4; c) 3; d) 4.

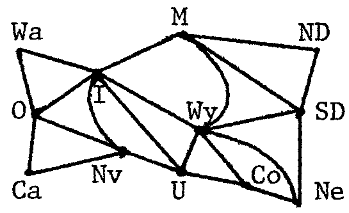
SOLUTIONS TO EXERCISES

6.1: Replace the map by a graph and find an Eulerian circuit.



Her trip could be: Ne,Wy,I,O, Wa,I,M,Wy,SD,M,ND,SD,Ne.

6.2: The graph is as follows.



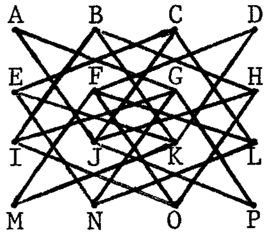
Colorado is the only other vertex of odd degree, so her trip must end there.

6.3: In the graph with vertices as states and edges connecting bordering states, there must be an even number of states with odd degree. (Theorem 6.5.)

6.4: Construct a graph with vertices corresponding to numismatists and two vertices connected by an edge if the corresponding people traded with each other. By Theorem 6.5, the graph must have an even number of vertices of odd degree.

6.5: Construct a graph with vertices representing the cells of the board and edges connecting two vertices if there is a knight's move between them.

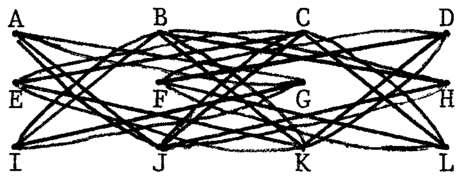
A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P



a) There are more than two vertices of odd degree, so it is not possible for the knight to travel in the manner indicated.

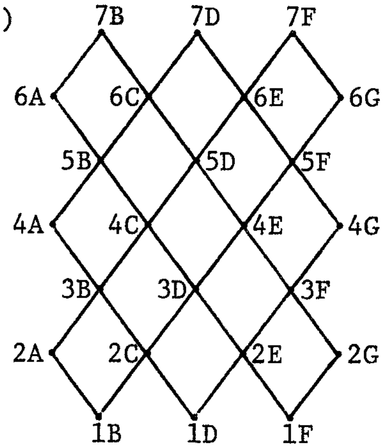
b) It is not possible on any n x n board with n ≥ 4, because there will be more than two vertices of odd degree. On the other hand, for n = 3 it is possible since there are no vertices of odd degree. (It is also possible for n = 1 and n = 2, but here such a trip involves no moves.)

c) The requested trip corresponds to an Eulerian circuit on the following graph:



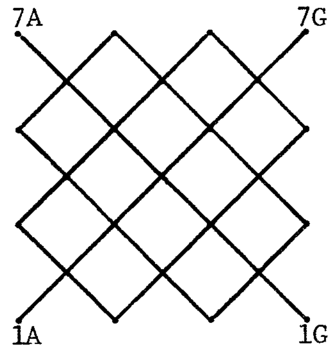
Since all vertices are even, it is possible to find such a circuit. For example, AGIBHJCLFDKEKBKDFLCECJHBIGAJA will do.

6.6: a)



It is possible, since all vertices are of even degree. One such trip is:
2C,1B,2A,3B,4A,5B,6A,7B,6C,7D,6E,7F,6G,5F,4G,3F,2G,1F,2E,1D,2C,3D,2E,3F,4E,5F,6E,5D,6C,5B,4C,5D,4E,3D,4C,3B,2C.

b)



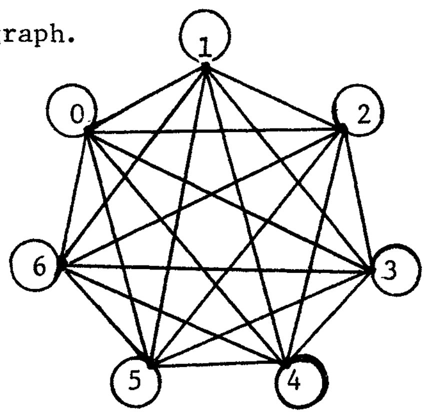
In this case, such a trip is not possible, because the four corner vertices are of odd degree.

6.8: a) There are 4 vertices of odd degree B,F,H, and D. To eliminate two, join two by as short a path as possible. For example, construct an extra edge BF (i.e., Maggie will walk

over BF twice.) For example, DABDEBC FBFEHDGHFIH will be a possible walk. Her distance will be 3350 meters.
b) We must make all vertices even, so repeat edge DH as well as BF. We get 3600 meters with the following walk: ABCFIHGDHFBDEBFEHDA.

6.9: We need a graph with no odd vertices, and so the odd vertices at WM, EM, SWM, and SEM must be eliminated. This could be done by repeating the SWM-SEM leg of the trip as well as the WM-EM leg. However, the latter adds an extra 40 km, whereas repeating the WM-M and M-EM legs would accomplish the same purpose with only 29 km added (in addition to the five from SWM to SEM). Therefore, one possible route would be
M-WM-EM-M-WM-SWM-SEM-EM-M-SWM-SEM-M.

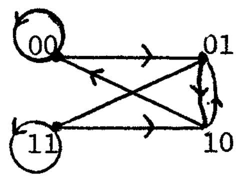
6.10: Since each pair of distinct numbers appears on one domino (doubles were removed), the set of dominos corresponds to the following graph.



This graph has all even vertices. Removal of a domino corresponds to removal of an edge, and creates two odd vertices. A chain corresponds to an Eulerian path, the endpoints of which are the vertices of the edge which was removed.

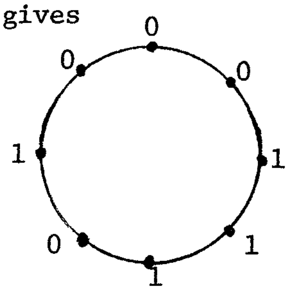
Therefore, the magician knew the 2-5 had been removed since there was a two at one end of the chain, and a five at the other.

6.12: As in the solution of Exercise 6.11, triples can be thought of as a merging of pairs (e.g., 101 can be thought of as 10--01, etc.). Consider the graph with vertices as pairs made up of 0's and 1's and with an edge from U to V if the last digit of U is the first digit of V.



This graph has an Eulerian circuit. 00,00,01,11,11,10,01,10,00.

This gives



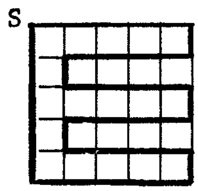
6.13: We want a Hamiltonian circuit. Note that F,K,I are vertices of degree 2 so CG cannot be traversed. Also note that to get to I and K, we must pass through J, so DJ and EJ cannot be traversed. Once DJ is removed there are only two edges to D. We thus must have ADHIJKGFC or its reverse. Therefore the route must be ADHIJKGFCBEA or ADHIJKGFCEBA or one of their reversals.

6.14: We want a Hamiltonian circuit. Note E,H,G and F are vertices of degree two. Therefore CHIEAFKGD or its reversal must be part of the route. Removal of other edges to vertices I,A, and K leaves only one way to get from D to C, passing through each remaining vertex once: DJBLC. This gives the circuit CHIEAFKGDJBLC which may be started at any point, or reversed.

6.15: We want to show that there is no Hamiltonian circuit for the graph: Since B,C,I,G are each of degree 2, a Hamiltonian circuit must contain AGHIFCDBA or its reversal. There is no room to insert E.

6.16: By considering vertices of degree two, we see that the required path must be THISISTOODIFFICULT, or THISISODIFFICULTOT.

6.19: a) Represent the chessboard as a graph:

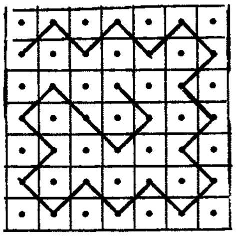


b) If you color a 5x5 chessboard in the normal black and white coloring, then there are 13 white cells and 12 black ones. To cover every vertex of the graph and end on your starting point requires 25 moves. But each move changes the color of the cell on which the rook is located. Therefore after 25 moves, the rook cannot end on a cell of the same color as the one on which it started. Therefore, the trip is impossible.

c) This difficulty arises with odd by odd boards. Even by even can be covered in a manner similar to (a).

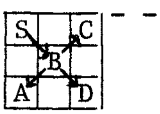
6.20: Consideration of vertices of degree two leads to a solution similar to the one shown below

a)



b) Since the four corners give four vertices of degree 1, a Hamiltonian path on the white cells is not possible.

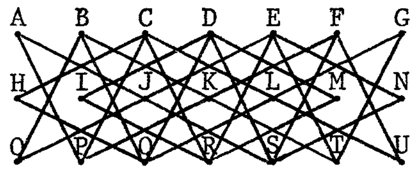
c) The two colors of the board are symmetric. The graph will have two vertices (at diagonally opposite corners) of degree 1. So the Hamiltonian path (if it exists) would have to start at one corner and end at the opposite corner. Consider the starting corner of the board

start→ 

From the starting corner, S, the path must proceed to B. Then to A or D or C. Therefore at least one of A or C is not covered and this vertex for path purposes becomes a vertex of degree 1, and the path will terminate at this vertex. But the path must terminate at the corner opposite to the start. So <u>a Hamiltonian path is impossible</u>.

d) Consider an n x n board, n greater than 5. If n is even then no bishop's tour is possible (see c). If n is odd, the corner cells are all of the same color--say, white--and then a tour on the white cells is impossible (see b). Try the black cells. Since the edge cells are all of degree two, a Hamiltonian circuit passing through a border cell would have to alternate between cells on the outer two rows and columns and the bishop would never get to the center.

<u>6.21</u>: Construct a graph:

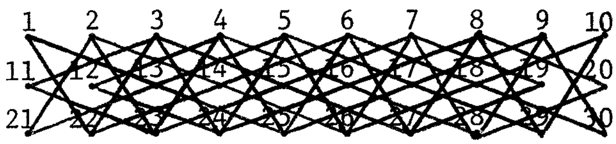


Vertices A,H,O,I,M,G,N and U are all of degree two. Hence, unless the tour begins or ends at one of these vertices, the tour must contain the segments BOJAP, CHQ, DIR, DMR, ENS, FULGT or their reversals.

Since DIRMD is a closed circuit, the tour must start at I or M.

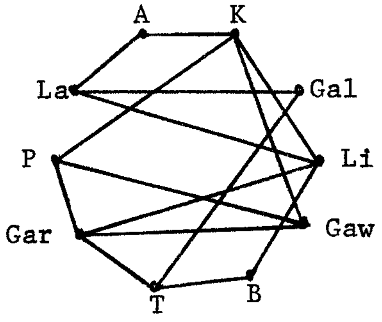
From here, it seems to be a matter of trial and error; it is not difficult to find a solution such as: <u>IDQHCPAJOBKTGLUFSNERM</u>.

<u>6.22</u>: Construct a graph:



Since 1,11,12,21,10,19,20,30 are of degree two, the reentrant tour must include 2-21-13-1-22, 3-11-23,4-12-24, 7-19-27, 8-20-28, and 9-30-18-10-29, or their reversals. With a little trial and error, we find a solution: <u>5-24-12-4-16-28-20-8-29-10-18-30-9-17-25-6-14-22-1-13-21-2-23-11-3-15-27-19-7-26</u>.

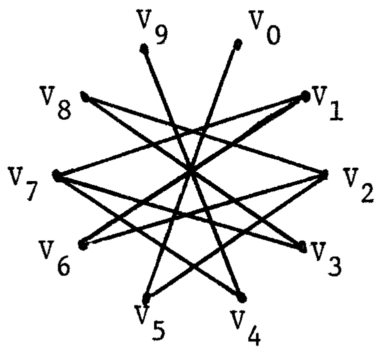
6.23: Following either method suggested in the hint, we obtain the following graph.



We are looking for a Hamiltonian circuit. One possibility is A-K-P-Gaw-Gar-Li-B-T-Gal-La-A.

6.24: Let f represent the farmer; w, the wolf; etc. There are 9 allowable states $(a_1,\ldots,a_k|b_1,\ldots,b_r)$, where $a_1,\ldots,a_k$ are located on the near shore of the river and $b_1,\ldots,b_r$ are on the far shore:

$V_0 = (f,w,g,c|\ )$; $V_1 = (f,w,g|c)$;
$V_2 = (f,w,c|g)$; $V_3 = (f,g,c|w)$;
$V_4 = (f,g|w,c)$; $V_5 = (w,c|f,g)$;
$V_6 = (w|f,g,c)$; $V_7 = (g|f,w,c)$;
$V_8 = (c|f,w,g)$; $V_9 = (\ |f,w,g,c)$.



The heavy lines are forced; there are two possible ways to connect V_2 to V_7 (via V_8 and V_3 or via V_6 and V_1) $V_0,V_5,V_2,V_6,V_1,V_7,V_4,V_9$ or $V_0,V_5,V_2,V_8,V_3,V_7,V_4,V_9$.

6.25: There are 21 allowable states:

$V_0 = (h_1h_2h_3w_1w_2w_3|\ )$
$V_1 = (h_2h_3w_1w_2w_3|h_1)$
$V_2 = (h_1h_3w_1w_2w_3|h_2)$
$V_3 = (h_1h_2w_1w_2w_3|h_3)$
$V_4 = (h_3w_1w_2w_3|h_1h_2)$
$V_5 = (h_2w_1w_2w_3|h_1h_3)$
$V_6 = (h_1w_1w_2w_3|h_2h_3)$
$V_7 = (h_2h_3w_2w_3|h_1w_1)$
$V_8 = (h_1h_3w_1w_3|h_2w_2)$
$V_9 = (h_1h_2w_1w_2|h_3w_3)$
$V_{10} = (w_1w_2w_3|h_1h_2h_3)$
$V_{11} = \overline{V}_{10} = (h_1h_2h_3|w_1w_2w_3)$
$V_{12} = \overline{V}_9 = (h_3w_3|h_1h_2w_1w_2)$
$V_{13} = \overline{V}_8 = (h_2w_2|h_1h_3w_1w_3)$
...
$V_{21} = \overline{V}_0 = (\ |h_1h_2h_3w_1w_2w_3)$

with the boat at the original bank (except for V_{21}).

The resulting graph is shown in Figure 6.1 on the top of the next page. From this figure, a path from V_0 to V_{21} is easily found. For example,

$V_0 \to V_2 \to V_5 \to V_9 \to V_{11} \to V_{13} \to V_{21}$.

Each step requires 2 crossings, (over and back), except for the last which only requires 1.

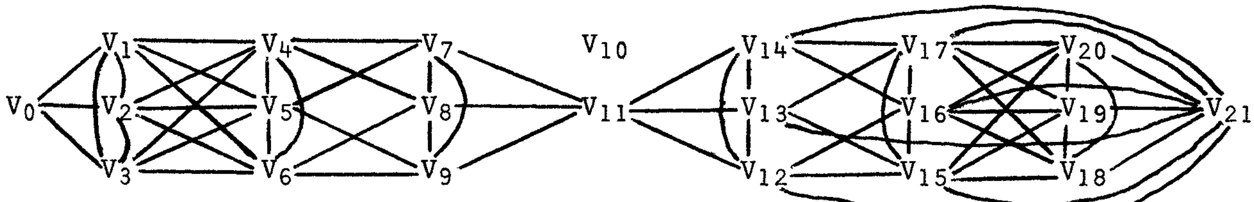


Figure 6.1

6.26:

$(h_1h_2h_3h_4h_5w_1w_2w_3w_4w_5| \ )$

$(h_1h_2h_3h_4h_5w_1w_2w_3|w_4w_5)$

$(h_1h_2h_3h_4h_5w_1w_2|w_3w_4w_5)$

$(h_1h_2h_3w_1w_2w_3|h_4h_5w_4w_5)$

$(w_1w_2w_3w_4w_5|h_1h_2h_3h_4h_5)$

$(w_1w_2w_3|w_4w_5h_1h_2h_3h_4h_5)$

$(\ |h_1h_2h_3h_4h_5w_1w_2w_3w_4w_5)$

6.27: One couple can ferry each of the other couples one at a time.

6.29: (Solve the problem for one man first; then two; generalize.)

Both boys cross; one brings the boat back; one man crosses; the second boy returns the boat. Repeat the procedure for each man.

Twenty trips are needed.

6.30: The safe situations for each to descend are:

descend	Hugo	Jon	Val	John
ascend		Hugo chest	Jon	Val Hugo chest

Clearly only Hugo can descend to begin with. Jon cannot descend unless the chest is also down, so the chest is lowered (Hugo has first stepped out of the basket at the bottom). Hugo enters the basket with the chest and Jon can now descend. Val is now able to descend, returning Jon to the tower; etc. (See the answer section of the book).

6.31: Who's on the far side of the river with the boat?

V_0 = {no one; no money}

V_1 = {Val; $10,000};

V_2 = {John;$20,000};

V_3 = {Jon;$30,000};

V_4 = {Val, John; $10,000, $20,000};

V_5 = {Val, John; $30,000};

V_6 = {Jon; $10,000, $20,000};

V_7 = {Val, Jon; $10,000, $30,000};

V_8 = {John, Jon; $20,000, $30,000};

V_9 = {everyone; $10,000};

V_{10} = {everyone; $20,000};

V_{11} = {everyone; $30,000};

V_{12} = {everyone; $10,000, $20,000}

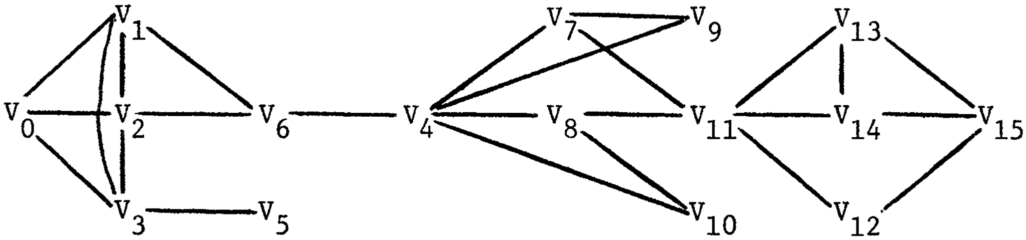
V_{13} = {everyone; $10,000, $30,000},

V_{14} = {everyone; $20,000, $30,000},

V_{15} = {everyone; all money}

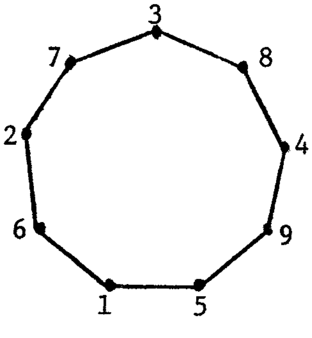
The solution may now be easily found from Figure 6.2 (next page).

Figure 6.2



6.33: The solution is almost identical to that of 6.32. Construct a bipartite graph, one set of vertices corresponding to the people, and the other corresponding to the numbers from 1 to n-1. (A person with no friends can be eliminated from consideration; the remaining people will each have at least one friend.) Apply the Cubby Hole Principle.

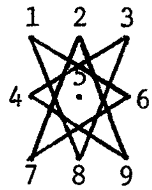
6.34: When we unravel the figure as is suggested in the hint, we obtain

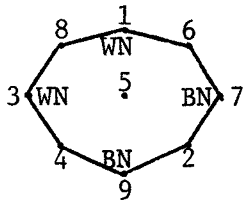


If a situation ever arises in which two uncovered points are separated, in both directions, by points which are covered, then it will be impossible to place all eight counters. Therefore, after the first counter is placed, each successive counter must end on a spot adjacent to a previously placed counter. One easy way to do this is: 6-1; 2-6; 7-2; 3-7; 8-3; 4-8; 9-4; 5-9.

6.35: Think of the checkerboard as a graph and unravel it as suggested in the hint:

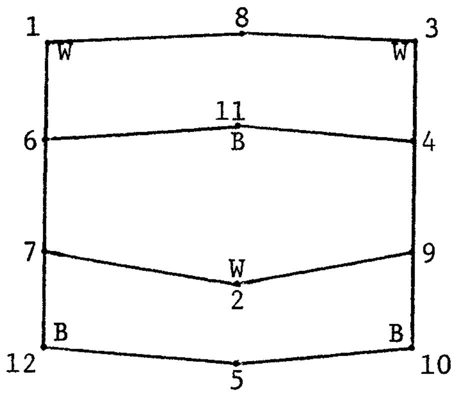
1	2	3
4	5	6
7	8	9





It is clear that all moves must be made in the same circular direction (i.e., all clockwise or all counterclockwise) in this graph to avoid two knights occupying the same cell at the same time. Each knight must make four moves, so sixteen moves are required in all: For example, 1-6; 3-8-1; 9-4-3-8; 7-2-9-4-3; 6-7-2-9; 1-6-7; 8-1.

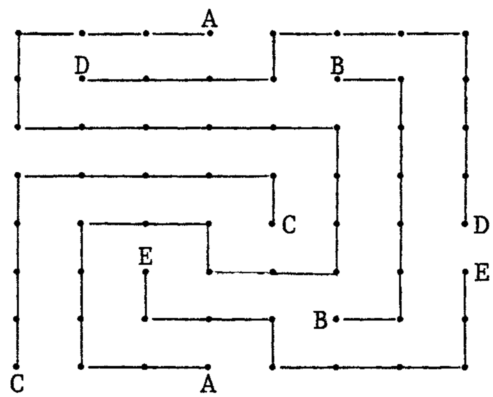
6.36: Unravel the graph, as suggested in the hint;



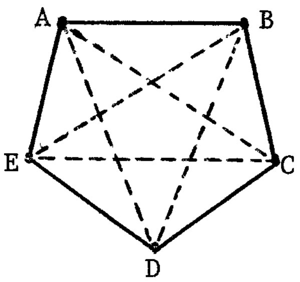
From here it is clear that a general clockwise (or counterclockwise) motion must be followed: There are four possible starting moves, after any of which all other moves are more or less forced (subject only to some slight variations in sequencing). For example, 1-8, 10-5, 2-9, 11-6, 3-4, 12-7, 8-3, 5-12, 9-10, 4-9, 3-4, 6-1, 7-6, 12-7, 10-5, 9-10, 1-8, 6-1, 4-11, 7-2, 8-3, 5-12. In any case, twenty-two moves are required.

6.37: For reference purposes, identify the cells of the grid by a coordinate system, with the lower left hand corner denoted by (1,1), the lower right by (8,1), etc.

Since the path connecting the A's divides the grid into two pieces, each pair of letters must be in one half of the board. Clearly the D's and E's must belong to the right half, and the C's must belong to the left half. The A-path must therefore pass to the left of the D on (2,7), to the right of the C on (5,4) and to the left of the E on (3,3). This implies that cells (2,2), (2,3) and (2,4) and cells (1,6), (1,7), (1,8), (2,8) and (3,8) are on the A-path, while the C-path must contain (1,2), (1,3), (1,4), (1,5), and (2,5). This in turn implies that the A-path contains (3,4), (4,4) and (4,3), (2,6) and (3,6); the C-path (3,5), (4,5) and (5,5); the E-path (3,2) and (4,2); etc. Since there is no longer room for the B-path to pass to the left of the E on (3,3), the E-path must pass below the B on (6,2). Continuing in this manner, we obtain the solution:

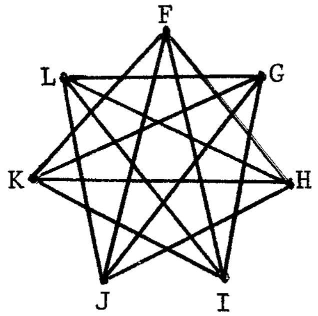


6.40: Constructing the graph suggested in the hint, we obtain



The dotted line graph gives the solution: A-C-E-B-D.

6.41: Constructing the complete graph on seven vertices and removing edges corresponding to the original seating arrangement, we obtain



The problem is now to decompose this graph into two disjoint subgraphs, each of which is a Hamiltonian circuit. This can be done in many ways. For example:
FHJLGIKF and FILHKGJF.

6.42: Graph theoretically, the problem is to decompose the complete graph on nine vertices into three disjoint triangles of one color, three disjoint triangles of a second color, three of a third and three of a fourth. This graph theoretical formulation is not really helpful though for the solution. To solve the problem, there is no loss in assuming that the grouping for the first day is {M,N,O}, {P,Q,R}, {S,T,U}. For the second day, each grouping must contain one person from each of the original groups. Again there is no loss in assuming they are {M,P,S}, {N,Q,T}, {O,R,U}. For the third and fourth days the grouping involving M must be {M,Q,U} and {M,R,T}. The other groupings may be determined similarly:

Third day: {M,Q,U}, {N,R,S}, {O,P,T}.

Fourth day: {M,R,T}, {N,P,U}, {O,Q,S}.

6.43: If the queen in the top row is moved horizontally, there are several possibilities. If it is moved to the first column, then the queen in the fifth row must be moved. But no matter where we move it, we will not be able to move only one more queen to cover every row and column and still maintain independence.

Continuing in this manner with structured trial and error, we find that the queen in the top row cannot be moved horizontally to solve the problem.

We next try the queen in the second row. This leads to a solution (see the solution section in the text).

6.44: The four queens already on the board dominate all cells except D7, A2, D2 and D1 (columns are labeled A to H from left to right; rows 1 to 8 from bottom to top). The only cell which dominates all of these is D2.

6.45: a) If there are more than n, then two must be in the same row, since there are only n rows.

b) One queen must be placed in each of the bottom four rows in the columns A,C,E and G. In column G, the queen must be in row 1 or 2 (from the bottom); in column E, it must be in row 1,3 or 4; in column C, row 1, 2 or 4; in Column A, row 2 or 3.

If G1, then E4 is forced and no possibility remains for C. Therefore, there is a queen in G2. This forces A3, C4 and E1.

6.46: Pair the cells of the board as indicated below

1	2	3	4	15	13
5	6	1	2	3	4
18	8	17	6	13	15
17	5	18	7	14	16
8	7	9	10	11	12
9	10	11	12	16	14

No two knights can occupy cells bearing the same number (otherwise they attack each other), so at most 18 knights may be placed on the board.

