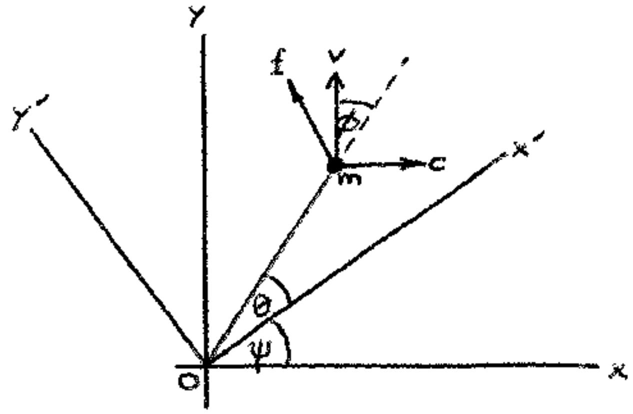
INTRODUCTION TO TENSOR CALCULUS, RELATIVITY AND COSMOLOGY

D F Lawden

Solutions to Exercises

No.1.



In the inertial frame Oxy, the polar coordinates of the particle are $(r, \theta+\psi)$ and the polar resolutes of the equation of motion are

$$m[\ddot{r} - r(\dot{\theta}+\dot{\psi})^2] = f_r, \quad m[r(\ddot{\theta}+\ddot{\psi}) + 2\dot{r}(\dot{\theta}+\dot{\psi})] = f_\theta.$$

The polar components of the particle's velocity relative to the frame Ox′y′ are $(v_r, v_\theta) = (\dot{r}, r\dot{\theta})$. Hence, the above equations can be rearranged into the form

$$m(\ddot{r} - r\dot{\theta}^2) = f_r + 2m\omega v_\theta + mr\omega^2, \quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = f_\theta - 2m\omega v_r - mr\dot{\omega}.$$

Since

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \quad v_r = v\cos\phi, \quad v_\theta = v\sin\phi,$$

these are the equations of motion asked for.

A force c perpendicular to the direction of v in the sense shown in the diagram, will have radial and transverse components $c\sin\phi$, $-c\cos\phi$, respectively. Taking $c = 2m\omega v$, this inertial force will accordingly yield the second terms in the right-hand members of the equations of motion; the two remaining terms in these members are supplied by a force $mr\omega^2$ acting along OP and a force $mr\dot{\omega}$ acting transversely in the clockwise sense.

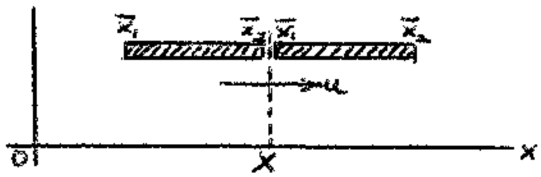
Thus, the motion of the particle relative to the non-inertial frame Ox′y′ will be correctly predicted by Newton's second law if the three inertial forces are taken to act upon it, in addition to the 'real' force f.

No.2. Let $\bar{x}_1, \bar{x}_2$ $(\bar{x}_2 > \bar{x}_1)$ be the coordinates of the two ends of the bar in $\bar{S}$, so that its length in this frame is $\bar{\ell} = \bar{x}_2 - \bar{x}_1$. Suppose the positions of the two ends of the bar on the x-axis are established to have coordinates x_1, x_2 at instants which are simultaneous in $\bar{S}$ with time $\bar{t}$. Then, these events have space-time coordinates (x_1,t_1) and (x_2,t_2) in S and space-time coordinates $(\bar{x}_1,\bar{t})$ and $(\bar{x}_2,\bar{t})$ in $\bar{S}$ (N.B. t_1 and t_2 will differ, since the events will not be simultaneous in S). Application of the inverse Lorentz equations now leads to the relationships

$$x_1 = \beta(\bar{x}_1 + u\bar{t}), \quad x_2 = \beta(\bar{x}_2 + u\bar{t})$$

and subtraction gives the result $x_2 - x_1 = \beta\bar{\ell}$. Thus, if $x_2 - x_1$ is accepted as the length ℓ of the bar in S (incorrectly, of course), then ℓ is greater than $\bar{\ell}$ by the factor β.

No.3.



Let X be the x-coordinate of the fixed point on the x-axis and suppose the front end of the bar passes this point at t_2 and the rear end passes the point at the later time t_1, both times being measured by the S-frame clock located at X which has been synchronized with the master at O. In the S-frame, these two events have space-time coordinates (X,t_2) and (X,t_1). In the $\bar{S}$-frame, the same events will take place at the points $\bar{x}_2$ and $\bar{x}_1$ (respectively) on the $\bar{x}$-axis and will have space-time coordinates $(\bar{x}_2,\bar{t}_2)$ and $(\bar{x}_1,\bar{t}_1)$. Employing the Lorentz transformation $\bar{x} = \beta(x-ut)$ to relate the space-time coordinates in the two frames, we find

$$\bar{x}_1 = \beta(X - ut_1), \quad \bar{x}_2 = \beta(X - ut_2).$$

Subtraction yields

$$\bar{\ell} = \bar{x}_2 - \bar{x}_1 = \beta u(t_1 - t_2) = \beta uT.$$

Accepting that $\ell_1 = uT$, this leads to the Fitzgerald contraction formula $\ell = \bar{\ell}(1 - u^2/c^2)^{\frac{1}{2}}$.

This analysis confirms our intuition that the two methods of measuring the length of a moving bar (viz. (i) by fixing the positions of its ends at simultaneous instants and (ii) by measuring the time it takes to pass a fixed point) lead to the same result. Had this not been the case, it would have been necessary to reach agreement amongst physicists, which of these two procedures defined the length of a moving body.

No.4. Although the chalk marks fixing the positions of the two ends of the bar on the x-axis are made simultaneously at time t in the S-frame, according to the $\bar{S}$ clocks, these marks are made at different times $\bar{t}_1$, $\bar{t}_2$. Applying the inverse Lorentz transformation $t = \beta(\bar{t} + u\bar{x}/c^2)$ to these two events, we get

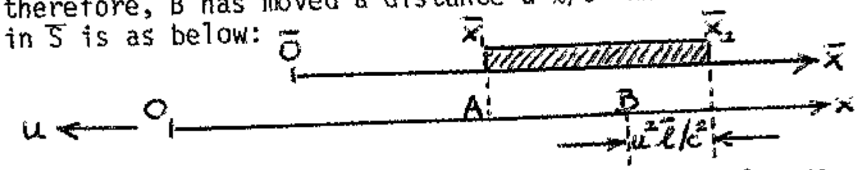
$$t = \beta(\bar{t}_1 + u\bar{x}_1/c^2) = \beta(\bar{t}_2 + u\bar{x}_2/c^2).$$

Whence

$$\bar{t}_1 - \bar{t}_2 = u(\bar{x}_2 - \bar{x}_1)/c^2 = u\bar{\ell}/c^2.$$

This means that the $\bar{S}$ observer will calculate that the mark fixing the $\bar{x}_2$ end of the bar is made a time $u\bar{\ell}/c^2$ before the mark fixing the $\bar{x}_1$ end.

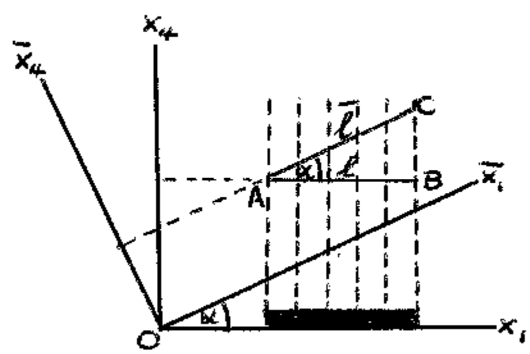
Thus, according to observations made from $\bar{S}$, a mark B first appears on the x-axis opposite the $\bar{x}_2$ end and this then moves with the x-axis in the negative $\bar{x}$ sense for a time $u\bar{\ell}/c^2$ at speed u before the second mark A appears opposite the $\bar{x}_1$ end. At this instant $\bar{t}_1$ in $\bar{S}$, therefore, B has moved a distance $u^2\bar{\ell}/c^2$ and the situation as it appears in $\bar{S}$ is as below:



We deduce that, for $\bar{S}$, $AB = \bar{\ell} - u^2\bar{\ell}/c^2 = (1 - u^2/c^2)\bar{\ell}$. The observer employing the frame S is now observed from $\bar{S}$ to measure the distance AB between the marks using a measuring rod of length $(1 - u^2/c^2)^{\frac{1}{2}}$ which he takes to be of unit length. Hence, according to $\bar{S}$, the result S will record for this measurement will be $AB/(1 - u^2/c^2)^{\frac{1}{2}} = (1 - u^2/c^2)^{\frac{1}{2}}\bar{\ell}$. This is in agreement with equation (6.3) and there is no inconsistency.

A frequent source of confusion should here be remarked. If we wish to be precise, we should always clearly distinguish between a physical event and our observation of the event. Thus, a nova outburst on a star distant 100 light years from the earth may be observed in our telescopes today, but the event itself will have occurred a century ago by terrestrial time. Nevertheless, in these circumstances it is normal to say that, in a terrestrial reference frame, the nova outburst 'was observed to occur' a century ago. Thus, when using a particular reference frame, observations will always be assumed to be made by local observers who are present at the events in question and who are equipped with standard instruments at rest in the frame, the clocks having already been synchronized with the master at the origin. Unless otherwise stated, no correction will be made for any time delay which may occur while light travels from an event to the eye of some particular observer in the frame. The direct visual impression which such an individual observer receives when he studies a sequence of phenomena through his telescope is of no fundamental importance and does not count for our purposes as the description of the phenomena relative to his reference frame. This description must always be couched in terms of space-time coordinates and other physical quantities defined locally at the events involved. For example, in the earlier argument, it was stated that "for $\bar{S}$, AB = $\bar{\ell} - u^2\bar{\ell}/c^2$"; however, if the segment AB of the moving x-axis were observed through a telescope fixed at some point of $\bar{S}$, the length seen would need correction to allow for the different times it takes light to reach the instrument from the two ends; such a correction would depend upon the position of the telescope and clearly has no fundamental significance - it is therefore ignored. In future, therefore, a phrase of the type 'as seen from S ...' must be interpreted as introducing a description of physical phenomena in terms of physical quantities defined in the frame S, local to the events involved. Once this description has been calculated, the view which an individual observer in the frame has when he turns his telescope on to the phenomena, can then be determined if needed.

No.5.



Since the bar is stationary on the x-axis, the world-lines of its particles will all be parallel to the x_4-axis as shown in the diagram. Taking a section AB of this band of world-lines by a perpendicular to the x_4-axis, we obtain a representation of the bar at some instant in the S-frame; thus AB = ℓ. Similarly, taking a section AC by a perpendicular to the $\bar{x}_4$-axis, a representation of the bar at some instant in the $\bar{S}$-frame is obtained; thus AC = $\bar{\ell}$. From the diagram it follows that

$$\bar{\ell} = \ell\sec\alpha = \ell(1 - u^2/c^2)^{\frac{1}{2}}.$$

This is the Fitzgerald contraction formula.

No.6. Defining α by the equation $\tanh\alpha = u/c$ (α will be real since $|u| < c$), it follows that $\cosh\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}$, $\sinh\alpha = u/c(1 - u^2/c^2)^{\frac{1}{2}}$. Equations (5.8) are now seen to be equivalent to the equations given in the exercise.

Since $\cosh\alpha + \sinh\alpha = e^{\alpha}$, $\cosh\alpha - \sinh\alpha = e^{-\alpha}$, by subtracting and adding the equations for $\overline{x}$ and $c\overline{t}$, we find

$$\overline{x} - c\overline{t} = (x - ct)e^{\alpha}, \quad \overline{x} + c\overline{t} = (x + ct)e^{-\alpha}.$$

Multiplying these equations, we then get

$$\overline{x}^2 - c^2\overline{t}^2 = x^2 - c^2t^2,$$

i.e. the quadratic form $x^2 - c^2t^2$ has the same value in both frames and so is an invariant. This is clearly a special case of the invariant which has been used to define the proper time in section 7.

Consider the $\overline{S}$ clock as it passes through the point (x,y,z) of S at time t in this latter frame. The $\overline{S}$ coordinates $(\overline{x},\overline{y},\overline{z})$ of this clock and the time $\overline{t}$ it indicates are all given by the transformation formulae; in particular

$$c\overline{t} = ct\cosh\alpha - x\sinh\alpha.$$

If $\overline{t} = t$, this equation gives

$$ct(\cosh\alpha - 1) = x\sinh\alpha.$$

Using half angle identities, we find $x = ct\tanh\frac{1}{2}\alpha$, defining a plane parallel to the yz-plane on which all the $\overline{S}$ clocks indicate t at this instant in the S-frame (i.e. the S and $\overline{S}$ clocks happen to be in synchrony in this plane). Clearly, this plane moves with velocity $c\tanh\frac{1}{2}\alpha$ as t increases in S.

No.7. By choosing $t = 0$ to be the instant in S when the rearwards pulse passes through O, the equations of motion of the pulses can be written

$$x = ct, \quad x = ct + d.$$

Translating to the $\overline{S}$-language by the inverse transformation equations

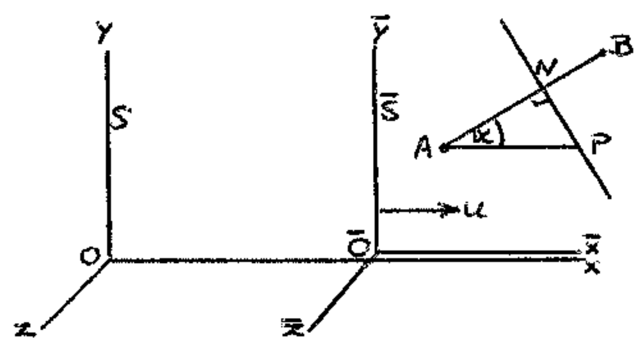
$$x = \beta(\overline{x} + u\overline{t}), \quad t = \beta(\overline{t} + u\overline{x}/c^2),$$

after some manipulation these equations become

$$\overline{x} = c\overline{t}, \quad \overline{x} = c\overline{t} + d\sqrt{[(c+u)/(c-u)]}.$$

These $\overline{S}$ equations of motion imply that the pulses move with velocity c (as expected) along the $\overline{x}$-axis, a distance $d\sqrt{[(c+u)/(c-u)]}$ apart.

No.8.



Choose axes in the two frames to be parallel and so that $\overline{O}$ moves along the x-axis of S with speed u. Then AP is parallel to Ox; denote the angle BAP in S by α. Also choose the axes so that, in S, the plane BAP is parallel to Oxy.

In S, the events at A and B can be taken to have coordinates (x_A,y_A,z_A,t_A) and $(x_A+d\cos\alpha, y_A+d\sin\alpha, z_A, t_A+T)$ respectively.

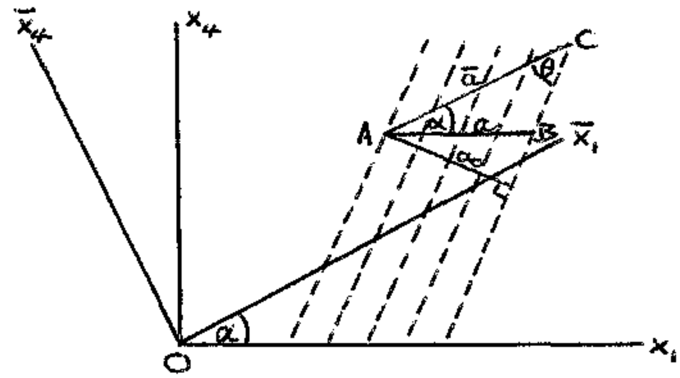
Let $\bar{t}_A$, $\bar{t}_B$ be the times of these events according to the clocks of $\bar{S}$. Then, using a Lorentz transformation equation, we deduce that

$$\bar{t}_A = \beta(t_A - ux_A/c^2), \quad \bar{t}_B = \beta[t_A + T - u(x_A + d\cos\alpha)/c^2].$$

But $\bar{t}_A = \bar{t}_B$; hence $AP = u = c^2 T\sec\alpha/d$.

Constructing PN perpendicular to AB (in frame S), we now find $AN = AP\cos\alpha = c^2T/d$. Thus P lies in the plane stated.

No.9. The band of world-lines of particles of the rod lying in the x_1x_4-plane in Minkowski space-time is shown in the diagram.



By taking sections of the band parallel to the x_1, $\bar{x}_1$ axes, we obtain the lengths a, $\bar{a}$. The rest length a_0 is the right-section of the band.

Equations (5.7) give $\cos\alpha = \beta$, $\sin\alpha = iu\beta/c$. Hence, using the cosine rule on ΔABC, we get

$$BC = \sqrt{(a^2 + \bar{a}^2 - 2\beta a\bar{a})}.$$

The sine rule applied to the same triangle shows that

$$\sin\theta = \frac{AB}{BC}\sin\alpha = \frac{a\beta u/c}{\sqrt{(2\beta a\bar{a} - a^2 - \bar{a}^2)}}.$$

Since $a_0 = \bar{a}\sin\theta$, the result now follows.

No.10. Assume first that the axes of the inertial frames S and $\bar{S}$ are as shown in Fig.2 of Chap.1. Then $\underline{u} = (-u,0,0)$. Thus, $\underline{u}\cdot\underline{r} = -ux$.

The three components of the first equation are therefore

$$\bar{x} = x - u\left[-\frac{x}{u}(\beta-1) + \beta t\right] = \beta(x - ut)$$

$$\bar{y} = y, \quad \bar{z} = z.$$

The second equation reduces to

$$\bar{t} = \beta(t - ux/c^2).$$

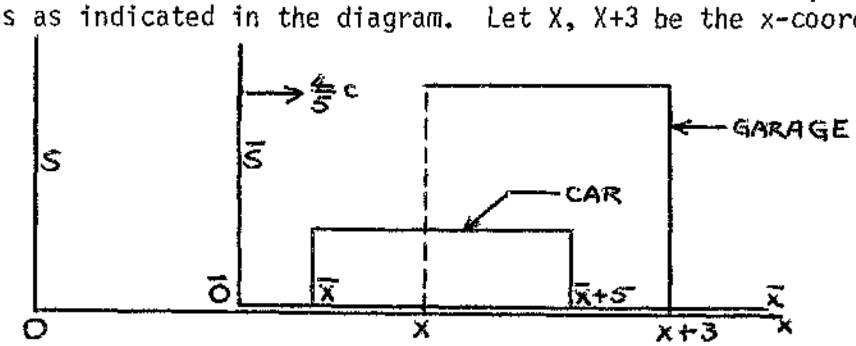
These equations are known to be valid and thus the stated equations have been justified with this choice of axes.

Being tensor equations in the $\mathcal{E}_3$ (t and $\bar{t}$ are invariants relative to rotations of axes in this space), the stated equations will remain valid when the axes of S and $\bar{S}$ are rotated in any manner which retains parallelism. Thus, Lorentz transformation equations valid for two parallel inertial frames, whose relative motion is not necessarily along common x and $\bar{x}$ axes, have been obtained

No.11. In the wife's frame, the car is subject to a Fitzgerald contraction factor of $\surd(1 - 4^2/5^2) = 3/5$ and is accordingly 3m long. She can therefore slam the doors just before the car is brought to rest. When the car comes to rest, it attempts to resume its rest length (in her frame) and thus, provided the garage wall and doors do not collapse, the car will crumple to fit the garage.

In the driver's frame, the car is stationary with length 5m and the garage is subject to the contraction factor of 3/5, which reduces its length to 9/5 m. As explained in the solution to problem 4 (earlier), to say that the driver 'sees' the garage to have this length, is to speak loosely - this is the length he will arrive at from his visual observations after he has allowed for the time light takes to travel from the parts of the garage to his eye and which, even in classical physics, would be accepted as the garage length appropriate to his frame.

Take S to be the wife's frame and $\overline{S}$ the driver's frame, choosing axes as indicated in the diagram. Let X, X+3 be the x-coordinates



of the front and rear of the garage and let $\overline{X}$, $\overline{X}+5$ be the $\overline{x}$-coordinates of the rear and front of the car respectively. Consider the two events: (i) front of car meets rear wall of garage and (ii) garage doors are slammed. In S, these events have coordinates (i) $(X+3,t)$, (ii) (X,t) (i.e. the events are simultaneous). In $\overline{S}$, the events have coordinates (i) $(\overline{X}+5,\overline{t}_1)$, (ii) $(\overline{X},\overline{t}_2)$. The transformation equation $t = \beta(\overline{t} + u\overline{x}/c^2)$ applied to these events yields

$$t = \beta[\overline{t}_1 + 4(\overline{X}+5)/5c] = \beta(\overline{t}_2 + 4\overline{X}/5c).$$

We deduce that $\overline{t}_2 = \overline{t}_1 + 4/c$.