On the other hand, 18 knights may be placed, e.g., on all the black cells (no two knights on the same color cell can attack each other).

CHAPTER 7

COMMENTS AND SUGGESTIONS

1. We have included this chapter not only because we, as well as the students, enjoy the material, but also because it deals with an important aspect of problem solving-case analysis. Here again, no mathematical background is needed, and the student must employ many of the same techniques of problem solving discussed in Chapter 1 - simplification, deja vu, etc., as well as case analysis.

2. In the past, we have tried two different approaches to this chapter. One is to define the notions of winning, losing, and drawing strategies when we first discuss the matchstick game in Chapter 1. With this approach, we introduce other games as they are relevant to the various chapters. Lectures on analyzing a game by working backward, on a frontal attack, on symmetry as a limiting factor and on déjà vu are presented as the opportunity arises. Chapter 7 is not really considered as a separate entity.

Under the second approach, the chapters are covered in the order in which they appear in the text, and the discussion of winning and losing strategies and analyzing games is delayed until Chapter 7 is considered. At this time, when appropriate, we recall concepts considered earlier, and we apply them to the analysis of some of the games in this chapter.

We have found that the first approach works better in a one semester course set up as ours is. On the other hand, in a two semester course we would probably prefer the second approach, spending a good part of the second semester on a variety of games.

3. Our students usually enjoy this chapter very much. When time permits, we include some classes devoted to individual tournaments in some of the games or divide the class into teams and set one team against another. In a full year course, we would include more activities of this type.

4. Many of our former class members who have had some experience with computer programming have enjoyed programming some of these games - either so that they can play against the computer or else to have the computer aid them to extend the analyses of some of the games beyond the cases proposed in the text.

5. The relationships between games of this chapter and various mathematical concepts (tree diagrams, congruences, the binary system, graph theory) discussed in previous chapters have already been noted in the cross references sections of the

earlier chapters of this manual. In addition, there are also relationships between some of these games and other mathematical notions. The game of Sprouts can be used to introduce a discussion of topology; some of the games (e.g., Exercise 7.52) are related to considerations of parity - a topic which has cropped up from time to time.

6. Problems that are triple starred and some that are double starred may be lengthy and difficult to analyze. In fact, in the triple starred cases, we have not completed the analyses ourselves.

7. Some of the arguments in the text and exercises are very interesting and worth discussing in some detail. Among these are

a) the argument in the text relating to the game of Hex which presents an existential proof of a winning strategy for A even though the actual strategy is not known.

b) the argument presented in the solution of Exercise 7.40b to the effect that a drawing strategy which does not result in a draw must be a winning strategy.

c) the use of symmetry not only to analyze a game but also as a playing strategy. Some examples of when it does work (Exercises 7.16 regular form, 7.39a, 7.44, 7.48, 7.52, 7.53, 7.54. 7.55, 7.64) and some of when it does not (Exercise 7.16 misère; 7.49d,e) should be included.

d) the discussion of the join of two games, considered in the solution of Exercise 7.19a. This can also be related to Exercises 7.8, 7.16, 7.19b, 7.56 and 7.61c.

8. When we first began teaching this course, we used to assign term papers for the students to hand in. For these papers,the students had to choose some game or game variation that we had not considered in class and analyze it or a limited version of it. If it was possible to find information on the history of the game or on variations of it, that was to be included in the paper. We found papers to be worthwhile projects for our students, but discontinued assigning them because checking the analyses of all the games considered became too time-consuming for us. We may try it again in the future, and would almost definitely do so if Chapter 7 were treated during the second semester of a two semester course.

9. We have found that the students also enjoy working on some of the current or new games which are discussed periodically in Martin Gardner's column in Scientific American magazine.

CROSS REFERENCES

1. Solutions of some of the matchstick games (Sample Problem 7.1 and Exercises 7.1, 7.2, 7.3, 7.4, 7.5, 7.7, 7.8, 7.20, 7.24) can be discussed more easily if congruence notation (Chapter 4) has been introduced.

2. The game of Nim (Sample Problem 7.4) and some of the matchstick games in the exercises (7.13 and 7.14) relate to the binary system discussed in Chapter 5.

3. The game of Sim (Exercises 7.68 and 7.69) relates to Exercise 6.39; and many of the other games (Dots, Hex, Bridgit, Slither, Sprouts) in the exercises of the chapter can be presented as graph-related problems.

PRACTICE PROBLEM ANSWERS

7.B:

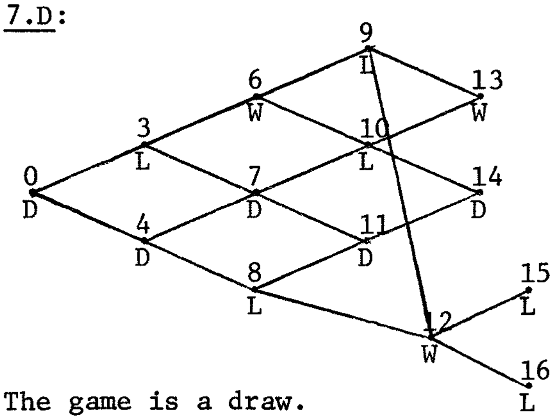
3. The argument does not apply since a player's earlier moves may hurt later on. That is, the opponent may complete four in a row by making use of an X originally placed by the player.

4. In Bridgit the same argument used for the game Hex may be applied. (That is, the game may not end in a draw and a player's earlier moves cannot hurt later on.)

7.C:

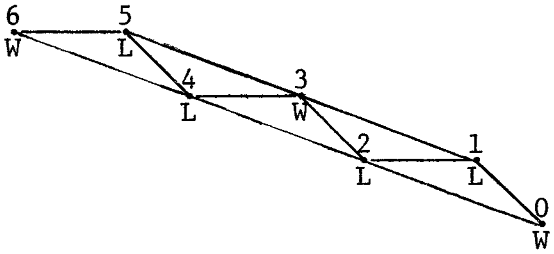
1. See Figure 7.1 (on the following page). A wins by leaving 7 and then leaving 2 after his next move.

7.D:



The game is a draw.

2.



The first player loses, since the initial position, 6, is a winning position.

7.E:

1. Our conjecture is:

I (winning positions): 4k

II (losing positions): 4k+1,4k+2,4k+3

No drawing positions.

We must check (1), (2), and (3).

(1): The only terminal position is 0 = 4·0 sticks. This position is on list I, and the player who leaves it does win the game. So (1) checks.

(2): a) A move of "take 1" from a position 4k leaves 4k-1 = 4(k-1)+3, a position on list II.

b) A move "take 2" from a position

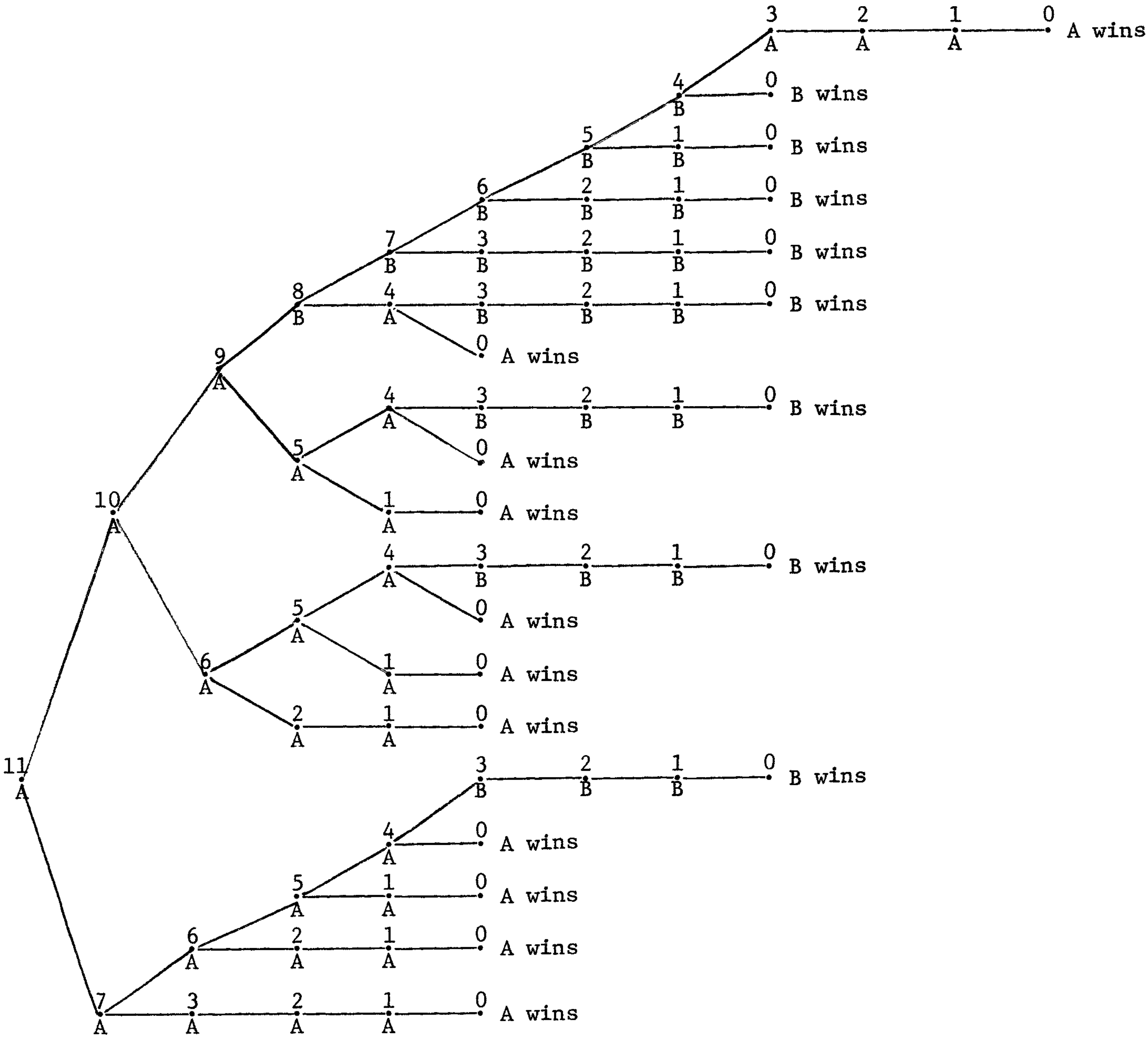

3 A 2 A 1 A 0 A wins
4 B 0 B wins
5 B 1 B 0 B wins
6 B 2 B 1 B 0 B wins
7 B 3 B 2 B 1 B 0 B wins
8 B 4 A 3 B 2 B 1 B 0 B wins
9 A 0 A wins
4 A 3 B 2 B 1 B 0 B wins
5 A 0 A wins
10 A 1 A 0 A wins
4 A 3 B 2 B 1 B 0 B wins
5 A 0 A wins
6 A 1 A 0 A wins
2 A 1 A 0 A wins
11 A 3 B 2 B 1 B 0 B wins
4 A 0 A wins
5 A 1 A 0 A wins
6 A 2 A 1 A 0 A wins
7 A 3 A 2 A 1 A 0 A wins

Figure 7.1

4k leaves 4k-2 = 4(k-1)+2, a position on list II.

c) A move of "take 3" from a position 4k leaves 4k-3 = 4(k-1)+1, a position on list II.

(3): A move of "take 1" from the position 4k+1 leaves a position on list I. A move of "take 2" from the position 4k+2 leaves a position on list I. A move of "take 3" from the position 4k+3 leaves a position on list I.

7.F:

2. (a), (b), and (d) are equivalent; (c) and (e) are not equivalent to any of the others.

4. Put an X in the bottom left corner of (b).

7.G:

2. The pair in (a) are not equivalent; the pairs in (b) and (c) are.

7.H:

2. a) You win by placing 3 checkers in column 2.

b) You win by placing a checker in the middle box of the top row.

c) You win by placing a checker in the center box.

d) You do not have a winning move.

7.I:

2. a) losing; b) losing; c) losing.

4. a) Take 9 from the 16 pile, leaving (3,7,8,12);

b) Take 6 from the 7 pile, leaving (1,8,9);

c) Take 1 from any pile, leaving (6,11,13), (7,10,13) or (7,11,12).

SOLUTIONS TO EXERCISES

7.1: Working backward, find winning, losing, and drawing positions:

regular:

win	0,6,12,18,...	6k
lose	2,3,4,8,9,10,14,15,16,	6k±2,6k+3
draw	1,5,7,11,13,17,...	6k±1

The pattern is clear and is easily justified.

misère:

win	2,8,14,...	6k+2
lose	0,4,5,6,10,11,12,...	6k,6k+4,6k+5
draw	1,3,7,9,13,...	6k+1,6k+3

7.2: Work backward:

regular:

win	0,9,18,...	9k
lose	3,4,5,6,12,13,14,15	9k+3,4,5,or 6
draw	1,2,7,8,10,11,16,17	9k+1,2,7,or 8

misère:

win	3,12,21,...	9k+3
lose	0,6,7,8,9,15,16,17,18	9k+0,6,7,or 8
draw	1,2,4,5,10,11,13,14	9k+1,2,4,or 5

7.3: Work backward:

regular:

winning: $(m+j)k$;

losing: $(m+j)k+j+r$, $0 \le r \le m$;

drawing: $(m+j)k+r$, $1 \le r \le j-1$ or $j+1 \le r \le m+j-1$

misère:

winning: $(m+j)k+j$;

losing: $(m+j)k+r$; $2j \le r \le j+m$;

drawing: $(m+j)k+r$, $1 \le r \le j-1$ or $j+1 \le r \le 2j-1$

7.4: Work backward:

regular:

win	0,2,7,9,14,16,21,23,28
lose	1,3,4,5,6,8,10,11,12,13,15,17,18, 19,20,22,24,26,27

winning: 7k, 7k+2;

losing: 7k+1, 7k+3, 7k+4, 7k+5, 7k+6.

misère:

win	1,3,8,10,15,17,22,24
lose	0,2,4,5,6,7,9,11,12,13,14,16,18 19,20,21,23,25,26,27,28

winning: 7k+1, 7k+3;

losing: 7k, 7k+2, 7k+4, 7k+5, 7k+6.

7.5: As is suggested in the hint, consider positions (r,t) where r represents the number of sticks remaining and t represents the number of sticks taken on the previous move. ((r,≠t) and (r) are also defined as in the hint.)

After making the chart shown in Figure 7.2 (at the top of the following page), we find that, starting with 28, A wins by taking 1,2 or 4.

7.6:

win	0,2,5,7,10,12,15,17,20,22,34,39, 44,52,57
lose	1,3,4,6,8,9,11,13,14,16,18,19,21, 23,24,25,26,27,28,29,30,31,32, 33,35,36,37,38,40,41,42,43,45, 46,47,48,49,50,51,53,54,55,56, 58,59,60

A wins by taking 16.

7.7: As suggested in the hint, consider positions (n,p) where n represents the number of sticks remaining and p = e (even) or o (odd), the parity of the number of sticks possessed by the player who has just moved. Note that if you leave (n,p) with n odd then your opponent possesses parity p; and if n is even, your opponent possesses the opposite parity.

regular:

win	(0,o),(1,o), (4,e),(5,e), (8,o),(9,o),...
lose	(0,e),(1,e),2,3,(4,o),(5,o),6,7, (8,e),(9,e),...

winning: (8k,o), (8k+1,o), (8k+4,e), (8k+5,e).

misère:

win	(0,e),(1,e), (4,o),(5,o) (8,e),(9,e),...
lose	(0,o),(1,o),2,3,(4,e),(5,e),6,7, (8,o),(9,o),...

winning: (8k,e), (8k+1,e), (8k+4,o), (8k+5,o).

win	(0),(1,1) (3,3) (4,4) (5,5) (6,3), (7) (11,4), (12,5),(13)
lose	(1,≠1),(2),(3,≠3),(4,≠4),(5,≠5),(6,≠3),(8),(9),(10),(11,≠4),(12,≠5),

win	(14,1), (16,3), (18,5), (19,3),(20),(21,1) (24,4),
lose	(14,≠1),(15),(16,≠3),(17),(18,≠5),(19,≠3), (21,≠1),(22),(23),(24,≠4),

win	(25,5), (26), (27,1)
lose	(25,≠5), (27,≠1), (28)

Figure 7.2

7.8: Work backward:

regular:

win	(0,0),(1,1),(0,4),(2,2),(1,5) (3,3),...
lose	(0,1),(0,2),(0,3),(1,2),(1,3), (0,5),(1,4),(2,3),(0,6), (2,4),...

winning: (m,n), where m ≡ n (mod 4).

misère:

win	(0,1),(2,2),(0,5),(1,4),(3,3),...
lose	(0,0),(0,2),(1,1),(0,3),(1,2), (0,4),(1,3),(2,3),(0,6),(1,5), (2,4),(0,7),(1,6),(2,5),(3,4),..

winning: (m,n), where m ≡ 0 and n ≡ 1, or m ≡ 1 and n ≡ 0, or m ≡ n ≡ 2, or m ≡ n ≡ 3 (mod 4).

7.9: regular: clearly (0,0) is a winning position and (0,a) and (a,a) are losing for a > 0; (1,2) is winning, and (1,a) and (a,a+1) are losing for a > 1. Continuing in this manner, we find that the winning positions are: (0,0), (1,2), (3,5), (4,7), (6,10), (8,13), (9,15), (11,18), (12,20).

Starting with (11,15), A wins by leaving (9,15) or (6,10); starting with (12,20), B wins.

misère: Here the winning positions are (0,1), (2,2), (3,5), (4,7), (6,10), (8,13), (9,15), (11,18), (12,20),... Therefore, we obtain the same answer as in the regular version.

7.10: If, at any point, only two piles remain, the game reduces to the game in Exercise 7.9. Since (3,5) is a winning position in both the regular and misère versions of that game, A can win both the regular and misère versions of the game in 7.10, by wiping out the 7 pile on his first turn, leaving (0,3,5).

7.11: Working backward, making use of the results of Exercise 7.9, obtain the following winning positions: (0,0,0), (0,1,2), (0,3,5), (0,4,7), (1,1,3), (1,4,4), (1,5,5), (1,6,6), (1,7,7), (2,2,3), (2,4,5), (2,6,7), (3,3,6), (3,4,8), (3,7,9).

Therefore, B wins.

7.12: Work backward to find winning positions:

regular: winning: (k,k,k)

misère: winning: (0,0,1) and (k,k,k), k > 1.

Therefore, in a), A wins by leaving (3,3,3); and, in b), A wins by leaving (m,m,m) if m > 1 and (0,0,1) if m ≤ 1.

7.13: The winning positions are those for which the column sums are all divisible by three when the numbers are expressed in the binary system.

In a), A can win in a number of different ways. He can leave (2,5,7,7), (3,4,7,7) or (3,5,6,7).

7.14: The winning positions are those for which the column sums are all divisible by m+1 when the numbers are expressed in the binary system.

7.15: regular: Working backward, we see that (0,0,0) is winning; (0,0,a), a > 0 is losing; (0,1,1) is winning; (0,1,2) is losing; (0,1,3) and (0,2,2) are winning. A pattern emerges: (0,odd,odd) and (0,even,even) are winning; (0,odd,even) are losing. From any starting three pile position, we can leave (0,odd,odd) or (0,even,even) by wiping out an appropriate pile. Therefore A always wins if we start with three piles.

misère: (0,0,1), (0,0,3),..., (0,0,odd), (0,even,even) are winning positions. With three piles, if two or more are even, the position is clearly losing. (1,1,1) is winning; (1,1,2), losing; (1,1,3) winning; (1,2,3) or (1,1,4) losing; etc. (odd,odd,even) is losing and (odd,odd,odd) winning. Therefore, starting with (3,5,7), B wins.

7.16: regular: Clearly A can win by taking the middle stick (or the middle two sticks if n is even) and then playing by symmetry.

misère: Here the situation is complicated by the fact that the first player to play in the four stick game can force the other player to win that game. This means that a symmetry strategy won't work, since from (4,4) the next player can leave (4,1,1) and win. As a result, a complete analysis of the general misère game seems difficult.

For the 13 stick game, the analysis may proceed by working backward. (Note that the number of sticks plus the number of piles remaining cannot exceed 14.)

We list the winning positions: (1); (1,1,1); (4); (2,2); (1,1,1,1,1); (5,1); (3,3); (4,1,1); (3,2,1); (2,2,1,1); (1,1,1,1,1,1,1); (5,1,1,1); (3,3,1,1); (2,2,2,2); (4,1,1,1,1); (3,2,1,1,1); (2,2,1,1,1,1); (9); (8,1); (7,2); (6,3); (5,4); (6,2,1); (4,4,1); (4,3,2); (4,2,2,1); (3,3,2,2); (9,1,1); (7,3,1); (5,5,1); (12); (10,2); (8,4); (6,6).

So A can win by leaving 12, (10,2), (8,4), or (6,6).

7.17: regular: Regardless of whether A takes one or two sticks, he leaves the game of Exercise 7.16, which the first player can win. Therefore B can always win the regular game (as long as n > 2).

misère: As in 7.16, A can win by leaving 12 sticks.

7.18: The game always lasts n-1 moves, since each move increases the number of piles by 1 and the game ends when there are n piles (note the connection with Exercise 9.22).

7.19: b) (Refer to the solution in the text for the notation and some important ideas.) A can win by leaving (12,3). B must then leave (12,2,1), (11,3,1), (10,3,2), (9,3,3), (8,4,3) or (7,5,3). The first and third of these each consists of two winning and one losing position, and hence these are losing positions.

From (9,3,3) or (7,5,3), A can leave (7,3,3,2) - working with one 3 pile when B works with the other, and otherwise winning the 7 game - to eventually win.

From (8,4,3), A leaves (6,4,3,2). B must then leave (6,4,2,2,1), (6,3,3,2,1), (5,4,3,2,1) or (4,4,3,2,2). From the first of these A wins by leaving (4,4,2,2,2,1), and from the others A wins by leaving (4,3,3,2,2,1).

From (11,3,1), A must leave (9,3,2,1). If B responds with (9,2,2,1,1) or (7,3,2,2,1), A leaves (7,2,2,2,1,1) - a winning position; if B responds with (6,3,3,2,1) or (5,4,3,2,1), A leaves (4,3,3,2,2,1) - a winning position for the same reason that (7,3,3,2) is; if B responds with (8,3,2,1,1), A must leave (6,3,2,2,1,1). B can then leave (6,2,2,2,1,1,1), (4,3,2,2,2,1,1) or (5,3,2,2,1,1,1), from the first two of which A wins with (4,2,2,2,2,1,1,1) and from the third A wins with (3,3,2,2,2,1,1,1).

7.20: a) Here A will win if n is even - he leaves two odd piles. B must choose one of them and leave an odd and an even. A will choose the even and leave two odd; etc. Since A will always have a move, A will win.

b) If one player leaves piles of m and n sticks, then the other player will win if either m or n is such that the first player wins a game starting with that number of sticks. This suggests working backward to find all n such that A wins the n stick game.

Doing so leads to the following conjecture: A wins unless n = 2 or $n \equiv 1 \pmod 3$. To prove this conjecture, note that if $n \equiv 0 \pmod 3$, A can break the pile into 2 and n-2 ≡ 1 (mod 3); and if n ≡ 2 (mod 3), A can break the pile into 1 and n-1 ≡ 1 (mod 3). In either case, B will have to choose the pile ≡ 1 (mod 3) and break it into two piles which are both ≡ 2 (mod 3) (and at least one of which is not equal to 2) or which are congruent to 0 and 1 (mod 3) respectively. In either case, A will be able to choose a pile ≡ 0 or 2 (mod 3) and to proceed as above. Since, by following this strategy, A will always have a move, and since the game is finite, A will win.

7.22-24: A position in this game may be represented by (n,k), where n is the number still needed to reach the goal and k is the number on the top face of the die. Clearly, (0,any) is a winning position. Working backward, we obtain Figure 7.3 below.

The pattern repeats itself modulo 9. (I.e., 12 is the same as 3, etc.)

By placing the die so that a 4 is on top, A leaves the position (22,4). Since 22 ≡ 4 (mod 9). A wins the game in Exercise 7.22.

Since 27 ≡ 0 (mod 9) is a winning position, B can win the game in 7.23 by choosing 27.

Since (2,any), (6,any), (7,any) and (10,any) are losing positions, A can win the game in Exercise 7.24 by choosing any number congruent to either 1,2,6 or 7 modulo 9.

7.25: Working backward, the winning dates are Nov. 30, Oct. 29, Sept. 28, Aug. 27, July 26, June 25, May 24, April 23, March 22, Feb. 21, and Jan. 20.

7.26: B can win by always moving toward A and ending on the same diagonal as A.

7.27: A wins - See the Answer section in the text.

7.28: As the hint suggests, this game is essentially Nim, with the numbers of spaces between the counters in a row taking the place of the number of sticks in a pile. The only difference is that the number of

win	(0,any);(1,1 or 6); (3,3 or 4);(4,3 or 4);(5,2 or 5); (8,3 or 4);(9,any);
lose	(1,other);(2,any);(3,other); (4,other); (5,other);(6,any);(7,any); (8,other); (10,any);(11,any)

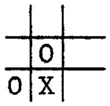
Figure 7.3

spaces in a row may be increased temporarily.

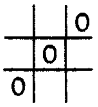
In the 3 × 8 game, A wins by moving one counter all the way to the right, leaving (0,6,6).

In the m × n game the starting position is (n-2,n-2,...,n-2) which is even (winning) if m is even and odd (losing) otherwise. Therefore, A wins if m is even.

7.31: A wins. Unless B allows A to win on his second turn, the position after three moves will be



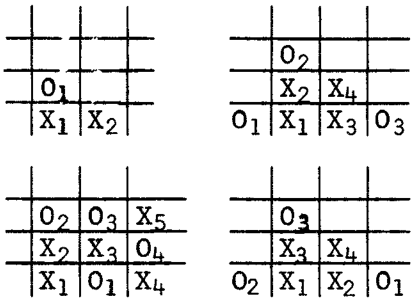
and A will eventually complete



unless B allows him to win sooner.