In the driver's frame, therefore, the garage approaches at speed $4c/5$ and its back wall strikes the front of the car at time $\overline{t}_1$. The car then begins to crumple as the garage continues to advance and, during the time $4/c$ seconds, the garage moves a distance 16/5 m, thereby reducing the length of the car to 5 - 16/5 = 9/5 m, i.e. to the length of the garage. The doors now close and the remainder of the car is brought to rest.

Thus, although the driver and his wife 'see' the sequence of events differently, there is no inconsistency between their two accounts of the process and the final state, when both are using the same frame S, is the same for both.

No.12. This is similar to problem 7. In the S frame, the equations of motion are

$$x = vt, \quad x = vt + d.$$

Transforming to the $\overline{S}$-language using the transformation equations

$$x = \beta(\overline{x} + u\overline{t}), \quad t = \beta(\overline{t} + u\overline{x}/c^2),$$

the equations become

$$\bar{x} = \frac{v - u}{1 - uv/c^2}\bar{t}, \qquad \bar{x} = \frac{v - u}{1 - uv/c^2}\bar{t} + \frac{d(1 - u^2/c^2)^{\frac{1}{2}}}{1 - uv/c^2} .$$

The results asked for now follow immediately.

Note that the transformed velocity also follows from the first of equations (15.11) derived in Chap.3.

No.13. The point's equation of motion in $\bar{S}$ is

$$\bar{x} = c\bar{t}^2/2\tau.$$

Transforming to the S-language by the Lorentz equations

$$\bar{x} = \sqrt{2}(x - ct/\sqrt{2}), \qquad \bar{t} = \sqrt{2}(t - x/\sqrt{2}c),$$

this equation takes the form

$$x^2 - 2\sqrt{2}c(t + \tau)x + 2c^2t(t + \tau) = 0.$$

Solving this quadratic for x, we find

$$x = \sqrt{2}c\left[(t + \tau) \pm \sqrt{\tau(t + \tau)}\right].$$

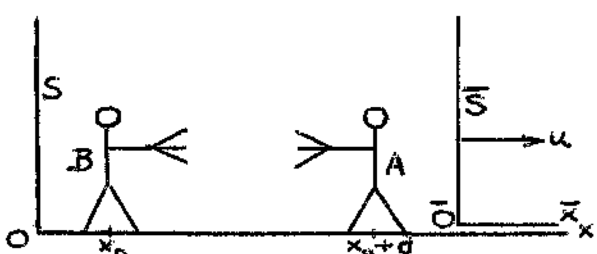
At the instant the point leaves $\bar{O}$, O and $\bar{O}$ coincide and the clocks fixed to these points are synchronized to zero. Hence, x = 0 at t = 0 and the negative sign must be taken in the last equation.

If t/τ is small, then

$$\begin{aligned} x &= \sqrt{2}c\tau\left[\frac{t}{\tau} + 1 - \left(\frac{t}{\tau} + 1\right)^{\frac{1}{2}}\right] \\ &= \sqrt{2}c\tau\left[\frac{t}{2\tau} + \frac{t^2}{8\tau^2} + O(t^3/\tau^3)\right] \\ &= \frac{ct}{\sqrt{2}}(1 + t/4\tau) \end{aligned}$$

if terms of order t^3/τ^3 are neglected.

No.14.



Let the two events (i) A fires, (ii) B fires, have coordinates (x_0+d,t), (x_0,t), respectively in S. In $\bar{S}$, the first event will have coordinates

$$\bar{x}_A = \beta(x_0 + d - ut), \qquad \bar{t}_A = \beta\left[t - u(x_0+d)/c^2\right]$$

and the second event coordinates

$$\bar{x}_B = \beta(x_0 - ut), \qquad \bar{t}_B = \beta(t - ux_0/c^2)$$

Since $\bar{t}_B - \bar{t}_A = \beta ud/c^2$, this shows that, in $\bar{S}$, A fires a time $\beta ud/c^2$ before B.

In $\bar{S}$, A fires from the point $\bar{x}_A$ in the negative $\bar{x}$ sense towards B. During the time $\beta ud/c^2$, his light pulse will travel a distance $\beta ud/c$ and will then be at the point P, where

$$\bar{x}_P = \bar{x}_A - \beta ud/c = \beta(x_0 + d - ut - ud/c).$$

At the instant the pulse arrives at P, B fires from the point $\bar{x}_B$; thus, the pulse's distance from B is then

$$\bar{x}_P - \bar{x}_B = \beta d(1 - u/c) = d\left(\frac{c - u}{c + u}\right)^{\frac{1}{2}}$$

It should be noted that, for all possible values of u, this distance is positive, i.e. $\bar{x}_P > \bar{x}_B$, so that A's pulse is still to the right of B and has not reached B. Thus, in no frame can it be observed that A's missile strikes B before the latter has time to fire. If this were not the case, in some frames A could kill B without himself being harmed and inconsistency would arise.

No.15. Substituting from the transformation equations

$$x = \beta(\bar{x} + u\bar{t}), \quad y = \bar{y}, \quad t = \beta(\bar{t} + u\bar{x}/c^2),$$

the equation describing the wave is found to take the form

$$\bar{y} = a \sin 2\pi f\beta(1 - u/w)\left(\bar{t} - \frac{1 - uw/c^2}{w - u}\bar{x}\right).$$

This describes a wave having frequency $\bar{f} = f\beta(1 - u/w)$ and wave velocity $\bar{w} = (w-u)/(1-uw/c^2)$.

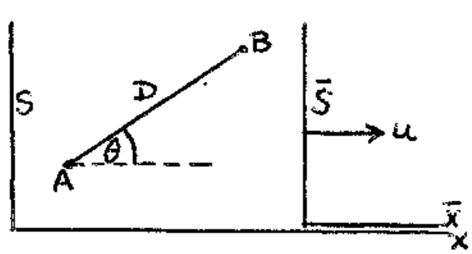
In the special case $u = w$, the transformed equation is $\bar{y} = a \sin(2\pi f\bar{x}/\beta u)$, i.e. a standing wave, since the frame then moves with the wave.

No.16. If τ is the proper time interval between the events and $\bar{D}$ is their distance apart in $\bar{S}$, then, by equation (7.4),

$$\tau^2 = -\frac{1}{c^2}D^2 = T^2 - \frac{1}{c^2}\bar{D}^2.$$

Thus, $\bar{D} = \sqrt{(D^2 + c^2T^2)}$.

It is helpful to note that the distance between events is dependent upon the frame being used even in classical theory. For example, the two events of entering and leaving a room are separated by zero distance in a terrestrial frame, but are separated by many miles in a frame fixed in the sun for which the earth is a moving body.



Taking axes S and $\bar{S}$ in the usual directions (see diagram), if the events take place at A and B in S, their coordinates can be assumed to be (x_A,t) and $(x_A+D\cos\theta,t)$ respectively. In $\bar{S}$, let $(\bar{x}_A,\bar{t})$ and $(\bar{x}_B,\bar{t}+T)$ be the coordinates of the events. Employing the transformation equation $\bar{t} = \beta(t - ux/c^2)$, we obtain the equations

$$\bar{t} = \beta(t - ux_A/c^2), \quad \bar{t} + T = \beta[t - u(x_A + D\cos\theta)/c^2].$$

Subtraction leads to

$$T = -\beta uD\cos\theta/c^2.$$

Since $\beta = (1 - u^2/c^2)^{-\frac{1}{2}}$, this equation can be solved for u to give the result stated.

No.17. If τ is the proper time interval between the events and $\bar{D}$ is the distance between them in $\bar{S}$, then equation (7.4) gives

$$\tau^2 = T^2 - \frac{1}{c^2}D^2 = T^2 - \frac{1}{c^2}\bar{D}^2.$$

Thus, $\bar{D} = D$.

Let the coordinates of the events in S be $A(x,t)$, $B(x+D,t+T)$, and in $\bar{S}$ be $A(\bar{x}_A,\bar{t})$, $B(\bar{x}_B,\bar{t}-T)$. The transformation equation $\bar{t} = \beta(t-ux/c^2)$ yields the equations

$$\bar{t} = \beta(t - ux/c^2), \quad \bar{t} - T = \beta[t + T - u(x+D)/c^2].$$

Subtracting, we find

$$T = \beta(-T + uD/c^2).$$

Squaring and solving for u, we get $u = 2c^2DT/(D^2 + c^2T^2)$. If this value of u is substituted back into the previous equation, it will be found that it satisfies the equation with positive β only provided $D > cT$.

No.18. In S, the particle's trajectory lies in the xy-plane and is determined by the equations

$$x = \tfrac{3}{4}ct, \quad y = \tfrac{1}{2}ct - \tfrac{1}{2}gt^2.$$

The transformation equations to $\bar{S}$ are

$$x = \frac{2}{\sqrt{3}}(\bar{x} + \tfrac{1}{2}c\bar{t}), \quad y = \bar{y}, \quad t = \frac{2}{\sqrt{3}}(\bar{t} + \bar{x}/2c).$$

Substituting in the first of the trajectory equations, we get $\bar{x} = 2c\bar{t}/5$. Putting this result in the third transformation equation, it follows that along the trajectory, $t = 4\sqrt{3}\bar{t}/5$. Substitution in the second trajectory equation then gives

$$\bar{y} = \frac{2\sqrt{3}}{5}c\bar{t} - \frac{24}{25}g\bar{t}^2.$$

Thus, at $\bar{t} = 0$ when the particle is first projected, $d\bar{x}/d\bar{t} = 2c/5$ and $d\bar{y}/d\bar{t} = 2\sqrt{3}c/5$; these $\bar{x}$ and $\bar{y}$ components of velocity correspond to a direction of projection making an angle $\tan^{-1}(\sqrt{3}) = 60^\circ$ with the $\bar{x}$-axis. There is zero acceleration in the $\bar{x}$ direction and $d^2\bar{y}/d\bar{t}^2 = -48g/25$ gives the acceleration parallel to the $\bar{y}$-axis.

No.19. At time t in S, all the clocks stationary in this frame register the time t. Thus, C registers t. At this instant, the clock $\bar{O}$ registering $\bar{t}$ passes C. The coordinates of this event in S are (x_c,t); the coordinates of the same event in $\bar{S}$ are $(0,\bar{t})$. The transformation equation $t = \beta(\bar{t} + u\bar{x}/c^2)$ applied to this event accordingly gives the result $t = \beta\bar{t}$ or $\bar{t} = \sqrt{(1 - u^2/c^2)}t$. Thus, as $\bar{O}$ moves along the line of S-clocks on the x-axis, it runs slow by comparison with these clocks.

We now observe the situation from the $\bar{S}$-frame. In this frame, at the instant $\bar{t}$, suppose the clock at O registers t_0. This event has coordinates $(0,t_0)$ in S and coordinates $(\bar{x}_0,\bar{t})$ in $\bar{S}$. Using the transformation $\bar{t} = \beta(t - ux/c^2)$, we find $\bar{t} = \beta t_0$. But we have proved that $t = \beta\bar{t}$. Hence $t_0 = \beta^{-2}t = t - u^2t/c^2$; i.e. as seen from $\bar{S}$ at the instant $\bar{t}$, the clock C^0 is u^2t/c^2 ahead of the clock at 0.

We now follow the situation as it develops in $\bar{S}$ and, whilst making allowance for C running slow in this frame, show that nevertheless C is ahead of the clock at $\bar{0}$ when they meet. Thus, at time $\bar{t} = 0$ in $\bar{S}$, 0 and $\bar{0}$ coincide. Subsequently, 0 recedes from $\bar{0}$, whilst C approaches, both with speed u; hence, the rates of clocks 0 and C remain equal but are slowed by a factor $\surd(1 - u^2/c^2)$. At time $\bar{t}$ after 0 leaves $\bar{0}$, the clock 0 therefore registers $\surd(1 - u^2/c^2)\bar{t} = (1 - u^2/c^2)t$, and C arrives at $\bar{0}$. Since C is ahead of 0 by u^2t/c^2, on arrival at $\bar{0}$, C registers t. The paradox has now disappeared. When C meets $\bar{0}$, C registers t and $\bar{0}$ registers $\bar{t}$, where $\bar{t} = \surd(1 - u^2/c^2)t$, i.e. C is ahead of $\bar{0}$ as calculated in the frame S.

No.20. At time t in S

$$x = \tfrac{5}{6}c(t - 1), \quad y = \tfrac{1}{3}c(t - 1).$$

To transform to the $\bar{S}$-language, we use the transformation equations

$$x = \tfrac{1}{4}(5\bar{x} + 3c\bar{t}), \ y = \bar{y}, \ ct = \tfrac{1}{4}(3\bar{x} + 5c\bar{t}).$$

Upon substitution and solution for $\bar{x}$, $\bar{y}$ in terms of $\bar{t}$, it is found that

$$\bar{x} = \frac{1}{15}c(7\bar{t} - 20), \quad \bar{y} = \frac{1}{15}c(8\bar{t} - 10).$$

If d is the particle's distance from $\bar{0}$ as measured in $\bar{S}$ at time $\bar{t}$, then

$$d^2 = \bar{x}^2 + \bar{y}^2 = \frac{1}{225}c^2(113\bar{t}^2 - 440\bar{t} + 500).$$

d is a minimum when $226\bar{t} = 440$, which is the result stated.

No.21. The event 0 passes $\bar{0}$ occurs at t = 0 in S and $\bar{t} = 0$ in $\bar{S}$.

In S, the coordinates of the event (ii) are (a,t_A). In $\bar{S}$, the coordinates of the same event are $(a,\bar{t}_A)$. The transformation equations $\bar{x} = \beta(x - ut)$, $x = \beta(\bar{x} + u\bar{t})$ applied to this event show that

$$a = \beta(a - ut_A), \quad a = \beta(a + u\bar{t}_A).$$

Clearly,

$$t_A = -\bar{t}_A = T = a(1 - \beta^{-1})/u.$$

If T = a/3c, this gives $1 - \surd(1 - u^2/c^2) = u/3c$. Solving for u, we find u = 3c/5.

No.1. If rectangular axes Oxy are rotated about O in their plane through an angle θ to give axes $O\overline{xy}$, it follows by simple trigonometry that a point having coordinates (x,y) in the first frame has coordinates $(\overline{x},\overline{y})$ in the rotated frame, where

$$\overline{x} = x\cos\theta + y\sin\theta, \quad \overline{y} = -x\sin\theta + y\cos\theta.$$

The matrix A of this orthogonal transformation is therefore as stated.

If the sense of the $\overline{y}$-axis is reversed, so that the transformation relates two frames which cannot be brought into coincidence by a simple rotation, the second of these equations must be replaced by

$$\overline{y} = x\sin\theta - y\cos\theta$$

and the matrix A amended accordingly.

In the first case, $|A| = \cos^2\theta + \sin^2\theta = 1$, but in the second case $|A| = -\cos^2\theta - \sin^2\theta = -1$. This verifies equation (8.10).

Solving for x and y in terms of $\overline{x}$ and $\overline{y}$, we find in the first case that

$$x = \overline{x}\cos\theta - \overline{y}\sin\theta, \quad y = \overline{x}\sin\theta + \overline{y}\cos\theta.$$

The matrix of this inverse transformation is A^{-1} and thus

$$A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A'.$$

That $A^{-1} = A'$ may be verified similarly in the second case.

The transformation equations for T_{ij} are

$$\overline{T}_{11} = a_{1i}a_{1j}T_{ij} = T_{11}\cos^2\theta + T_{12}\cos\theta\sin\theta + T_{21}\cos\theta\sin\theta + T_{22}\sin^2\theta$$

$$\overline{T}_{12} = a_{1i}a_{2j}T_{ij} = -T_{11}\cos\theta\sin\theta + T_{12}\cos^2\theta - T_{21}\sin^2\theta + T_{22}\cos\theta\sin\theta$$

$$\overline{T}_{21} = a_{2i}a_{1j}T_{ij} = -T_{11}\cos\theta\sin\theta - T_{12}\sin^2\theta + T_{21}\cos^2\theta + T_{22}\cos\theta\sin\theta$$

$$\overline{T}_{22} = a_{2i}a_{2j}T_{ij} = T_{11}\sin^2\theta - T_{12}\cos\theta\sin\theta - T_{21}\cos\theta\sin\theta + T_{22}\cos^2\theta$$

It follows that

$$\overline{T}_{ii} = \overline{T}_{11} + \overline{T}_{22} = (T_{11} + T_{22})(\cos^2\theta + \sin^2\theta) = T_{11} + T_{22} = T_{ii}$$

i.e. T_{ii} is an invariant. (The second case is dealt with similarly.)

No.2. Since A and B are orthogonal, $AA' = BB' = I$. It follows that

$$BA(BA)' = BA(A'B') = BIB' = BB' = I.$$

Thus, BA is orthogonal.

In subscript notation, if a_{ij} and b_{ij} are elements of A and B respectively, then

$$\overline{\overline{x}}_i = b_{ij}\overline{x}_j, \quad \overline{x}_j = a_{jk}x_k.$$

Eliminating $\overline{x}_j$, we find

$$\overline{\overline{x}}_i = b_{ij}a_{jk}x_k = c_{ik}x_k,$$

where $c_{ik} = b_{ij}a_{jk}$ are the coefficients of the resultant transformation.

Since T_{ij} is a tensor with respect to the transformations with matrices B and A, then

$$\overline{\overline{T}}_{ij} = b_{ip}b_{jq}\overline{T}_{pq}, \quad \overline{T}_{pq} = a_{pr}a_{qs}T_{rs}.$$

Eliminating $\overline{T}_{pq}$, we find

$$\overline{\overline{T}}_{ij} = b_{ip}a_{pr}b_{jq}a_{qs}T_{rs} = c_{ir}c_{js}T_{rs},$$

proving that T_{ij} transforms like a tensor with respect to the resultant transformation.

No.3. Since A_i and B_i are vectors, their transformation equations for the change of frame represented by the orthogonal transformation $\overline{x}_i = a_{ij}x_j + b_i$, are

$$\overline{A}_i = a_{ij}A_j, \quad \overline{B}_i = a_{ij}B_j.$$

The inverse transformation equations (11.4) show that

$$A_i = a'_{ij}\overline{A}_j = a_{ji}\overline{A}_j, \quad B_i = a_{ji}\overline{B}_j.$$

Since $X_{ij}A_iB_j$ is an invariant,

$$\overline{X}_{ij}\overline{A}_i\overline{B}_j = X_{ij}A_iB_j = X_{rs}A_rB_s.$$

Substituting for A_r and B_s from the inverse transformation equations, this gives

$$\overline{X}_{ij}\overline{A}_i\overline{B}_j = X_{rs}a_{ir}\overline{A}_ia_{js}\overline{B}_j.$$

Since the components $\overline{A}_i$ and $\overline{B}_i$ can take arbitrary values, we can equate the coefficients of the terms in $\overline{A}_i\overline{B}_j$ on the two sides of this equation to yield

$$\overline{X}_{ij} = a_{ir}a_{js}X_{rs},$$

proving that X_{ij} transforms as a tensor.

No.4. It is easily verified that the matrix A of the transformation satisfies the condition AA' = I and is therefore orthogonal.

Using the inverse transformation, it is found that

$$A_1 = x_1^2 = \frac{1}{9}(\overline{x}_1 - 2\overline{x}_2 + 2\overline{x}_3)^2$$

$$A_2 = x_2^2 = \frac{1}{225}(-14\overline{x}_1 - 5\overline{x}_2 + 2\overline{x}_3)^2$$

$$A_3 = x_3^2 = \frac{1}{225}(2\overline{x}_1 - 10\overline{x}_2 - 11\overline{x}_3)^2$$

Also, since A_i is a vector

$$\overline{A}_1 = \frac{1}{15}(5A_1 - 14A_2 + 2A_3)$$

$$\overline{A}_2 = -\frac{1}{3}(2A_1 + A_2 + 2A_3)$$

$$\overline{A}_3 = \frac{1}{15}(10A_1 + 2A_2 - 11A_3)$$

These two sets of equations now permit the $\overline{A}_i$ to be expressed in terms of the $\overline{x}_i$.

In the x-frame,

$$\text{div}A = A_{i,i} = 2(x_1 + x_2 + x_3).$$

In the $\bar{x}$-frame,

$$\text{div}A = \partial\bar{A}_i/\partial\bar{x}_i = \frac{1}{15}(-14\bar{x}_1 - 50\bar{x}_2 + 2\bar{x}_3).$$

Substituting for the $\bar{x}_i$ in this last expression from the original transformation equations, it can be shown that it reduces to the form $2(x_1 + x_2 + x_3)$, thus verifying the invariance of divA.

No.5. The vector A_{iji} has components

$$A_{i1i} = A_{111} + A_{212} + A_{313} = -1$$

$$A_{i2i} = A_{121} + A_{222} + A_{323} = 1$$

$$A_{i3i} = A_{131} + A_{232} + A_{333} = 0$$

If A is the matrix of the given transformation equations, it is easily verified that AA' = I and hence that the transformation is orthogonal. Then,

$$\bar{A}_{123} = a_{1i}a_{2j}a_{3k}A_{ijk}$$

$$= a_{11}a_{21}a_{31}A_{111} + a_{12}a_{22}a_{32}A_{222} + a_{12}a_{21}a_{32}A_{212}$$

having omitted the zero terms in the sum. Substituting in this expression, we find $\bar{A}_{123} = 120/343$.