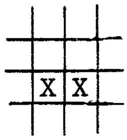
7.32: A wins. He starts in the bottom row second column.

B has four possible responses. In each case, A can essentially force a win, by establishing a double threat.

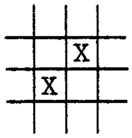


7.33: A wins by placing his first X in the center and continuing as follows, depending on whether B places her first O in a corner or a side.

7.34: a) A wins by establishing a double threat:

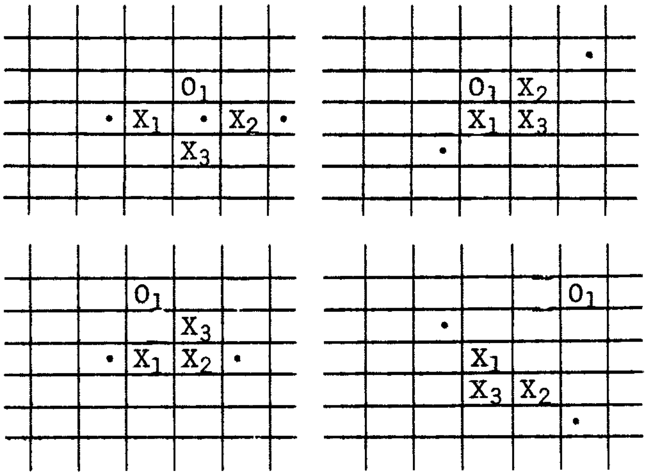


or



b) B can ensure a draw by using central symmetry. We believe that A can also ensure a draw, but we have no proof.

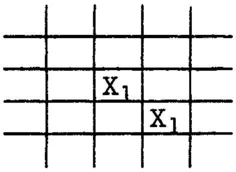
7.35: A wins by starting in the center. B must try to prevent A from getting three in a row with both ends free, but cannot succeed. Some sample situations are:



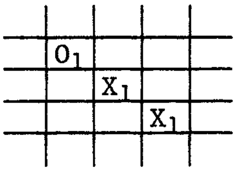
etc.

O_2 must be in a cell with a dot to prevent A from making an open 3 on his third move. But now A can't be stopped from making one on his fourth (or fifth) move.

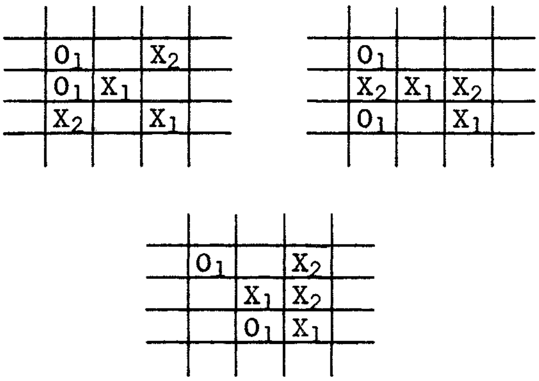
7.36: A wins by starting as follows:



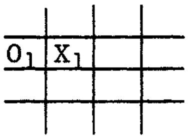
B must place one O as follows:



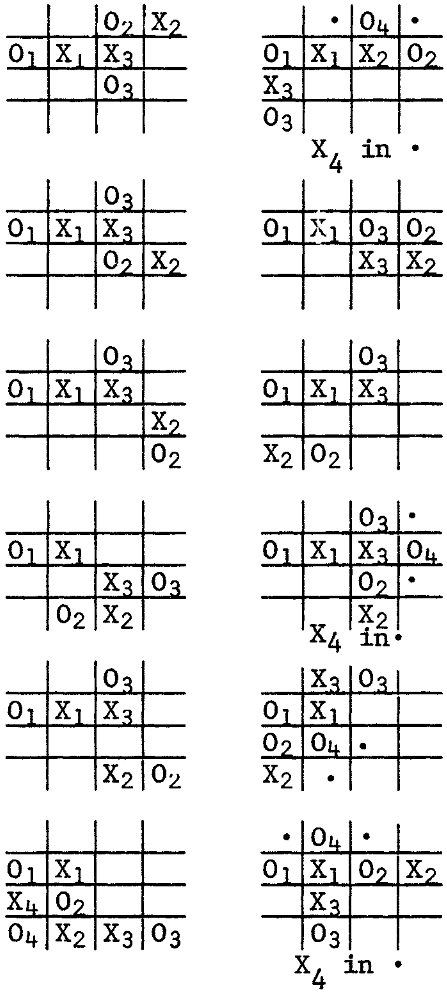
Regardless of where B places her second O, A can block (if necessary) and at the same time set up two double threats. For example,



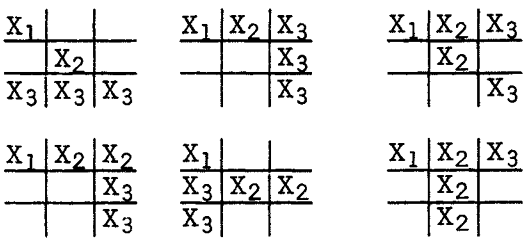
7.37: A wins by starting as follows:



B has several possible responses. For each, either A can win immediately or else A can force a win. The latter cases are shown below:



7.38: This game is the misère version of the game in Sample Problem 7.2, and has the same symmetries as that game. A wins by taking one box.



These are all clearly winning positions.

7.39: a) A wins by taking the middle row and then playing by central symmetry.

b) A wins by taking the middle row and then playing as follows:

X_2	X_2	X_2
X_1	X_1	X_1
X_3	X_3	

X_2	X_2	
X_1	X_1	X_1
X_3	X_3	X_3

X_2		
X_1	X_1	X_1
X_3		

	X_2	
X_1	X_1	X_1
X_3	X_3	

7.40: a) A wins by starting in the center. By symmetry, B has three possible responses. A wins in each case as follows.

Case 1:

O_1		

bottom

X_2	X_1	O_2
X_3		

middle

top

Case 2:

	X_2	X_3

bottom

	X_1	
O_1		

middle

	O_2	

top

Case 3:

bottom

X_3		O_2
	X_1	
X_2	O_1	

middle

top

7.41: a) A wins by starting in the middle box of the bottom level. B must play in a corner on that level or else A wins on the bottom level.

The game then proceeds as follows:

	O_3	
	X_1	
O_1	X_3	

bottom

	X_2	
	X_4	

middle

	O_2	

top

and A wins on his next move.

b) A wins by forcing B to complete three in a row on the second level. A starts in the middle of the bottom level and then plays by central symmetry on that level. Whenever B plays on the second level, A plays on top of B.

The bottom level will end in a draw, but B will complete three in a row on the second level.

7.42: A wins by playing as follows: (Note: B's first move is forced, or A will win immediately.)

	O_1	

bottom

	X_1	

middle

O_3	X_2	O_2
X_3		

top

A now has a double threat.

7.43: A wins as follows:

	X_2	
X_3	O_1	
		X_5

bottom

	O_2	
O_3	X_1	
		O_5

middle

	O_4	
	X_4	

top

The order of moves 2,3,4 and 5 is not significant, but, after the fifth move, the position will be essentially as above. B now has no safe move.

7.44: A wins -- He places an X on the middle box and then plays using symmetry until B gives him an opportunity to win.

<u>7.45</u>: a) <u>A wins</u> by starting in the fifth box.

There is at most one safe move on the left and one on the right. Therefore, on her second move, B will have to leave A the opportunity to win.

b) <u>B wins</u>. No matter what A's first move is, B can ensure that there are exactly two safe moves remaining by placing her first X six boxes away from A's.

<u>7.46</u>: a) <u>B wins</u>. By symmetry, we may assume that A starts in 1,2,3,4 or 5. If A starts in 5, B plays in 6 and continues by symmetry.

If A starts in 3 or 4, B plays in the other of the two.

B's next move will be in 8 unless A plays in 8 in which case B will play in 7. In all cases, two safe moves will remain.

If A starts in 2, B can play in 3.

This essentially leaves a 1 × 6 game (in boxes 5-10). Such a game has four safe moves regardless of the order of play.

If A starts in 1, B can play in 8.

Unless A now plays in 7 or 10, B can play in 9 -- again leaving a 1 × 6 game (columns 1-6) in which two moves have been made and in which two moves remain. If A's second move is in 7, B plays in 3 and only two safe moves remain. Finally, if A's second move is in 10, B plays in 3 and again two safe moves are left.

b) <u>B wins</u>. Again we may assume, by symmetry, that A starts in 1,2,3,4 or 5.

If A starts in 1 or 2, B plays in the other. This leaves a 1 × 6 game (columns 4-9), which B wins in four more moves.

If A starts in 3 or 4, B can play in the other. B's next move will be in either 6 or 9 and then two more safe moves will remain.

If A starts in 5, B plays in 4. This leaves a 1 × 2 game (columns 1 and 2) and a 1 × 3 game (columns 7,8 and 9). In each of these games, two safe moves remain.

Thus B makes the last safe move in all cases, and so B wins.

c) <u>A wins</u> by starting in column 6. By symmetry, we may assume B's first move is in column 1,2,3,4 or 5.

If B plays in 2 or 4, A plays in the other, leaving

.

Columns 6-11 may be considered as a 1 × 6 game in which the first move has been made.

No matter how play continues, there will be three more safe moves in that game and a safe move in column 1 -- a total of four safe moves, so A will make the final safe move.

If B plays in 3 or 5, A plays in the other, leaving

His next move will be in either 8 or 11, and then only two safe moves will remain.

If B plays in 1, A plays in 5, leaving

His next move will again be in 8 or 11, and then two safe moves will remain.

Thus A makes the last move and will eventually win.

7.47: a) B wins on her first turn by putting her two X's next to one placed by A.

b) A wins by starting in the two central cells. This essentially leaves two 1 × 6 games (columns 1-6 and 11-16), each of which can accommodate exactly four X's safely regardless of the sequence of play.

7.48: a) and b) A wins by starting in the center and then playing by central symmetry until B leaves an opportunity for A to win.

7.49: a) B wins on the fourth move.

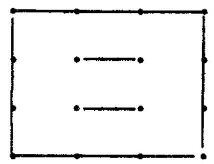
b) Either player can ensure a draw by connecting the two middle dots at the first opportunity.

c) B can ensure a win by drawing one of the central vertical lines at her first opportunity. The best A can then do is win one box but he will then have to give up two.

d) and e) We believe that the game is a draw in both cases, but we don't have complete proofs.

In d), a drawing strategy for B seems to be to use central symmetry until A gives her an opportunity to take a box. The same seems true for A in e), if he starts by taking the central horizontal line.

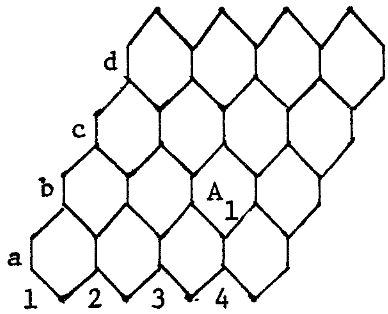
It's interesting to note that this symmetry idea does not generalize to larger boards. For example, if B uses symmetry on a 4 × 4 board then the following position arises.



A now gives up the central box and wins the outer eight.

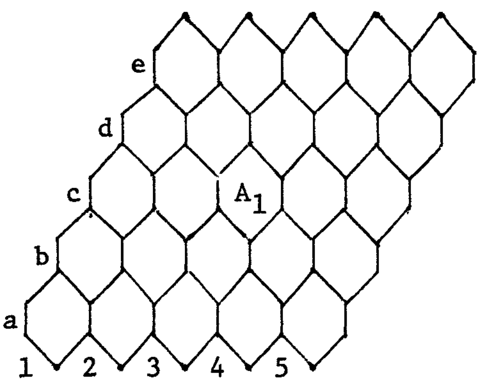
7.50: a) i) A wins by placing his first marker in the center. He can't be cut off from either the top or bottom of the board and wins on his third turn.

ii) A wins by placing his first marker in cell b3. A cannot be cut off from the bottom of the board.



If B does not play in d1, d2, or c2 then A can play in c2; and if B does not play in d2, d3, or c3 then A can play in c3. Therefore, B's only hope to cut A from the top is to play in d2. But then A can play in c4, obtaining an unstoppable chain.

iii) A wins by starting in the center. As in ii, B must play in a4 to cut A from the bottom, but then A plays in b2. Similarly B must play in e2; but then A plays in d4.

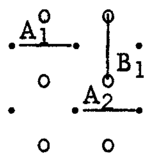


b) Solutions for i) and ii) are given in the text. It has been proved by R. Winder that B wins the n × n game when n is odd and that A wins when n is even, but we do not know what the winning strategy is.

7.51: a) This is a game like Hex, which cannot end in a draw and for which a player's early moves cannot hurt later on, so A can win in all cases. A winning, pairing, strategy, due to O. Gross, may be found in Martin Gardner's New Mathematical Diversions from Scientific American.

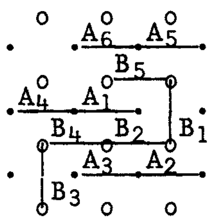
On these small boards, A may also win as follows:

i)

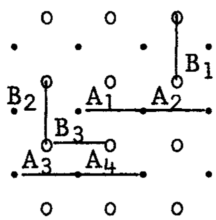


B_1 is forced. Now A has a double threat.

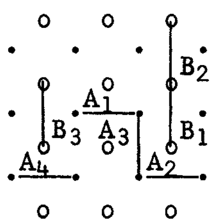
ii) A starts in the center:



or

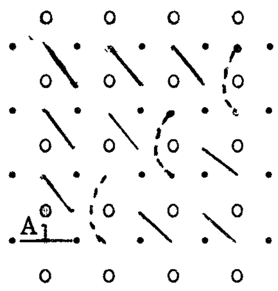


or



If B_1 does not block A_1 immediately, then A completes his connection to the side of the board on which B played. Each subsequent move of B is forced, until A establishes a double threat. The same is true if B_2 deviates, etc.

iii) To show that A wins in this case, we consider the strategy of Gross, mentioned above. Pair the moves of the game as indicated below.



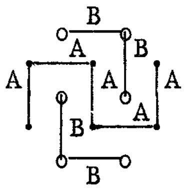
where A_1 indicates the recommended opening move for A. Each time that B makes a move which passes through one end of one of the lines above, A is to make the move which passes through the other end of that line.

Since the game cannot end in a draw (by essentially the same argument used for Hex) either A or B will win. Suppose that B has a winning path connecting the top of the board to the bottom. This path must pass through one of the three dots which lie to the right of the leftmost points of the curved dotted lines in the figure. We will refer to the imaginary line through these dots as "the diagonal." The path must reach the bottom side of the board to the right of the diagonal and the top of the board to its left.

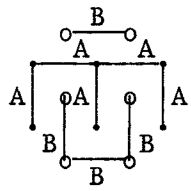
We follow the "winning path," starting from the top. Because of A's strategy, this path cannot turn right at least until it reaches the diagonal. It therefore must reach the diagonal by means of a top to bottom vertical move. By A's strategy, the way down will then be blocked and so the path will have to turn left or right. If it turns left, it will again lie above the diagonal; if it turns right, the way down will again be blocked by A's strategy. In fact, the path will never be able to move downward below the main diagonal and will therefore never reach the bottom of the board. It is therefore not a winning path.

Therefore, A must be the winner.

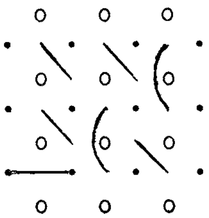
b) i) B wins. After each player makes two safe moves around the sides, A will have to enter the center of the board. B will win by playing once in each of the two central columns.



or



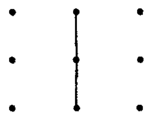
ii) Again B wins. Here we can make use of the ideas presented in the solution of Exercise 7.50b in the text, together with Gross' pairing pattern mentioned above.



Whenever A makes a move for which the paired move has not yet been made, B makes that paired move. Otherwise, B makes a free move. At the end, A will have made one move in each pair as well as the move shown by the horizontal line above. (Recall that A makes the first move and hence an extra move.) By the argument in a) iii), A will have a path from bottom to top and so B will win.

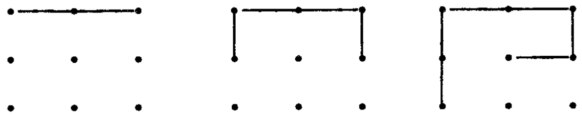
Using Gross' pairing pattern for the general $n \times (n+1)$ game, we see, by the argument above, that B wins the misère version.

7.52: a) i) B wins by preventing A from isolating any points. By symmetry, there are two possible opening moves for A. If A plays using the central dot, then B leaves

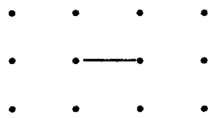


and continues by central symmetry.

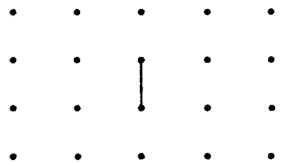
If A starts on a side, then B leaves the following positions (or equivalents) after the 2nd, 4th and 6th moves respectively.



ii) and iv) A wins by starting in the center and continuing by central symmetry.

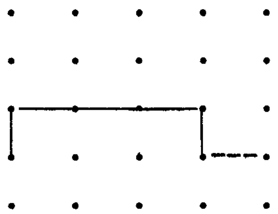


or



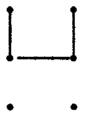
iii) A wins by starting in the middle of any row (column) and then playing by vertical (horizontal) symmetry.

v) B can win by preventing A from isolating an odd number of dots. However, she must play carefully to do so. For example, from a position such as the one shown below, B must not move as the dotted line indicates, or else A will be able to isolate the eleven top dots by playing down from the end of B's move.

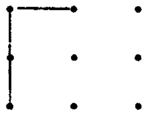


We believe that no situation can arise in which B has no safe move, especially if B tries to keep the configuration as compact as possible subject to the above constraints. But we have not been able to supply a proof. (It is not hard to see that if A's first move touches the center dot then B can win by central symmetry.)

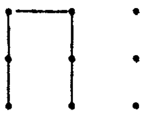
b) i) A wins by starting in a corner and preventing B from isolating any dots. After A's next turn, the position will be

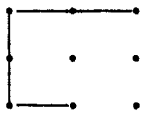


or



In the former case, all dots must end up used; in the latter, A proceeds to leave one of the following:

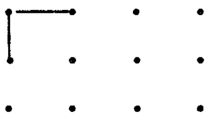


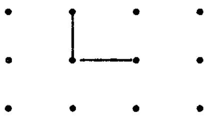


In either case, it is no longer possible to isolate any dots.

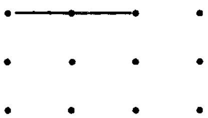
ii) B wins by not allowing A to isolate an odd number of dots.

If B starts with

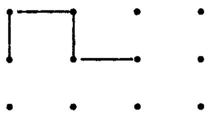


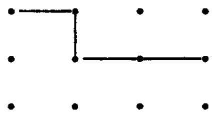


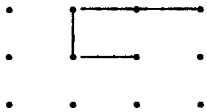
or

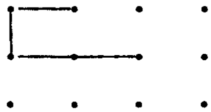


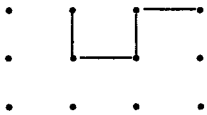
she can always leave one of the following positions after her second turn.

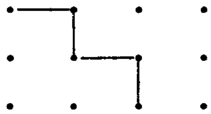


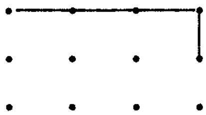


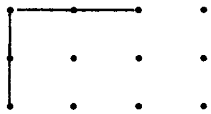




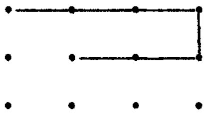


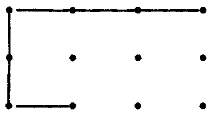


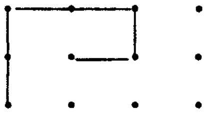


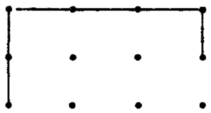


All but the last two clearly result in a favorable outcome for B regardless of how the game continues. In the last two cases, B can leave one of the following after her next move.

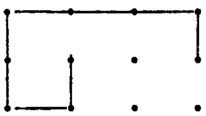


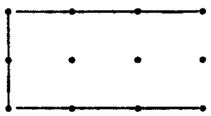






Again, the two on the left leave no doubt about the outcome, and the other two lead to one of the following

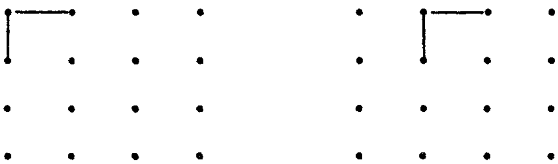




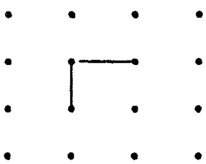
all of which result in all dots being used.

iii) <u>B can</u> again <u>win</u> by preventing A from isolating an odd number of dots.

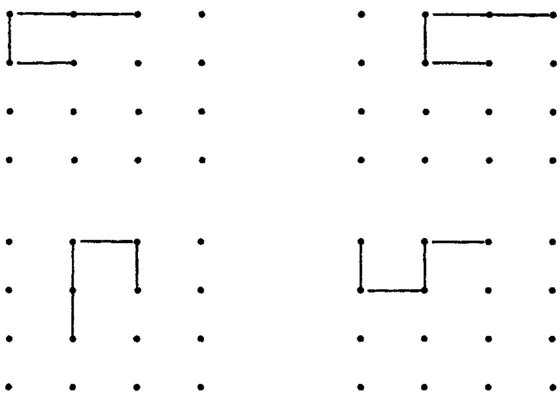
After her first move, B leaves

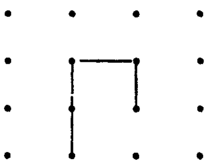


or

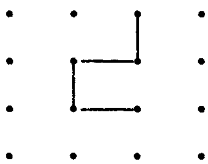


After the next move

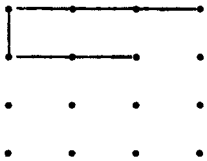




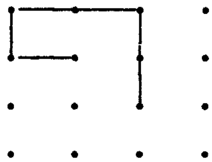
or



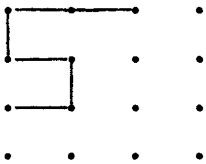
And, after her third move, she leaves one of the following or an equivalent.



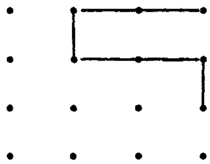
(a)



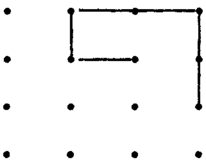
(b)



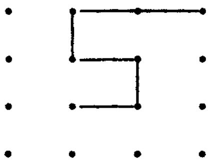
(c)



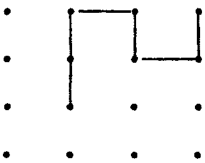
(d)



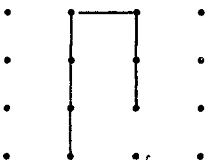
(e)



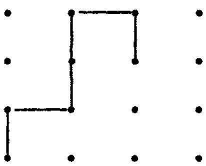
(f)



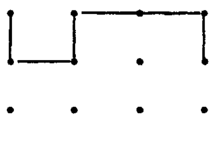
(g)



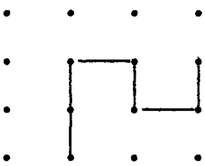
(h)



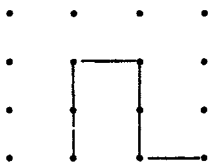
(i)



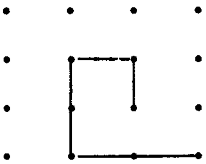
(j)



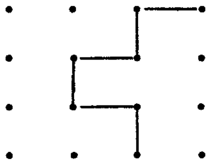
(k)



(l)

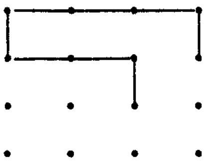


or

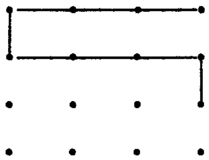


(m) (n)

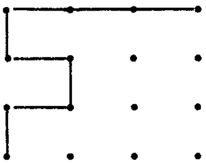
In (b), (f), (g), (h), (k), (l), (m) and (n), A will make the last move no matter how the game continues, and B will win. In the other cases, B can be sure of leaving one of the following:



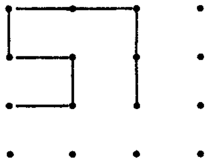
(a)



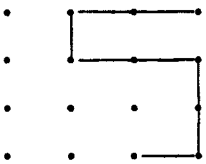
(b)



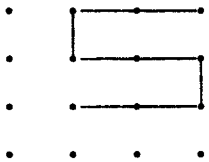
(c)



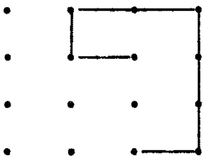
(d)



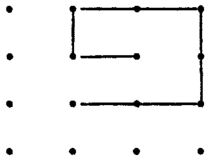
(e)



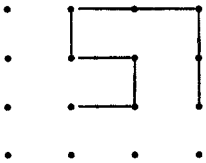
(f)



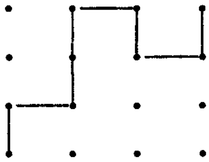
(g)



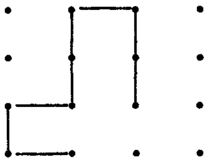
(h)



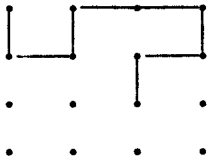
(i)



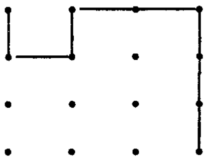
(j)



(k)



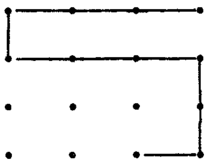
(l)



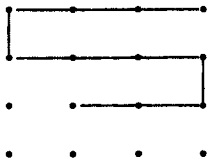
(m)

Again, all but (b), (c), (e), (g) and (m) end in wins for B regardless of how play continues.