The matrix of the inverse transformation is the transpose A'. Thus, the equations are

$$x_1 = \frac{1}{7}(-3\bar{x}_1 - 2\bar{x}_2 + 6\bar{x}_3)$$

$$x_2 = \frac{1}{7}(-6\bar{x}_1 + 3\bar{x}_2 - 2\bar{x}_3)$$

$$x_3 = \frac{1}{7}(-2\bar{x}_1 - 6\bar{x}_2 - 3\bar{x}_3)$$

Since B_{ij} is a tensor,

$$B_{ij} = a'_{ir}a'_{js}\bar{B}_{rs},$$

where a'_{ij} are the coefficients of the inverse transformation. Thus,

$$B_{12} = a'_{1r}a'_{2s}\bar{B}_{rs} = a'_{11}a'_{23}\bar{B}_{13} = (-3/7)(-2/7)1 = 6/49.$$

No.6. Taking transposes

$$A' = (I' + B')^{-1}(I' - B') = (I - B)^{-1}(I + B).$$

Hence,

$$A'A = (I - B)^{-1}(I + B)(I - B)(I + B)^{-1}$$

$$= (I - B)^{-1}(I - B^2)(I + B)^{-1}$$

$$= (I - B)^{-1}(I - B)(I + B)(I + B)^{-1}$$

$$= I$$

proving that A is orthogonal.

If B is the 3×3 matrix given, then

$$I + B = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \qquad (I + B)^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}$$

Thus

$$A = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -7 & -4 & -4 \\ 4 & 1 & -8 \\ 4 & -8 & 1 \end{bmatrix}$$

and the transformation equations are

$$\bar{x}_1 = \frac{1}{9}(-\,7x_1 - 4x_2 - 4x_3)$$
$$\bar{x}_2 = \frac{1}{9}(4x_1 + x_2 - 8x_3)$$
$$\bar{x}_3 = \frac{1}{9}(4x_1 - 8x_2 + x_3)$$

Since C_{ij} is skew-symmetric, $C_{21} = C_{31} = -1$ and the remaining components vanish. Thus,

$$\bar{C}_{12} = a_{1i}a_{2j}C_{ij} = a_{11}a_{22}C_{12} + a_{11}a_{23}C_{13} + a_{12}a_{21}C_{21} + a_{13}a_{21}C_{31}$$

omitting the zero terms. Substituting the values of the a_{ij} and C_{ij}, we find $\bar{C}_{12} = 1$.

The coefficients of the inverse transformation are $a'_{ij} = a_{ji}$. Thus,

$$D_{111} = a'_{1i}a'_{1j}a'_{1k}\bar{D}_{ijk} = a_{i1}a_{j1}a_{k1}\bar{D}_{ijk}$$
$$= a_{11}a_{21}a_{11}\bar{D}_{121} + a_{11}a_{21}a_{21}\bar{D}_{122} + a_{11}a_{21}a_{31}\bar{D}_{123}$$

omitting the zero terms. Substitution of given values now yields the result $D_{111} = -980/729$.

In the $\bar{x}$-frame, the vector $D_{ijj} = E_i$ (say) has components

$$\bar{E}_1 = \bar{D}_{1jj} = \bar{D}_{111} + \bar{D}_{122} + \bar{D}_{133} = 2$$
$$\bar{E}_2 = \bar{D}_{2jj} = \bar{D}_{211} + \bar{D}_{222} + \bar{D}_{233} = 0$$
$$\bar{E}_3 = \bar{D}_{3jj} = \bar{D}_{311} + \bar{D}_{322} + \bar{D}_{333} = 0$$

Thus, in the x-frame, this vector has components

$$E_1 = a'_{1i}\bar{E}_i = a'_{11}\bar{E}_1 = a_{11}\bar{E}_1 = -14/9$$
$$E_2 = a'_{2i}\bar{E}_i = a'_{21}\bar{E}_1 = a_{12}\bar{E}_1 = -8/9$$
$$E_3 = a'_{3i}\bar{E}_i = a'_{31}\bar{E}_1 = a_{13}\bar{E}_1 = -8/9$$

In the $\bar{x}$-frame, the vector $C_{ij}D_{ijk} = F_k$ has components

$$\bar{F}_1 = \bar{C}_{ij}\bar{D}_{ij1} = \bar{C}_{12}\bar{D}_{121} = -1$$
$$\bar{F}_2 = \bar{C}_{ij}\bar{D}_{ij2} = \bar{C}_{12}\bar{D}_{122} = 2$$
$$\bar{F}_3 = \bar{C}_{ij}\bar{D}_{ij3} = \bar{C}_{12}\bar{D}_{123} = 5$$

Thus,

$$F_1 = a'_{1i}\bar{F}_i = a_{11}\bar{F}_1 + a_{21}\bar{F}_2 + a_{31}\bar{F}_3 = 35/9$$
$$F_2 = a'_{2i}\bar{F}_i = a_{12}\bar{F}_1 + a_{22}\bar{F}_2 + a_{32}\bar{F}_3 = -34/9$$
$$F_3 = a'_{3i}\bar{F}_i = a_{13}\bar{F}_1 + a_{23}\bar{F}_2 + a_{33}\bar{F}_3 = -7/9$$

No.7. By substitution of the given x_i values, we find

$$A_{11} = A_{12} = A_{13} = A_{21} = A_{31} = 0, \quad A_{22} = A_{23} = A_{32} = A_{33} = 1.$$

If a_{ij} are the coefficients of the orthogonal transformation

$$\bar{A}_{11} = a_{1i}a_{1j}A_{ij} = a_{12}a_{12}A_{22} + a_{12}a_{13}A_{23} + a_{13}a_{12}A_{32} + a_{13}a_{13}A_{33},$$

where we have omitted the zero terms. Substitution of the calculated values of the A_{ij} at P, we find $\bar{A}_{11} = 64/49$.

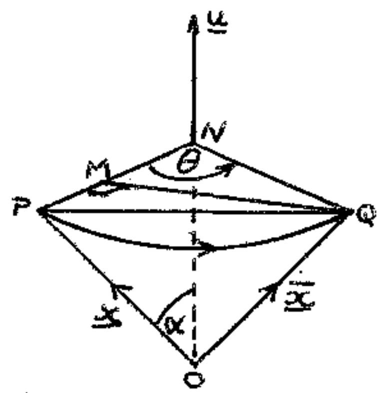
We have

$$A_{1j,j} = \frac{\partial}{\partial x_1}(x_1^2) + \frac{\partial}{\partial x_2}(x_1x_2) + \frac{\partial}{\partial x_3}(x_1x_3) = 4x .$$

This verifies $A_{ij,j} = 4x_i$ in the case i = 1. The equation may be verified similarly for the cases i = 2,3. Then

$$A_{ij,ij} = (A_{ij,j})_{,i} = (4x_i)_{,i} = 4+4+4 = 12.$$

No.8.



As OP is rotated, the point P describes an arc of a circle with centre N on the axis of rotation, arriving ultimately at Q. Thus, the triangle PNQ is isosceles with angle θ at the vertex N. Construct QM perpendicular to PN. Then QM = QNsinθ = PNsinθ = xsinαsinθ, where α is the angle shown in the figure. The vector $\underline{x} \times \underline{u}$ has magnitude xsinα and direction parallel to MQ. Hence, $\underline{MQ} = \underline{x} \times \underline{u}\sin\theta$. Also, NM = NQcosθ = NPcosθ; hence, $\underline{NM} = \underline{NP}\cos\theta$. Since $\underline{ON} = x\cos\alpha\underline{u} = (\underline{x}\cdot\underline{u})\underline{u}$, then $\underline{NP} = \underline{OP} - \underline{ON} = \underline{x} - (\underline{x}\cdot\underline{u})\underline{u}$. It follows that $\underline{NM} = \{\underline{x} - (\underline{x}\cdot\underline{u})\underline{u}\}\cos\theta$. Thus

$$\underline{\bar{x}} = \underline{ON} + \underline{NM} + \underline{MQ}$$
$$= (\underline{x}\cdot\underline{u})\underline{u} + \{\underline{x} - (\underline{x}\cdot\underline{u})\underline{u}\}\cos\theta + \underline{x} \times \underline{u}\sin\theta$$

which is equivalent to the result stated.

If the Cartesian frame is kept fixed and the position vector of the point with coordinates x_i is rotated about the stated axis through an angle $\theta = -\sin^{-1}(4/5)$, the new coordinates of the point will be $\bar{x}_i$. Hence, the $\bar{x}_i$ can be expressed in terms of the x_i by putting $\theta = -\sin^{-1}(4/5)$ in the equation just obtained. The unit vector along the axis of rotation has components (1/3,2/3,2/3); hence, we get

$$(\bar{x}_1,\bar{x}_2,\bar{x}_3) = \frac{3}{5}(x_1,x_2,x_3) + \frac{2}{5}\cdot\frac{1}{9}(x_1 + 2x_2 + 2x_3)(1,2,2)$$
$$- \frac{4}{5}\cdot\frac{1}{3}(2x_3 - 2x_2, 2x_1 - x_3, x_2 - 2x_1)$$

After rearrangement and separation into three component equations, this leads to the transformation stated.

No.9. The only non-zero components of A_{ij} are $A_{12} = A_{13} = 1$, $A_{21} = A_{31} = -1$. Thus

$$\bar{A}_{12} = a_{1i}a_{2j}A_{ij} = a_{11}a_{22}A_{12} + a_{11}a_{23}A_{13} + a_{12}a_{21}A_{21} + a_{13}a_{21}A_{31}$$
$$= 1/3.$$

Using the transposed matrix for the inverse transformation, we have

$$B_{111} = a'_{1i}a'_{1j}a'_{1k}\bar{B}_{ijk} = a_{11}a_{21}a_{11}\bar{B}_{121} + a_{11}a_{21}a_{21}\bar{B}_{122} + a_{11}a_{21}a_{31}\bar{B}_{123}$$
$$= 188/729$$

In the $\bar{x}$-frame, the vector $B_{ijj} = C_i$ has components

$$\bar{C}_1 = \bar{B}_{1jj} = \bar{B}_{111} + \bar{B}_{122} + \bar{B}_{133} = 2$$
$$\bar{C}_2 = \bar{B}_{2jj} = \bar{B}_{211} + \bar{B}_{222} + \bar{B}_{233} = 0$$
$$\bar{C}_3 = \bar{B}_{3jj} = \bar{B}_{311} + \bar{B}_{322} + \bar{B}_{333} = 0$$

In the x-frame, therefore, its components are

$$C_1 = a'_{1i}\bar{C}_i = a_{11}\bar{C}_1 = 2/9$$
$$C_2 = a'_{2i}\bar{C}_i = a_{12}\bar{C}_1 = -16/9$$
$$C_3 = a'_{3i}\bar{C}_i = a_{13}\bar{C}_1 = 8/9$$

In the $\bar{x}$-frame, the vector $A_{ij}B_{ijk} = D_k$ has components

$$\bar{D}_1 = \bar{A}_{ij}\bar{B}_{ij1} = \bar{A}_{12}\bar{B}_{121} = -1/3$$
$$\bar{D}_2 = \bar{A}_{ij}\bar{B}_{ij2} = \bar{A}_{12}\bar{B}_{122} = 2/3$$
$$\bar{D}_3 = \bar{A}_{ij}\bar{B}_{ij3} = \bar{A}_{12}\bar{B}_{123} = 5/3$$

In the x-frame, its components are

$$D_1 = a'_{1i}D_i = a_{11}D_1 + a_{21}D_2 + a_{31}D_3 = 47/27$$
$$D_2 = a'_{2i}D_i = a_{12}D_1 + a_{22}D_2 + a_{32}D_3 = 11/27$$
$$D_3 = a'_{3i}D_i = a_{13}D_1 + a_{23}D_2 + a_{33}D_3 = -10/27$$

No.10. The vector $A_{iji} = B_j$ has components

$$B_1 = A_{i1i} = A_{111} + A_{212} + A_{313} = 8x_1^2 + 2x_2^2 + 2x_3^2$$
$$B_2 = A_{i2i} = A_{121} + A_{222} + A_{323} = 2x_1^2 + 8x_2^2 + 2x_3^2$$
$$B_3 = A_{i3i} = A_{131} + A_{232} + A_{333} = 2x_1^2 + 2x_2^2 + 8x_3^2$$

Its divergence is therefore $B_{i,i} = 16x_1 + 16x_2 + 16x_3$.

The vector $A_{ijj} = C_i$ has components

$$C_1 = A_{1jj} = A_{111} + A_{122} + A_{133} = 6x_1^2 + 3x_2^2 + 3x_3^2$$

$$C_2 = A_{2jj} = A_{211} + A_{222} + A_{233} = 3x_1^2 + 6x_2^2 + 3x_3^2$$

$$C_3 = A_{3jj} = A_{311} + A_{322} + A_{333} = 3x_1^2 + 3x_2^2 + 6x_3^2$$

Its curl has x_1-component $C_{3,2} - C_{2,3} = 6x_2 - 6x_3$. The x_2- and x_3-components are, similarly, $6(x_3 - x_1)$ and $6(x_1 - x_2)$ respectively.

No.11. The components of $A_{ijj} = C_i$ in the x-frame are

$$C_1 = A_{1jj} = A_{111} + A_{122} + A_{133} = 6$$

$$C_2 = A_{2jj} = A_{211} + A_{222} + A_{233} = 15$$

$$C_3 = A_{3jj} = A_{311} + A_{322} + A_{333} = 0$$

In the $\bar{x}$-frame, therefore,

$$\bar{C}_1 = \frac{1}{15}(5C_1 - 14C_2 + 2C_3) = -12$$

$$\bar{C}_2 = -\frac{1}{3}(2C_1 + C_2 + 2C_3) = -9$$

$$\bar{C}_3 = \frac{1}{15}(10C_1 + 2C_2 - 11C_3) = 6$$

Also,

$$\bar{A}_{123} = a_{1i}a_{2j}a_{3k}A_{ijk}$$

$$= a_{11}a_{21}a_{31}A_{111} + a_{12}a_{22}a_{32}A_{222} + a_{11}a_{22}a_{32}A_{122} + a_{12}a_{23}a_{33}A_{233}$$

$$= -1396/225$$

Using the inverse transformation,

$$B_{11} = a'_{1i}a'_{1j}\bar{B}_{ij} = a_{11}a_{21}\bar{B}_{12} + a_{21}a_{31}\bar{B}_{23} = -2/3$$

The first equation of the inverse transformation is

$$x_1 = \frac{1}{3}(\bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3)$$

and, hence,

$$\bar{V} = \frac{1}{9}(\bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3)^2$$

The components of ∇V in the $\bar{x}$-frame are accordingly

$$\partial\bar{V}/\partial\bar{x}_1 = \frac{2}{9}(\bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3) = 2$$

$$\partial\bar{V}/\partial\bar{x}_2 = -\frac{4}{9}(\bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3) = -4$$

$$\partial\bar{V}/\partial\bar{x}_3 = \frac{4}{9}(\bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3) = 4$$

No.12. $\bar{A}_{12} = a_{1i}a_{2j}A_{ij} = a_{11}a_{22}A_{12} + a_{12}a_{21}A_{21}$

$$= \cos^2\alpha + \sin^2\alpha = 1$$

No.13. Taking $i = 1$,

$$A_{1j,j} = \partial A_{11}/\partial x_1 + \partial A_{12}/\partial x_2 + \partial A_{13}/\partial x_3 = 4x_1 + 2x_2 + 2x_3$$

which verifies (a) for this value of i. Similarly, (a) can be verified for $i = 2,3$. It now follows that

$$A_{ij,ij} = \partial A_{1j,j}/\partial x_1 + \partial A_{2j,j}/\partial x_2 + \partial A_{3j,j}/\partial x_3 = 4 + 4 + 4 = 12$$

No.14. $\bar{A}_{311} = a_{3i}a_{1j}a_{1k}A_{ijk} = a_{31}a_{12}a_{13}A_{123} + a_{32}a_{12}a_{12}A_{222} = -192/25$

Using the inverse transformation

$$B_{i2} = a'_{ij}a'_{2k}\bar{B}_{jk} = a_{ji}a_{k2}\bar{B}_{jk} = a_{ji}a_{32}\bar{B}_{j3} = 0$$

since $a_{32} = 0$.

No.15. (i) Taking transposes,

$$A' = I' - (2xx')' = I - 2xx'$$

Thus

$$AA' = (I - 2xx')(I - 2xx') = I - 4xx' + 4x(x'x)x'$$

$$= I - 4xx' + 4xx' = I$$

and A must be orthogonal, therefore.

For the given x',

$$x'x = \alpha^2(1,-2,3)\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \alpha^2(1^2 + 2^2 + 3^2) = 14\alpha^2$$

Thus, $x'x = 1$ provided $\alpha = 1/\sqrt{14}$. Then

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{14}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}(1,-2,3) = \frac{1}{7}\begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ -3 & 6 & -2 \end{bmatrix}$$

Using this transformation matrix,

$$\bar{B}_{2221} = a_{2i}a_{2j}a_{2k}a_{1\ell}B_{ijk\ell} = a_{21}a_{21}a_{23}a_{13}B_{1133} + a_{21}a_{22}a_{23}a_{12}B_{1232}$$

$$= 72/49$$

B_{iijj} is an invariant. Hence

$$\bar{B}_{iijj} = B_{iijj} = B_{1133} + \text{zero terms} = -20$$

We find that $|A| = -1$. Hence

$$\bar{\mathfrak{C}}_1 = |A|a_{1i}\mathfrak{C}_i = -\frac{1}{7}(6\mathfrak{C}_1 + 2\mathfrak{C}_2 - 3\mathfrak{C}_3) = 0$$

$$\bar{\mathfrak{C}}_2 = |A|a_{2i}\mathfrak{C}_i = -\frac{1}{7}(2\mathfrak{C}_1 + 3\mathfrak{C}_2 + 6\mathfrak{C}_3) = -7$$

$$\bar{\mathfrak{C}}_3 = |A|a_{3i}\mathfrak{C}_i = -\frac{1}{7}(-3\mathfrak{C}_1 + 6\mathfrak{C}_2 - 2\mathfrak{C}_3) = 0$$

(ii) The non-zero components of A_{ij} are $A_{11} = A_{22} = A_{33} = x_2^2$. Thus, the non-zero components of $A_{ij,k}$ are $A_{11,2} = A_{22,2} = A_{33,2} = 2x_2$. Since $A_{ij,k}$ is a tensor,

$$\begin{aligned}\bar{A}_{11,1} &= a_{1i}a_{1j}a_{1k}A_{ij,k}\\ &= a_{11}a_{11}a_{12}A_{11,2} + a_{12}a_{12}a_{12}A_{22,2} + a_{13}a_{13}a_{12}A_{33,2}\\ &= 4x_2/7\end{aligned}$$

Using the inverse transformation,

$$x_2 = \frac{1}{7}(2\bar{x}_1 + 3\bar{x}_2 + 6\bar{x}_3) = 2$$

Hence, $\bar{A}_{11,1} = 8/7$.

No.16. $\bar{A}_{32} = a_{3i}a_{2j}A_{ij} = a_{31}a_{21}A_{11} + a_{32}a_{22}A_{22} = 6$

Using the inverse transformation,

$$B_{111} = a'_{1i}a'_{1j}a'_{1k}\bar{B}_{ijk} = a_{i1}a_{j1}a_{k1}\bar{B}_{ijk} = a_{21}a_{31}a_{21}\bar{B}_{232} = 24$$

The determinant of the orthogonal transformation is $|A| = -1$.

Hence

$$\bar{\mathfrak{C}}_{12} = |A|a_{1i}a_{2j}\mathfrak{C}_{ij} = |A|a_{13}a_{23}\mathfrak{C}_{33} = -24$$

No.17. Since A is skew-symmetric, $A' = -A$. Thus

$$(A^2)' = (AA)' = A'A' = (-A)(-A) = A^2,$$

showing that A^2 is symmetric. If, also, $A^3 = -A$, then

$$\begin{aligned}BB' &= (I + 2A^2)(I + 2A^2)' = (I + 2A^2)(I + 2A^2)\\ &= I + 4A^2 + 4A^4 = I + 4A^2 + 4A(A^3)\\ &= I + 4A^2 - 4A^2 = I\end{aligned}$$

Thus B is orthogonal.

By multiplication of A by itself, we find

$$A^3 = (a^2 + b^2 + c^2)\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}$$

Thus, if $a^2 + b^2 + c^2 = 1$, then $A^3 = -A$.

If $a = 1/3$, $b = c = 2/3$, then $a^2 + b^2 + c^2 = 1$ and

$$B = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \frac{2}{9}\begin{bmatrix}-5 & -4 & 2\\ -4 & -5 & -2\\ 2 & -2 & -8\end{bmatrix} = \frac{1}{9}\begin{bmatrix}-1 & -8 & 4\\ -8 & -1 & -4\\ 4 & -4 & -7\end{bmatrix}$$

Thus,

$$\bar{C}_{23} = b_{2i}b_{3j}C_{ij} = b_{21}b_{31} + b_{21}b_{32} + \text{etc.} = 91/81$$

Since $x_1^2 + x_2^2 + x_3^2$ is the square of the distance of a point from the origin of the x-frame and the origins of the two frames coincide, the transformation must be such that

$$x_1^2 + x_2^2 + x_3^2 = \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2$$

Hence

$$\bar{A}_1 = b_{1i}A_i = b_{11}A_1 = -\frac{1}{9}(\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)$$

$$\bar{A}_2 = b_{2i}A_i = b_{21}A_1 = -\frac{8}{9}(\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)$$

$$\bar{A}_3 = b_{3i}A_i = b_{31}A_1 = \frac{4}{9}(\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)$$

In the x-frame, the divergence of the vector field is

$$A_{1,1} + A_{2,2} + A_{3,3} = A_{1,1} = 2x_1$$

In the $\bar{x}$-frame, the divergence is

$$\bar{A}_{1,1} + \bar{A}_{2,2} + \bar{A}_{3,3} = \frac{2}{9}(-\bar{x}_1 - 8\bar{x}_2 + 4\bar{x}_3)$$

The first of the equations defining the inverse coordinate transformation is

$$x_1 = \frac{1}{9}(-\bar{x}_1 - 8\bar{x}_2 + 4\bar{x}_3)$$

This establishes that the two expressions for the divergence are identical.

No.18. (i) The orthogonality conditions are

$$a^2 + 8^2 + 4^2 = 81, \quad 4a + 8b + 28 = 0,$$

$$4^2 + b^2 + 7^2 = 81, \quad 8a + 8 + 4c = 0,$$

$$8^2 + 1^2 + c^2 = 81, \quad 32 + b + 7c = 0.$$

All these equations are satisfied if $a = 1$, $b = -4$, $c = -4$.

We have, with the usual notation,

$$\bar{A}_{31} = a_{3i}a_{1j}A_{ij} = a_{32}a_{13}A_{23} = 4/9.$$

(ii) Putting $B_{jk} = A_{ijki}$, we have

$$\bar{B}_{jk} = \bar{A}_{ijki} = a_{ir}a_{js}a_{kt}a_{iu}A_{rstu} = \delta_{ru}a_{js}a_{kt}A_{rstu}$$

$$= a_{js}a_{kt}A_{rstr} = a_{js}a_{kt}B_{st}$$

having used the orthogonality condition $a_{ir}a_{iu} = \delta_{ru}$. This proves B_{jk} is a tensor.

For the component values given

$$B_{13} = A_{i13i} = A_{1131} + A_{2132} + A_{3133} = 9$$

$$B_{22} = A_{i22i} = A_{1221} + A_{2222} + A_{3223} = 18$$

All other components of B_{jk} vanish in the x-frame. Thus

$$\bar{B}_{11} = a_{1i}a_{1j}B_{ij} = a_{11}a_{13}B_{13} + a_{12}a_{12}B_{22} = 44/3.$$

No.19. $\bar{A}_{123} = a_{1i}a_{2j}a_{3k}A_{ijk} = a_{11}a_{21}a_{31}A_{111} = 432/3125$

If $C_j = B_{iji}$, then

$$\bar{C}_1 = \bar{B}_{i1i} = \bar{B}_{111} + \bar{B}_{212} + \bar{B}_{313} = 3$$

$$\bar{C}_2 = \bar{B}_{i2i} = \bar{B}_{121} + \bar{B}_{222} + \bar{B}_{323} = 4$$

$$\bar{C}_3 = \bar{B}_{i3i} = \bar{B}_{131} + \bar{B}_{232} + \bar{B}_{333} = 0$$

Thus, using the inverse transformation

$$C_1 = a'_{1i}\bar{C}_i = a_{i1}\bar{C}_i = 3$$

$$C_2 = a'_{2i}\bar{C}_i = a_{i2}\bar{C}_i = 0$$

$$C_3 = a'_{3i}\bar{C}_i = a_{i3}\bar{C}_i = 4$$

No.20. By use of the identity $\cos^2\alpha + \sin^2\alpha = 1$, it is easy to verify that all the orthogonality conditions are satisfied. It is also easy to check that the transformation is the resultant of the pair of transformations

$$\bar{x}_1 = x'_1\cos\alpha - x'_3\sin\alpha \qquad x'_1 = x_1\cos\beta + x_2\sin\beta$$

$$\bar{x}_2 = x'_2 \qquad x'_2 = -x_1\sin\beta + x_2\cos\beta$$

$$\bar{x}_3 = x'_1\sin\alpha + x'_3\cos\alpha \qquad x'_3 = x_3$$

The right-hand member of this pair represents a rotation of the x-frame through an angle β about the x_3-axis to generate the x'-frame. The left-hand transformation represents a further rotation of the x'-frame about the x'_2-axis, through an angle α(in the negative sense), to generate the $\bar{x}$-frame.

If $\alpha = \beta = \frac{1}{4}\pi$, the transformation equations become

$$\bar{x}_1 = \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{\sqrt{2}}x_3$$

$$\bar{x}_2 = \frac{1}{\sqrt{2}}(-x_1 + x_2)$$

$$\bar{x}_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{\sqrt{2}}x_3$$

With this transformation

$$\bar{A}_{1123} = a_{1i}a_{1j}a_{2k}a_{3\ell}A_{ijk\ell} = a_{11}a_{11}a_{22}a_{33}A_{1123} + a_{12}a_{12}a_{21}a_{33}A_{2213}$$

$$= 0$$

No.21. Taking the transpose of AB,

$$(AB)' = B'A' = B^{-1}A^{-1} = (AB)^{-1}$$

which is the condition for AB to be orthogonal.

Giving A, B the values stated, we find

$$C = AB = \frac{1}{25}\begin{bmatrix} 9 & 20 & 12 \\ -12 & 15 & -16 \\ -20 & 0 & 15 \end{bmatrix}$$

Thus,

$$\bar{A}_{321} = c_{3i}c_{2j}c_{1k}A_{ijk} = c_{32}c_{22}c_{11}A_{221} + c_{31}c_{22}c_{12}A_{122}$$
$$= -48/5$$

Writing $D_\ell = A_{ijk}B_{ijk\ell}$, we have

$$\bar{D}_1 = \bar{A}_{ijk}\bar{B}_{ijk1} = \bar{A}_{321}\bar{B}_{3211} = 48$$
$$\bar{D}_2 = \bar{A}_{ijk}\bar{B}_{ijk2} = \bar{A}_{321}\bar{B}_{3212} = -96$$
$$\bar{D}_3 = \bar{A}_{ijk}\bar{B}_{ijk3} = \bar{A}_{321}\bar{B}_{3213} = -144$$

Transforming this vector to the x-frame,

$$D_1 = c'_{1i}\bar{D}_i = c_{i1}\bar{D}_i = c_{11}\bar{D}_1 + c_{21}\bar{D}_2 + c_{31}\bar{D}_3 = 4464/25$$
$$D_2 = c'_{2i}\bar{D}_i = c_{i2}\bar{D}_i = c_{12}\bar{D}_1 + c_{22}\bar{D}_2 + c_{32}\bar{D}_3 = -96/5$$
$$D_3 = c'_{3i}\bar{D}_i = c_{i3}\bar{D}_i = c_{13}\bar{D}_1 + c_{23}\bar{D}_2 + c_{33}\bar{D}_3 = -48/25$$

No.22. Relative to an orthogonal transformation $\bar{x}_i = a_{ij}x_j + b_i$, we have transformation equations

$$\bar{\mathfrak{A}}_{ij} = |A|a_{ir}a_{js}\mathfrak{A}_{rs}, \quad \bar{\mathfrak{B}}_i = |A|a_{it}\mathfrak{B}_t, \quad \bar{E}_j = a_{ju}E_u.$$

Thus,

$$\bar{\mathfrak{A}}_{ij}\bar{\mathfrak{B}}_i\bar{E}_j = |A|^2 a_{ir}a_{js}a_{it}a_{ju}\mathfrak{A}_{rs}\mathfrak{B}_tE_u$$
$$= \delta_{rt}\delta_{su}\mathfrak{A}_{rs}\mathfrak{B}_tE_u, \text{ since } |A|^2 = 1$$
$$= \mathfrak{A}_{rs}\mathfrak{B}_rE_s$$

proving $\mathfrak{A}_{ij}\mathfrak{B}_iE_j$ is an invariant.

No.23. At $x_1 = x_2 = x_3 = \frac{1}{2}\sqrt{\pi}$,

$$(\text{curl}A)_1 = A_{3,2} - A_{2,3} = x_1\sec^2(x_1x_2) + x_1\sin(x_3x_1)$$
$$= \sqrt{\pi}(1 + 1/2\sqrt{2})$$
$$(\text{curl}A)_2 = A_{1,3} - A_{3,1} = x_2\cos(x_2x_3) - x_2\sec^2(x_1x_2)$$
$$= -\sqrt{\pi}\,(1 - 1/2\sqrt{2})$$
$$(\text{curl}A)_3 = A_{2,1} - A_{1,2} = -x_3\sin(x_3x_1) - x_3\cos(x_2x_3)$$
$$= -\sqrt{(\pi/2)}$$

The transformation to the $\bar{x}$-frame is determined by the equations

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = -x_2, \quad \bar{x}_3 = x_3$$

In matrix notation, $\bar{x} = Ax$, where $a_{11} = -a_{22} = a_{33} = 1$, all other elements of A being zero. Clearly, $|A| = -1$.

Since curl$A = \mathfrak{C}$ is a pseudovector, in the $\bar{x}$-frame its components are

$$\overline{\mathfrak{C}}_1 = |A|a_{1i}\mathfrak{C}_i = -\mathfrak{C}_1 = -\sqrt{\pi}(1 + 1/2\sqrt{2})$$
$$\overline{\mathfrak{C}}_2 = |A|a_{2i}\mathfrak{C}_i = \mathfrak{C}_2 = -\sqrt{\pi}(1 - 1/2\sqrt{2})$$
$$\overline{\mathfrak{C}}_3 = |A|a_{3i}\mathfrak{C}_i = -\mathfrak{C}_3 = \sqrt{(\pi/2)}$$

No.24. The orthogonality conditions require that

$$\alpha^2(a^2 + 2^2 + 5^2) = 1, \quad a + 2b + 10 = 0,$$
$$\beta^2(1^2 + b^2 + 2^2) = 1, \quad 2 - 11b + 2c = 0,$$
$$\gamma^2(2^2 + 11^2 + c^2) = 1, \quad 2a - 22 + 5c = 0.$$

Solving, we find $a = -14$, $b = 2$, $c = 10$, $\alpha = 1/15$, $\beta = 1/3$, $\gamma = 1/15$.

If $\overline{x} = Ax$ represents the transformation, then

$$\overline{A}_{3121} = a_{3i}a_{1j}a_{2k}a_{1\ell}A_{ijk\ell} = a_{31}a_{13}a_{21}a_{13}A_{1313} + a_{33}a_{12}a_{22}a_{13}A_{3223}$$
$$= 2/5$$

In the $\overline{x}$-frame, the vector $C_j = A_iB_{ij}$ has components

$$\overline{C}_1 = \overline{A}_i\overline{B}_{i1} = \overline{A}_1\overline{B}_{11} + \overline{A}_2\overline{B}_{21} + \overline{A}_3\overline{B}_{31} = 0$$
$$\overline{C}_2 = \overline{A}_i\overline{B}_{i2} = \overline{A}_1\overline{B}_{12} + \overline{A}_2\overline{B}_{22} + \overline{A}_3\overline{B}_{32} = 3$$
$$\overline{C}_3 = \overline{A}_i\overline{B}_{i3} = \overline{A}_1\overline{B}_{13} + \overline{A}_2\overline{B}_{23} + \overline{A}_3\overline{B}_{33} = 15$$

Transforming to the x-frame,

$$C_1 = a'_{1i}\overline{C}_i = a_{i1}\overline{C}_i = a_{11}\overline{C}_1 + a_{21}\overline{C}_2 + a_{31}\overline{C}_3 = 3$$
$$C_2 = a'_{2i}\overline{C}_i = a_{i2}\overline{C}_i = a_{12}\overline{C}_1 + a_{22}\overline{C}_2 + a_{32}\overline{C}_3 = -9$$
$$C_3 = a'_{3i}\overline{C}_i = a_{i3}\overline{C}_i = a_{13}\overline{C}_1 + a_{23}\overline{C}_2 + a_{33}\overline{C}_3 = 12$$

No.25. We are given $A' = -A$, $A^3 = -A$. Thus

$$B' = I - A + A^2$$

and

$$BB' = (I + A + A^2)(I - A + A^2) = I + A^2 + A^4 = I + A^2 + A(A^3) = I$$

Thus, B is orthogonal.

The given matrix A is clearly anti-symmetric and, by repeated multiplication of A by itself, we find

$$A^2 = \frac{1}{9}\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix}, \quad A^3 = \frac{1}{3}\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}$$

Thus, $A^3 = -A$. We now find that

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4 & -1 & 8 \\ -7 & 4 & 4 \\ -4 & -8 & 1 \end{bmatrix}$$

Thus,

$$\bar{A}_{12} = b_{1i}b_{2j}A_{ij} = b_{13}b_{22}A_{32} = 32$$

$$B_{321} = b'_{3i}b'_{2j}b'_{1k}\bar{B}_{ijk} = b_{13}b_{22}b_{31}\bar{B}_{123} = -128$$

No.26. Considering the ith component

$$(\text{curl grad}V)_i = \epsilon_{ijk}(\text{grad}V)_{k,j} = \epsilon_{ijk}V_{,kj} = 0$$

since terms in which j, k are the same vanish, and other terms cancel in pairs, e.g.

$$\epsilon_{i12}V_{,12} + \epsilon_{i21}V_{,21} = 0.$$

Also

$$\text{div curl}A = (\text{curl}A)_{i,i} = (\epsilon_{ijk}A_{k,j})_{,i} = \epsilon_{ijk}A_{k,ji} = 0$$

since terms in which i, j are the same vanish, and other terms cancel in pairs as before.

No.27. (i) There are four cases to consider:

Case I: The pair (k,ℓ) is not identical with the pair (m,n). Then, each term in the left-hand member must contain an e-factor, two of whose suffices are equal; this member is accordingly zero. Also, each term in the right-hand member must contain a δ-factor whose suffices are different; this member is also zero therefore. The identity is thus verified in this case.

Case II: $k = m$, $\ell = n$, but $k \neq \ell$. Then one term in the left-hand member will be non-zero, this being the term for which $(ik\ell)$ and (imn) are the same permutations of (123); this term has the value +1. Since $\delta_{km} = \delta_{\ell n} = 1$, $\delta_{kn} = \delta_{\ell m} = 0$, the identity is again verified.

Case III: $k = n$, $\ell = m$, but $k \neq \ell$. Again, one term in the left-hand member will be non-zero, this being the term for which $(ik\ell)$ and (imn) are permutations of (123); however, one of these permutations will be even and one odd and the value of the member will be -1. In the right-hand member, $\delta_{km} = \delta_{\ell n} = 0$, $\delta_{kn} = \delta_{\ell m} = 1$, thus verifying the identity.

Case IV: $k = \ell = m = n$. All terms in the left-hand member are clearly zero. Since $\delta_{km} = \delta_{\ell n} = \delta_{kn} = \delta_{\ell m} = 1$, the right-hand member is also zero and the identity is verified.

$$\text{(ii)}\quad \epsilon_{ik\ell}\epsilon_{ikm} = \delta_{kk}\delta_{\ell m} - \delta_{km}\delta_{\ell k} = 3\delta_{\ell m} - \delta_{\ell m} = 2\delta_{\ell m}$$

No.28. By definition of divergence

$$\text{div grad}V = (\text{grad}V)_{i,i} = (V_{,i})_{,i} = V_{,ii} = \frac{\partial^2 V}{\partial x_i \partial x_i}$$

No.29. By definition of curl, the ith component of the left-hand member is

$$\epsilon_{ijk}(\text{curl}A)_{k,j} = \epsilon_{ijk}(\epsilon_{krs}A_{s,r})_{,j} = \epsilon_{kij}\epsilon_{krs}A_{s,rj}$$

$$= (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})A_{s,rj}$$

$$= A_{j,ij} - A_{i,jj}$$
$$= (A_{j,j})_{,i} - \nabla^2 A_i$$
$$= (\mathrm{div}A)_{,i} - (\nabla^2 A)_i$$

This is the ith component of the right-hand member and the identity has therefore been established.

No.30. (i) Taking the ith component

$$[A \times (B \times C)]_i = \epsilon_{ijk} A_j (B \times C)_k$$
$$= \epsilon_{ijk} A_j \epsilon_{krs} B_r C_s$$
$$= \epsilon_{kij} \epsilon_{krs} A_j B_r C_s$$
$$= (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) A_j B_r C_s$$
$$= A_j B_i C_j - A_j B_j C_i$$
$$= (A \cdot C) B_i - (A \cdot B) C_i$$

which is the ith component of the right-hand member.

(ii) $A \cdot (B \times C) = A_i (B \times C)_i = A_i \epsilon_{ijk} B_j C_k = \epsilon_{ijk} A_i B_j C_k$

This last expression is the expansion of the determinant and so verifies the identity.

No.31. $\mathrm{div}(VA) = (VA_i)_{,i} = VA_{i,i} + V_{,i} A_i = V\,\mathrm{div}A + A \cdot \nabla V$

No.32. (i) $(\mathrm{curl}\ VA)_i = \epsilon_{ijk}(VA_k)_{,j} = \epsilon_{ijk} V A_{k,j} + \epsilon_{ijk} V_{,j} A_k$

$$= V(\mathrm{curl}A)_i + (\mathrm{grad}V \times A)_i$$

(ii) $\mathrm{div}(A \times B) = (A \times B)_{i,i} = (\epsilon_{ijk} A_j B_k)_{,i}$

$$= \epsilon_{ijk} A_{j,i} B_k + \epsilon_{ijk} A_j B_{k,i}$$
$$= \epsilon_{kij} A_{j,i} B_k - \epsilon_{jik} B_{k,i} A_j$$
$$= (\mathrm{curl}A)_k B_k - (\mathrm{curl}B)_j A_j$$
$$= B \cdot \mathrm{curl}A - A \cdot \mathrm{curl}B$$

(iii) The jth component of the operator ∇ is $\partial/\partial x_j$. Thus, the formal scalar product of A and ∇ yields the operator $A \cdot \nabla = A_j \partial/\partial x_j$. Hence, the ith component of $(A \cdot \nabla)B$ is $(A \cdot \nabla)B_i = A_j B_{i,j}$. Thus

$$\mathrm{curl}(A \times B)_i = \epsilon_{ijk}(A \times B)_{k,j} = \epsilon_{ijk}(\epsilon_{krs} A_r B_s)_{,j}$$
$$= \epsilon_{kij} \epsilon_{krs}(A_{r,j} B_s + A_r B_{s,j})$$
$$= (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})(A_{r,j} B_s + A_r B_{s,j})$$
$$= A_{i,j} B_j + A_i B_{j,j} - A_{j,j} B_i - A_j B_{i,j}$$
$$= (B \cdot \nabla) A_i + A_i \mathrm{div}B - B_i \mathrm{div}A - (A \cdot \nabla) B_i$$

verifying the identity.

(iv) $(A \times \mathrm{curl}B)_i = \epsilon_{ijk}A_j(\mathrm{curl}B)_k = \epsilon_{ijk}A_j\epsilon_{krs}B_{s,r}$

$$= \epsilon_{kij}\epsilon_{krs}A_jB_{s,r} = (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})A_jB_{s,r}$$

$$= A_jB_{j,i} - A_jB_{i,j}$$

Similarly

$$(B \times \mathrm{curl}A)_i = B_jA_{j,i} - B_jA_{i,j}$$

Adding

$$(A \times \mathrm{curl}B + B \times \mathrm{curl}A)_i = A_jB_{j,i} + B_jA_{j,i} - A_jB_{i,j} - B_jA_{i,j}$$

$$= (A_jB_j)_{,i} - (A\cdot\nabla)B_i - (B\cdot\nabla)A_i$$

$$= [\mathrm{grad}(A\cdot B) - (A\cdot\nabla)B - (B\cdot\nabla)A]_i$$

which is equivalent to the identity required.

No.33. Transforming to an $\overline{x}$-frame by the orthogonal transformation $\overline{x}_i = a_{ij}x_j + b_i$, we find

$$\overline{B}_{ij} = \overline{A}_{ji} = a_{jr}a_{is}A_{rs} = a_{is}a_{jr}B_{sr}$$

showing that B_{ij} transforms as a tensor.

It now follows that $A_{ij} = A_{ji} = B_{ij}$ is a tensor equation and hence is valid in all frames if it is true in one.

This result is easily generalised: Thus, if A_{ijk} is a tensor and $B_{ijk} = A_{kij}$, then B_{ijk} is a tensor; i.e. the components of a tensor can be rearranged to provide components of a new tensor.