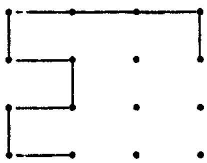
Continuing, in these five cases, B can leave one of the following after her fifth move:



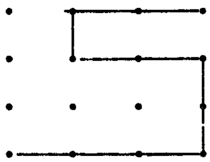
(a)



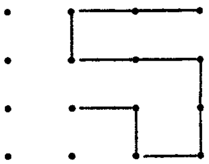
(b)



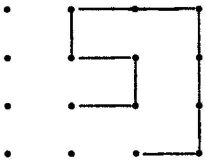
(c)



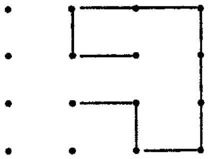
(d)



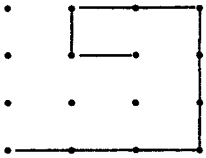
(e)



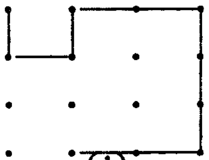
(f)



(g)

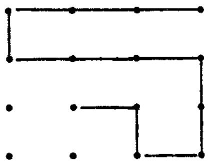


(h)

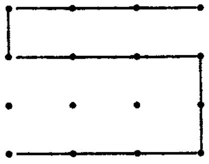


(i)

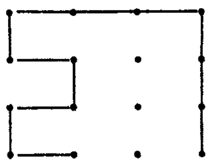
All of these but (a) and (c) must lead to wins for B regardless of the continuation. In (a), B can leave either



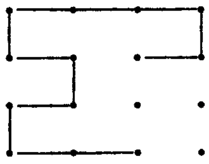
or



and, in (c) B can leave



or



all of which lead to wins for B.

iv) and v) These board may be analyzed in a similar manner. Clearly, there are a large number of cases which must be considered; and we have not carried out a complete analysis. However, we believe that B wins the 4 × 5 game and that A wins the 5 × 5.

7.53: On a 2 × n board with n odd, A wins by placing her domino vertically on the middle column. She then plays by central symmetry.

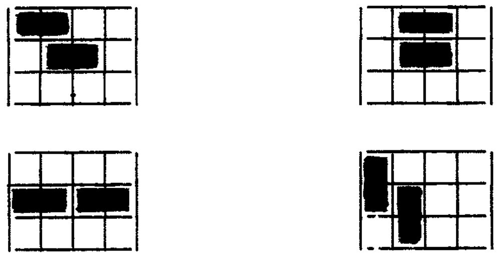
On a 2 × n board with n even, B uses central symmetry and B wins.

ii) On a 2 × n board with n odd. A makes a move; if vertical, B plays vertically. If horizontal, B plays over or under A's domino and B wins.

On a 2 × n board with n even, A plays vertically in the first column, leaving a 2 × n board with n odd in which he is the second player. A wins.

7.54: i) A wins by placing his first domino horizontally on the middle of the middle row and then playing by central symmetry.

ii) Since the board has 12 cells, the game can last at most 6 moves. In order to win, B must force the game to end in 5 moves. B can win by leaving one of the following positions (or a symmetrical equivalent) after her first move. In each case, B will be able to guarantee that the game ends at the fifth move with two isolated cells remaining.



7.55: a) A wins. He plays his first domino horizontally on the center cells of the middle row, and then plays by central symmetry.

b) B wins by playing by central symmetry.

7.56: i) We first show that, regardless of who goes first on a 2 × 4 board, A can always be sure of winning.

If A goes first, he begins with

and then wins by leaving

or its symmetrical counterpart.

If B goes first, then A will be able to make his second move directly under his first, completely filling the board and winning.

As the next step in the solution, we observe that, as a result of the observation above, if A wins on a 2 × n board, then he will win on a 2 × (n+4k) board, for any k. He simply considers the board as partitioned into one 2 × n section, with the balance of the board partitioned into 2 × 4 sections. For example, he thinks of the 2 × 19 board partitioned as follows:

(Note that A can win the 2 × 3 game, as we'll see below.) A begins with his "winning move" in the 2 × n section. Thereafter, he plays in whatever section B does, using the winning strategy in each. Since A can be sure of the last move in each section, he can be sure of the last move of the game.

We now apply the reasoning above: Since A wins the 2 × 2 game,

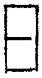
A wins the 2 × (4k+2) game, k ≥ 0.

Since A wins the 2 × 3 game,

A wins the 2 × (4k+3) game for any k ≥ 0.

Since A wins the 2 × 4 game (see above), A wins the 2 × 4k game for any k ≥ 0.

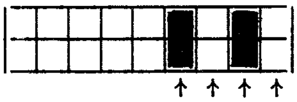
The 2 × (4k+1) case is a little more difficult. B wins the 2 × 1 game:



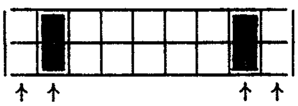
B wins the 2 × 5 game:

Since if B chooses her first move properly she cannot be stopped from playing at least twice, A can play in at most three columns and therefore cannot make more than two moves. Therefore B wins. Note, however, that A can force all of the columns to be used, leaving B no move to spare.

B wins the 2 × 9 game: Since there are nine columns, if B is able to place at least four vertical dominoes, then A will be limited to at most five columns. But, in five columns, A can play at most four (horizontal) dominoes, so B will win. B can be sure of being able to place at least four dominoes by placing her first domino in the second column from one edge of the board and then playing properly. After A's second move, there will be at least three vacant columns in addition to the one B has insured for herself at the edge of the board. No matter where these three columns are located, B can play to be sure of getting two of them: Suppose the columns of the board are numbered from left to right, and suppose (by symmetry) that B's first move has been in column 8. Then, if either column 2 or 6 is blank after A's second move then B can take that column and assure herself four moves:



or



Thus A must have a domino in column 2 and one in column six. This leaves four cases to consider. In each, B wins as indicated below:

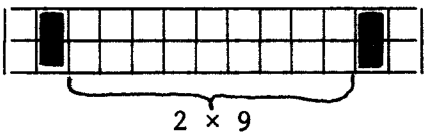
(B gets 4,8 and 9 and either 3 or 7.)

(B gets 4,8 and 9 and either 3 or 5.)

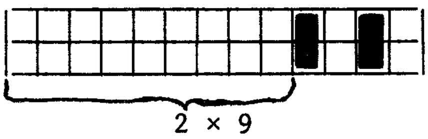
(B gets 4,8 and 9 and either 1 or 7.)

(B gets 4,8 and 9 and either 1 or 5.)

B wins the 2 × 13 game: If B can take six columns, this leaves at most seven columns for A. But A can place at most six dominoes in seven columns, so B makes the last move and wins. To see that B can always take six columns, we argue as follows: By symmetry, we may assume that A's first move lies in columns 1-7. B should then move in column 12. If A allows B to make her second move in column 2 or column 10

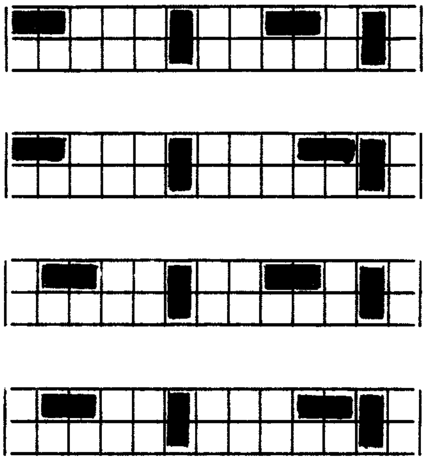


or



then B is already assured of four columns and, when next it is B's turn to move, A will have made three moves in the remaining 2 × 9 game. In three moves, A can capture at most six columns, so at least three columns in the 2 × 9 game will remain for B. B can take one of these (ensuring her fifth column) in such a way that at least two nonadjacent free columns remain. Since A cannot cover two non-adjacent columns in one move, B will still have a free column in the 2 × 9 game for her fourth move, and then will win with six columns in all.

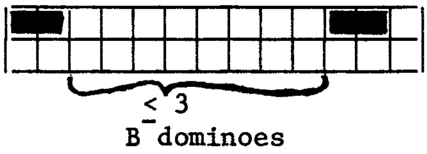
To prevent this from happening, A's first two moves must capture columns 2 and 10. This leads to four cases, in all of which B makes her next move in column 6.

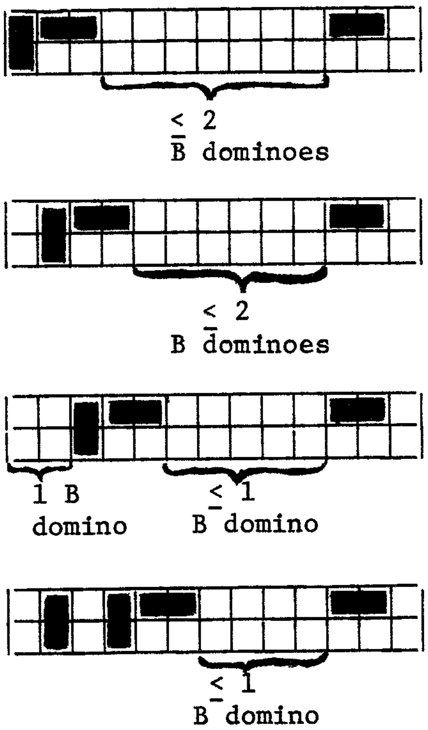


Note that, in all cases, columns 4, 5, 7 and 8 are still vacant so that, regardless of A's next move, B can move in 4 or 8, ensuring herself at least five columns. In addition, at this point, regardless of A's third move, at least two nonadjacent vacant columns will remain in addition to those which B has ensured. Since A will be unable to take both of these, B will be able to take one on her fourth turn and thereby ensure herself of six columns and a win.

A wins the 2 × 17 game by thinking of the game as the join of a 2 × 4 game and a 2 × 13 game. A starts in the middle of the 2 × 4 game and does not make another move in that game unless B does, in which case A finishes off the game. This forces B to go first in the 2 × 13 game. If we can show that A wins the 2 × 13 game when B goes first, then A wins the 2 × 17 game.

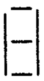
To show that A wins the 2 × 13 game when B moves first, it is enough to show that A can capture seven columns in six moves. (This would leave B only six columns and so six moves.) Regardless of B's first move (which, by symmetry, we may assume lies in columns 1-7), A moves in columns 11 and 12. A's next move is in as low numbered columns as possible. After B's third move, one of the following will remain:





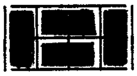
Regardless of where B's remaining dominoes are located, A will be able to place his third domino somewhere in the columns between his first two, in such a way that he will still have a double threat for capturing a seventh column. He will therefore win

Since A wins the 2 × 17 game, he wins the 2 × (4k+1) game for k ≥ 4.

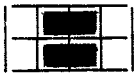
ii) 2 × 1 board:  A wins.

2 × 2 board: , B wins

2 × 3 board: , B wins.

2 × 4 board: , A wins.

2 × 4k board: In 2k moves A covers 2k columns by ultimately playing under himself



k times. B, then has 2k moves and fills the other 2k columns. A wins.

2 × (4k+1) board, k ≥ 1: A plays in the 2nd and 3rd columns at his first move. At his second move A plays either

 or  depending on

B's first move. Thereafter, A plays under himself as follows: .

So,in 2k moves, A covers 2k+1 columns. B has 2k moves to cover 2k columns and A wins.

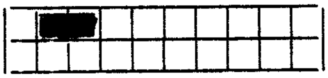
2 × (4k+2) board, k ≥ 3: A has time to make two moves of this type

. Then A plays as .

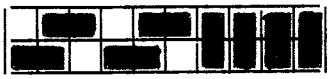
So in 2k moves, A covers 2k+2 cells, B has 2k and A wins.

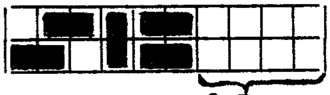
2 × 6 board: B wins by playing in column 2 or 5 and then in column 1 or 6 respectively. After A's third move, B will have no move remaining.

2 × 10 board: A wins as follows:

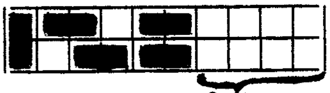


If B plays in column 7,8,9 or 10, then A wins by playing in columns 5 and 6 in the top row and then arriving at one of the following configurations.





3 B dominoes



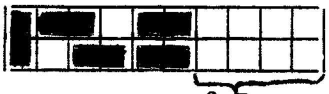
3 B dominoes

If B's first move is in column 5 or 6, A plays in columns 8 and 9 and cannot be prevented from occupying six columns after his fourth move. B's fourth move will then end the game.

If B's first move is in column 4, A plays in the bottom row in columns 1 and 2 and makes his fourth move under his third (anywhere on the board). The game will end after B's fourth move.

If B's first move is in column 1, A plays in columns 3 and 4:

A will now win by filling in columns 5 and 6.

 .

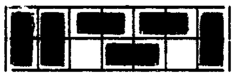
3 B dominoes

Or, if B plays in 5 or 6, A will play in 8 and 9 and then play in 7 and 8 or 9 and 10 of the bottom row.

2×7 board: A wins by playing

 ,

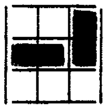
and then forcing this formation



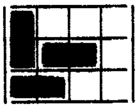
or an equivalent.

$2 \times (4k+3)$ board, $k \geq 1$: A wins by considering it as one section of 7 cells and (k-1) sections of 4 cells. If A starts in the first section of 7 as in the 2×7 game, either B will play in that section, B having the last move and A will start the 2×4 sections and A will win. Or, B will play first in a 2×4 section, in which case A will have the last move in it so that B will eventually have to return to the 2×7 section and A will win anyway.

7.57: i) a) 3×3 board: A wins by playing in the middle row. B only has essentially one move.



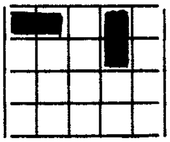
b) 3 × 4 board: A wins by playing in the center. After A's second move, the position will be

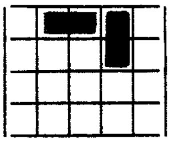


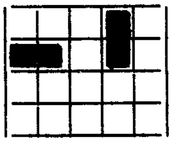
or a symmetrical equivalent.

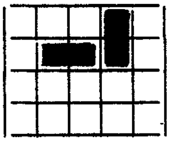
B has only one move left.

d) 4 × 5 board: B wins by countering A's move in each case as shown.





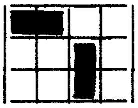


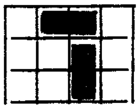


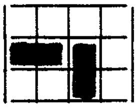
B then plays in a lower corner block of 4 in the 2nd or 4th column. This limits A to 4 moves and gives B at least 4 moves.

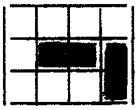
ii) a) 3 × 3 board: B wins by playing in the column not used by A.

b) 3 × 4 board: B wins by countering A's first move as follows.





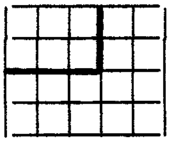




In each of the first three cases, B makes her second move in the lower right corner.

In the last case, B's second move is made in the left column.

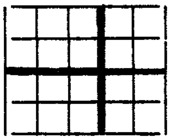
d) 4 × 5 board: B wins by viewing the board as made up of two 2 × 3 rectangles and two 2 × 2's, depending on A's first move. By symmetry, we may assume that A's first move is in the upper left 2 × 3:



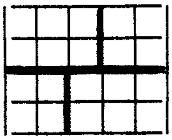
B's first move is also in this 2 × 3.

If A's second move is also in one of the top two rows then B's second move is in the middle column of rows 3 and 4.

If A's second move is in one of the bottom two rows then B plays in the same bottom 2 × 3 as A:



or



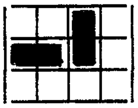
In either case, B's third and fourth moves are in an as yet untouched 2 × 2. After A's next move only two uncovered boxes will remain, but they will not be vertically connected, so B will be unable to move and will win.

7.58: i) B wins by countering A's possible opening moves as shown.

ii) B wins by countering A's 2 possible opening moves as follows.

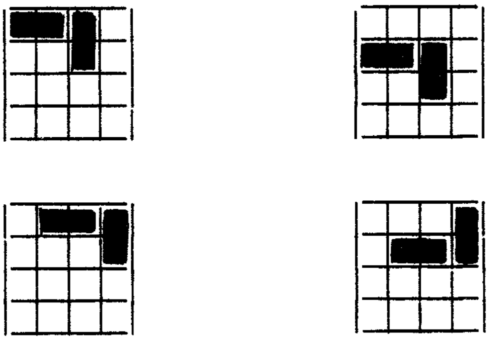
<u>7.59</u>: i) a) <u>B wins</u> by completing the row in which A plays.

b) <u>A wins</u> by placing his domino on the left in the center row. B counters as follows:

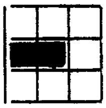


A wins by placing his domino on the lower right, completing the third column.

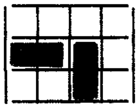
c) <u>B wins</u> by countering A's first move as follows:



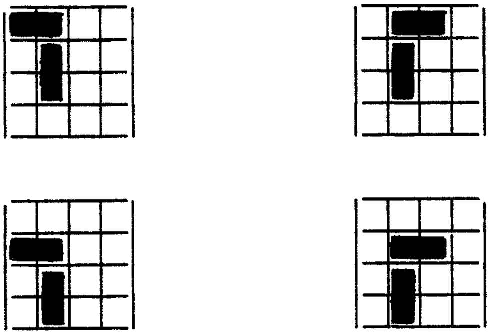
ii) a) <u>A wins</u> by playing in the center row.



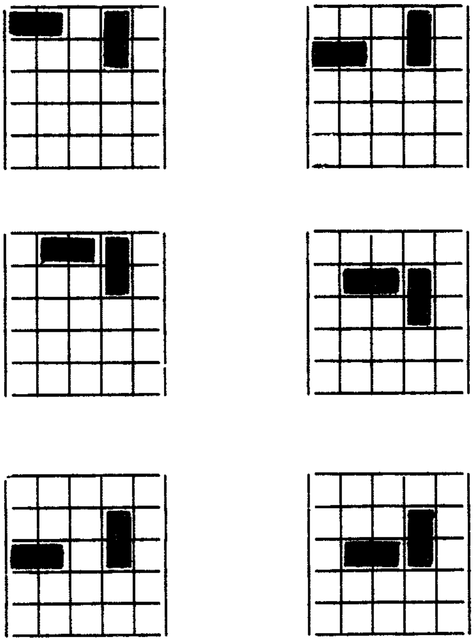
b) A wins by opening in the center row on the left.



c) <u>B wins</u> by countering A's opening move as shown and then playing above or below A's second move:



<u>7.60</u>: <u>B wins</u> by making the following countermoves to A's possible first moves:



In each case B can win in two more moves no matter what A does next.

7.61: a) B wins. There are two possible opening moves for A.

If A pushes on a side, then B captures. A must recapture (or else B wins on the next turn), but then B pushes and wins.

If A starts by pushing in the center, B captures, and A must recapture. B now pushes leaving

A		
B	A	
	B	

or

		A
B	A	
	B	

.

In either case, B wins.

b) A wins by pushing in column 1.

B	B	B	B
A			
	A	A	A

We consider all possible responses by B.

Case I: B pushes in columns 1 or 4: Then A pushes in column 4 leaving

	B	B	B
B			
A			A
	A	A	

or its symmetrical counterpart.

If B now pushes in column 2, then A pushes in column 4.

		B	B
B	B		A
A			
	A	A	

B must move her pawn in column 3 (either to capture or to push). But then A will capture B's pawn in column 2 and cannot be stopped.

If B's second move is to push in column 3, then A captures leaving

	B		B
B		A	
A			
	A	A	

.

B must recapture. Regardless of whether she does so from the right or left, A pushes in column 3, pinning B's remaining pawn.

Finally, if B's second move is in 4, then A pushes in column 2, leaving

	B	B	
B			B
A	A		A
		A	

.

B cannot push her piece in column 2 (A will win by capturing B's pawn in column 1), nor can she push in column 3 (A will capture, B will have to recapture, and A will capture again thereby assuring a win). Therefore B's only hope is to capture. A recaptures, leaving

	B	B	
			B
A	A		A

.

B now has no safe move. She cannot push in column 2 or A will push and win in column 1. If B pushes in column 3, A captures, B must recapture, and A captures again winning on the next turn.

Case II: B's first move is to push in column 2 or 3. Then A pushes in column 4, leaving

B		B	B
	B		
A			A
	A	A	

or its symmetrical equivalent.

The cases where B now pushes in column 1 or 4 may be discarded by déjà vu -- the resulting positions or symmetrical equivalents were considered in Case I.

If B captures, then A recaptures leaving

B		B	B
A			A
		A	

.

There are now three possibilities. If B pushes in column 1, then A pushes in column 3; if B pushes in column 3, then A pushes in column 4; and if B pushes in column 4, then A pushes in column 1. In all cases, A has an easy win.

If B's second move is to push in column 3, then A captures and B must recapture leaving

B			
	B	B	
A			
	A	A	

.

A now pushes in column 1 and wins on his next turn.

Finally, if B's second move is to push in column 2, then A captures leaving

B		B	B
A	A		A
	A		

.

If B pushes in column 1, A captures and can't be stopped. If B pushes in column 3, A captures, B must recapture, A captures again, winning on his next turn. If B pushes in column 4, A pushes in column 1, leaving B to push in column 3. A then captures and wins.

c) As we observed in the solution of Exercise 7.19 in the text, in a game in which the object is to make the last move, if (m) is a winning position and (n) is a losing position, then (m,n) is a losing position; and if (m) and (n) are both winning positions then (m,n) is a winning position.

Applying this same reasoning to 3 × n Hexapawn, we observe that if the game has been reduced to two disjoint games, one of which is a win for the first player and the other of which is a win for the second player, then the player who is about to move in the reduced game will win. On the other hand, if the 3 × n game has been reduced to two disjoint games and if the second player is the winner in each of these games, then the player who is about to move in the reduced game will lose.

We make use of this and the following observations to analyze the 3 × n game for some of the larger values of n.

Observation 1: If the second player wins the 3 × (n-2) game and the first player wins the 3 × (n-3) game, then A can win the 3 × n game by pushing in column 1.

Note that if B captures, then A recaptures -- leaving himself as the second player in the 3 × (n-2) game; and if B does not capture then she must push in column 2, in which case A captures and B must recapture from column 3 -- leaving A to start the remaining 3 × (n-3) game.

Observation 2: For odd n, if the second player wins the 3 × $(\frac{n-3}{2})$ game and the first player wins the 3 × $(\frac{n-5}{2})$ game, then A can win the 3 × n game by pushing in the center.

Note that B is forced to capture and A can recapture from the same side. This leaves B two options. If B captures, then A recaptures, leaving two disjoint 3 × $(\frac{n-3}{2})$ games with A as the second player; whereas, if B pushes, then A can capture -- forcing B to recapture and leave disjoint 3 × $(\frac{n-3}{2})$ and 3 × $(\frac{n-5}{2})$ games in which A has the first move.

Observation 3: If A starts any 3 × n game, n > 3, by pushing in the second column, then B can force a win by capturing from column 3. A is forced to recapture from column 3. (If he recaptures from column 1, then B can push in column 1 and win on the next turn.) B now has a choice. If the 3 × (n-3) game is a win for the first player, then B captures and A will have to recapture leaving B to start the 3 × (n-3) game. On the other hand, if the 3 × (n-3) game is a win for the second player, then B can push in column 1, leaving A to start the 3 × (n-3) game.

We are now ready to consider the 3 × n game. Clearly <u>A wins the 3 × 1</u> game, and <u>B can win the 3 × 2 game</u> by pushing after A's move rather than capturing.

We saw above that <u>B wins the 3 × 3</u> game.

<u>3 × 4</u>: By Observation 1, <u>A wins</u> by pusing in column 1. (Note that B wins the 3 × 2 game and A wins the 3 × 1.)

3 × 5: A wins by pushing in the center. B must capture and A recaptures leaving

B	B	B		B
		A		
A	A			A

If B captures, A recaptures; and if B pushes (in column 2), then A captures. In either case, A gets the last move.

3 × 6: B wins.
If A begins by pushing in column 1 (or 6), then B can capture, forcing A to recapture and leave a 3 × 4 game in which B moves first.

If A begins by pushing in column 2 (or 5), then B will win by Observation 3.

If A begins by pushing in column 3 (or 4), then B captures, and A must recapture. B then captures again and A must again recapture, leaving

B		B		B	B
		A			
A				A	A

B can now push in column 1, forcing A to start the remaining 3 × 2 game (in columns 5 and 6).

3 × 7: By Observation 2, A wins by pushing in the center. (Note that B wins the 3 × 2 game and A wins the 3 × 1 game.)

3 × 8: By Observation 1, A wins by pushing in column 1. (Note that B wins the 3 × 6 game and A wins the 3 × 5.)

3 × 9: B wins.

If A begins by pushing in column 1 (or 9), B captures and A must recapture, leaving B to move first in the 3 × 7 game.

If A begins by pushing in column 2 (or 8), B wins by Observation 3.

If A begins by pushing in column 3 (or 7), B captures from the left and A must recapture from the left. (If A recaptures from column 4, then B can win quickly by pushing in column 4.) B now pushes in column 4, leaving

B		B		B	B	B	B	B
		A	B					
A			A	A	A	A	A	A

A now has two choices. If he pushes in column 5, then B captures, and A must recapture. B then pushes in column 1, leaving A to go first in the remaining 3 × 3 game. On the other hand, if A captures, then B recaptures -- leaving disjoint 3 × 1 and 3 × 4 games. A cannot afford to play in the 3 × 1 game, for this will leave B to play first in the remaining 3 × 4 game. Therefore A will have to move in the 3 × 4 game (columns 6-9). If he pushes in column 6 (or 9), B recaptures and A must recapture, after which B pushes in column 1 and leaves A to move first in the remaining 3 × 2 game; if A pushes in column 7 (or 8), then B captures from column 8 and A must recapture from column 8, after which B pushes in column 6, leaving two 3 × 1 games in which A must move first.