No.34. Since $\delta_{ij}a_{ik} = a_{jk}$ follows from the properties of the Kronecker delta, the first identity follows by putting $a_{ik} = \delta_{ik}$.

If (ijk) are all different and (ℓmn) is an even permutation of (ijk), then (ℓmn) is obtained from (ijk) by an even number of transpositions and hence ϵ_{ijk} and $\epsilon_{\ell mn}$ have the same sign and their product is +1. If (ℓmn) is an odd permutation of (ijk), the number of transpositions is odd and ϵ_{ijk} and $\epsilon_{\ell mn}$ have opposite signs and their product is -1. In all other cases, one of the triads contains repeated values and one of ϵ_{ijk} , $\epsilon_{\ell mn}$ is zero.

Suppose (ijk) are all different and (ℓmn) is an even permutation of (ijk). Then one of the three even permutations of (ℓmn) will be (ijk). Hence, one of the first three terms of the right-hand member of the next identity to be proved will be a product of three delta symbols, each with an identical pair of suffices and this term will have value +1. The remaining five terms of this member must each contain at least one delta symbol with non-identical suffices and so must all be zero. Since the left-hand member has already been proved to equal +1 in this case, the identity is verified.

If (ijk) are all different and (ℓmn) is an odd permutation of (ijk), it will be one of the last three terms of the right-hand member which is non-zero with value -1, all other terms in this member vanishing. In this case, the left-hand member has been shown to be -1 and again the identity is verified.

If (ℓmn) is not a permutation of (ijk), then at least one pair of (ℓmn) or one pair of (ijk) are the same and both sides of the identity vanish.

Next, suppose (ℓmn) is a permutation of (ijk), but a pair of indices are equal in each case. The left-hand member vanishes. In the right-hand member, one only of the first three terms will be non-vanishing, with the value +1. By transposition in this term of the pair of equal indices in the group (ℓmn), one of the last three terms will be generated; this term will accordingly contribute -1 to the member and cause it to vanish. Again, the identity is verified.

Finally, suppose ($\ell mnijk$) are all equal. Both members of the identity then clearly vanish and the identity is verified.

Contracting by putting $\ell = i$, the identity gives

$$\begin{aligned} \epsilon_{ijk}\epsilon_{imn} &= \delta_{ii}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} \\ &\quad - \delta_{in}\delta_{jm}\delta_{ki} - \delta_{ii}\delta_{jn}\delta_{km} - \delta_{im}\delta_{ji}\delta_{kn} \\ &= 3\delta_{jm}\delta_{kn} + \delta_{km}\delta_{jn} + \delta_{jn}\delta_{km} \\ &\quad - \delta_{kn}\delta_{jm} - 3\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} \\ &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \end{aligned}$$

No.35. (i)

$$\begin{aligned} (a \times b)\cdot(c \times d) &= (a \times b)_i(c \times d)_i = \epsilon_{ijk}a_j b_k \epsilon_{irs}c_r d_s \\ &= (\delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr})a_j b_k c_r d_s \\ &= a_j b_k c_j d_k - a_j b_k c_k d_j \\ &= (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c) \end{aligned}$$

(ii)

$$\begin{aligned} [(a \times b) \times (c \times d)]_i &= \epsilon_{ijk}(a \times b)_j(c \times d)_k \\ &= \epsilon_{ijk}\epsilon_{jrs}a_r b_s \epsilon_{kmn}c_m d_n \\ &= (\delta_{kr}\delta_{is} - \delta_{ks}\delta_{ir})\epsilon_{kmn}a_r b_s c_m d_n \\ &= (\epsilon_{kmn}a_k c_m d_n)b_i - (\epsilon_{kmn}b_k c_m d_n)a_i \\ &= [acd]b_i - [bcd]a_i \end{aligned}$$

having used the result in No.30(ii) above. The alternative identity can be established by associating a different pair of e-symbols together in a contracted product.

No.1. The successive space-time points occupied by a moving particle have coordinates in the two frames S, $\bar{S}$ which are related by the equations (5.8). Differentiating these equations, we find

$$d\bar{x} = \beta(dx - udt), \quad d\bar{y} = dy,$$

$$d\bar{t} = \beta(dt - udx/c^2), \quad d\bar{z} = dz.$$

Thus,

$$\bar{v}_x = d\bar{x}/d\bar{t} = \frac{dx - udt}{dt - udx/c^2} = \frac{v_x - u}{1 - uv_x/c^2}$$

since $v_x = dx/dt$. Also

$$\bar{v}_y = d\bar{y}/d\bar{t} = \frac{dy}{\beta(dt - udx/c^2)} = \frac{v_y}{\beta(1 - uv_x/c^2)}$$

since $v_y = dy/dt$. $\bar{v}_z$ is obtained similarly.

No.2. Differentiating the equation for $\bar{v}_x$ obtained in the last exercise, we get

$$d\bar{v}_x = \frac{1 - u^2/c^2}{(1 - uv_x/c^2)^2}dv_x$$

Since

$$d\bar{t} = \beta(1 - uv_x/c^2)dt$$

also follows from the previous exercise, division of this pair of equations leads to the stated equation for $\bar{a}_x = d\bar{v}_x/d\bar{t}$.

Differentiating the equation for $\bar{v}_y$ from the previous exercise, gives

$$d\bar{v}_y = \frac{dv_y}{\beta(1 - uv_x/c^2)} + \frac{uv_y/c^2}{\beta(1 - uv_x/c^2)^2}dv_x$$

Again, division of the last two equations provides the equation for $\bar{a}_y = d\bar{v}_y/d\bar{t}$.

The equation for $\bar{a}_z$ follows similarly.

Clearly, if a_x, a_y, a_z are all constant, in general $\bar{a}_x$, $\bar{a}_y$, $\bar{a}_z$ will not be constant, since these quantities will vary with the velocity components v_x, v_y, v_z.

No.3. Take the nucleus at the origin of the $\bar{S}$-frame. In this frame, we can assume the electron to move in the $\bar{x}\bar{y}$-plane with velocity components $\bar{v}_x = 3c/7$, $\bar{v}_y = 3\sqrt{3}c/7$. Taking S to be the stationary frame, the velocity transformation equations give for the electron's velocity components in S

$$v_x = \frac{\frac{3}{7}c + u}{1 + \frac{3}{7}uc}, \quad v_y = \frac{(1 - u^2/c^2)^{\frac{1}{2}}3\sqrt{3}c/7}{1 + 3uc/7},$$

where u is the velocity of $\bar{S}$ and the nucleus. Hence

$$1/\sqrt{3} = \tan 30^{\circ} = v_y/v_x = \frac{(1 - u^2/c^2)^{\frac{1}{2}}3\sqrt{3}c}{3c + 7u}$$

Squaring this equation and solving for u, we find $u = 3c/5$ or $-12c/13$. The second root is spurious, since it fails to satisfy the last equation.

No.4. Take S to be the frame of the stationary observer and $\bar{S}$ the frame moving with the nucleus. In $\bar{S}$, take the velocity components of the β-particle to be $\bar{v}_x = 0$, $\bar{v}_y = 3c/4$, $\bar{v}_z = 0$. Since $u = 3c/5$, the transformation equations give

$$v_x = 3c/5, \quad v_y = 3c/5, \quad v_z = 0.$$

Thus, in S, $v = 3\sqrt{2}c/5$ and the direction of motion makes 45^o with the direction of motion of the nucleus.

In the second case, let θ be the angle in $\bar{S}$ between the line of motion of the β-particle and the $\bar{x}$-axis. Then, we can take $\bar{v}_x = 3c\cos\theta/4$, $\bar{v}_y = 3c\sin\theta/4$. Thus

$$v_x = \frac{\frac{3}{4}c\cos\theta + \frac{3}{5}c}{1 + \frac{9}{20}\cos\theta}, \quad v_y = \frac{\frac{3}{5}c\sin\theta}{1 + \frac{9}{20}\cos\theta}$$

Since $v_x = 0$, the first equation shows that $\cos\theta = -4/5$, i.e. θ is obtuse. The second equation then gives $v_y = 9c/16$; this is the velocity of the β-particle in S.

No.5. The square of the magnitude of a vector is defined by equation (12.4). In the case of the 4-velocity, its magnitude V is given by

$$V^2 = \underline{V}\cdot\underline{V} = V_iV_i = (1 - v^2/c^2)^{-1}(\underline{v},ic)\cdot(\underline{v},ic)$$
$$= (1 - v^2/c^2)^{-1}(v^2 - c^2) = -c^2$$

Hence, $V = ic$

No.6. Consider a photon in the light beam. Its velocity components in S along the x- and y-axes are $c\cos\alpha$, $c\sin\alpha$ respectively. In $\bar{S}$, the corresponding velocity components are $c\cos\bar{\alpha}$, $c\sin\bar{\alpha}$. The velocity transformation equations show that

$$c\cos\bar{\alpha} = \frac{c\cos\alpha - u}{1 - uc\cos\alpha}, \quad c\sin\bar{\alpha} = \frac{\sqrt{(1 - u^2/c^2)}c\sin\alpha}{1 - uc\cos\alpha}$$

Division now yields the equation stated.

Clearly, if u is small, α and $\bar{\alpha}$ are approximately equal. Writing $\bar{\alpha} = \alpha + \Delta\alpha$, where Δα is small, a Taylor expansion shows that

$$\cot\bar{\alpha} = \cot(\alpha + \Delta\alpha) = \cot\alpha - \Delta\alpha\,\mathrm{cosec}^2\alpha$$

approximately. Further, an approximation to the right-hand member to order (u/c) is clearly $\cot\alpha - (u/c)\mathrm{cosec}\alpha$. Substituting these approximations for the two members of the aberration formula just found, we deduce that $\Delta\alpha = (u/c)\sin\alpha$.

No.7. The first result is obtained by differentiating out the right-hand member of the equation

$$\underline{f} = \frac{d}{dt}\left[\frac{m_0\underline{v}}{(1 - v^2/c^2)^{\frac{1}{2}}}\right]$$

If the acceleration is parallel to $\underline{v}$, it is directed along the tangent to the particle's trajectory and the component of the acceleration along the principal normal to the trajectory (viz. κv^2) vanishes. Thus, the component of the acceleration along the tangent (viz. $\dot{v}$) gives the magnitude of the acceleration. The unit vector along the tangent is $\underline{v}/v$ and, hence, $d\underline{v}/dt = \dot{v}\underline{v}/v$. Substituting $\dot{v}\underline{v} = v\,dv/dt$ into the second term of the expression for $\underline{f}$ now gives the second equation.

If the acceleration is perpendicular to $\underline{v}$, the tangential component must vanish and so $\dot{v} = 0$. The first expression for $\underline{f}$ therefore reduces to the last equation.

In these two special cases, the force can be expressed in the Newtonian form: mass acceleration. In the first case, the mass is $m_0(1 - v^2/c^2)^{-3/2}$ and is sometimes termed the longitudinal mass. In the second case, the mass is $m_0(1 - v^2/c^2)^{-1/2}$ and is sometimes called the transverse mass.

No.8. The velocities of the particles in $\bar{S}$ are

$$\bar{v} = \frac{v_1 - u}{1 - uv_1/c^2}\,, \qquad -\bar{v} = \frac{v_2 - u}{1 - uv_2/c^2} \qquad \text{(i)}$$

Elimination of $\bar{v}$ leads to the equation

$$(v_1 + v_2)u^2 - 2(v_1v_2 + c^2)u + (v_1 + v_2)c^2 = 0$$

Solving this quadratic for u, we obtain

$$u = \frac{c^2 + v_1v_2 - (c^2 - v_1^2)^{\frac{1}{2}}(c^2 - v_2^2)^{\frac{1}{2}}}{v_1 + v_2}$$

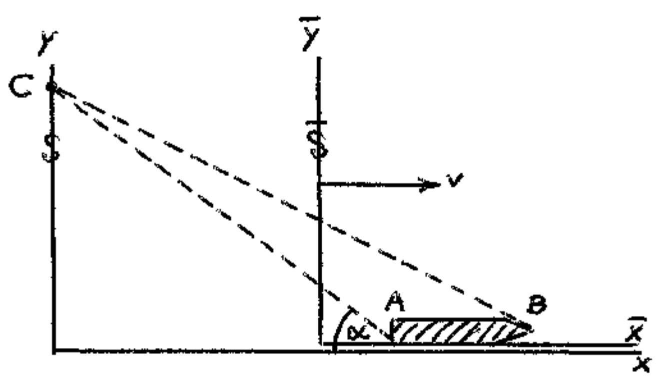
The negative sign must be taken before the root since $(c - v_1)(c - v_2) > 0$ implies that $c^2 + v_1v_2 > c(v_1 + v_2)$ and u would therefore exceed c if the positive sign were taken.

To obtain $\bar{v}$, it is simplest to eliminate u between the equations (i) to give the equation

$$(v_1 - v_2)\bar{v}^2 - 2(c^2 - v_1v_2)\bar{v} + c^2(v_1 - v_2) = 0$$

Solving this quadratic for $\bar{v}$ gives the result stated.

No.9.



Suppose the bullet AB is attached to the $\bar{x}$-axis of $\bar{S}$ as shown. The camera is placed at C on the y-axis of S. The fixed scale is the x-axis.

Suppose the light entering C at the instant the shutter is opened leaves A at time T and B at time t as measured by the clocks of S. At time T, let A have x-coordinate x_A and at time t let B have x-coordinate x_B. Then the difference in the distances travelled by the light from A and B is $(x_B - x_A)\cos\alpha$ (assuming the length of the bullet is small compared with the distance of the camera from the bullet). Since the light from A and from B must enter the camera at the same instant, we must have

$$T - t = (x_B - x_A)\cos\alpha/c \qquad (i)$$

Consider the event of the light leaving A. In S, the coordinates of this event are (x_A, T). In $\overline{S}$, let the coordinates be $(\overline{x}_A, \overline{T})$. Then

$$\overline{x}_A = \beta(x_A - vT),$$

where $\beta = (1 - v^2/c^2)^{-\frac{1}{2}}$. Similarly, if the $\overline{x}$-coordinate of B is $\overline{x}_B$, then

$$\overline{x}_B = \beta(x_B - vt)$$

Subtracting, we get

$$d = \overline{x}_B - \overline{x}_A = \beta(x_B - x_A) + \beta v(T - t)$$

Substituting for $(T - t)$ from (i), we find that

$$x_B - x_A = (1 - v^2/c^2)^{\frac{1}{2}}d/(1 + v\cos\alpha/c)$$

Since x_A, x_B are the coordinates of the points on the x-axis (fixed scale) from which the light leaves to enter the camera, this will give the apparent length on the photograph.

No.10. If S is attached to the table and $\overline{S}$ to the rth cart, the velocity of $\overline{S}$ relative to S is cv_r. The velocity of the (r+1)th cart in S is cv_{r+1} and in $\overline{S}$ is kc. Using the velocity transformation equation, we have

$$cv_{r+1} = \frac{kc + cv_r}{1 + kv_r}$$

The last equation is equivalent to the recurrence relationship

$$u_{r+1} = \frac{1 - k}{1 + k}\, u_r$$

where $u_r = (1 - v_r)/(1 + v_r)$. By repeated application of this relationship, we find

$$u_n = \frac{1 - k}{1 + k}\, u_{n-1} = \dots = \left(\frac{1 - k}{1 + k}\right)^{n-1} u_1$$

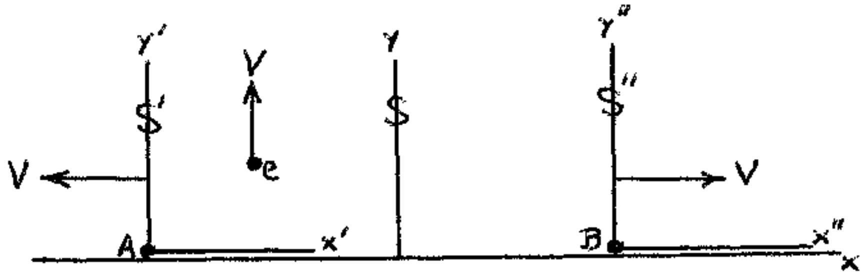
But $v_1 = k$ and, hence, $u_1 = (1 - k)/(1 + k)$. Thus,

$$u_n = \left(\frac{1 - k}{1 + k}\right)^{n}$$

Since $v_n = (1 - u_n)/(1 + u_n)$, this leads immediately to the result stated.

Since $0 < (1 - k)/(1 + k) < 1$, $u_n \to 0$ as $n \to \infty$. It follows that $v_n \to 1$ as $n \to \infty$.

No.11.



S is the laboratory frame, S' is a frame moving with A and S" is a frame moving with B as indicated in the diagram.

In S, B's velocity is $v_x = V$ and S' has velocity $-V$. Hence, in S' B's velocity is

$$v'_x = \frac{v_x + V}{1 + Vv_x/c^2} = \frac{2V}{1 + V^2/c^2}$$

In S', the electron's velocity has components $v'_x = 0$, $v'_y = V$ and S" has velocity $u = 2V/(1 + V^2/c^2)$. Hence, in S" the electron's velocity has components

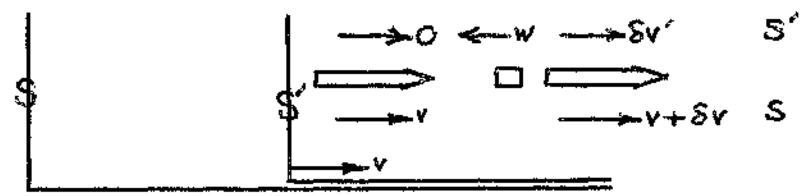
$$v''_x = \frac{v'_x - u}{1 - uv'_x/c^2} = -\frac{2V}{1 + V^2/c^2}$$

$$v''_y = \frac{(1 - u^2/c^2)^{\frac{1}{2}}v'_y}{1 - uv'_x/c^2} = \frac{1 - V^2/c^2}{1 + V^2/c^2}\,V$$

Thus $\tan\alpha = v''_y/(-v''_x) = \frac{1}{2}(1 - V^2/c^2)$. The magnitude of the electron's velocity in S" is

$$\sqrt{(v''^2_x + v''^2_y)} = V\left[4 + (1 - V^2/c^2)^2\right]^{\frac{1}{2}}/(1 - V^2/c^2)$$

No.12.



At time t, let v be the rocket velocity in S; at time $t + \delta t$, the velocity has increased to $v + \delta v$. At time t in S, S' is an inertial frame relative to which the rocket is instantaneously at rest, i.e. S' has velocity v relative to S and is the crew's natural inertial frame. When the rocket has velocity $v + \delta v$ in S, let $\delta v'$ be its velocity in S'. In S', as the rocket velocity increases from 0 to $\delta v'$, suppose its rest mass decreases from m_0 to $m_0 + \delta m_0$ ($\delta m_0 < 0$); let Δm_0 be the rest mass of the particles ejected with velocity w into the jet stream during this change. Then, conservation of inertial mass requires that

$$m_0 = \frac{m_0 + \delta m_0}{\sqrt{(1 - \delta v'^2/c^2)}} + \frac{\Delta m_0}{\sqrt{(1 - w^2/c^2)}}$$

and conservation of linear momentum that

$$0 = \frac{(m_0 + \delta m_0)\delta v'}{\sqrt{(1 - \delta v'^2/c^2)}} - \frac{\Delta m_0 w}{\sqrt{(1 - w^2/c^2)}}$$

Approximating to the first order in all small quantities, these equations reduce to

$$\delta m_0 + \Delta m_0/\sqrt{(1 - w^2/c^2)} = 0$$

$$m_0\delta v' = w\Delta m_0/\sqrt{(1 - w^2/c^2)}$$

We now deduce that

$$m_0\delta v' = -w\delta m_0 \qquad \text{(i)}$$

Transforming the velocity $v + \delta v$ in S into $\delta v'$ in S', we obtain

$$\delta v' = \frac{v + \delta v - v}{1 - v(v+\delta v)/c^2} = \frac{\delta v}{1 - v^2/c^2} \qquad \text{(ii)}$$

to the first order in δv. Equations (i) and (ii) now yield in the limit

$$\frac{dv}{1 - v^2/c^2} = -w\frac{dm_0}{m_0}$$

Integration over the range from $v = v_0$ to $v = v_1$ gives

$$\tfrac{1}{2}c \log\left[\frac{(c+v_1)(c-v_0)}{(c-v_1)(c+v_0)}\right] = w \log R,$$

where $R = m_{01}/m_{00}$ = mass ratio. Solving for R, we get the result stated.

Since $(1 + v/c)^c \to e^v$ as $c \to \infty$, we calculate that

$$R \to \left[e^{2(v_1-v_0)}\right]^{1/2w} = \exp[(v_1-v_0)/w]$$

This is the well-known classical formula for the mass ratio.

If the jet is a stream of photons, then $w = c$. (This would be the case if the propellants were electrons and positrons which mutually annihilate to generate photons.) Then, setting $v_0 = 0$, $v_1 = v$, we find $R = \sqrt{[(c+v)/(c-v)]}$. If $R = 6$, this gives $v = 35c/37$.

No.13. Equation (5.6) shows that

$$\tan\alpha = iu/c, \quad \tan\beta = iv/c, \quad \tan\gamma = iw/c,$$

where w is the velocity of S" relative to S. Thus

$$\tan\gamma = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\,\tan\beta}$$

is equivalent to

$$w = \frac{u + v}{1 + uv/c^2}.$$

No.14.

$$f = \frac{d}{dt}(mv) = m\frac{dv}{dt} + \frac{dm}{dt}v$$

The result stated now follows by use of equation (17.7).

No.15. The force acting on the charge due to the field is of magnitude eE and along the x-axis. The equation of motion is accordingly

$$\frac{d}{dt}\left[\frac{m_0 v}{\sqrt{(1 - v^2/c^2)}}\right] = eE$$

Since $v = 0$ at $t = 0$, integration leads to the result

$$\frac{v}{\sqrt{(1 - v^2/c^2)}} = kt$$

Solving for v, we get

$$v = \frac{dx}{dt} = \frac{kt}{\sqrt{(1 + k^2t^2/c^2)}}.$$

Integrating again with the initial condition $x = 0$ at $t = 0$, the result stated is derived.

If c is taken to be large by comparison with kt, the binomial theorem shows that

$$x = \frac{c^2}{k}\left[1 + k^2t^2/2c^2 + O(k^4t^4/c^4) - 1\right] \to \tfrac{1}{2}kt^2$$

as $c \to \infty$. According to classical theory, the particle is subjected to a constant acceleration $eE/m_0 = k$ and this result is accordingly consistent with such motion.

If kt is large compared with c, the relativistic equation can be approximated by taking $\sqrt{(1 + k^2t^2/c^2)} = kt/c$. Then $x = c(t - c/k)$; i.e. when a long time has elapsed, the particle will appear to move uniformly with the velocity of light.

No. 16. Take A and B to be at the origins of the frames S and $\bar{S}$ respectively, these frames being related as in section 5.

Suppose B receives the tachyon from A at time $\bar{t}$ by the $\bar{S}$ clocks and at time t by the S clocks. Then this event has coordinates (d,t) in S and coordinates $(0,\bar{t})$ in $\bar{S}$. The transformation equations $x = \beta(\bar{x} + u\bar{t})$ and $\bar{x} = \beta(x - ut)$ now show that $\bar{t} = d/\beta u$ and $t = d/u$. It now follows that the tachyon must leave A at time $t = d/u - d/v$.

Consider the motion of the tachyon transmitted back towards A from B. It moves along the negative $\bar{x}$-axis with speed v, starting from $\bar{0}$ at $\bar{t} = d/\beta u$. Its equation of motion in $\bar{S}$ is accordingly

$$\bar{x} = -v(\bar{t} - d/\beta u)$$

Transforming this equation to the S-frame, it yields

$$x(1 - uv/c^2) = -(v - u)t + vd/\beta^2 u$$

as the equation of motion in this frame. Thus, the tachyon arrives at A $(x = 0)$ at time

$$t = \frac{vd}{\beta^2 u(v - u)} = \frac{vd(1 - u^2/c^2)}{u(v - u)}$$

Thus, the S time elapsing between the first tachyon leaving A and the second tachyon arriving back at A is

$$\frac{vd(1 - u^2/c^2)}{u(v - u)} - \frac{d}{u} + \frac{d}{v} = \frac{d}{v(v - u)}(2v - u - v^2u/c^2)$$

This time is negative, and reception by A occurs before transmission from A, provided

$$uv^2 - 2c^2v + c^2u > 0$$

The quadratic equation for v: $uv^2 - 2c^2v + c^2u = 0$, has roots

$$v = \frac{c}{u}[c \pm \sqrt{(c^2 - u^2)}].$$

By consideration of the parabolic graph of $uv^2 - 2c^2v + c^2u$ for varying v, it is clear that the above inequality will be satisfied provided

$$v < \frac{c}{u}[c - \sqrt{(c^2 - u^2)}] \text{ or } v > \frac{c}{u}[c + \sqrt{(c^2 - u^2)}]$$

The first inequality is equivalent to

$$v < \frac{u}{1 + \sqrt{(1 - u^2/c^2)}}$$

i.e. $v < u$ and the tachyon's speed is less than c; moreover, in this case, the particle transmitted by A would not overtake B. The second inequality is the only possibility, therefore.

Differentiation of equation (15.6) yields

$$\underline{A} = d\underline{V}/d\tau = \left[\frac{v\dot{v}}{c^2}(1 - v^2/c^2)^{-3/2}(\underline{v},ic) + (1 - v^2/c^2)^{-1/2}(d\underline{v}/dt,0)\right]\frac{dt}{d\tau}$$

Using equation (15.1), this leads to the result

$$\underline{A} = (1 - v^2/c^2)^{-2}\left[(1 - v^2/c^2)\underline{a} + v\dot{v}\underline{v}/c^2,\ iv\dot{v}/c\right]$$

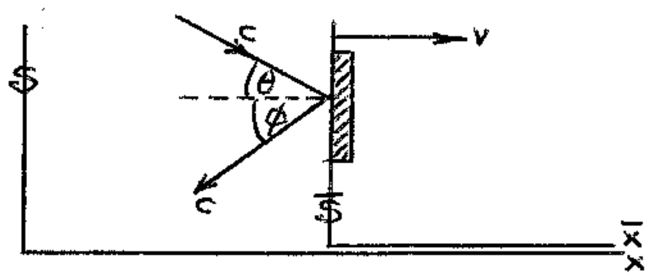
Thus

$$A^2 = \underline{A}\cdot\underline{A} = (1 - v^2/c^2)^{-4}\left[(1 - v^2/c^2)^2a^2 + 2v\dot{v}(1 - v^2/c^2)\underline{a}\cdot\underline{v}/c^2 + v^4\dot{v}^2/c^4 - v^2\dot{v}^2/c^2\right]$$

$$= (1 - v^2/c^2)^{-3}\left[(1 - v^2/c^2)a^2 + 2v\dot{v}\underline{a}\cdot\underline{v}/c^2 - v^2\dot{v}^2/c^2\right]$$

which is equivalent to the result stated.

No.18. We imagine the mirror to be fixed to the $\bar{S}$-frame which is moving with velocity v relative to the laboratory frame S as indicated in the diagram.



In the frame $\bar{S}$, the mirror is stationary, and the angles of incidence and reflection will be equal (say α). Thus, photons in the incident and reflected beams will have $\bar{x}$ velocity components $c\cos\alpha$, $-c\cos\alpha$ respectively. These same photons observed from S will have x velocity components $c\cos\theta$, $-c\cos\phi$. The velocity transformation equation requires that

$$c\cos\alpha = \frac{c\cos\theta - v}{1 - v\cos\theta/c}, \qquad -c\cos\alpha = \frac{-c\cos\phi - v}{1 + v\cos\phi/c}$$

Eliminating $\cos\alpha$ between these equations and solving for $\cos\phi$, we obtain the result stated.

No.19. The classical result is $T = L/U$.

Using the relativistic velocity transformation, the approach velocity of one train as seen by the other is $2U/(1 + U^2/c^2)$. Thus, allowing for the Fitzgerald contraction, the length of this approaching train as seen by the driver of the other is

$$L\sqrt{\left[1 - 4U^2/c^2(1 + U^2/c^2)^2\right]} = L\,\frac{1 - U^2/c^2}{1 + U^2/c^2}$$

Working in the driver's frame, in order to complete the passage the approaching train must move through its own length together with the length of the driver's train, i.e. a total length of

$$L + L\,\frac{1 - U^2/c^2}{1 + U^2/c^2} = \frac{2L}{1 + U^2/c^2}$$

At the speed $2U/(1 + U^2/c^2)$, this will take a time $L/U = T$.

No.20. The equations should first be corrected to read

$$\beta\bar{\underline{v}} = Q\left[\underline{v} + \underline{u}\left\{\frac{\underline{u}\cdot\underline{v}}{u^2}(\beta-1) - \beta\right\}\right] \qquad \text{(i)}$$

$$Q = 1/(1 - \underline{u}\cdot\underline{v}/c^2)$$

$$u^2\beta\bar{\underline{v}} = Q\left[(1-\beta)\underline{u}\times(\underline{v}\times\underline{u}) + \beta u^2(\underline{v}-\underline{u})\right] \qquad \text{(ii)}$$

$$\bar{v}^2 = Q^2\left[(\underline{v}-\underline{u})^2 - (\underline{v}\times\underline{u})^2/c^2\right] \qquad \text{(iii)}$$

In the independent $\mathcal{E}_3$, we can take $\underline{v} = (v_x, v_y, v_z)$, $\bar{\underline{v}} = (\bar{v}_x, \bar{v}_y, \bar{v}_z)$, $\underline{u} = (u,0,0)$. Then $Q = 1/(1 - uv_x/c^2)$. Hence, the first component of equation (i) gives $\bar{v}_x = Q(v_x - u)$, which is the first of equations (15.11). The second and third components of (i) give the remaining velocity transformation equations. This verifies equation (i).

Since $\underline{u}\times(\underline{v}\times\underline{u}) = u^2\underline{v} - (\underline{u}\cdot\underline{v})\underline{u}$, equation (ii) is equivalent to (i).

To obtain equation (iii), we square (ii), noting that

(a) $[\underline{u}\times(\underline{v}\times\underline{u})]^2 = u^2(\underline{v}\times\underline{u})^2$

(b) $\underline{u}\cdot[\underline{u}\times(\underline{v}\times\underline{u})] = 0$

(c) $\underline{v}\cdot[\underline{u}\times(\underline{v}\times\underline{u})] = (\underline{v}\times\underline{u})\cdot(\underline{v}\times\underline{u}) = (\underline{v}\times\underline{u})^2$

(d) $\beta^2 - 1 = u^2\beta^2/c^2$

No.21. In the $\bar{S}$-frame, a ray of light from a point on the edge of the disc to $\bar{O}$ makes an angle $\bar{\alpha}$ with the $\bar{x}$-axis, where $\tan\bar{\alpha} = a/\bar{x}$. In the S-frame, this ray will be observed to make an angle α with the x-axis, where

$$\cot\alpha = \frac{\cot\bar{\alpha} - (u/c)\operatorname{cosec}\bar{\alpha}}{(1 - u^2/c^2)^{\frac{1}{2}}}$$

(This equation follows from the result quoted in No.6 above by writing down the inverse transformation (i.e. exchange α, $\bar{\alpha}$ and replace u by -u) and then substituting $\pi+\alpha$, $\pi+\bar{\alpha}$ for α, $\bar{\alpha}$ respectively to allow for the reversal in the sense of the light ray.)

If $\bar{x} \gg a$, we can approximate $\operatorname{cosec}\bar{\alpha} = \cot\bar{\alpha} = \bar{x}/a$ and this then leads to the stated result.

No.22. By the velocity transformation equations (15.11)

$$\bar{v} = \frac{v - u}{1 - uv/c^2}$$

Differentiating with respect to $\bar{t}$, we get

$$\bar{a} = \frac{d\bar{v}}{d\bar{t}} = \frac{d\bar{v}}{dv}\frac{dv}{dt}\frac{dt}{d\bar{t}} = \frac{1 - u^2/c^2}{(1 - uv/c^2)^2}\, a\, \frac{dt}{d\bar{t}}$$

Differentiating the Lorentz transformation equation $\bar{t} = \beta(t - ux/c^2)$, gives $d\bar{t}/dt = \beta(1 - uv/c^2)$. The first result stated now follows immediately.

At some instant t in S, take $\bar{S}$ to be the frame in which the particle is instantaneously at rest. Then u = v and, hence, $\bar{a} = a/(1 - v^2/c^2)^{3/2}$. Thus, if $\bar{a} = \alpha$,

$$(1 - v^2/c^2)^{-3/2}\frac{dv}{dt} = \alpha$$

This is equivalent to the second stated result.

Integration of the equation of motion now proceeds as explained in the solution to No.15 earlier.

It is helpful to imagine the particle to be a space rocket accelerating away from the solar system. α will then be the acceleration experienced by the crew; thus, if $\alpha = g$, conditions in the ship will be similar to gravity on the surface of the earth.

No.23. In $\bar{S}$, $\bar{\bar{S}}$ has velocity (0,v,0). Using the inverse of the velocity transformation equations (15.11), the velocity of $\bar{\bar{S}}$ as seen from S is found to have components

$$v_x = u, \quad v_y = (1 - u^2/c^2)^{\frac{1}{2}}v, \quad v_z = 0$$

Hence,

$$\tan\theta = v_y/v_x = \frac{v}{u}(1 - u^2/c^2)^{\frac{1}{2}}.$$

The velocity transformation equations relating $\bar{S}$ and $\bar{\bar{S}}$ can be derived from equations (15.11) by exchanging the roles of the x- and y-axes and replacing u by v; they are

$$\bar{\bar{v}}_x = \frac{(1 - v^2/c^2)^{\frac{1}{2}}\bar{v}_x}{1 - v\bar{v}_y/c^2}, \quad \bar{\bar{v}}_y = \frac{\bar{v}_y - v}{1 - v\bar{v}_y/c^2}, \quad \bar{\bar{v}}_z = \frac{(1 - v^2/c^2)^{\frac{1}{2}}\bar{v}_z}{1 - v\bar{v}_y/c^2}.$$

In $\bar{S}$, S has velocity (-u,0,0). Substituting $\bar{v}_x = -u$, $\bar{v}_y = \bar{v}_z = 0$ in these transformation equations, we obtain

$$\bar{\bar{v}}_x = -(1 - v^2/c^2)^{\frac{1}{2}}u, \quad \bar{\bar{v}}_y = -v, \quad \bar{\bar{v}}_z = 0$$

for the components of the velocity of S as seen from $\bar{\bar{S}}$. Thus, if ϕ is the angle made by this velocity with the $\bar{\bar{x}}$-axis, then

$$\tan\phi = \bar{\bar{v}}_y/\bar{\bar{v}}_x = \frac{v}{u}(1 - v^2/c^2)^{-\frac{1}{2}}$$

If u and v are small compared with c, the binomial theorem shows that

$$\tan\theta = \frac{v}{u}(1 - u^2/2c^2), \quad \tan\phi = \frac{v}{u}(1 + v^2/2c^2)$$

approximately. Hence

$$\tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1+\tan\phi\,\tan\theta} = uv/2c^2$$

to the second order of small quantities.

No.24. $\ell = \bar{\ell}(1 - u^2/c^2)^{\frac{1}{2}}$ is derived as in section 6.

Let v be the velocity of S' as seen from S. Then -v is the velocity of S along the x'-axis of S'.

The velocity of $\bar{S}$ as seen from S' follows from the first of equations (15.11) to be $(u - v)/(1 - uv/c^2)$. This must equal v, thus leading to the equation

$$uv^2 - 2c^2v + c^2u = 0$$

Solving this quadratic for v and rejecting the root > c, we now find

$$v = \frac{c^2}{u}\left[1 - (1 - u^2/c^2)^{\frac{1}{2}}\right] \qquad (i)$$

Two applications of the Fitzgerald contraction formula yield the equations

$$\ell = \bar{\ell}(1 - u^2/c^2)^{\frac{1}{2}}, \qquad L = \bar{\ell}(1 - v^2/c^2)^{\frac{1}{2}}$$

Thus, equation (i) is equivalent to

$$(\bar{\ell}^2 - L^2)^{\frac{1}{2}}/\bar{\ell} = \frac{\bar{\ell}}{(\bar{\ell}^2 - \ell^2)^{\frac{1}{2}}}(1 - \ell/\bar{\ell})$$

Solving for L, the stated result follows.

No.25. It follows from the symmetry of the situation that the combined particle's direction of motion bisects the right angle between the velocities of the colliding particles. Thus, if v is the speed of the combined particle and M_0 is its rest mass, momentum is conserved provided

$$\frac{2m_0}{\sqrt{(1 - 1/2^2)}}\,\tfrac{1}{2}c\cos(\pi/4) = \frac{M_0 v}{\sqrt{(1 - v^2/c^2)}}$$

Since energy (or inertial mass) is also conserved, we must have

$$\frac{2m_0}{\sqrt{(1 - 1/2^2)}} = \frac{M_0}{\sqrt{(1 - v^2/c^2)}}$$

Dividing these equations, we find $v = c/2\sqrt{2}$. Thus $M_0 = \sqrt{(14/3)}m_0$.

No.26. Since momentum is conserved, the particle M must follow the same line of motion as m . Thus, if M has speed u, conservation of momentum requires that

$$\frac{m_1 v}{\sqrt{(1 - v^2/c^2)}} = \frac{Mu}{\sqrt{(1 - u^2/c^2)}} \qquad (i)$$

Conservation of inertial mass is expressed by the equation

$$\frac{m_1}{\sqrt{(1 - v^2/c^2)}} + m_2 = \frac{M}{\sqrt{(1 - u^2/c^2)}} \qquad (ii)$$

Division of equation (i) by (ii) gives a formula for u.

Dividing both sides of (i) by c, squaring and subtracting from the square of (ii), we obtain the equation

$$\frac{m_1^2(1 - v^2/c^2)}{1 - v^2/c^2} + \frac{2m_1m_2}{\sqrt{(1 - v^2/c^2)}} + m_2^2 = \frac{M^2(1 - u^2/c^2)}{1 - u^2/c^2}$$

This is equivalent to the result asked for.

Note that $M^2 > m_1^2 + m_2^2 + 2m_1m_2$ and, hence, $M > m_1 + m_2$. The increase in rest mass corresponds to the mass of the heat generated by the collision.

No.27. Choose an inertial frame $\bar{S}$ whose velocity relative to the laboratory frame S is w in a direction parallel to the line of motion of the first mentioned particle. Choose w so that in $\bar{S}$ the two particles have equal and opposite velocities before collision. Then, as in No.24 above, we can show that

$$w = \frac{c^2}{u}\{1 - (1 - u^2/c^2)^{\frac{1}{2}}\}$$

Then, in $\bar{S}$, both particles must have the same speed w.

Appealing to the symmetry of the particles' motions in $\bar{S}$, it is clear that after collision they will be moving apart with equal and opposite velocities along another line of motion. Since inertial mass is conserved, these velocities will still have magnitude w. Let α be the angle made by the new line of motion with the original line of motion (as measured in $\bar{S}$).

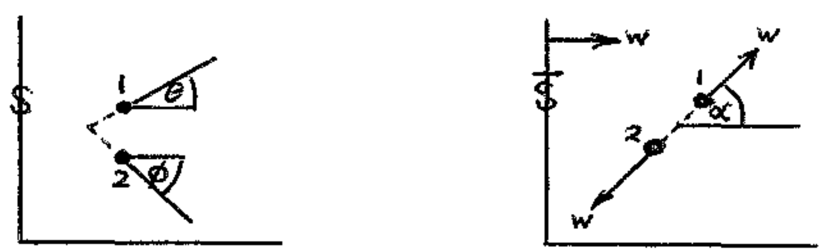
Choosing parallel axes in S and $\bar{S}$ in the usual way and assuming the motion occurs in the xy-plane and is along the x-axis before collision, after collision the first particle will have velocity components ($w\cos\alpha$, $w\sin\alpha$,0) in $\bar{S}$; hence,

$$\tan\theta = v_y/v_x = \frac{(1 - w^2/c^2)^{\frac{1}{2}} w\sin\alpha}{w\cos\alpha + w}$$

by use of the velocity transformation equations. The second particle has velocity components ($-w\cos\alpha$,$-w\sin\alpha$,0) in $\bar{S}$ after collision and it follows similarly that

$$\tan\phi = - v_y/v_x = \frac{(1 - w^2/c^2)^{\frac{1}{2}} w\sin\alpha}{- w\cos\alpha + w}$$

(assuming ϕ is measured from the line of motion in the opposite sense to θ).



Hence

$$\tan\theta\tan\phi = \frac{(1 - w^2/c^2)w^2\sin^2\alpha}{w^2(1 - \cos^2\alpha)} = 1 - w^2/c^2$$

$$= \frac{2}{\gamma + 1}$$

No.28. Since momentum is conserved, the two parts will have equal and opposite momenta of magnitude p. Using equation (19.1),

$$E_1 = c\sqrt{(p^2 + M_1^2c^2)}, \qquad E_2 = c\sqrt{(p^2 + M_2^2c^2)}$$

Thus

$$E_1^2 - E_2^2 = (M_1^2 - M_2^2)c^4 \qquad \text{(i)}$$

Energy is conserved and therefore

$$E_1 + E_2 = Mc^2 \qquad \text{(ii)}$$

Dividing (ii) into (i), we deduce

$$E_1 - E_2 = c^2(M_1^2 - M_2^2)/M \qquad \text{(iii)}$$

Equations (ii) and (iii) now yield the results required.

No.29. Take m_1 to be moving along the x-axis of S, m_2 to be moving along a line making an angle α with this axis and the combined particle m to move along a line making an angle β with the x-axis.