If A's first move is to push in column 4 (or 6), then B captures, A recaptures, B captures and A must recapture, leaving

B	B		B		B	B	B	B
			A					
A	A				A	A	A	A

Since the first player wins the 3 × 4 game and the second wins the 3 × 2 game and since it is now B's turn, B will win, by our opening remarks.

Finally, if A's first move is to push in column 5, then B captures and A must recapture from the same side. B now pushes in column 6, leaving

B	B	B		B		B	B	B
				A	B			
A	A	A			A	A	A	A

This is essentially the same situation as if there were disjoint 3 × 3 and 3 × 4 games and B made the first (winning) move in the 3 × 4 game. Therefore, by our opening remarks, B will win.

<u>3 × 10</u>: <u>B wins:</u>

If A begins by pushing in column 1 (or 10), B captures and A must recapture, leaving B to move first in the remaining 3 × 8 game.

If A begins in column 2 (or 9), B wins by Observation 3.

If A begins in any other column, B captures, A must recapture, B captures again and A must again recapture. This leaves two disjoint games -- either 3 × 1 and 3 × 6, or 3 × 2 and 3 × 5, or 3 × 3 and 3 × 4 (depending on the column in which A made his first move). In all cases, one game is a win for the first player and the other is a win for the second player. Therefore, since it is B's move, it follows from our opening remarks that B will win.

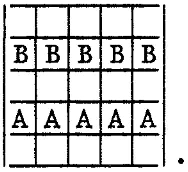
d) <u>A wins</u>, but analysis of this game requires consideration of many cases. There are however some principles which govern A's strategy:

i) Don't make the first move to the third row unless forced to do so.

ii) Moves to the second row should be in adjacent columns, beginning with the center and continuing toward the side on which B moves.

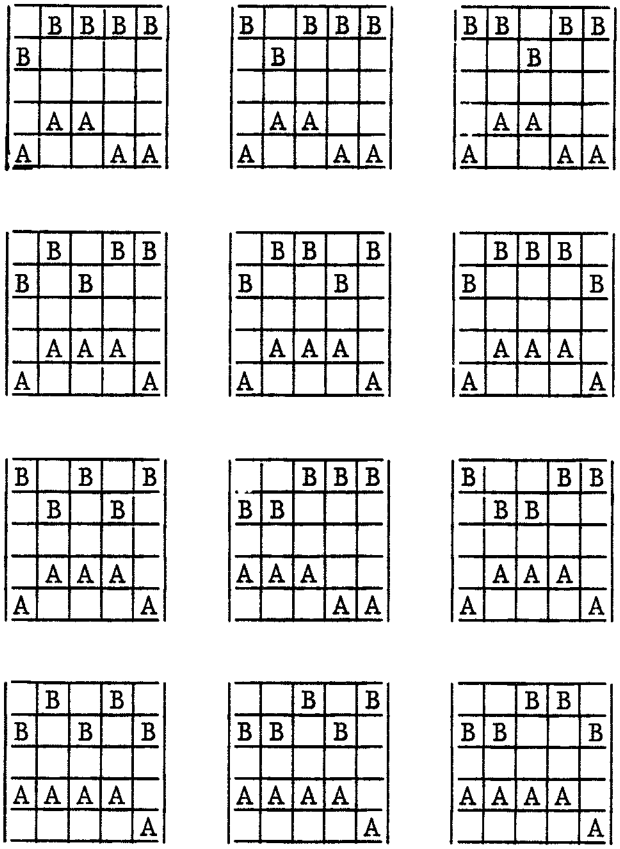
iii) A's third move should be one of the following: Capture, if B's second move has been to the third row; move in the (adjacent) end column if both of B's moves have been on the same side of the board; fill in the second row of columns 2, 3 and 4 otherwise.

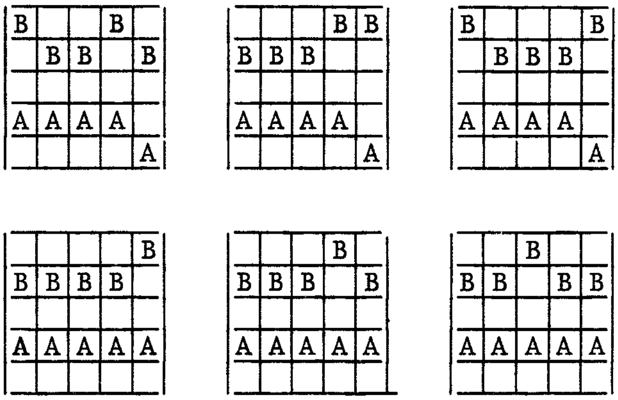
As a result of i), A will not make the first move to the third row unless the situation is



As this is essentially the 3 × 5 game, A will now win by pushing in the center.

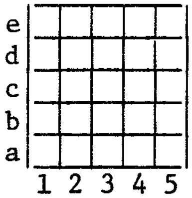
In any other case, B will make the first move to the third row. By ii) and iii), the position just prior to her doing so will be one of the following (or an equivalent).





We now consider cases, according to the move on which B first moves to the third row.

To facilitate our discussion, we use coordinate notation to denote the cells of the board. That is, we label the rows and columns of the board as indicated below.



Moves and captures will be denoted in the obvious manner. That is b2-b3 means that the pawn on b2 has been moved to b3; and b2×c3 means that a pawn that was on b2 captured a pawn on c3.

<u>Case I</u>: If B moves to row c on her second move, A will capture. The position will then be one of the following:

	B	B	B	B
A				
		A		
A			A	A

Case Ia

B		B	B	B
	A			
	A			
A			A	A

Case Ib

B	B		B	B
		A		
		A		
A			A	A

Case Ic

In Case Ia, A's strategy is to force B to move the pawn on e2, permitting the pawn on c1 to advance to victory.

If e2-d2, then c1-d1 wins.

If e3-d3, then b3-c3. If B now pushes in column 5, then the pawns in columns 2 and 4 are both pinned and A will win easily. Therefore, instead, B must play e4-d4. A plays a4-b4. B must again move the piece in column 4. If she captures, then A recaptures after which B can only delay the inevitable by pushing in column 5. A's free move a1-b1 will eventually force B to move the pawn on e2 and A will win. Therefore, B must play d4-c4, leaving

	B			B
		B		
A		A	B	
			A	
A				A

.

This is followed by a1-b1, e5-d5; c1-d1, e2×d1 (forced), b1-c1, d5-c5 (forced); b4×c5, c4-b4 (forced); a5×b4 and A wins.

If B's first move in Case Ia is e4-d4, then A plays a4-b4. e3-d3, b3-c3 results in deja vu, and d4-c4 is clearly losing (b3×c4), so B must play e5-d5. The situation is now decided by b4-c4. If B captures, then A recaptures; and if B pushes in column 5, then A plays a5-b5. In either case, play now proceeds e3-d3; c4×d3, e2×d3, c1-d1 and A wins on his next move.

The remaining possibility for B in Case Ia is to make her first move in column 5. If A plays a4-b4, then B is lost. e3-d3, b3-c3 or d5-c5, b4×c5 are both disastrous, and e4-d4 results in déjà vu.

Therefore A wins Case Ia.

In Case Ib, A takes advantage of his numerical superiority in columns 1 and 2 as well as of the free moves that are available to his pawn in column 1.

Clearly B cannot move the pawn in e1, and if she moves the pawn in e3 then c2-d2, e1×d2 (forced); a1-b1 wins for A. That is, unless B plays d3-c3, A pushes the pawn in column 1 (recapturing if B captures) and cannot be stopped. Even d3-c3 does not help B. A simply captures, forcing the pawn on d2 to move, and again A marches up column 1.

If B begins Case Ib with e4-d4, then the pawns on e1 and e3 are pinned forever. A plays a5-b5, and eventually the game in columns 4 and 5 will end with neither player being able to move. A still has the free move a1-b1, if he needs it, and B will be forced to move in columns 1 or 3, permitting A to capture and win.

The case which begins e5-d5, a4-b4 is similar. However, there are pitfalls which A must avoid. If B follows with d5-c5, then A can capture or push; but, if he captures, he must change his strategy in column 2, as column 3 is no longer protected. That is, if e3-d3, then A must capture and not push. A more interesting variation is e5-d5; a4-b4, e3-d3. Here, A can stick to his original strategy, but must be careful after c2-d2, e1×d2; a1-b1, d5-c5; rather than capture, he must push b1-c1 (otherwise B takes the initiative in column 3). Naturally, d2×c1 must be answered by b2×c1 and c5×b4 must be answered by a5×b4. Otherwise, A just keeps pushing in column 1, and cannot be stopped.

In Case Ic, A tries to take advantage of his numerical superiority on the right side of the board.

If B starts in column 1, A plays a4-b4. B can't afford to push in column 5 or the pawns in columns 2 and 4 will both be pinned. B can continue to push in column 1, but eventually, she will have to play e4-d4. A will capture, and B will have to recapture. This frees B's pawn in column 2 to try to force a breakthrough, but A's pawns in columns 1 and 3 are sufficient to stop the threat, and A's two pawns in columns 4 and 5 are sufficient to force a breakthrough for A.

If B starts in column 5, then the game proceeds a4-b4, e2-b2 (essentially forced); c3×d2, e1×d2; a5-b5, e4-d4; b4-c4, d5-c5; a1-b1 and A wins.

If B starts in column 2 or 4, A captures and B must recapture. A plays a4-b4 and the analysis proceeds essentially as above. Thus, A wins if B moves to row c on her second move.

Case II: B first moves to row c on her third move. In all cases but one (IIj below), A captures as indicated. The position will then be equivalent to one of the following:

	B		B	B
		B		
A				
		A	A	
A				A

IIa

	B	B		B
			B	
A				
		A	A	
A				A

IIb

	B	B	B	
				B
A				
		A	A	
A				A

IIc

B		B		B
			B	
	A			
	A		A	
A				A

IId

B		B	B	
				B
	A			
	A		A	
A				A

IIe

	B		B	B
B				
		A		
	A	A		
A				A

IIf

		B	B	B
	B			
A				
A		A		
			A	A

IIg

		B	B	B
B				
	A			
A	A			
			A	A

IIh

B			B	B
	B			
		A		
	A	A		
A				A

IIi

B			B	B
		B		
	B	A		
	A		A	
A				A

IIj

Cases IIb, IIc and IIe are positions which are encountered in the analysis of Cases Ia and Ib. Case IIf is a symmetric counterpart of a position arising in Case Ic. Case IIa never actually arose; but B's only reasonable choices are to push in columns 4 or 5, in which case c3-d3 results in déjà vu. Similarly, Case IId will result in dejà vu unless B plays d4-c4. But then A's three free moves in column 1 are more than enough to counter B's one safe move in column 5.

In Case IIg, B must capture. A then plays a4-b4, and B must push a pawn in columns 3, 4 or 5. A plays a5-b5 and B must push a second pawn (pushing the same pawn a second time can clearly not succeed). The position now is equivalent to one of the following:

				B
		B	B	
B				
A		A	A	A

or

			B	
		B		B
B				
A		A	A	A

In the latter case, A wins easily by pushing in column 3. In the former case, A also pushes in column 3. If B captures, then A recaptures and the game ends after A's next move. On the other hand, if B plays d4-c4, then A captures and B is lost anyway.

In IIh, if B captures, A can march up column 1; and if B pushes, A will capture and march up column 1 anyway.

In IIi, if B captures, A recaptures. The timing is now such that B will be forced to push the pawn in column 4 before A will have to push the pawn in column 3. When he does so, A wins by c3-d3. If, instead of capturing, B plays d2-c2, then A captures, and c2-d2 on his next turn forces the win. Finally, if B's first move in Case IIi is to push in column 4 or 5, then A captures c3×d2 and B must recapture. A now marches up column 1, until eventually B is forced to capture. A then recaptures and cannot be stopped.

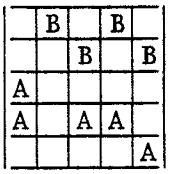
Case IIj is the first in which A does not capture when B first moves to row c. Instead, he pushes b3-c3. If B now plays e1-d1, A responds a5-b5, B is now forced to play e4-d4, (since e5-d5; b5-c5 pins B's pieces). A captures, B must recapture, and then b5-c5 wins.

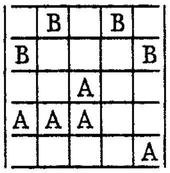
If B starts Case IIj with e4-d4, then A captures and B must recapture, and A then plays a5-b5. If B now pushes in column 1, then we have déjà vu, so B must push in column 3 or 4. If she plays d3-c3, then A responds b2×c3 and whether B recaptures or not, A will win in column 3. Similarly, if B plays d4-c4, then b5-c5 wins. (B can further delay A's victory by d3-c3; but A plays b2×c3 and then continues to victory in column 5.)

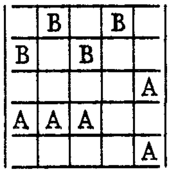
Finally, if B's first move in Case IIj is e5-d5, then A plays a5-b5. A move by B in column 1 would be followed by b5-c5 and a win for A, so, instead, B must try d5-c5. A captures, and now B pushes in column 1. Play continues c5-d5, e4×d5; b5-c5, and now B is forced to play again in column 1, permitting A to capture and win.

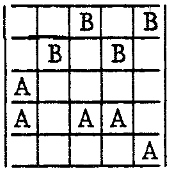
Thus A wins if B moves to row c on her third move:

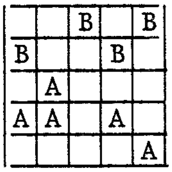
Case III: B first moves to row c on her fourth move. After A's turn, the position will be equivalent to one of the following:

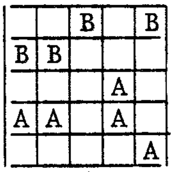

IIIa

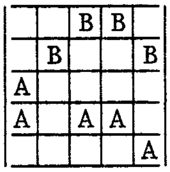

IIIb

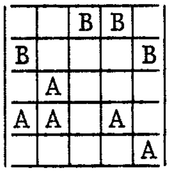

IIIc

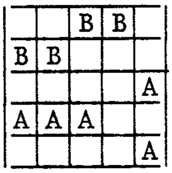

IIId

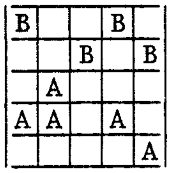

IIIe

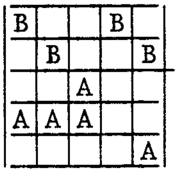

IIIf

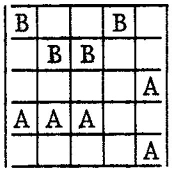

IIIg

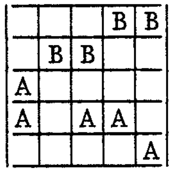

IIIh

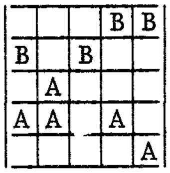

IIIi

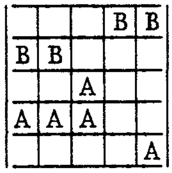

IIIj

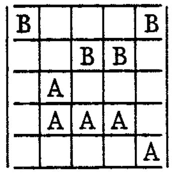

IIIk

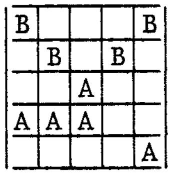

IIIℓ

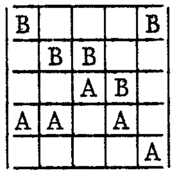

IIIm


IIIn


IIIo


IIIp


IIIq


IIIr

In Case IIIa, B's only reasonable move is e4-d4, to prevent b3-c3. A responds a5-b5, forcing B to move first in the 3 × 3 game of columns 3, 4 and 5. A wins that game and B is forced to play e2-d2. A then captures and wins.

In IIIb, after d5-c5; a5-b5 and d1-c1; b2×c1, B will be forced to push in column 2 or 4. A will capture and win.

In IIIc, B's only reasonable move is e2-d2, and then a5-b5 results in déjà vu with a symmetrical equivalent of Case IIIa.

In IIId, B must capture d2×c1 or else A will march up column 1 and win. A then plays b4-c4, and B is lost.

In IIIe and IIIh, B must capture (then A pushes in column 1) or push (then A captures). In either case, A will be able to march up column 1 to win.

In IIIf, B must push in column 1 or 2. A then pushes in the other of the two, ensuring the win.

In IIIg, B must capture. A plays a5-b5. B's only reasonable move is e4-d4. (Otherwise A can pin the pawn at e4.) A then ensures the win with b3-c3. (If B captures, A recaptures; if B pushes d4-c4 then A captures; and if B pushes d5-c5, then A captures.)

In IIIi, B has a number of alternatives. If d1-c1, then b2-c2 wins for A (e3-d3; c2×d3, e4×d3; c5-d5, etc.; or e4-d4; c5-d5, etc.). If d2-c2, then b3×c2 leaves B in the predicament of Case IIIe. If e3-d3, then a5-b5 forces B to move first in the 3 × 3 game. And if e4-d4, then c5-d5 wins.

In IIIj, B must capture. (e1-d1 permits c2-d2; e4-d4 permits c2×d3; and pushing in column 3 or 5 sacrifices a second piece without gaining anything.) A recaptures and B's only reasonable move is e4-d4. A plays a5-b5, forcing B to move first in the 3 × 2 game.

In IIIk, if B captures, then A can win by recapturing or by pushing. If A plays b2-c2, the best B can do is e4-d4. Play proceeds b1-c1, d4-c4; c2-d2, e1×d2; c1×d2, and A wins on his next turn. Similarly, if B pushes in column 2 or 5, then A captures in column 2 and can't be stopped in columns 1 and 2.

In IIIℓ, if B plays e1-d1, then a5-b5 forces B to play first in the 3 × 3 game. If B starts with d2-c2, then A captures and captures again on his next turn, winning easily. Finally, if B starts with d3-c3, then b2-c2 wins.

In IIIm, B must capture or A can march up column 1. A responds with b3-c3. B must push in column 4 (if she plays e5-d5, then a5-b5 leaves her no move). A plays a5-b5, and regardless of whether B captures or pushes, A captures and wins.

In IIIn, B must play d3×c2 or either column 1 or 2 will free for A to pass. A then recaptures and B must recapture. Then a5-b5 followed by e5-d5, b5-c5 (or e4-d4; b4-c4) wins for A.

In IIIo, B must capture or push. Either way, A plays to c2 and cannot be stopped.

In IIIp, if B captures then A plays b3-c3, and if B pushes then A plays b2×c3. In either case, A breaks through in column 3.

In IIIq, B must capture, A recaptures, and B must recapture again (otherwise A can march up column 3). A then pushes a5-b5 and wins after two more turns.

In IIIr, A has pushed instead of capturing. B cannot afford to push in column 1. If she pushes in column 2, A captures and cannot be stopped from breaking through in column 3. If B begins by capturing, A recaptures b4×c3. B must play e1-d1 (otherwise b2-c2 forces a breakthrough in column 3). A plays b2-c2 anyway. The game proceeds d1×c2; b1×c2, d3×c2; c3-d3 and A wins. (Note, B does not have enough time to win in column 5.) Finally, if B starts with e5-d5, then A captures, B recaptures and A pushes b1-c1 and cannot be stopped in column 1.

Therefore A wins if B moves to row c on her fourth move.

<u>Case IV</u>: B moves to row c on her fifth move. After A's turn the position will be equivalent to one of the following:

				B
	B	B	B	
A				
A		A	A	A

IVa

				B
B		B	B	
	A			
A	A		A	A

IVb

				B
B	B		B	
		A		
A		A	A	A

IVc

				B
B	B	B		
			A	
A	A		A	A

IVd

			B	
	B	B		B
A				
A		A	A	A

IVe

			B	
B		B		B
	A			
A	A		A	A

IVf

			B	
B	B			B
		A		
A	A	A		A

IVg

			B	
B	B	B		
				A
A	A	A		A

IVh

		B		
B			B	B
	A			
A	A		A	A

IVi

		B		
	B		B	B
B	A			
A		A	A	A

IVj

In Case IVa, B must capture and A plays b5-c5. Regardless of what B does, A captures and wins.

In IVb, B must capture d3×c2 (otherwise A marches up column 1 or 2). A recaptures and B must capture again. A pushes b4-c4 and wins.

In IVc, B must capture, A recaptures and B must capture again. A then plays b1-c1 and wins.

In IVd, if B captures then A recaptures and B must make the first move in the 3 × 2 game. If B pushes, A captures b4×c3, and cannot be stopped from breaking through in column 3.

In IVe, B must capture, and then b3-c3 wins for A.

In IVf, B must capture, A recaptures and B must capture again. Then b5-c5 wins for A.

In IVg, if B pushes, A captures and cannot be stopped. If B captures, A recaptures and again wins easily.

In IVh, B must play first in the 3×3 game, and A wins.

In IVi, B must capture and then A wins by pushing in column 1.

Finally, in IVj, A has pushed rather than capture. If B pushes in column 4, A captures, B recaptures and A captures again, winning. If B pushes in column 5, A wins with b4-c4.

Therefore, A wins if B moves to row c on her fifth move. This completes the case analysis, and since A wins in all cases, A wins the 5×5 game.

<u>7.62:</u> <u>Regular</u>:

a) <u>A wins</u>: If A pushes in the center, then B captures, A recaptures, B must recapture, and A recaptures and wins.

It is worthwhile noting, however, that A can also force B to win if he so chooses. A pushes on a side. Then B must capture and A recaptures, after which B must push and B wins. We will use this fact in parts of the solution of the $3 \times n$ game.

b) <u>B wins</u>: By symmetry, A can start by pushing in column 1 or in column 2.

If A pushes in column 1, B also pushes in column 1. If A now pushes in column 2, then B captures and A must recapture leaving

	B	B	B
A	A		
			A

.

B now pushes in column 2, and A must capture and B recaptures. B wins after her next move.

If A's second move is to push in column 3, then B also pushes in column 3 pinning A's remaining pawns and winning.

If A's second move is to push in column 4, B also pushes in column 4. By symmetry, A now must push in column 2. B captures, and A recaptures, leaving

	B	B	
			B
A	A		A

.

B now wins by pushing in column 2. A must capture and B wins by recapturing.

If A's first move is to push in column 2, then B also pushes in column 2. If A now pushes in column 1, then B captures and wins. If A's second move is to push in column 3, then

B captures and A recaptures leaving

B		B	B
	A	A	
A			

B now pushes in column 3, A captures and B recaptures, winning on her next move.

Finally, if A's second move is to push in column 4, then B pushes in column 4 and wins easily.

c) 3 × 1: A wins since B cannot move.

3 × 2: A wins: He pushes, B captures and A recaptures.

3 × n: As in Exercise 7.61 c), a number of observations will help us analyze the general case.

Observation 1: If A pushes in column 1, then B captures and A recaptures, leaving the 3 × (n−2) game with B making the first move.

Observation 2: If A pushes in column 2, then, after a series of captures and recaptures, the 3 × (n−3) game remains with B making the first move.

Observation 3: If A pushes in column k, $3 \le k \le n-2$, then, after a series of captures and recaptures, what remains is the join of a 3 × (k−2) game and a 3 × (n−k−1) game with B to make the first move.

3 × 4: B wins by Observations 1 and 2. That is, by symmetry, A pushes in column 1 or 2, leaving the 3 × 2 or 3 × 1 game with B to make the first move.

3 × 5: A wins by pushing in column 3. This leaves two 3 × 1 games with B to move first.

3 × 6: A wins by pushing in column 1. By Observation 1, this leaves the 3 × 4 game with B to make the first move.

3 × 7: A wins by pushing in column 2. Again this leaves the 3 × 4 game.

3 × 8: B wins: If A pushes in column 1 or 2, then B is left to make the first move in the resulting 3 × 6 or 3 × 5 game.

If A pushes in column 3, then B is left with a 3 × 1 and a 3 × 4 game. B plays in the 3 × 1 game, forcing A to start the 3 × 4.

If A pushes in column 4, then B is left with a 3 × 2 and a 3 × 3 game. B plays in the 3 × 3 forcing A to take the last move, then B starts the 3 × 2 and wins. (It makes no difference if A starts the 3 × 2 game rather than completing the 3 × 3.)

3 × 9: A wins by pushing in the center. This leaves two 3 × 3 games. If B pushes in the middle of either 3 × 3 game, then B wins that game, but A begins the remaining 3 × 3 game

and wins. If B starts on the side of a 3 × 3 game then, after A captures and B recaptures, A starts on the side of the other 3 × 3 game (rather than completing the first). After B captures and A recaptures, two 3 × 1 games remain, with B to make the first move. Thus A wins.

3 × 10: A wins by pushing in column 1. By Observation 1, this leaves the 3 × 8 game with B to make the first move.

Misère:

a) A wins by pushing on the side. B must capture, A recaptures, and then B pushes, leaving A with no move.

Again note that if A pushes in the middle then he can force B to win the 3 × 3 game.

b) B wins: By symmetry, A has two possible opening moves.

If A pushes in column 1, then B pushes in column 2. A must capture and B recaptures from the right leaving

B			B
	B		
	A	A	A

.

If A now pushes in column 2, then B pushes in 1, A captures, B pushes in 2, A captures, and B must win. If, instead, A pushes in column 3, then B captures, A recaptures, and B pushes in column 4. A must capture and B will eventually win. Finally, if A's second move is to push in column 4, then B pushes in 2, A captures, B pushes in 1, A captures, and B pushes in 4 and wins.

If, on the other hand, A begins by pushing in column 2, then B also pushes in 2, leaving

B		B	B
	B		
	A		
A		A	A

.

If A now pushes in column 3, then B captures, A recaptures and B pushes in column 1. A must capture, B pushes in 4, A captures, B recaptures and B wins since A must reach the opponent's side of the board first.