Since momentum is conserved, by resolving along and perpendicular to the x-axis, we get the equations

$$\frac{m_1u_1}{\sqrt{(1 - u_1^2/c^2)}} + \frac{m_2u_2}{\sqrt{(1 - u_2^2/c^2)}}\cos\alpha = \frac{mu}{\sqrt{(1 - u^2/c^2)}}\cos\beta$$

$$\frac{m_2u_2}{\sqrt{(1 - u_2^2/c^2)}}\sin\alpha = \frac{mu}{\sqrt{(1 - u^2/c^2)}}\sin\beta$$

where u is the speed of m. Squaring and adding, we find

$$\frac{m_1^2u_1^2}{1 - u_1^2/c^2} + \frac{m_2^2u_2^2}{1 - u_2^2/c^2} + \frac{2m_1m_2u_1u_2\cos\alpha}{\sqrt{\{(1 - u_1^2/c^2)(1 - u_2^2/c^2)\}}} = \frac{m^2u^2}{1 - u^2/c^2} \quad \text{(i)}$$

Since inertial mass is conserved,

$$\frac{m_1}{\sqrt{(1 - u_1^2/c^2)}} + \frac{m_2}{\sqrt{(1 - u_2^2/c^2)}} = \frac{m}{\sqrt{(1 - u^2/c^2)}}$$

which after squaring yields

$$\frac{m_1^2}{1 - u_1^2/c^2} + \frac{m_2^2}{1 - u_2^2/c^2} + \frac{2m_1m_2}{\sqrt{\{(1 - u_1^2/c^2)(1 - u_2^2/c^2)\}}} = \frac{m^2}{1 - u^2/c^2} \quad \text{(ii)}$$

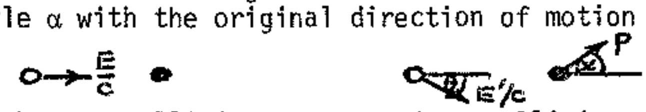
Multiplying (i) through by $1/c^2$ and subtracting from (ii), we now obtain the result required.

We can assume u_1, u_2 to be positive and therefore

$$(c^2 - u_1u_2\cos\alpha)^2 > (c^2 - u_1u_2)^2 > (c^2 - u_1^2)(c^2 - u_2^2)$$

the second inequality being justified since $u_1^2 + u_2^2 > 2u_1u_2$. It now follows that $m^2 > m_1^2 + m_2^2 + 2m_1m_2$, i.e. $m > m_1 + m_2$. The increase in rest mass is due to the inertia of the heat generated in the collision.

No.30. Let p be the momentum given to the electron and suppose its direction makes an angle α with the original direction of motion of the photon.



before collision after collision

Then, since momentum is conserved, by resolving along and perpendicular to the line of motion of the incident photon, we get

$$\frac{E}{c} + \frac{E'}{c}\cos\theta = p\cos\alpha$$

$$\frac{E'}{c}\sin\theta = p\sin\alpha$$

Squaring and adding these equations, we find

$$E^2 + E'^2 + 2EE'\cos\theta = c^2p^2$$

Since energy is conserved

$$E + m_0c^2 = E' + c\surd(p^2 + m_0^2c^2)$$

Elimination of p between the last two equations now gives the result stated.

It is well-known that the frequency ν of a photon is related to its energy E by Einstein's formula $E = h\nu$. Since $\lambda\nu = c$, we deduce that $\lambda = hc/E$ and the result just obtained is therefore equivalent to

$$\frac{m_0c}{h}(\lambda' - \lambda) = 1 - \cos\theta = 2\sin^2\tfrac{1}{2}\theta$$

Since $\Delta\lambda = \lambda' - \lambda$, this now yields the second result.

No.31. Let P be the original momentum of P and p, q, the momenta of P and Q respectively after the collision. Since momentum is conserved in the direction of the original line of motion of P and perpendicular to this direction, we must have

$$P = p\cos 30^0 + q\cos 30^0$$

$$p\sin 30^0 = q\sin 30^0$$

Hence, $P = \surd 3p$ and $p = q$.

Conservation of energy requires that

$$c\surd(P^2 + 4m_0^2c^2) + m_0c^2 = c\surd(p^2 + m_0^2c^2) + c\surd(q^2 + m_0^2c^2)$$

Solving these equations, we find that $p = q = \surd 15m_0c$, $P = 3\surd 5m_0c$.

Then, if v is the original velocity of P, $2m_0v/\surd(1 - v^2/c^2) = P = 3\surd 5m_0c$. Hence, $v = 3\surd 5c/7$.

No.32. Resolving momenta in directions perpendicular to the lines of motion of the photons and using the principle of momentum conservation, we derive the equations

$$\frac{m_0v}{\surd(1 - v^2/c^2)}\sin\alpha = \frac{E_2}{c}\sin(\alpha+\beta),$$

$$\frac{m_0v}{\surd(1 - v^2/c^2)}\sin\beta = \frac{E_1}{c}\sin(\alpha+\beta),$$

m_0 being the rest mass of the original particle. Conservation of energy requires that

$$\frac{m_0c^2}{\surd(1 - v^2/c^2)} = E_1 + E_2$$

Eliminating E_1, E_2, from these equations, we find that

$$\frac{c}{v} = \frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)} = \frac{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)}$$

$$= \frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta + \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta - \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta} = \frac{1 + \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}{1 - \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}$$

This is equivalent to the result stated.

In the special case when the original particle is a photon, $v = c$ and $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta = 0$. Only $\alpha = \beta = 0$ permits conservation of both energy and momentum in this case.

No.33. Since momentum must be conserved in a direction perpendicular to the neutrino's line of motion, the lines of motion of the particles m_0 must be equally inclined at an angle θ to the line of motion of the neutrino and all lines of motion must lie in a plane.

Then, if p is the momentum of each particle m_0 and E is the energy of the neutrino, momentum is conserved along the line of the neutrino's motion if

$$E/c = 2p\cos\theta = 2p/3.$$

For conservation of energy, we require

$$3m_0c^2 = E + 2c\sqrt{(p^2 + m_0^2c^2)}$$

Solving these equations for E and p, we find $p = 3m_0c/4$ and $E = \frac{1}{2}m_0c$.

If v is the speed of either of the particles m_0, then

$$\frac{m_0v}{\sqrt{(1 - v^2/c^2)}} = p = \tfrac{3}{4}m_0c$$

Hence $v = 3c/5$.

No.34. Let θ be the angle through which the line of motion of the particle is deflected and let p be its new momentum. All lines of motion must lie in a plane. The original momentum of the particle is $\frac{3}{4}m_0c$. Momentum is conserved along and perpendicular to the original line of motion provided

$$\tfrac{3}{4}m_0c = p\cos\theta + \tfrac{1}{4}m_0c\cos60^0, \qquad p\sin\theta = \tfrac{1}{4}m_0c\sin60^0$$

It follows that $p = \sqrt{7}m_0c/4$, $\tan\theta = \sqrt{3}/5$.

The original energy of the particle is $5m_0c^2/4$. If m_0' is the new rest mass of the particle, energy is conserved if

$$\frac{5}{4}m_0c^2 = \frac{1}{4}m_0c^2 + c\sqrt{(p^2 + m_0'^2c^2)}$$

Substituting for p and solving for m_0', we find $m_0' = \frac{3}{4}m_0$.

If v is the final speed of the particle,

$$\frac{m_0'v}{\sqrt{(1- v^2/c^2)}} = p = \frac{\sqrt{7}}{4}m_0c$$

Thus, $v = \sqrt{7}c/4$.

No.35. Let p, p' be the magnitudes of the momenta of the proton and electron respectively and let E be the energy of the neutrino. The equations of conservation of momentum and energy are

$$p = p' + E/c \qquad \text{(i)}$$

$$m_Nc^2 = E + (T + m_Pc^2) + (T' + m_Ec^2) \qquad \text{(ii)}$$

where T, T' are the kinetic energies of the proton and electron respectively. Thus

$$T + m_Pc^2 = c\sqrt{(p^2+ m_P^2c^2)} \qquad \text{(iii)}$$

$$T' + m_Ec^2 = c\sqrt{(p'^2 + m_E^2c^2)} \qquad \text{(iv)}$$

Eliminating E between (i) and (ii), we find

$$(m_N - m_p)c - \frac{T}{c} = p - p' + \frac{T'}{c} + m_E c \qquad (v)$$

Solving (iii) for p gives $p = \sqrt{(2m_p T + T^2/c^2)}$ and, hence, equation (v) is equivalent to

$$c(p' + k) = T' + m_E c^2$$

Using (iv), this can be written

$$p' + k = \sqrt{(p'^2 + m_E^2 c^2)}$$

Squaring and solving for p', we get

$$p' = (m_E^2 c^2 - k^2)/2k$$

Substitution for p' in (iv) now yields

$$T' = c(m_E c - k)^2/2k$$

No.36. Let 2θ be the angle between the photon tracks and let p be the momentum of the recoiling particle. Since each photon has momentum $\frac{1}{4}m_o c$, momentum is conserved provided

$$p = \tfrac{1}{2}m_o c \cos\theta$$

For energy to be conserved, it is necessary that

$$m_o c^2 = \tfrac{1}{2}m_o c^2 + c\sqrt{(p^2 + m_o^2 c^2/16)}$$

Hence, $p = \sqrt{3}m_o c/4$ and, thus, $\cos\theta = \sqrt{3}/2$, i.e. $\theta = 30^o$.

If v is the velocity of recoil,

$$\frac{\frac{1}{4}m_o v}{\sqrt{(1 - v^2/c^2)}} = p = \sqrt{3}m_o c/4$$

It follows that $v = \sqrt{3}c/2$.

Take S to be the laboratory frame and $\bar{S}$ the frame moving with the recoiling particle, such that the x- and $\bar{x}$-axes are in the opposite direction to the particle's motion. For one of the photons, $v_x = c\cos 30^o = \sqrt{3}c/2$, $\bar{v}_x = c\cos\alpha$. Hence, by equations (15.11) with $u = -\sqrt{3}c/2$,

$$c\cos\alpha = \frac{\sqrt{3}c}{1 + \frac{3}{4}}$$

i.e. $\cos\alpha = 4\sqrt{3}/7$. This is equivalent to $\sin\alpha = 1/7$.

No.37. The rest mass of the nucleus after emission of the photon is $M - E_o/c^2$ and the magnitude of its momentum must equal that of the photon, viz. E/c. For conservation of energy, it is necessary that

$$Mc^2 = E + c\left[\frac{E^2}{c^2} + \left(M - \frac{E_o}{c^2}\right)^2 c^2\right]^{\frac{1}{2}}$$

Solving for E, we get the result stated.

No.38. Let p be the momentum of m_o before it absorbs the photon and P be its momentum afterwards. Then, if α is the angle through which the line of motion of m_0 is deflected by the photon, resolving along the original lines of motion of particle and photon, we find that momentum and energy are conserved provided

$$p = P\cos\alpha, \quad \bar{E}/c = P\sin\alpha, \quad E + \bar{E} = c\sqrt{(P^2 + M_0^2c^2)}$$

Also, $E = c\sqrt{(p^2 + m_0^2c^2)}$.

Eliminating p, P, α and solving for M_0, we now get the result stated.

No.39. Let α be the angle made by the final line of motion of the particle with the x-axis and let p be the magnitude of its momentum. Momentum is conserved in the x and y directions provided

$$E/c = p\cos\alpha, \quad E/c = p\sin\alpha$$

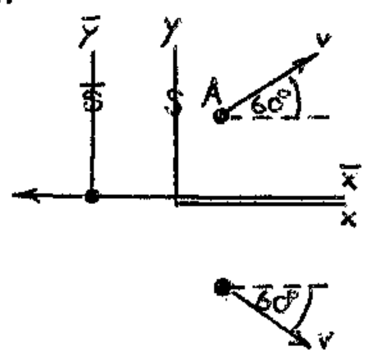
Thus, $\alpha = 45^{\circ}$ and $p = \sqrt{2}E/c$.

Conservation of energy requires that

$$E + m_0c^2 = E + c\sqrt{(p^2 + M_0^2c^2)},$$

where M_0 is the final rest mass of the particle. Solving for M_0, we now get the result stated.

No.40.



Let v be the speed of each fragment in the laboratory frame S (see figure). Momentum is clearly conserved. Inertial mass is conserved provided

$$3m_0/\sqrt{(1 - v^2/c^2)} = 3\lambda m_0 \qquad \text{(i)}$$

$\bar{S}$ is a frame moving with one of the fragments (see figure). The components of A's velocity in $\bar{S}$ are given by

$$\bar{v}_x = \frac{v\cos 60^{\circ} + v}{1 + v\cos 60^{\circ}/c^2} = \frac{3v/2}{1 + v^2/2c^2}$$

$$\bar{v}_y = \frac{\sqrt{(1 - v^2/c^2)}v\sin 60^{\circ}}{1 + v^2\cos 60^{\circ}/c^2} = \frac{\sqrt{(1 - v^2/c^2)}\sqrt{3}v/2}{1 + v^2/2c^2}$$

Thus, if α is the angle made by the line of motion of A with the $\bar{x}$-axis as seen from $\bar{S}$, then

$$\cot\alpha = \bar{v}_x/\bar{v}_y = \sqrt{3}/\sqrt{(1 - v^2/c^2)} = \sqrt{3}\lambda$$

by equation (i). This is the result asked for.

No.41. Let E_1, E_2 be the energies of the photons, E_2 referring to the photon whose motion is in the same direction as that of the positron. The energy and momentum of the positron are $5mc^2/4$ and $3mc/4$ respectively. Hence, energy and momentum are conserved provided

$$E_1 + E_2 = mc^2 + 5mc^2/4, \quad E_2 - E_1 = 3mc^2/4$$

These equations now give the values for E_1 and E_2 stated.

No.42. For momentum to be conserved along a line at right angles to the positron's motion, it is necessary that the photons should have equal momenta and energies; let E denote the common energy. Then, momentum is conserved along the line of the photon's motion if $p = 2E\cos\alpha/c$.

For conservation of energy, we require that

$$c\surd(p^2 + m^2c^2) + mc^2 = 2E$$

Eliminating E between these equations and solving for p, we find $p\sin\alpha\tan\alpha = 2mc$.

If $\alpha = 60^{\circ}$, then $p = 4mc/3$. Thus $v = 4c/5$ is the positron's velocity.

No.43. Let p be the momentum of the moving particle before collision and let q be its momentum afterwards. Let r be the momentum generated in the stationary particle and let ϕ be the angle made by its line of motion with the line of motion of the incident particle. Then, momentum is conserved provided

$$p = q\cos\theta + r\cos\phi, \quad 0 = q\sin\theta - r\sin\phi$$

Elimination of ϕ yields

$$r^2 = p^2 + q^2 - 2pq\cos\theta \qquad \text{(i)}$$

The KE of the incident particle before and after collision is given by

$$T = c\surd(p^2 + m_0^2c^2) - m_0c^2, \quad T' = c\surd(q^2 + m_0^2c^2) - m_0c^2$$

It follows from these equations that

$$p^2 = 2m_0T + T^2/c^2, \quad q^2 = 2m_0T' + T'^2/c^2 \qquad \text{(ii)}$$

For conservation of energy, we require that

$$T + m_0c^2 - T' = c\surd(r^2 + m_0^2c^2) \qquad \text{(iii)}$$

Squaring (iii) and eliminating p, q, r, by use of (i) and (ii), we get

$$T'(T + 2m_0c^2) = \surd\{TT'(T + 2m_0c^2)(T' + 2m_0c^2)\}\cos\theta$$

Cancelling the factor $\surd\{T'(T + 2m_0c^2)\}$, squaring and solving for T', we now obtain the result quoted.

No.44. Defining the momenta p, q, r, as in the previous exercise, equation (i) can be derived as before. Energy is conserved provided

$$E + m_2c^2 - E' = c\surd(r^2 + m_2^2c^2)$$

Squaring this equation and eliminating p, q, r, by equation (i) and the equations

$$c^2p^2 = E^2 - m_1^2c^4, \quad c^2q^2 = E'^2 - m_1^2c^4$$

we are led to the result stated.

Let θ be the angle asked for, E the energy of the neutrino and p the momentum of the muon. For conservation of momentum and energy, it is necessary that

$$4m_0c/3 = p\cos\theta, \quad E/c = p\sin\theta, \quad 5m_0c^2/3 = c\surd(p^2 + 4m_0^2c^2/9) + E$$

Thus, $p^2 = 16m_0^2c^2/9 + E^2/c^2$ and substitution in the last equation gives

$$5m_0c^2/3 - E = \surd(E^2 + 20m_0^2c^4/9)$$

Squaring and solving for E, we find $E = m_0c^2/6$. Hence, $\tan\theta = 3E/4m_0c^2 = 1/8$.

No.46. To conserve momentum, the fragments must have the same speed v in a laboratory frame S. To conserve inertial mass, it is required that

$$m_0 = \frac{\lambda m_0}{\surd(1 - v^2/c^2)}$$

This gives $v = c\surd(1 - \lambda^2)$.

Taking $\bar{S}$ to be a frame moving with one of the fragments and using the velocity transformation equations (15.11), we calculate the speed of the other fragment in $\bar{S}$ to be

$$\frac{2v}{1 + v^2/c^2} = \frac{2c\surd(1 - \lambda^2)}{2 - \lambda^2}$$

If λ is small, the binomial theorem permits us to approximate this speed by

$$c(1 - \tfrac{1}{2}\lambda^2 - \tfrac{1}{8}\lambda^4 + \ldots)(1 + \tfrac{1}{2}\lambda^2 + \tfrac{1}{4}\lambda^4 + \ldots)$$
$$= c(1 - \tfrac{1}{8}\lambda^4)$$

to $O(\lambda^4)$; i.e. speed is less than c by a fraction $\lambda^4/8$.

No.47. Let p be the momentum of the positron and E_1, E_2 the energies of the photons associated with the angles 30^0 and 90^0 respectively.

The equations of conservation of energy and momentum are:

$$c\surd(p^2 + m_0^2c^2) + m_0c^2 = E_1 + E_2,$$

$$p = E_1\cos 30^0/c, \quad E_2/c = E_1\sin 30^0/c.$$

Eliminating p and E_2, we find

$$\surd(\tfrac{3}{4}E_1^2 + m_0^2c^4) + m_0c^2 = \tfrac{3}{2}E_1$$

Thus, $E_1 = 2m_0c^2$. This gives $E_2 = m_0c^2$ and $p = \surd 3m_0c$. To have momentum $\surd 3m_0c$, the positron's velocity must be $\surd 3c/2$.

No.48. The photons must have the same energy E for momentum to be conserved. Then, if p is the positron's momentum, equations of conservation of energy and momentum are

$$c\surd(p^2 + m_0^2c^2) + m_0c^2 = 2E, \quad p = 2E\cos 60^0/c = E/c$$

Eliminating p and solving for E, we get $E = 4m_0c^2/3$.

Let α be the angle through which the nucleus velocity is deflected and let p be the momentum after deflection. Before emission of the photon, the nucleus has energy $5m_0c^2/3$ and momentum $4m_0c/3$. If μ_0 is its rest mass after the emission, energy and momentum are conserved provided

$$\frac{5}{3}m_0c^2 = c\sqrt{(p^2 + \mu_0^2c^2)} + \frac{1}{3}m_0c^2$$

$$\frac{4}{3}m_0c = \frac{1}{3}m_0c\cos 60^0 + p\cos\alpha$$

$$0 = \frac{1}{3}m_0c\sin 60^0 - p\sin\alpha$$

The last two equations show that $\tan\alpha = \sqrt{3}/7$ and $p = \sqrt{13}m_0c/3$. The first equation then gives $\mu_0 = m_0/\sqrt{3}$.

If the frame $\overline{S}$ moves with the nucleus before deflection, its $\overline{x}$-axis being in the direction of motion, let θ be the angle made by the photon's track with the $\overline{x}$-axis. S is the parallel laboratory frame. Then, for the photon, $v_x = c\cos 60^0 = \frac{1}{2}c$, $\overline{v}_x = c\cos\theta$. Since $u = 4c/5$ is the velocity of $\overline{S}$ relative to S,

$$c\cos\theta = \frac{\frac{1}{2}c - \frac{4}{5}c}{1 - \frac{2}{5}} = -\frac{1}{2}c$$

Hence, $\theta = 120^0$.

No.50. Equations of conservation of energy and momentum are:

$$c\sqrt{(p^2 + m_0^2c^2)} = E + E', \quad p\cos\alpha = E/c, \quad p\sin\alpha = E'/c$$

Thus,

$$\sqrt{(p^2 + m_0^2c^2)} = p(\cos\alpha + \sin\alpha)$$

Squaring and solving for p, we get the result stated. The results for E and E' then follow immediately.