If, instead, A pushes in column 4, B wins by successively playing in column 1, then 2, and then 3.

Finally, if A's second move is to push in column 1, then B must capture leaving

B		B	B
B	A		
		A	A

.

A again has three alternatives.

If A pushes in column 3 or 4, then B pushes in 1, A captures and B pushes in the same column as A did. A can force B to capture two pawns, but will eventually have to move to the top row and B will win.

On the other hand, if A pushes in

column 2, then B captures from the right leaving

B			B
	B		
B			
		A	A

If A now pushes in column 4, then B pushes in 2, A captures, B pushes in 1, A captures, and B pushes in 4 and wins.

If A pushes in 3, then B captures, A recaptures, B pushes in 4, A captures and B will win.

c) The same observations hold as in the regular 3 × n game.

3 × 1: B wins, since she cannot move.

3 × 2: B wins: A pushes, B captures and A recaptures.

3 × 3: A wins as in part a) above.

3 × 4: A wins by pushing in column 1. By Observation 1, this leaves the 3 × 2 game with B to move first.

3 × 5: A wins by pushing in column 2. By Observation 2, this leaves the 3 × 2 game with B to move first.

3 × 6: B wins: If A pushes in column 1 or 2, then, by Observation 1 or 2, B is left to go first in either the 3 × 4 or the 3 × 3 game. If A pushes in 3, he leaves the join of a 3 × 1 and a 3 × 2 game. B plays in either game and wins.

3 × 7: B wins: If A pushes in column 1 or 2, he leaves the 3 × 5 or 3 × 4 game with B to move first. If he pushes in column 3, he leaves the join of a 3 × 1 and a 3 × 3 game. B plays in the 3 × 3 game to force A to win that game (see (a) above). Then A must play in the 3 × 1 game and B wins.

Finally, if A begins the 3 × 7 game by pushing in column 4, then B is left to move first in the join of two 3 × 2 games. B will now win easily.

3 × 8: A wins by pushing in column 1. By Observation 1, this leaves B to move first in the 3 × 6 game.

3 × 9: A wins by pushing in column 2. Again this leaves B to move first in the 3 × 6 game.

3 × 10: A wins by pushing in column 2. This leaves the 3 × 7 game with B to make the first move.

7.63: (i) a) The game is a draw:

B can be sure of ending with two corner and two side cells, which cannot form a square.

On the other hand, A can be sure of getting the cell labeled A, and

A	D	C
E	D	B
E	B	C

one cell labeled B, one labeled C, one labeled D and one labeled E in the diagram above. No such selection forms a square.

b) We believe that the game is a draw, but do not have a complete proof. B can ensure a draw by using central (or some other) symmetry.

(ii) a) The game is a draw.

A has a drawing strategy by playing in the center and then playing by central symmetry.

If B has at least one corner, one side, and at least one piece in each of the four quadrants of the board, A cannot win. B can guarantee this by the end of her third move.

B's first move is in a side in a corner 2 × 2 that A is in. B's second move is in a corner not in the same 2 × 2 cell as her first move. Her third move is chosen so that she has a piece in each 2 × 2 subsection of the board.

b) A wins by playing in the center. A has enough time and room to set up a double threat such as:

		X		X
	O		X	
		X		

or

		X		
	O	X	X	
		X		

He does this by playing his second move diagonally adjacent to the center in a 3 × 3 corner diagonally opposite to the one in which B made her first move. From A's third move on, B's move will be forced, until A eventually can establish a double threat.

c) A wins by using the strategy of the 5 × 5 game.

7.64: a) A wins by starting in the center and using central symmetry, until B completes the third vertex of a square.

b) B wins by using central symmetry until A completes the third vertex of a square.

c) A wins as in part (a).

d) B wins as in part (b).

7.65: i) m × n, where n is the number of columns:

If n is even, B wins. A places the checker and B moves to the side with an odd number of vacant cells. Each time A moves up, B moves to the side with an odd number of cells remaining. B never moves up until A moves to next to the top row.

If n is odd, A can always win by placing the checker properly: If m is odd, A places the checker in a corner; if m is even, A places the checker in the cell next to a corner. At each subsequent move, A moves the checker up the board, until the (m-2)nd row is reached. If A is the player who moves the checker to the (m-2)nd row, then, in that row, there will be an even number of cells on each side of the checker; if B is the player in question, then there will be an odd number of cells on each side. In either case, B will eventually be forced to move to the row below the top one, and A will win.

ii) The misère $m \times n$ is equivalent to the regular $(m+1) \times n$ game, since whoever moves to the m^{th} row first loses.

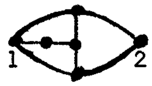
7.66: A wins by starting in the center and playing by central symmetry.

7.67: a) B wins: If A starts as  B wins by 

If A starts by 

B counters by

After B's next move, the board is equivalent to either

 or 

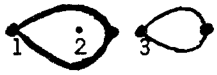
In both cases, B wins.

b) A can win as follows:

Up to equivalence B has only three responses:

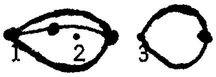
or



In each case A wins as follows:

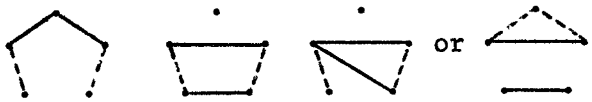
or



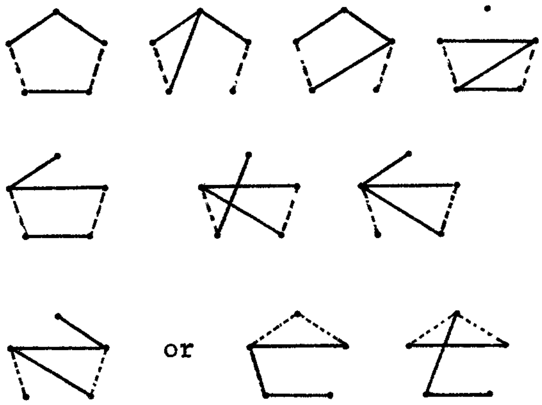
In all cases the game will end after four more moves, with proper play.

7.68: a) i) The game is a theoretical draw.

B's drawing strategy is as follows: Her first move should connect with A's first move; and, after her second move, she should leave one of the following (B's moves shown by dotted lines):

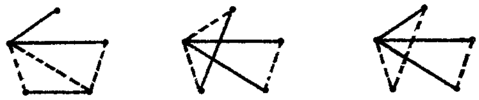


A will then have to respond with one of the following or an equivalent

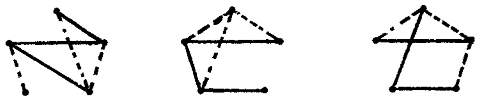


to which B responds

(a) (b) (c) (d)



(e) (f) (g)



(h) (i) (j)

Note that (j) is equivalent to (a) and may therefore be eliminated.

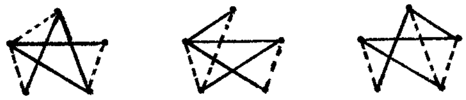
In (b) and (c), A cannot prevent B from completing a quadrilateral on her next turn, so his only hope is to prevent B from completing the tail. (See the hint to the problem in the text.) However, he is unable to do so without completing a triangle on his fifth turn. Therefore, B wins in cases (b) and (c).

In (a) and (e), B is again threatening to complete a quadrilateral on her fourth turn. If A allows her to do so, he will again be unable to prevent B from completing the tail, and B will win. Therefore, in these cases, A's only hope is to prevent B from completing the quadrilateral. That is, A's fourth move is forced.

In the remaining cases, B has no immediate threat to complete a quadrilateral, so A has a number of different possible moves in each.

The positions arising as a result of these moves are shown below. In each case, the letter below the figure indicates which of the positions above could have led to that configuration or an equivalent one.

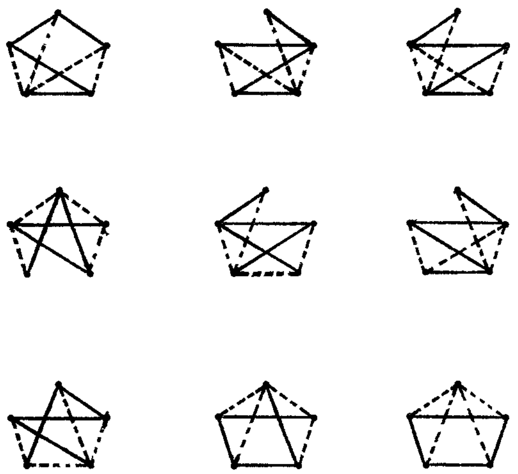
(a) (d) (d) or (e)



(f) (g) or (h) (h)

(h) (i) (j)

B then responds respectively as follows:

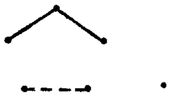


In all but the third, fourth, and sixth cases, A must either complete a triangle immediately, or he will be unable to prevent B from making a quadrilateral with a tail, in which case he will complete a triangle on his fifth turn, so B will win. In the remaining three cases, A is able to complete a pentagon, and the game ends in a draw.

Therefore, B has at least a drawing strategy.

Similarly, A can force at least a draw: On his second move, he leaves

or



.

B must then leave

or

to which A responds

or

.

In all cases but the last, B's move is forced if she is to prevent A from forming a quadrilaterial with a tail. In the last case she cannot prevent the quadrilateral and therefore must prevent the tail.

and

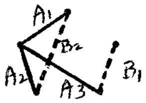
A now responds with

In all but the second case, A cannot be stopped from winning by completing a quadrilaterial with a tail. In the second case, B can block the quadrilateral, but then A can obtain a draw by completing a pentagon.

Therefore, A has a drawing strategy, and so the game is a draw.

ii) A wins as follows:

or



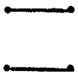
B's second move is forced and, after A's third move, A has a double threat.

iii) A wins using the same strategy as in ii).

(b) No. See the solution to Exercise 6.39 in the Hints and Solution section of the text.

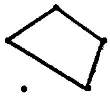
7.69: Regular:

(a) B wins by making her first line nonadjacent to A's.

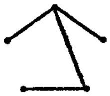


and her second move completes a quadrilaterial.

(b) B wins by placing her first line adjacent to A's. After B's second move, B can force the position to be one of the following:



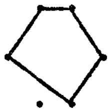
or



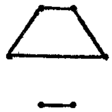
In each case there are only two safe moves remaining. A takes one, B takes the other.

(c) A wins. At the end of A's second move, the position is equivalent to

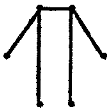
After A's third move, the position will be equivalent to



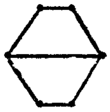
,



or



In first case, at most two safe moves are left. In the second and third cases, after A's next move, we have positions equivalent to



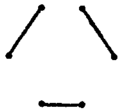
There are two moves left and A wins.

Misère:

a) B wins by

b) B wins by 

c) A wins. B's first move cannot be adjacent to A's. A wins by making



Whatever B does, A can complete a triangle.

7.70 - 7.72: Complete solutions to these games involve too many cases to be presented here.

In 7.70, it is difficult for the geese to lose unless the person playing them is very careless. A slow but steady controlled advance, without the sacrifice of any geese, is the winning strategy.

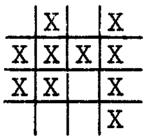
For a complete discussion of 7.71 including winning positions, see F. Schuh, The Master Book of Mathematical Recreations, Dover, New York, 1968, under the heading "Dwarfs."

In 7.72, regardless of who starts, Fred should head for B2, which he reaches after his eighth move. At his next move, he determines his distance to Sam. If an odd number of blocks, he heads directly for Sam; if an even number, his next three moves are A2, B1, B2. Fred now retains the opposition (a chess and checker term), forcing Sam into one of the three corners of the board. The best Sam can do is head for H7, but Fred will catch him on G7 or H6 on his 21st move.

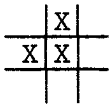
7.73: To restore an even position take 12 sticks from the pile of 18:

6	:	110
13	:	1101
15	:	1111
41	:	101001
45	:	101101

7.74: a) If you play either b3 or c3 you leave



which is equivalent to the position



in the 3 × 3 game. We saw in the solution of Sample Problem 8.2 in the text that this is a winning position.

b) If you play c3 or d3 then you leave a position equivalent to . On your next move, you leave , , or . In each case, you win. Starting in a3 is also a winning move.

7.75: If you play in d3, then the only safe moves remaining are a4, a6, e1, e6, f2, f4, or f6. Since,in addition to e1, only one of a4-a6, one of f2-f4, one of f4-f6, and one of e6-f6 may be played, the game will last exactly four more moves before your opponent will have to leave you an opportunity to win. (You also win by starting with f2.)

7.76: You must play JK or KO to win. This gives your opponent one box (which he or she may or may not take), but then you gain control.

Suppose you play JK and your opponent takes the box with KO. If he or she then plays KL, hoping to give up three boxes but to win the rest, you take OP, but then play CD rather than GH. Your opponent gets two more boxes, but then you get the rest.

Note, you cannot win by starting with OP - a "safe" move. Your opponent will then play BF, giving up two boxes (AE and FG) and winning the rest.

7.77: In the given position, Black cannot be cut off from the left side of the board. (Even if you play a1, Black responds b2. This forces b1. c2 then gives Black a double threat at c1 and d1.)

You therefore must cut Black from the right side of the board. If you don't play in a5, b5 or b6, then Black will play in b6 and can no longer be cut off. Even a5 (Black plays in b5) or b5 (Black plays in a5) are not enough. So you must play in b6.

Observe that you cannot be prevented from creating a chain from the bottom of the board to b4, and your piece at b6 can't be cut from the top, so Black must play in b5 -- to try to separate your two chains. You then play in c6, forcing Black to play in c5. You next play in d6. Since you can't be prevented from forming a chain from the bottom to d4 (via e3), nor can you now be prevented from connecting d6 to the top, Black must again cut your chain with d5. You next play in e6 -- forcing Black to play in f6. But now you play e5 and can be cut from neither the top nor bottom of the board.

7.78: Play i2 - i4. You now have a chain from the bottom of the board to e4, and are threatening e4-c4 and a double threat. Black is therefore forced to play d3-d5. (Playing b3-b5 does not really help as you play c2-c4; and b3-d3 you answer with a4-c4.)

You now play e4-e6. Black must prevent c4-c6, and so plays b5-d5. You play c6-c8; Black must play b7-b9. You play a10-c10, and Black must play b9-d9. You then play e8-e10, giving you a double threat (e6-e8 or g4-g6).

7.79: Since you are the second player in a game with an even number of dots, you want to isolate an odd number of dots. You therefore play 1d-1e. You will now win if you prevent your opponent from isolating any more dots. With careful play, you will be able to do this (or, at least, you will be able to ensure that if any other dots are isolated, there will be an even number of them).

7.80: Push the pawn in column 6 (or column 7). Black is forced to capture (or else you will capture on your next turn and win in two more moves).

Regardless of which way Black captures, you push in column 7. Black will have to capture again. This time, you recapture. Black is now forced to recapture again or to push the threatened pawn.

One of your pawns will now have an unhindered path to Black's side of the board.

7.81: Push the pawn in column 4. Black is forced to capture (or else you will continue to capture until Black does). Now push in column 3. Again Black must capture. You now push in column 2. Black must capture again. But now you push in column 1 and win on the next move.

No other first move for white will ensure a win, as Black has a similar threat in columns 7, 8 and 9. Even pushing your pawn to h7 and then g7 will not help as Black can capture the pawn in column 4.

7.82: a) Move to c4. The hound on d5 cannot move or you will escape. No matter what other move your opponent makes, move to b5. The hound on d5 is forced to move to c6. Move to a6 and the hound must move to b7. Return to b5 and you cannot be stopped from breaking through.

b) Move d5-c4. If your opponent responds d3-e2, then e4-d3 forces e2-d1. The game proceeds: f1-e2, d1-c2; c4-b3, any; d3-c2. You have now either won or the fox is on b1 and moves to a2 in which case you win with c2-b1.

If the fox's first move is d3-c2, you play e4-d3. The fox must now play c2-b3, (otherwise you play f1-e2 and, on your next turn, c4-b3, to reach the same ending described above). You play f1-e2, and the fox must play b3-a4 or again the ending is as above. The game now proceeds c4-b5, a4-b3; d3-c4, any; e2-d3, any; f3-e2. The fox is now trapped in the corner and, if they are careful, the hounds will have no trouble winning.

7.83: There are several different ways in which you can force a win. One is as follows:

Play an 0 in box c2 on level I.

This forces an X in c2 on level IV. Now place an 0 in c3 on level I. This forces an X in c4 on level I. Finally, you place an 0 in c3 on level II, establishing a double threat.

CHAPTER 8

COMMENTS AND SUGGESTIONS

1. This chapter contains some of the most beautiful mathematical ideas of the entire course. We have included it because we find it exciting to see how diverse mathematical concepts can be applied to the solution of puzzles which, on the surface at least, do not seem to have much to do with mathematics. And we want to share this feeling of excitement with our students.

2. Although there is easily enough material in this chapter for several weeks of class discussion, we are usually forced (by the selection of which other topics we have previously included) to try to squeeze it in to the last week or two of the semester. As a result, we frequently omit the material on geometrical dissections and polynominoes, and concentrate on the four other puzzles discussed in the chapter - the Tower of Brahma, Peg Solitaire, the Fifteen Puzzle and the Colored Cubes Puzzle. Most of our students have usually seen these puzzles before and are intrigued by the role mathematics plays in their solution.

3. The Tower of Brahma is an excellent vehicle for introducing mathematical induction. Although we occasionally discuss induction when we are considering Chapter 5 (in which case, we may introduce the Tower at that time), most often, we leave a careful consideration of the subject until we reach Chapter 8. We solve the Tower of Brahma puzzle for the cases of 1,2 and 3 rings, and have the class conjecture what will happen for 4 rings. Usually some members of the class suggest the incorrect result 13 [based on the recursion relationship $M_n = M_{n-1}+2(n-1)$]. This leads to the explicit formula $M_n = n^2-n+1$. We naturally also consider the correct formula $M_n = 2^n-1$. We discuss what we would have to do to establish either of these formulas and then we introduce the method of mathematical induction. We show that the correct formula can be proven by induction, whereas the other formula cannot. If time permits, we then consider some of the problems in Appendix B, some of which refer back to topics we've considered in previous chapters.

4. In discussing coloring arguments, we emphasize that they are used to produce a negative inference - they show what cannot be done rather than what can be done.

CROSS REFERENCES

1. The Tower of Brahma and Exercise 8.1 are tied to Appendix B - Mathema-

tical Induction. The Tower of Brahma is also related to the binary system - Chapter 5.

2. The number theory solution of the colored cubes relates to the discussion of prime factors - Chapter 4. In fact, instead of multiplying out the products as in Figures 8.58-8.60, the products can be represented in terms of the prime factors, and products containing more than two 2's, more than two 3's, etc. can be eliminated quickly.

3. The graph representation of the colored cubes relates to Chapter 6.

4. The coloring arguments and the solution of the Fifteen Puzzle involve parity arguments, which can be related to other parity arguments as in Chapters 6 and 7.

PRACTICE PROBLEM ANSWERS

8.A:

2. a) Twelve inversions; even;
   b) Fifteen inversions; odd;
   c) Fifteen inversions; odd.

8.B:

2. a) One possibility is: 7,6,2,3, 4,7,6,2,5,1,3,5,2,4,5,3,1,2,4,6;

   b) It is not possible.

   c) One possibility is: 4,3,2,1, 5,6,7,4,3,2,1,5,6,7,4,3,2,1,5,6,7,4,3, 2,1,5,6,7.

8.C:

1. a) One possibility is: 12,8,4,3, 7,11,10,9,5,6,11,10,9,14,15,12,8,4,3, 7,10,9,12,8,4,3,7,10,9,7,3,4,8,12,7,3, 4,7,14,11,3,4,7,14,11,5,6,3,4,11,12,15, 13,6,3,4,11,12,14,7,12,11,4,3,6,13,5, 14,15,8,7,15,14,6,13,5,8,7,15,14,6,8,7, 6,8,13,5,7,6,8,13,6,8,15.

   b) One possibility is: 15,14,13, 9,5,1,2,3,4,8,7,6,3,4,6,3,10,11,14,13, 11,5,1,10,3,14,12,7,14,12,5,3,12,14.

8.D:

1. With the line scanner listing order, any move results in a change of permutation of one of the following types: (...,a,16,...) to (...,16,a,...); (...,a,b,c,16,...) to (...,16,b,c,a,...); (...,a,b,c,d,e,16,...) to (...,16,b,c,d,e,a,...); or (...,a,b,c,d,e,f,g,16,...) to (...,16,b,c,d,e,f,g,a,...); or vice versa. In the first case, one inversion is created or destroyed, so the two permutations are of opposite parity. In the remaining cases the change may be broken up into two steps - first bring 'a' to the left of '16', and then bring '16' to the left of 'b'. In each case, 'a' is moving past an even number of numbers, and so the first step does not change the parity of the permutation; then '16' is moving past an odd number of numbers so that the net effect is to change the parity of the permutation.

SOLUTIONS TO EXERCISES

8.1: If n is odd, transfer the first ring to post C; if n is even, transfer it to B. (Note that if the rings are numbered 1,2,...,n from smallest to largest, then the optimal solution requires that a ring is always placed on a blank post or on a larger ring of opposite parity.)

8.2: Let M_n and G_n be defined as in the hint. To turn off n switches, we must first turn off n-2, then turn off the n^{th} switch and then turn off the remaining switch in G_{n-1} moves. Therefore $M_n = M_{n-2} + 1 + G_{n-1}$.

Also, to turn off the n^{th} switch, we must first turn on the (n-1)st, then turn off the n^{th}, and then turn off the (n-1)st. That is,

$$G_n = G_{n-1}+1+G_{n-1} = 2G_{n-1}+1.$$

This is the recursion relation we encountered in the Tower of Hanoi problem. Therefore, $G_n = 2^n-1$. This gives

$$M_n = M_{n-2}+1+2^{n-1}-1 = M_{n-2}+2^{n-1}$$

$$= M_{n-4}+2^{n-3}+2^{n-1}$$

$$= \ldots$$

e) If n is odd,

$$M_n = 2^{n-1}+\ldots+2^2+1 = \tfrac{1}{3}(2^{n+1}-1);$$

f) If n is even,

$$M_n = 2^{n-1}+2^{n-3}+\ldots+2^3+2$$

$$= 2(2^{n-2}+\ldots+2^2+1)= \tfrac{2}{3}(2^n-1).$$

The answers to parts a), b), c) and d) may now easily be found using these formulas.

8.3: Let M_n be defined as in the hint for Exercise 8.2. Clearly $M_1 = 1$ and $M_2 = 2$.

To describe the procedure for n switches, consider n-tuples, where 1 in the i^{th} position indicates that the i^{th} switch is on and 0 indicates that it is off. (For brevity, we omit commas.)

For n = 3,
(111) → (110) → (010) → (001) → (000), so $M_3 \le 4$.

For n = 4,
(1111) → (1101) → (1100) → (0100) → (0010) → (0001) → (0000), So $M_4 \le 6$.

For n = 5,
(11111) → (11101) → (11100) → (10100) → (01100) → (00100) → (00010) → (00001) → (00000), so $M_5 \le 8$.

For n = 6,

(111111) $\xrightarrow{M_3 \le 4}$ (111000) → (101000) → (011000) → (001000) → (000100) → (000010) → (000001) → (000000), so

$M_6 \leq 11$.

For n = 7,

$$(1111111) \xrightarrow{M_4 \leq 6} (1110000) \to (1010000) \to (0110000) \to (0010000) \xrightarrow{5} (0000000),$$ so $M_7 \leq 14$.

For n = 8,

$$(11111111) \xrightarrow{M_5 \leq 8} (11100000) \to (10100000) \to (01100000) \to (00100000) \xrightarrow{6} (00000000),$$ so $M_8 \leq 17$.

For n = 9,

$$(111111111) \xrightarrow{M_5 \leq 8} (111100000) \to (110100000) \to (101100000) \to (011100000) \to (010100000) \to (001100000) \to (000100000) \xrightarrow{6} (000000000),$$ so $M_9 \leq 20$.

For n = 10,

$$(1111111111) \xrightarrow{M_6 \leq 11} (1111000000) \to (1101000000) \to (1011000000) \to (0111000000) \to (0101000000) \to (0011000000) \to (0001000000) \xrightarrow{7} (0000000000),$$ so $M_{10} \leq 24$.

That these bounds are best possible is not obvious. To turn off n switches, we first turn off k of them, then work with the remaining n-k to leave only the rightmost on, and then, one switch at a time, we move down the line until all switches are off. Note that, when working with the n-k switches, we can turn the second off at will but can only turn other switches off by using the relay procedure described in rule (i). Therefore, the procedure above requires $M_k + \sum_{i=1}^{n-k-1} i + k+1$ moves. Minimizing this over all k, $0 \leq k \leq n-1$,

$$M_n \geq \min_k (M_k + \frac{(n-k)(n-k-1)}{2} + k+1).$$

Therefore

$$M_3 \geq \min\{4, M_1+1+2, M_2+3\} = 4;$$

$$M_4 \geq \min\{7, M_1+3+2, M_2+1+3, M_3+4\} = 6;$$

$$M_5 \geq \min\{11, M_1+6+2, M_2+3+3, M_3+1+4, M_4+5\} = 8;$$

$$M_6 \geq \min\{16, M_1+10+2, M_2+6+3, M_3+3+4, M_4+1+5, M_5+6\} = 11;$$

$$M_7 \geq \min\{22, M_1+15+2, M_2+10+3, M_3+6+4, M_4+3+5, M_5+1+6, M_6+7\} \geq 14;$$

$$M_8 \geq \min\{29, M_1+21+2, M_2+15+3, M_3+10+4, M_4+6+5, M_5+3+6, M_6+1+7, M_7+8\} \geq 17;$$

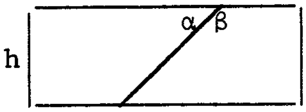
$$M_9 \geq \min\{37, M_1+28+2, M_2+21+3, M_3+15+4, M_4+10+5, M_5+6+6, M_6+3+7, M_7+1+8, M_8+9\} \geq 20;$$

$$M_{10} \geq \min\{46, M_1+36+2, M_2+28+3, M_3+21+4, M_4+15+5, M_5+10+6, M_6+6+7, M_7+3+8, M_8+1+9, M_9+10\} \geq 24.$$

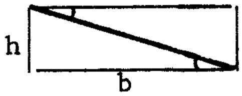
Therefore, we have equality in all cases.

8.4:

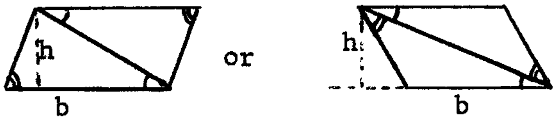
a) Cutting along the indicated altitude, the two pieces may be reassembled to form a rectangle of area bh. Note that α and β are supplementary angles since they are adjacent angles in a parallelogram.



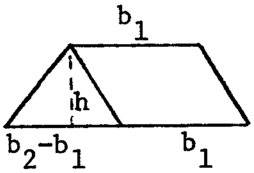
b) Using a congruent triangle, form a rectangle of area bh. The triangle has half the total area.



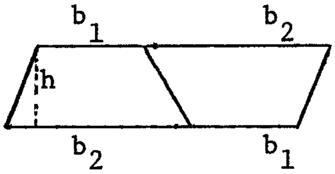
c) Using a congruent triangle form a parallelogram and then use the result of a).



d) Break the trapezoid into a triangle and a parallelogram and use (a) and (c) above.

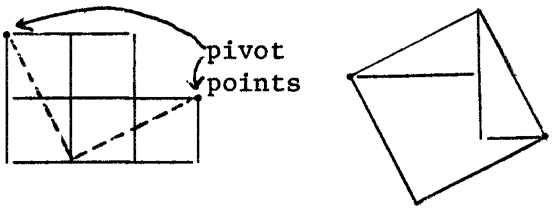


Or, using a congruent trapezoid, form a parallelogram and then use the result of a).

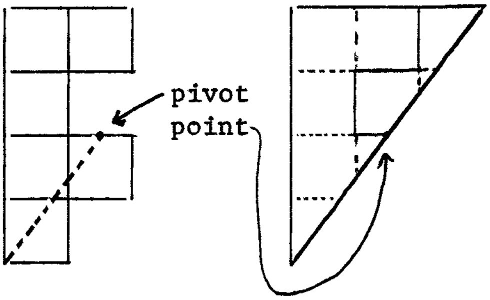


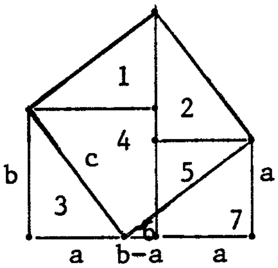
$A = (b_1+b_2)h$

8.5: a)



b)

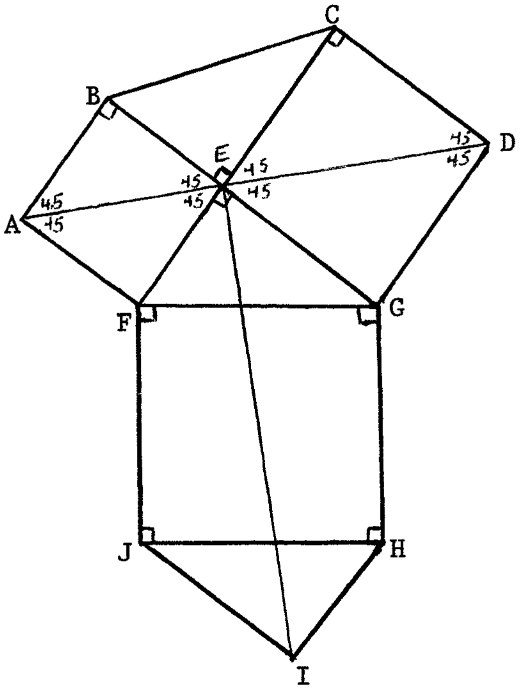




Regions 2 and 3 are congruent, and region 1 is congruent to the union of regions 6 and 7. Therefore,

$$
\begin{aligned}
c^2 &= A_1+A_2+A_4+A_5 \\
&= A_6+A_7+A_3+A_4+A_5 \\
&= (A_5+A_7)+(A_3+A_4+A_6) \\
&= a^2+b^2.
\end{aligned}
$$

ii) Label the regions as suggested in the hint, or as indicated below:



Here ABEF, CDGE and FGHJ are squares of sides a,b and c respectively, and triangles BCE, FEG and HIJ are congruent a,b,c right triangles.

$AB = AF = EF = HI = a$

$BC = FG = FJ = HG = c$

$CD = GD = JI = GE = b$

$\angle EBC = \angle EFG = \angle IHJ = \alpha$, so

$\angle ABC = \angle AFG = \angle EFJ = \angle IHG = 90 + \alpha$

and $\angle ECB = \angle EGF = \angle IJH = \beta$, so

$\angle BCD = \angle FGD = \angle FJI = \angle HGE = 90 + \beta$.

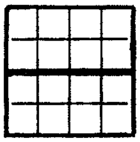
Therefore, quadrilaterals ABCD, AFGD, EFJI and IHGE are all congruent (by side, angle, side, angle, side). Hence the areas of ABCDGF and EFJIHG are equal:

$$a^2 + b^2 + ab = c^2 + ab$$

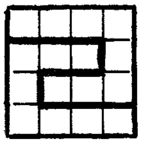
$$a^2 + b^2 = c^2.$$

8.6: a) Regardless of whether i), ii) or iii) holds, there is only one way of cutting the checkerboard:

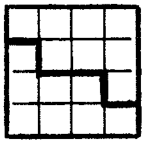
b) As indicated in the hint, there are six shapes into which the board may be cut:



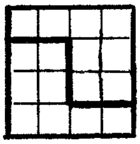
(1)



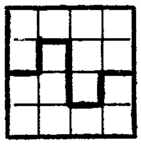
(2)



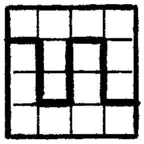
(3)



(4)



(5)

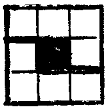


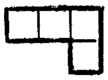
(6)

Each of these gives rise to a number of different cuts depending on whether i), ii) or iii) applies. The number is indicated in the following table.

	(1)	(2)	(3)	(4)	(5)	(6)	total
i	2	4	4	4	4	4	22
ii	1	2	2	2	2	2	11
iii	1	1	1	1	1	1	6

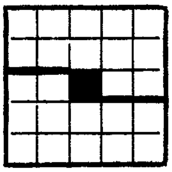
c) There is only one possible shape for each piece.

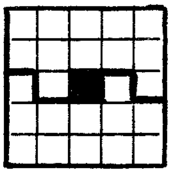


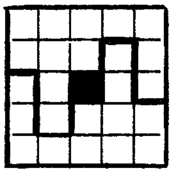


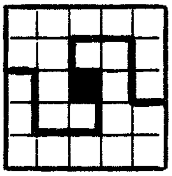
This gives rise to two different cuts in cases i) and ii) and only one in iii).

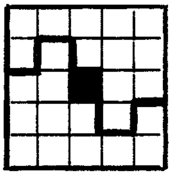
d) The only possible shapes are

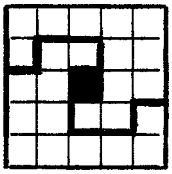

(1)

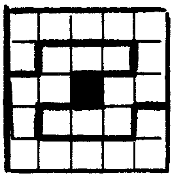

(2)

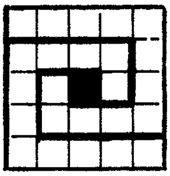

(3)

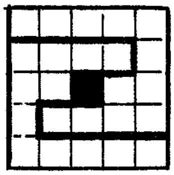

(4)

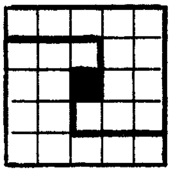

(5)

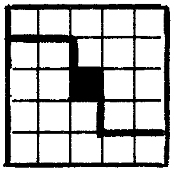

(6)

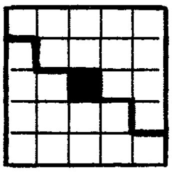

(7)

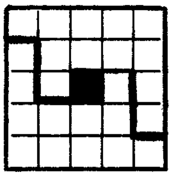

(8)

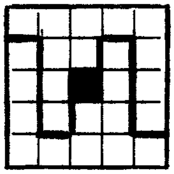

(9)

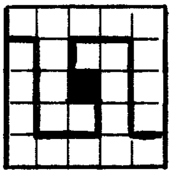

(10)


(11)


(12)


(13)

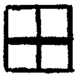

(14)


(15)

Each shape gives rise to 2 different cuts in cases i) and ii) and only one in case iii). Therefore there are 30 possible cuts in cases i) and ii) and 15 in case iii).

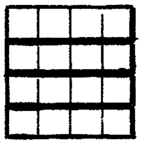
e) No cut is possible in any of the three cases.

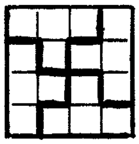
8.7: a) Only one cut is possible if color doesn't count.

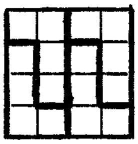


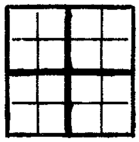
If color does count, then no cut is possible.

b) There are four shapes into which the board may be cut:


(1)


(2)

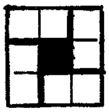

(3)


(4)

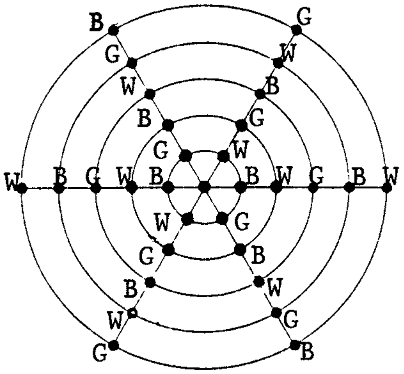
Each shape gives rise to a number of cuts, as indicated in the following table:

	(1)	(2)	(3)	(4)	total
i	1	0	4	1	6
ii	1	1	2	1	5
iii	1	1	1	1	4

c) Only one cut is possible in all cases.



d) The following shapes are possible.



Each move, except the final one, preserves the relative parity of the colors. To begin with, G = B = W = 6. But, to take a final jump into the center, two colors must be 1 and the third 0. Since 0 and 1 are of opposite parities, <u>this is impossible</u>.

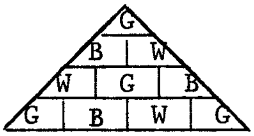
j) <u>33-53-35-13-31, 42-22, 34-12-32, 31-33</u>.

k) <u>No</u>. If the cells are colored

G	B	W	G	B
W	G	B	W	G
B	W	G	B	W
G	B	W	G	B
W	G	B	W	G

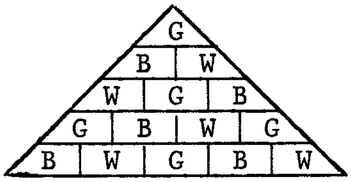
then any horizontal or vertical move preserves the relative parities of the colors. In the starting position, B = G = W = 3. Therefore, <u>it is not possible to leave a single peg anywhere</u>.

ℓ) Color as follows:



Any horizontal or diagonal move preserves relative parities. Therefore, since, with a corner cell empty, G = B = W = 3, <u>it is not possible to leave a single peg</u>.

m) Color as follows:



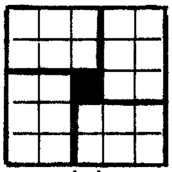
With the top corner cell empty, B = W = 5 and G = 4. Therefore, <u>if a single peg is to remain, it must be in one of the cells</u> labeled G. These cells are <u>11,32,41,44 and 53</u>.

A sequence of moves which actually leaves a peg in 53 is: 31-11, 33-31, 11-33, 41-21, 52-32, 33-31, 21-41, 51-31, 44-42, 54-52, 31-53, 52-54, 55-53.

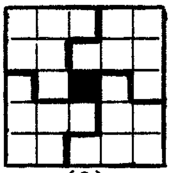
A sequence which leaves a peg in 11 is: 31-11, 33-31, 11-33, 52-32, 54-52, 51-53, 41-21, 44-22, 53-33, 22-44, 55-33, 33-31, 31-11.

Changing the final move of this sequence to 21-41, leaves a single peg in 41; and, by vertical symmetry, we could leave a single peg in 44.

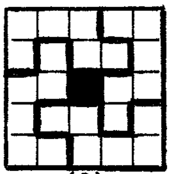
Although the coloring argument does not rule out 32 as the final resting place of a single peg, we have not been able to find a sequence of moves which



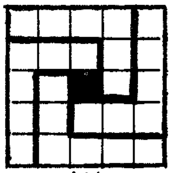
(1)



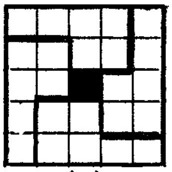
(2)



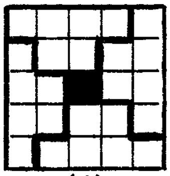
(3)



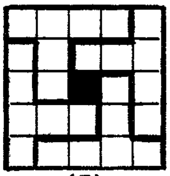
(4)



(5)



(6)

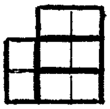


(7)

The number of cuts giving rise to these shapes are

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	total
i	1	2	2	2	2	2	2	13
ii	1	2	2	2	2	1	2	12
iii	1	1	1	1	1	1	1	7

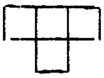
e) Only one cut is possible in all cases



8.9: a) Since the missing cells are of opposite color, the rectangle determined by the missing cells (i.e., with the missing cells in diagonally opposite corners) must be of odd by even dimensions. Suppose the even dimension is horizontal (the vertical case is analogous). Then, first use vertical dominoes to fill all columns to the left and right of the rectangle. Then fill the rows above and below the rectangle, using horizontal dominoes. (This is possible since each row is of even length.) Then fill the end columns of the rectangle. (This is possible since each column contains an even number of cells in addition to the missing one). This leaves an odd by even rectangle, with no missing cells, which can be covered with horizontal dominoes.

b) If the missing cell is of the "extra" color, then the answer is yes. The procedure is similar to that above. If the missing cell is of the other color, then the covering can clearly not be done.

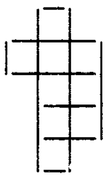
c) No. Each tetromino, if placed on a checkerboard, must cover two cells of each color, with the exception of the tetromino



which covers three cells of one color and one of the other color. Since a 2×10 or 4×5 rectangle has the same number of cells of each color, the five tetrominoes cannot cover such a rectangle.

d) (Solved in the text)

e)

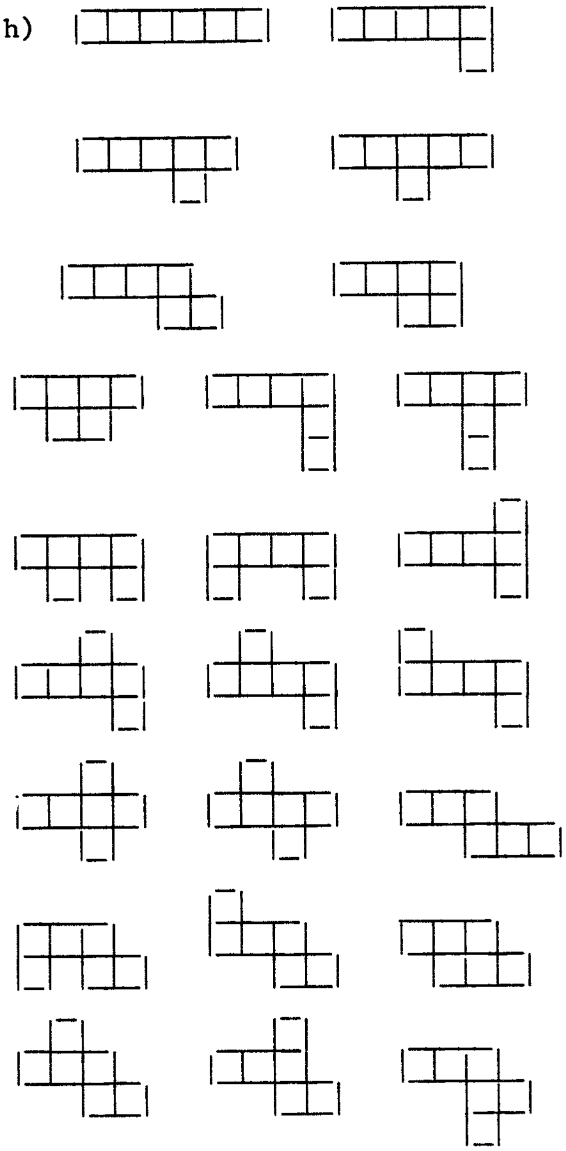


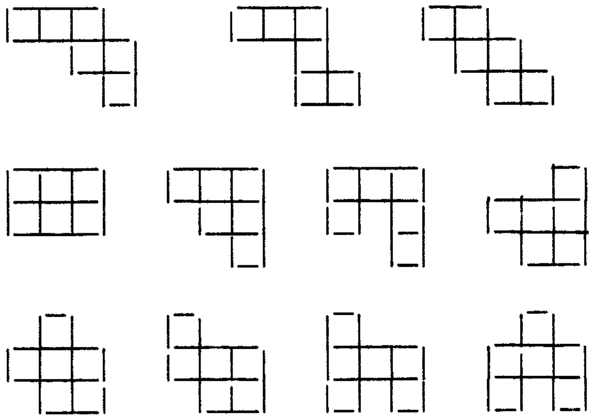
f) (See the Answer section of the text)

g) Following the hint, the maximal number of edge cells covered by each pentomino in an allowable position is shown in the following table.

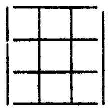
pentomino	F	I	L	P	N	T	U	V	W	X	Y	Z	total
border cells	3	1	1	2	2	1	1	1	3	3	2	1	21

Thus a maximum of 21 border cells can be covered. Since there are 22 border cells in the region, the region cannot be covered by the twelve pentominoes.

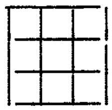
h) 



8.10: a) Using a coloring argument, the removed cube must be a corner or the middle of a face. In either of these cases, the covering is possible - cover the two adjoining layers without the missing cube and then cover the remaining layer as



or



b) One way of doing it is as follows:

bottom

5	5	3
5	4	2
4	4	2

middle

7	3	3
6	1	1
6	1	2

top

7	8	8
7	8	9
6	9	9

c) Color the solid in two colors, like a checkerboard. Polycubes (1), (2), (3), (5), and (7) each cover two cells of each color, no matter how they are placed. But polycube (4) covers three cells of one color and only one of the other color. Therefore these six polycubes will cover eleven cells of one color and thirteen of the other. Since the 2 × 3 × 4 solid contains twelve cells of each color, it cannot be covered by the six polycubes in question.

8.11: For reference purposes, label the cells of the board as in the text.

a) To land in 44, a peg must jump from 24,42,64 or 46. But to land in one of these holes, a peg can only come from 44. Thus, only the pegs that start in 24,42,46 or 64 could end in the center.

b) Using a coloring scheme similar to that in the text, if the center cell is empty then

		G	B	W		
	B	W	G	B	W	
W	G	B	W	G	B	W
B	W	G		W	G	B
G	B	W	G	B	W	G
	G	B	W	G	B	
		G	B	W		

G = B = W = 12.

Therefore, since relative parities are still preserved by each jump, it is not possible for only one peg to remain.

c) Imagine the center hole filled (B) above, and another hole vacant. If it is a hole which is colored black, then again G = B = W and one peg alone cannot be left in any hole.

If the vacant hole is gray, then G = 11, B = 13, W = 12, so G and B are of the same parity, and a single remaining peg must end up in a white hole. Similarly, if the vacant hole is white, then a single remaining peg must end in a gray hole. In either case, it cannot end in the center.

d) The same arguments in b) and c) apply to any hole labeled B in the coloring above or in the coloring obtained by applying any symmetry of the board. That is, it is true of any cell which is labeled with an x in the following figure.

			x			
	x	x		x	x	
	x	x		x	x	
x			x			x
	x	x		x	x	
	x	x		x	x	
			x			

e) 34-32, 54-34, 35-33, 32-34, 24-44.

f) 34-14, 54-34, 46-44-24, 42-44, 14-34-54, 64-44.

g) 44-64, 52-54, 55-35-33-53-55, 56-54, 64-44.

h) No. Color the cells of the board as follows:

G	G	G	G	G	G	G	G
W	W	W	W	W	W	W	W
B	B	B	B	B	B	B	B
G	G	G	G	G	G	G	G
W	W	W	W	W	W	W	W
B	B	B	B	B	B	B	B
G	G	G	G	G	G	G	G
W	W	W	W	W	W	W	W

Any move preserves the relative parities of the three colors. In the given position, G = 10, W = 10, B = 4. Since all are even, we cannot end with just one peg.

i) Color the cells of the board (with the exception of the center cell) as in the figure on the next page.

leaves a peg in 32. We suspect that it may not be possible to do so, although we have no proof.

8.12: a) 15 moves are required. Labeling the cells 1-7 from left to right, the only possible sequence of moves (up to symmetry) is: 5-4, 3-5, 2-3, 4-2, 6-4, 7-6 (leaving |o|•|o|•|o|•| |), 5-7, 3-5, 1-3 (leaving | |•|o|•|o|•|o|), 2-1, 4-2, 6-4, 5-6, 3-5, 4-3.

b) As above, there is only one way (up to symmetry) to produce the desired result. If there are ever adjacent black pegs anywhere to the left of the vacant cell and if there is at least one white peg to the left of these black pegs, then an impasse is established. The same is true if adjacent white pegs to the right of the vacant cell have a black peg to their right. If black starts, white must jump and then move a second peg. Black must jump twice and move the next peg, white must jump three times and move the next peg, and so on, until finally you make n-1 consecutive jumps of one color followed by moving the n^{th} peg of that color. This leaves

|o|•|o|•|...|•|o|•|o|•| |

if n is odd

or

| |o|•|o|•|o|...|o|•|o|•|

if n is even.

Now make n jumps leaving

| |•|o|•|o|...|o|•|o|•|o|

or

|•|o|•|o|...|•|o|•|o| |.

From here, the original procedure is followed in reverse. (I.e., a move and n-1 jumps of one color followed by a move and n-2 jumps of the next color, and so on, until the desired position is reached.

In all, you have made

$$(1+2+\ldots+n)+n+(n+n-1+\ldots+2+1)$$

$$= \frac{n(n+1)}{2} + n + \frac{n(n+1)}{2}$$

$$= n(n+1)+n$$

$$= n(n+2) \quad [= (n+1)^2-1]$$

moves.

c) Interchange the fourth column as in a) above; but each time there is a vacant cell in a new row in the fourth column, interchange the white and black pegs in that row. This will require 7 × 15 + 15 = 120 moves in all.

d) See the answer section of the book.

8.13: a) Label the checkers 1-8 from left to right. Move 5 over 4 and 3 and place it on 2; move 3 over 4 and 6 and place it on 7; move 8 over 7 and 3 and place it on 6; move 1 over 2 and 5 and place it on 4.

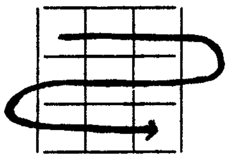
b) Move 7 over 8 and 9 and place it on 10. This leaves 8 single checkers in a row, and we can proceed as in a).

c) Proceed as in b) to reduce n by 1. Continue doing so until 8 single checkers in a row remain. Then proceed as in a).

8.14: Label the O's and T's as below

R	O_1	T_1
N		A
O_2	I	T_2

Using a listing order



which preserves parity, the given arrangement corresponds to the permutation $(R,O_1,T_1,A,N,O_2,I,T_2)$. Since the permutation $(N,R,O_1,T_1,O_2,I,T_2,A)$ is of the opposite parity, it is not obtainable from the arrangement above. However, the permutation $(N,R,O_2,T_1,O_1,I,T_2,A)$ is. Namely, the sequence of moves $O_1,R,N,O_2,I,T_2,A,O_1,O_2,N,R,O_2,O_1,T_1,O_2,R,N,O_1$ leaves the desired arrangement.

N	R	O_2
O_1		T_1
I	T_2	A

8.15: The original placement of the people was

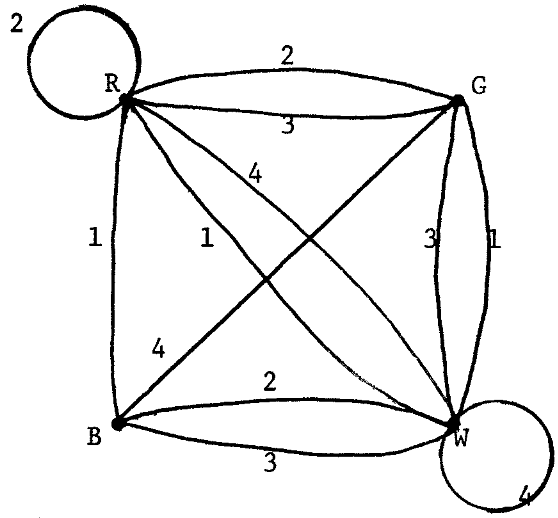
D	J	N
A		V

and the desired placement is

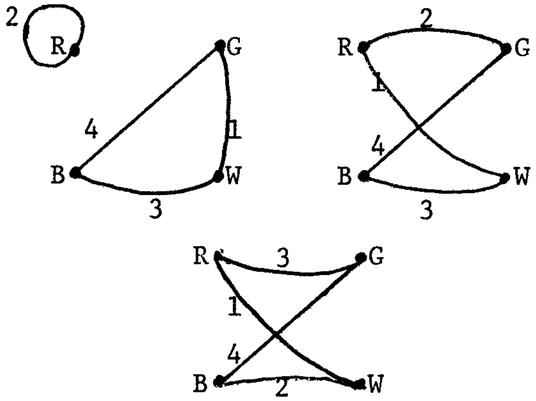
?	?	?
V	?	A

One sequence of moves which yields the desired result is V,N,J,V,A,D,V,J,N,A, D,V.

8.16: a) For our convenience, we label the cubes 1,2,3,4 rather than I,II,III, IV, as in the text. The graph of the set of cubes is

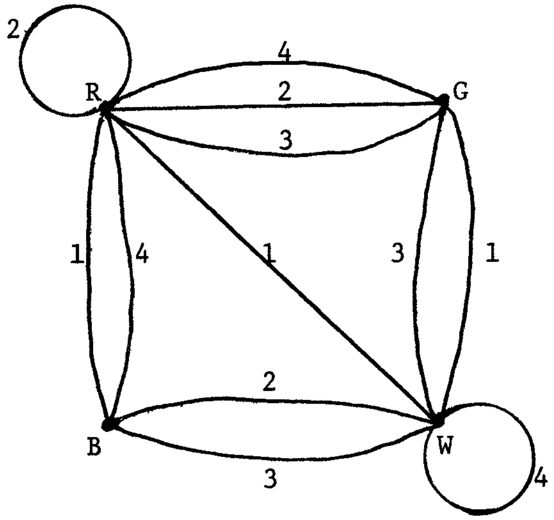


The only circuits satisfying the conditions of the problem are

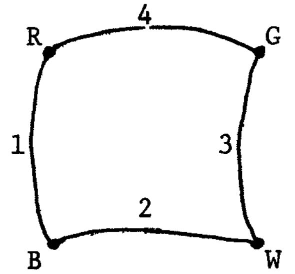


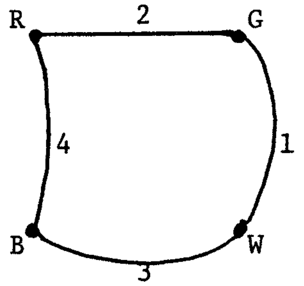
Since all three use the same cube 4 pair, it is not possible to have both a front-back and a side-side pair from cube 4. Therefore, there are no solutions.

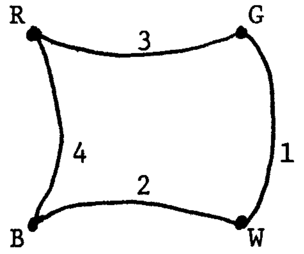
b) The graph of the set of cubes is



The circuits satisfying the conditions of the problem are







The only solution arises from (i) and (ii) (since (i) and (iii) use the same pair from cube 2, and (ii) and (iii) use the same pair from cube 4).

cube	front	back	left	right
1	R	B	G	W
2	B	W	R	G
3	W	G	W	B
4	G	R	B	R

8.17: Following the hint in the text, the eight cubes may be labeled as in the following chart.

cube	top	bottom	front	back	left	right
UFL	B	z	R	t	G	x
UFR	B	u	R	v	x	O
UBL	B	q	t	W	G	p
UBR	B	s	v	W	p	O
LFL	z	Y	R	m	G	n
LFR	u	Y	R	ℓ	n	O
LBL	q	Y	m	W	G	k
LBR	s	Y	ℓ	W	k	O

There are two possibilities for x: W or Y.

If x = W, then z = O, t = Y, u = G, v = Y, p = R, q = O, s = G, m = B, n = W, ℓ = B and k = R; if x = Y, then t = O, z = W, v = G, u = W, q = R, p = Y, s = R, n = B, m = O, ℓ = G and k = B.

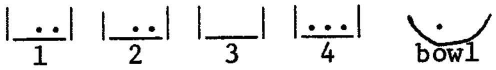
Note that, in both cases, the same eight cubes are used, but they are arranged differently: (B-O,R-Y,G-W), (B-G,R-Y,W-O), (B-O,Y-W,G-R), (B-G,Y-W,R-O), (O-Y,R-B,G-W), (G-Y,R-B,W-O), (O-Y,B-W,G-R) and (G-Y,B-W,R-O).

<u>8.18</u>: If the board has an odd number of cells, then a reentrant knight's tour would require an odd number of moves. If the board is colored like a checkerboard, then each knight's move changes the color of the cell on which the knight is found. An odd number of moves brings a knight from a cell of one color to a cell of the other color - not back to the original cell.

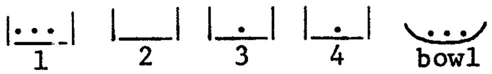
<u>8.19</u>: <u>The game can only be won in parts a and e of the problem</u>. In the other parts, it is necessary to consider all possible starting positions and several subcases to see that you cannot win.

In a), you can win by sowing from cup 3, the cup just before the bowl. Then sow from cup 1 and you will automatically win.

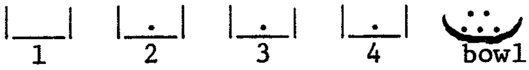
In e), start sowing from cup 3, two before the bowl. This ends in the bowl, leaving



Now sow from cup 4, ending in 2; and then from 2, ending in the bowl and leaving



Now sow from 3, then from 4 and then from 1. This ends in the bowl and leaves



This is essentially case a). You win by sowing first from 4, then from 2, then from 3 and finally from 4 again.

CHAPTER 9

COMMENTS AND SUGGESTIONS

This chapter contains an assortment of recreational problems. We have included it as a light dessert for those who have read this book.

The problems require basic reasoning skills and we assign them occasionally during the semester, but do not spend much class time on them.

CROSS REFERENCES

1. Problems of syllogism are related to logic - Chapter 2. This is an especially good topic for students who will take the LSAT's or GRE's.

2. Some of the coin weighing problems can be related to the binary or ternary systems (Chapter 5).

3. Shunting problems are also discussed in Chapter 1.

SOLUTIONS TO EXERCISES

9.1: Arrange the numbers from 1 to 41 in a circle and strike out every third number. The last two numbers remaining are 16 and 31.

9.2: Arrange the children in a circle and follow the instructions. The eventual winner is the person to the immediate left of the person who starts.

9.3: Write 10 blanks and work circularly:

A	C	E	1	T	W	O	2	T	H
R	E	E		3	F	O		U	R
4	F	I			V	E		5	S
	I	X			6	S			E
	V	E				N			7
	E	I				G			
	H	T				8			
	N	I							
	N	E							
	9	10							

The original arrangement of the cards was therefore 4,9,10,A,3,6,8,2,5,7.

9.4: Arrange the 12 children in a circle, labeling them ABCDEFGHIJKL, where A represents Helen. Then try values of n, starting with n = 11.

n = 15 is the first value that works, where Jeffrey is in position K, two seats to Helen's right.

9.5: Label the players in a circle, A,B,C,D,E,F,G,H,where A represents Burt, the first player counted. The good players must be D,E,A, and G.

Now, trial and error shows that "6" counts out the same four players for Lisa, when Lauren takes the first, third, fifth and seventh players.

9.6: Weigh three coins against three others. If they balance, then the bad coin is one of the remaining three. If they don't balance, then the bad coin is one of the lighter three. In either case, there are now only three possibilities for the bad coin. Weighing one of these against another determines which coin is bad. If they balance, then the third coin is bad; if they don't, then the lighter one is the bad coin.

9.8: Label the coins A,B,C,D,E,F,G,H, I,J,K,L. Weigh A,B,C,D against E,F,G,H.

Case 1: If A+B+C+D = E+F+G+H, then weigh I,J against A,K.

If I+J = A+K, then L is bad and weighing L against A tells us whether it is light or heavy. If I+J < A+K, then either I or J is light or K is heavy. Weighing I against J tells which is which. The case I+J > A+K is similar.

Case 2: If A+B+C+D < E+F+G+H, then A,B,C or D is light or E,F,G or H is heavy. Now weigh A,B,E against F,C,D. If A+B+E = F+C+D, then G or H is heavy. Weighing G against H tells which.

If A+B+E < F+C+D, then A or B is light or F is heavy. Weighing A against B tells which. Finally, if A+B+E > F+C+D, then E is heavy or C or D is light. Weighing C against D tells which.

Case 3: A+B+C+D > E+F+G+H. This case is the same as Case 2, with < replaced by > and heavy and light interchanged.

9.9: Weigh one coin against another and then weigh a third against a fourth. Next weigh the lighter coin in the first weighing against the lighter in the second. Call the lighter of these A, the other C, and the other two coins B and D, so that A < C < D and A < B. Let E represent the fifth coin.

For the fourth weighing, weigh E against C.

Case 1: If E < C, then A < E < C < D or E < A < C < D.

Weighing A against E determines which. Next weigh B against C. If B > C, weigh B against D. If B < C, then, if necessary, weigh B against E. (This isn't necessary if E < A, since A < B.) In either case, you will know the exact order of the weights.

Case 2: If C < E, then A < C < E < D or A < C < D < E. Weighing D against E determines which.

If A < C < E < D, weigh B against E and then against C or D as the case requires.

If A < C < D < E, weigh B against D and then against C or E as the case requires.

In either case, you will now know the exact order of the weights.

9.10: Choose four of the coins and weigh each pair of them against each other pair. There are three possible outcomes of these weighings: one of them is a perfect balance; one coin is always on the light side; one coin is always on the heavy side. Label the coins A,B,C and D so that

Case I: A+B = C+D, A+C < B+D,
A+D < B+C;

Case II: A+B < C+D, A+C < B+D,
A+D < B+C;

Case III: A+B < C+D, A+C < B+D,
B+C < A+D.

Label the fifth coin E.

If E is odd, then some combination of pairs will balance (10+40 = 20+30, 10+50 = 20+40, 20+50 = 30+40), and conversely. Therefore, E is odd in Case I and even in Cases II and III. Also, in Case I, A must be the lightest of the four remaining coins, and B the heaviest. (If, for example, C were heavier than B, then A+B < A+C < B+D < C+D - contradiction, etc.) Similarly, in Case II, A must be the lightest of the remaining coins, and, in Case III, D must be the heaviest. We therefore have the following possibilities:

Case I:

A	10	10	10	10	20	20
B	40	40	50	50	50	50
C	20	30	20	40	30	40
D	30	20	40	20	40	30
E	50	50	30	30	10	10

Case II: A = 10, E = 20, {C,D,E} = {30,40,50}. (Note that A = 10, E = 40 doesn't work since 10+50 > 20+30.)

Case III: D = 50, E = 40, {A,B,C} = {10,20,30}. (Again D = 50, E = 20 does not work since 10+50 < 30+40.)

In Cases I and II, weighing C against D now limits the possibilities to three.

In Case I, if C < D then (A,B,C,D,E) = (10,40,20,30,50), (10,50,20,40,30), or (20,50,30,40,10). Now, weighing B+C against D+E determines which. On the other hand, if C > D, then weighing B+D against C+E makes the determination.

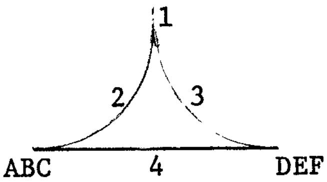
Similarly, in Case II, if C < D, then (A,B,C,D,E) = (10,30,40,50,20), (10,40,30,50,20), or (10,50,30,40,20). Weighing A+B against C+E determines which. If C > D, then weighing A+B against D+E makes the determination.

Finally, in Case III, weighing B against C cuts the possibilities to three. If B < C, then weighing B+D against C+E determines the weight of each coin; and if B > C, then weighing C+D against B+E does the job.

9.11: a) Take one coin from the first sack, two from the second, three from the third, four from the fourth and five from the fifth, and put them on the scale. The number of ounces over 15 tells us which sack contains the counterfeit coins. That is, if the coins weigh 16 ounces, then the first sack contains the counterfeits; if 17, then the second sack does; etc.

b) Take 16 coins from the first sack, 32 from the second, 49 from the third, etc.

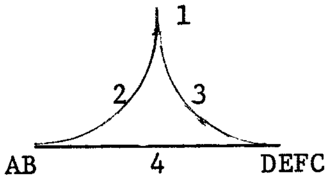
9.12: a)



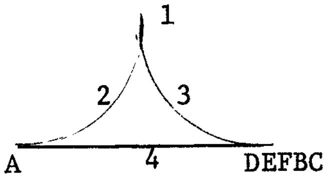
Since the cars are to be left exactly as they started, they must end up facing the same direction as initially. Therefore, they cannot enter track 1 from one side and leave from the other.

We therefore proceed as follows:

1) D goes via 4 to pick up C,backs up 2 to 1 (one reversal) leaving C on 2. D then leaves 1 via 3 (second reversal), picks up EF and brings EF via 4 (third reversal) to the left. Backing EF up 2 (fourth reversal) to pick up C, and returning to the main track (fifth reversal), D then pushes EFC to the right (sixth reversal), leaving

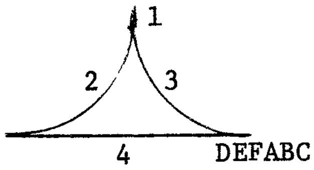


2) Leaving C at the right, the process is now repeated with B. But this time 7 reversals are needed, since D is initially moving in the wrong direction. This leaves



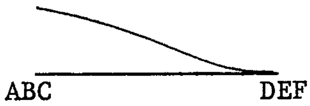
3) Seven more reversals would move A to the right of DEF, but then D would end up facing the wrong direction. This can be avoided by having D first pick up A by approaching via 3, 1 and then 2.

This introduces an extra reversal (at 1), but leaves

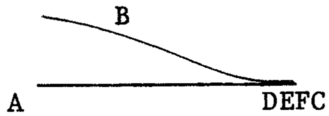


after a total of 21 reversals. One more reversal leaves the desired position.

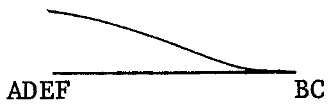
9.13:



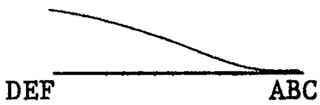
C moves to the right and up the siding (1 reversal). DEF moves to the left (no reversal), and C returns to the right (2nd reversal). BDEF moves to the right, up the siding and back, leaving B on the siding (5th reversal).



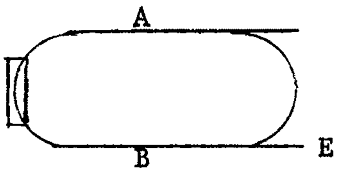
DEF moves to the left (6th reversal)
C picks up B and returns (8th reversal).



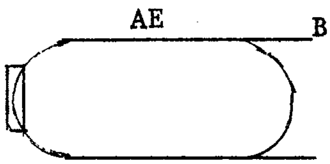
ADEF moves to the right (9th reversal), and leaves A on the siding. DEF returns to the main track and continues on its way. (12th reversal). BC picks up A and returns (14th reversal) continuing on its way.



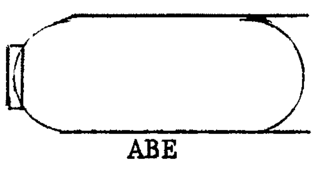
9.14:



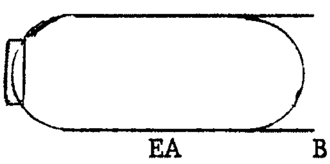
E moves to pick up B, pulls B around the right curve, hooks up to A, and then pushes B to the right side track on top.



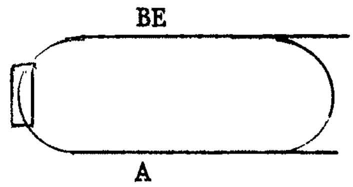
E now pulls A around the right curve, leaves A on the bottom and continues through the tunnel to pick up B. E then pushes B around the right curve to A.



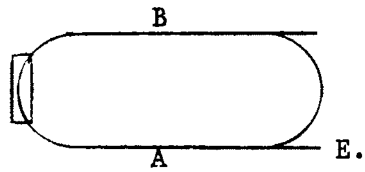
E then goes around the track and through the tunnel and pushes B to the right bottom side track.



E again goes through the tunnel, continues all the way around, picks up B, and pushes B around to the top.



Finally E comes around to the right and moves back to its original position.

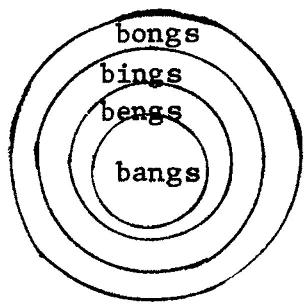


9.15: If something is a bang, then it's a beng; if it's a beng then it's a bing; if it's a bing, then it's a bong. Therefore, if something is a bang, then it's a bong, and so all bangs are bongs.

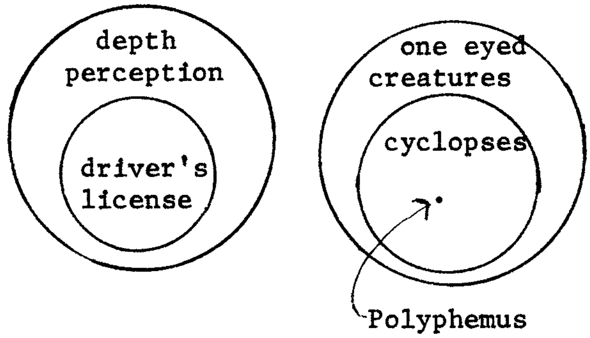
Symbolically,

$$a \to e$$
$$e \to i$$
$$\underline{i \to o}$$
$$\therefore a \to o$$

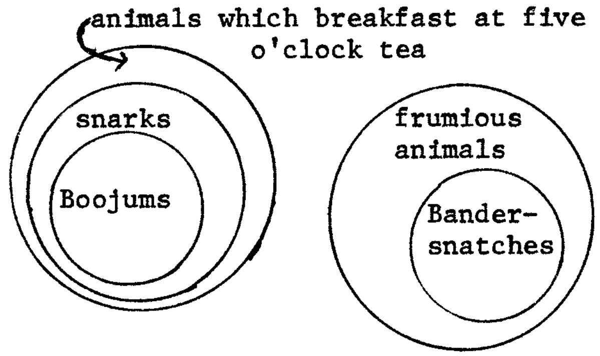
Or, pictorially,



9.17: Since Polyphemus is a Cyclops, he has only one eye. Therefore, since no one eyed creatures have depth perception, Polyphemus does not have depth perception. But then, since people without depth perception cannot get a driver's licence, Polyphemus cannot get a driver's license.



9.18: Since every Bandersnatch is a frumious animal and since frumious animals do not breakfast at five o'clock tea, Bandersnatches do not breakfast at five o'clock tea. But then, since only animals which breakfast at five o'clock tea can be snarks, Bandersnatches are not snarks. Since Boojums are snarks, no Bandersnatches are Boojums.



9.20: By statements 2 and 6, people who believe that knowledge is obtainable through reason do not believe in the supernatural. Therefore, by statement 4, they are not afraid of ghosts. Combining this with statement 7, we find that people who believe that knowledge is obtainable through reason would be willing to walk in a cemetery after midnight. But then, by statement 1, they are not superstitious and so, by statement 5, they do walk under ladders. Since people who walk under ladders take a chance that a bucket of paint might fall on their heads, people who believe that knowledge is obtainable through reason take a chance that a bucket of paint might fall on their heads.

9.21: By statements 4 and 1, people who have seen a mermaid have visited Atlantis. But then, by statements 7 and 9, such people have survived a shipwreck and therefore are protected by Neptune. Combining this with statements 2 and 8, people who have seen a mermaid have found favor with the gods and therefore have been invited to Mount Olympus. But then, by statement 6, people who have seen a mermaid have partaken food with the gods and therefore, by 3, have tasted real ambrosia. Since people who have tasted real ambrosia have hangovers, everyone who has seen a mermaid has a hangover.

9.22: Since each move joins two sections and reduces the number of sections by one, completion of the puzzle requires 499 moves, no matter how you proceed.

9.23: It takes 23 hours and 40 minutes, since in the next 20 minutes each bacterium will split and the dish will be full.

9.24: Let a_n, b_n and W_n respectively denote the number of adult worms, the number of newborn worms, and the total number of worms owned by Mr. Bonacci after n weeks. Clearly $W_n = a_n + b_n$. But $a_n = W_{n-1}$, the number of worms who were alive the previous week; and $b_n = W_{n-2}$, the number of worms that are at least two weeks old (and therefore ready to reproduce). Therefore

$$W_n = W_{n-1} + W_{n-2}$$

for $n \geq 2$.

Making a table

n	0	1	2	3	4	5	6	7	8	9	10	11	12
a_n	0	1	1	2	3	5	8	13	21	34	55	89	144
b_n	1	0	1	1	2	3	5	8	13	21	34	55	89
W_n	1	1	2	3	5	8	13	21	34	55	89	144	233

Therefore, Mr. Bonacci now owns 233 worms.

9.25: The needle does not travel around the record, the needle only travels from the outer groove of the record to the inner groove.

This is the radius of the record (15 cm), minus the distance from the edge of the record to the first groove (5 mm), minus the distance from the last groove to the center of the record (5 cm). Therefore, the needle travels 150-5-50 = 95 mm.

(Note that this is not quite correct. The needle actually travels in a slight circular arc rather than a straight line. However, not given the length of the needle arm, this is the best we can do.)

9.26: At the end of 16 minutes, the worm has progressed only 2 inches. At the end of 64 = 4·16 minutes, the worm has progressed 8 inches. In the next ten minutes, it climbs 2 more inches, and is out of the hole, so the climb takes 74 minutes in all.

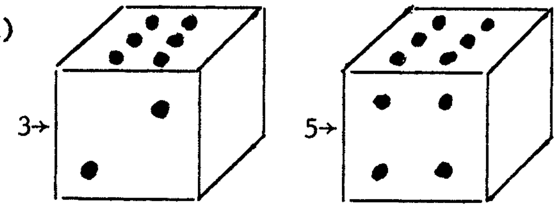
9.27: a) He must pull 14 socks to be guaranteed two black ones, since he might pull 12 brown ones first.

b) He must pull 3 socks to be guaranteed a matching pair.

9.28: The worst that can happen is that Paula first picks three red, three green and three white socks. The tenth sock will give four of one color.

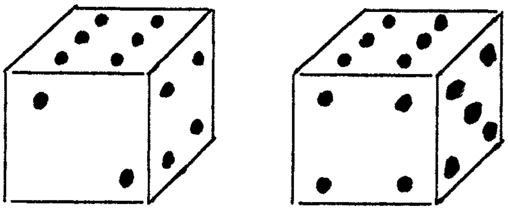
9.29:

a)



(1) (2)

b)



(3) (4)

The above dice are the given ones oriented so that the 6 is on top. The dice in (a) are supposed to be the same as those in (b). But if each number appears only once on each die, then die (2) is not the same as (4), since the 5 is on the opposite side. Also (2) is not the same as (3), since the dots of the 6 are oriented differently on the two dice with respect to the 4.

Therefore, either some number appears more than once on one of the dice, or else Steve switched dice. In either case, Steve is cheating.

9.30: The set {50,51,52,...,99} satisfies the condition and contains 50 elements. Consider all pairs of distinct numbers less than 100 whose sum is 100

(1,99), (2,98),..., (49,51).

Any set of 51 elements must contain at least 2 members of one of these pairs.

9.31: a) Answered in the text.

b) One possibility is to make the knots at the 1,3,7 and 12 inch marks on the string.

APPENDIX A - ANSWERS

A.2: a) $x = -1, y = 1/2$

b) $x = 4$

c) $x = -3 \pm \sqrt{10}$

d) $x = 2, y = -1, z = 3$

e) $x > -\frac{5}{6}$

f) no solution

g) $x = 5, y = 4$ or $x = -\frac{41}{3}, y = \frac{40}{3}$

APPENDIX B - ANSWERS

B.2: S_1 is true: $1^2 = \frac{1\cdot 2\cdot 3}{6}$.

Suppose S_k is true:

$$1^2+2^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Add $(k+1)^2$ to both sides of the equation equation:

$$1^2+2^2+\ldots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{2k^2+k}{6} + k+1\right]$$

$$= (k+1)\ \frac{2k^2+7k+6}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

This last equality is S_{k+1}, and so the induction is complete.

B.4: We proceed by proving a slightly stronger result:

S_n: at most 2^n-1 positive integral weights may be balanced by using a set of n given weights.

Clearly the desired result follows if S_n is true for all n, since the set of all positive integers not exceeding 2^n contains more than 2^n-1 elements.

S_1 is obviously true.

Suppose S_k is true.

Given a set of k+1 weights, consider any k of them: $w_1,\ldots,w_k$. By the induction hypothesis, these may be used to balance at most 2^k-1 weights. Using the (k+1)st weight with any of these combinations yields at most 2^k-1 additional weights which may be balanced. Using the (k+1)st weight alone yields one more possibility. Therefore, in all, k+1 weights may be used to balance at most $2^k-1+2^k-1+1 = 2^{k+1}-1$ integral weights.

This is just S_{k+1}, and the induction is complete.

B.5: S_n: There are 2^n cases in the truth table involving n distinct variables: $p_1,p_2,\ldots,p_n$.

S_1 is true since there are only 2 cases for one variable (p_1 is T or F).

Assume that S_k is true. Consider k+1 variables: $p_1,p_2,\ldots,p_k,p_{k+1}$. There are 2^k cases for the k variables $p_1,p_2,\ldots,p_k$. For each of these cases p_{k+1} is T or F. Therefore, altogether there are $2^k+2^k = 2^{k+1}$ cases for the k+1 variables, and so S_{k+1} is true.

Therefore, by the principle of mathematical induction, S_n is true for all $n \geq 1$.

B.6: Let S_n be the more general statement:

n guesses will suffice to determine a previously chosen positive integer chosen from among 2^n-1 distinct positive integers.

S_1 is obviously true.

Assume S_k is true. Consider a set of $2^{k+1}-1$ distinct positive integers. Let the first guess be the middle (median) integer. Either the guess is correct or the set of integers to be considered is now of order 2^k-1. The chosen number can now be identified in at most k guesses since S_k is assumed true.

Therefore, S_{k+1} is true, and the desired result follows by the principle of mathematical induction.