No.51. If p is the original and P is the final momentum of the particle, equations of conservation of energy and momentum are:

$$c\sqrt{(p^2 + m_0^2c^2)} + \frac{3}{2}m_0c^2 = c\sqrt{(P^2 + 4m_0^2c^2)}$$

$$p + \frac{3}{2}m_0c\cos\alpha = P\cos\beta, \quad \frac{3}{2}m_0c\sin\alpha = P\sin\beta.$$

Squaring and adding the last pair of equations, we get

$$P^2 = p^2 + m_0cp + \frac{9}{4}m_0^2c^2$$

Substituting for P^2 in the first equation and solving for p, we obtain $p = \frac{3}{4}m_0c$. Thus, the original velocity of the particle was 3c/5. The value of $\tan\beta$ now follows from the above equations to be $4\sqrt{2}/5$.

No.52. The work done by the force as the particle moves from 0 to a point with coordinate x is

$$\int_0^x \frac{2m_0c^2a}{(a - x)^2}dx = \frac{2m_0c^2x}{a - x}$$

Equating this to the increase in the particle's energy, we obtain

$$mc^2 - m_0c^2 = 2m_0c^2x/(a - x)$$

where $m = m_0/\surd(1 - v^2/c^2)$ is the particle's inertial mass and v is its velocity. Solving for v, we get

$$v = \frac{dx}{dt} = \frac{2c(ax)^{\frac{1}{2}}}{a + x}$$

It now follows that

$$2a^{\frac{1}{2}}ct = \int(ax^{-\frac{1}{2}} + x^{\frac{1}{2}})dx = 2ax^{\frac{1}{2}} + \frac{2}{3}x^{3/2}$$

the constant of integration being zero since t = 0 when x = 0. This is equivalent to the result stated.

No.53. Resolving the equation of motion (17.1) along the x- and y-axes, we get

$$dp_x/dt = 0, \quad dp_y/dt = f$$

Integrating under the initial conditions $p_x = p_0$, $p_y = 0$ at t = 0, these give

$$p_x = \frac{m_0v_x}{\surd(1 - v^2/c^2)} = p_0, \quad p_y = \frac{m_0v_y}{\surd(1 - v^2/c^2)} = ft \qquad \text{(i)}$$

Squaring and adding, these lead to

$$\frac{m_0^2v^2}{1 - v^2/c^2} = \frac{m_0^2c^2}{1 - v^2/c^2} - m_0^2c^2 = p_0^2 + f^2t^2$$

which is equivalent to the equation

$$\frac{m_0c^2}{\surd(1 - v^2/c^2)} = \surd(w_0^2 + f^2c^2t^2)$$

Equations (i) can now be rewritten in the form

$$v_x = \frac{dx}{dt} = \frac{p_0c^2}{\surd(w_0^2 + f^2c^2t^2)}, \quad v_y = \frac{dy}{dt} = \frac{fc^2t}{\surd(w_0^2 + f^2c^2t^2)}$$

Integrating under the initial conditions x = y = 0 at t = 0, we now get

$$x = \frac{p_0c}{f}\sinh^{-1}(fct/w_0), \quad y = \frac{1}{f}\{\surd(w_0^2 + f^2c^2t^2) - w_0\}$$

Elimination of t between these equations now gives the result stated.

No.54. The particle's equation of motion can be written

$$\frac{d}{dt}\left[\frac{m_0v}{\surd(1 - v^2/c^2)}\right] = m_0(1 - v^2/c^2)^{-3/2}\frac{dv}{dt} = -m_0v/k$$

or

$$\frac{dv}{v(1 - v^2/c^2)^{3/2}} = -dt/k$$

Integration (change the variable by $1 - v^2/c^2 = u^2$) under the initial condition v = 4c/5 at t = 0 now yields

$$\frac{t}{k} = \frac{1}{2}\log\left[\frac{1 + \sqrt{(1 - v^2/c^2)}}{1 - \sqrt{(1 - v^2/c^2)}}\right] - \log 2 - (1 - v^2/c^2)^{-\frac{1}{2}} + \frac{5}{3}$$

Putting $v = 3c/5$ into this equation, gives the result stated.

No.55. In moving from the point $x = 2$ to a point with coordinate x, the applied force does work $m_0c^2(2/x - 1)$. Hence, the equation of work is

$$m_0c^2(1 - v^2/c^2)^{-\frac{1}{2}} - m_0c^2 = m_0c^2(2/x - 1)$$

Solving for v^2, we obtain

$$v^2 = \tfrac{1}{4}c^2(4 - x^2)$$

This is the equation for simple harmonic motion of amplitude 2 and period $4\pi/c$. Since the particle's inertial mass becomes infinite at $x = 0$, a complete oscillation could not, in practice, be accomplished.

No.56. Using the equation of motion derived in No.22 above, we obtain

$$\frac{d}{dt}(\beta v) = -\alpha v^2$$

where $\beta = (1 - v^2/c^2)^{-\frac{1}{2}}$. If y is the distance moved by the ship, then $dy/dt = v$ and the equation of motion can be written in the form

$$\frac{d}{dy}(\beta v) = -\alpha v, \qquad \text{or} \qquad \frac{dv}{v(1 - v^2/c^2)^{3/2}} = -\alpha dy$$

Putting $x = (1 - v^2/c^2)^{\frac{1}{2}}$, the last equation transforms to

$$\frac{dx}{x^2(1 - x^2)} = \alpha dy$$

Integration now yields the result stated.

No.57. Since $\underline{P}$ and $\underline{V}$ are 4-vectors, all the expressions quoted are invariants and have the same values in all inertial frames.

Take $\overline{S}$ to be an inertial frame in which the observer is at rest. Then $\underline{\overline{V}} = (0,0,0,ic)$ and, hence,

$$-\underline{P}\cdot\underline{V} = -\underline{\overline{P}}\cdot\underline{\overline{V}} = -ic\overline{P}_4 = \overline{E}$$

since $\overline{P}_4 = i\overline{E}/c$, where $\overline{E}$ is the energy of the particle in $\overline{S}$. This proves (i).

Since $\underline{\overline{P}} = (\underline{\overline{p}}, i\overline{E}/c)$, then $\underline{\overline{P}}^2 = \overline{p}^2 - \overline{E}^2/c^2$. It now follows that

$$\sqrt{\{\underline{P}^2 + (\underline{P}\cdot\underline{V})^2/c^2\}} = \overline{p}$$

proving (ii).

Also,

$$c\sqrt{\{1 + c^2\underline{P}^2/(\underline{P}\cdot\underline{V})^2\}} = c\sqrt{\{1 + (c^2\overline{p}^2 - \overline{E}^2)/\overline{E}^2\}} = c^2\overline{p}/\overline{E}$$

But $\overline{p} = \overline{m}\overline{v}$ and $\overline{E} = \overline{m}c^2$, so this expression finally reduces to $\overline{v}$, proving (iii).

No.58. The work done by the force as the particle moves from 0 to a point x is

$$-\int_0^x \frac{m_0c^3\omega^2 x\,dx}{(c^2 - \omega^2a^2 + \omega^2x^2)^{3/2}} = m_0c^3\{(c^2 - \omega^2a^2 + \omega^2x^2)^{-\frac{1}{2}} - (c^2 - \omega^2a^2)^{-\frac{1}{2}}\}$$

Equating this to the energy increase, viz.

$$\frac{m_0 c^2}{\sqrt{(1 - v^2/c^2)}} - \frac{m_0 c^2}{\sqrt{(1 - \omega^2 a^2/c^2)}}$$

and solving for v^2, we derive the equation $v^2 = \omega^2(a^2 - x^2)$. This indicates that the particle's motion is simple harmonic with centre O and amplitude a.

No.59. As the particle moves from O to x, the force does work

$$\int_0^x \frac{m_0 c^2 dx}{2(1 + 2x^{1/2})} = m_0 c^2 \left[\frac{1 + x^{\frac{1}{2}}}{(1 + 2x^{\frac{1}{2}})^{\frac{1}{2}}} - 1\right]$$

(Change the variable by $u = 1 + 2x^{\frac{1}{2}}$.) Equating this to the energy increase, viz.

$$m_0 c^2 (1 - v^2/c^2)^{-\frac{1}{2}} - m_0 c^2$$

and solving for the velocity v, we obtain

$$v = \frac{dx}{dt} = \frac{cx^{\frac{1}{2}}}{1 + x^{\frac{1}{2}}}$$

Taking the reciprocal of both sides of this equation and integrating with respect to x, we then get

$$ct = 2x^{\frac{1}{2}} + x$$

the integration constant being zero since $x = 0$ at $t = 0$. This is a quadratic equation for $x^{\frac{1}{2}}$ whose positive root is

$$x^{\frac{1}{2}} = \sqrt{(1 + ct)} - 1$$

Squaring, we find the equation stated.

No.60. If v is the particle's speed, since $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$, equation (20.4) shows that the Lagrangian for the system is

$$L = - m_0 c^2 \{1 - (\dot{r}^2 + r^2\dot{\theta}^2)/c^2\}^{\frac{1}{2}} - V$$

(r,θ) are the generalised coordinates of the particle and $(\dot{r},\dot{\theta})$ are the generalised components of its velocity. Thus, its Lagrange equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Substituting for L, we obtain the equations stated.

Since the energy $E = m_0 c^2 (1 - v^2/c^2)^{-\frac{1}{2}} = m_0 c^2 \gamma$, there is an energy integral

$$m_0 c^2 \gamma + V = C \qquad \text{(i)}$$

Also, integrating the second Lagrange equation, we get

$$\gamma r^2 \dot{\theta} = h \qquad \text{(ii)}$$

Thus, using (ii),

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{h}{\gamma r^2}\frac{dr}{d\theta} = -\frac{h}{\gamma}\frac{du}{d\theta}$$

where $u = 1/r$. Hence

$$\gamma = \left[1 - \frac{h^2}{c^2\gamma^2}\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\}\right]^{-\frac{1}{2}}$$

and it follows that

$$h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = c^2(\gamma^2 - 1) = \frac{(C - V)^2}{m_0^2c^2} - c^2$$

making use of equation (i).

Differentiating the last equation with respect to θ and cancelling a factor $2du/d\theta$, we now get the second equation stated. (Note: $V' = dV/dr$.)

If $V = -\mu/r$, the equation determining the orbit takes the form

$$\frac{d^2u}{d\theta^2} + \eta^2 u = \mu C/m_0^2h^2c^2$$

By appropriate choice of the line $\theta = 0$, the general solution of this equation can always be expressed in the form

$$u = A\cos\eta\theta + \mu C/(m_0^2h^2c^2\eta^2)$$

where A is a constant of integration. This last equation can then be written

$$\ell u = 1 + e\cos\eta\theta$$

where $\ell = m_0^2h^2c^2\eta^2/\mu C$, $e = A\ell$.

If $\eta = 1$, the last equation is the polar equation of an ellipse with eccentricity e and semi-latus rectum ℓ. If η is just less than unity, the orbit will be approximately an ellipse, but will not 'close' exactly, i.e. the periapse at $\theta = 2\pi/\eta$ will not coincide with the periapse at $\theta = 0$. Since $2\pi/\eta = 2\pi + \pi\mu^2/(m_0^2h^2c^2)$ approximately, the major axis for a second revolution of the orbit will have rotated through an angle $\pi\mu^2/(m_0^2h^2c^2)$ relative to the major axis for the first revolution.

It is shown in texts devoted to Newtonian mechanics, that the orbit for a particle moving under a central force of attraction f per unit mass is determined by the equation

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = f$$

It follows that the special relativistic equation is the same as the classical equation if we take $f = (C-V)V'/m_0^2c^2$. In the inverse square law case, $V = -\mu/r$, this gives $f = \alpha/r^2 + \beta/r^3$, where α, β are positive constants, i.e. the relativistic effect is to supplement the inverse square law attraction by a small inverse cube perturbing attraction.

In section 53, it is shown that general relativity theory leads, in similar circumstances, to a supplementary inverse fourth power term. This, also, causes the elliptical orbit to rotate, but through an angle $6\pi\mu^2/m_0^2h^2c^2$ per revolution, i.e. six times the special relativistic effect. Needless to say, it is the general relativistic effect which is confirmed by observation of Mercury's orbit.

No.61. The equation of work is

$$\frac{d}{dt}\left[\frac{m_0c^2}{\sqrt{(1 - v^2/c^2)}}\right] = fv = m_0kc^2$$

Integration under the initial condition $v = 0$ at $t = 0$ yields

$$(1 - v^2/c^2)^{-\frac{1}{2}} = 1 + kt = \sec\theta$$

Whence, $v = c\sin\theta$.

Differentiating $\sec\theta = 1 + kt$, we find

$$\sec\theta \tan\theta \, \dot{\theta} = k$$

Thus,

$$c\sin\theta = v = \frac{dx}{dt} = \frac{dx}{d\theta}\dot{\theta} = k\cos\theta \cot\theta \frac{dx}{d\theta}$$

This leads to the equation

$$dx/d\theta = \frac{c}{k}\tan^2\theta = \frac{c}{k}(\sec^2\theta - 1)$$

Integrating this equation under the initial condition $x = 0$ at $\theta = 0$ (i.e. $t = 0$), we obtain the equation quoted.

No.62. The first result follows from differentiation of the identity $\underline{v}\cdot\underline{v} = v^2$.

Also, differentiating the product $\beta\underline{v}$, we get

$$\frac{d}{dt}(\beta\underline{v}) = \frac{1}{c^2}\beta^3 v\dot{v}\underline{v} + \beta\dot{\underline{v}}$$

and, hence,

$$\underline{v}\cdot\frac{d}{dt}(\beta\underline{v}) = \frac{1}{c^2}\beta^3 v^3\dot{v} + \beta\underline{v}\cdot\dot{\underline{v}} = \beta^3 v\dot{v}$$

using $\underline{v}\cdot\dot{\underline{v}} = v\dot{v}$.

Multiplying this identity by m_0 and using $m_0\beta = m$, we obtain

$$\underline{v}\cdot\frac{d}{dt}(m\underline{v}) = m_0\beta^3 v\dot{v} = \dot{m}c^2$$

Since $\underline{f} = d(m\underline{v})/dt$, this is the equation of work.

No.63. The work done by the attractive force as the particle moves from $x = a$ to the origin is

$$-\int_a^0 m_0\omega^2 x \, dx = \tfrac{1}{2}m_0\omega^2 a^2$$

Equating this to the increase in the particle's kinetic energy, we are led to the equation

$$m_0c^2(1 - v^2/c^2)^{-\frac{1}{2}} - m_0c^2 = \tfrac{1}{2}m_0\omega^2 a^2,$$

where v is the speed of arrival at O. Solving for v, we get the result stated.

No.64. If $\underline{f}$ is directed along a normal, $\underline{f}$ and $\underline{v}$ will be perpendicular and, hence, $\underline{f}\cdot\underline{v} = 0$. From the equation of work we deduce that $\dot{m} = 0$ and, therefore, v is constant.

The equation of motion of the particle is $d(m\underline{v})/dt = \underline{f}$ and, since m is constant, this is equivalent to

$$m\frac{d\underline{v}}{dt} = \underline{f}$$

Resolving this equation along the normal (we are assuming plane motion), we get $m\kappa v^2 = f$ as stated.

In the case of circular motion, $\kappa = 1/a$ and the equation of motion gives

$$\frac{m_0 v^2}{\sqrt{(1 - v^2/c^2)}} = fa$$

Solving for v^2, we obtain the result stated.

No.65. Take S to be the stationary frame and $\bar{S}$ a parallel frame moving with the nucleus. Suppose the nucleus moves along the x-axis of S with speed u and that the electron moves in the xy-plane. Then, the velocity components of the electron in S are $v_x = \frac{1}{2}c \cos 60^\circ = \frac{1}{4}c$, $v_y = \frac{1}{2}c \sin 60^\circ = \sqrt{3}c/4$. Hence, in $\bar{S}$, the electron has velocity components

$$\bar{v}_x = \frac{c - 4u}{4c - u}c, \qquad \bar{v}_y = \frac{\sqrt{(1 - u^2/c^2)}\sqrt{3}c}{4c - u}c$$

Since we are given $\bar{v}_x^2 + \bar{v}_y^2 = \frac{1}{4}c^2$, an equation for u can now be constructed and its solution proves to be $u = 8c/17$.

No.66. The work done by f as the particle moves from 0 to a point x is

$$\tfrac{1}{2}m_0 c^2 \int_0^x \frac{dx}{\sqrt{(1 + x)}} = m_0 c^2\{\sqrt{(1 + x)} - 1\}$$

Equating this to the increase in kinetic energy, we get

$$\frac{m_0 c^2}{\sqrt{(1 - v^2/c^2)}} - m_0 c^2 = m_0 c^2\{\sqrt{(1 + x)} - 1\}$$

Solving for v, we find

$$v = \frac{dx}{dt} = \frac{x^{\frac{1}{2}}}{(1 + x)^{\frac{1}{2}}}c$$

An integration then yields

$$ct = \int x^{-\frac{1}{2}}(1 + x)^{\frac{1}{2}}dx = \int (1 + \cosh 2\theta)d\theta$$

$$= \theta + \tfrac{1}{2}\sinh 2\theta$$

since $t = 0$ when $\theta = 0$ (i.e. $x = 0$). This is equivalent to the result stated.

No.67. The particle's equation of motion is

$$\frac{d}{dt}\left[\frac{m_0 v}{\sqrt{(1 - v^2/c^2)}}\right] = -\frac{\alpha m_0^2}{1 - v^2/c^2}$$

which is equivalent to the equation

$$(1 - v^2/c^2)^{-\frac{1}{2}}dv = -\alpha m_0 dt$$

Integrating over the interval from $v = \frac{1}{2}c$ to $v = 0$, we calculate for the time taken

$$\alpha m_0 t = \pi c/6$$

No.68. In an inertial frame in which the fluid at the point under consideration is at rest, $\underline{V} = (0, ic)$ and equations (22.19) reduce to

$T_{\alpha\beta} = \tau_{\alpha\beta}$, $T_{\alpha 4} = T_{4\alpha} = 0$, $T_{44} = -c^2\mu_{oo}$. It is now easy to verify, by direct substitution, that the equation $T_{ij}V_j = -c^2\mu_{oo}V_i$ is valid in this frame at the point. Being a tensor equation, this is sufficient to establish its validity in all inertial frames.

Expressing the equation $T_{\alpha j}V_j = -c^2\mu_{oo}V_\alpha$ in terms of three-dimensional quantities using equations (15.6) and (22.19), after cancellation of a factor $(1 - v^2/c^2)^{-1/2}$ we get

$$(\mu v_\alpha v_\beta + \tau_{\alpha\beta} + \frac{1}{c^2}\tau_{\gamma\alpha}v_\gamma v_\beta)v_\beta + ic(\mu v_\alpha + \frac{1}{c^2}v_\beta\tau_{\beta\alpha})ic = -c^2\mu_{oo}v_\alpha$$

Since $v_\beta v_\beta = v^2$, this last equation is equivalent to

$$(1 - v^2/c^2)(\mu v_\alpha + \frac{1}{c^2}v_\beta\tau_{\beta\alpha}) = \mu_{oo}v_\alpha + \frac{1}{c^2}\tau_{\alpha\beta}v_\beta$$

Equation (22.16) now leads to the first result required.

Similarly, the equation $T_{4j}V_j = -c^2\mu_{oo}V_4$ can be written

$$g_\alpha v_\alpha - c^2\mu = -c^2\mu_{oo}$$

giving the second result.

The stated formula for $T_{\alpha\beta}$ is equivalent to

$$T_{\alpha\beta} = (1 - v^2/c^2)^{-1}\mu_{oo}v_\alpha v_\beta + \tau_{\alpha\beta} + \frac{1}{c^2}(1 - v^2/c^2)^{-1}\tau_{\alpha\gamma}v_\gamma v_\beta$$
$$= g_\alpha v_\beta + \tau_{\alpha\beta}$$

having used the equation for g_α just derived. This is the first of equations (22.19).

The stated formula for $T_{\alpha 4}$ may be written

$$T_{\alpha 4} = (1 - v^2/c^2)^{-1}\mu_{oo}v_\alpha ic + (1 - v^2/c^2)^{-1}\tau_{\alpha\beta}v_\beta ic/c^2$$
$$= icg_\alpha$$

which is the second of equations (22.19).

Finally, the stated formula for T_{44} implies

$$T_{44} = (1 - v^2/c^2)^{-1}(-c^2\mu_{oo} - \frac{1}{c^2}\tau_{\alpha\beta}v_\alpha v_\beta)$$

But, equation (22.16) shows that

$$\frac{1}{c^2}\tau_{\beta\alpha}v_\beta v_\alpha = g_\alpha v_\alpha - \mu v^2 = c^2(\mu - \mu_{oo}) - \mu v^2$$

Hence, $T_{44} = -c^2\mu$, which is the third of equations (22.19).

Thus, the stated equations are equivalent to equations (22.19).

No.69. The components of the energy-momentum tensor for an ideal fluid are given by equation (22.21). Taking axes through O, since the flow is radial, the components of the 3-velocity flow vector are given by $v_\alpha = vx_\alpha/r$. Thus, the first of equations (21.20) can be written

$$\frac{\partial}{\partial x_1}\{(\mu_{oo} + p/c^2)(1 - v^2/c^2)^{-1}v^2x_1^2/r^2 + p\} + \frac{\partial}{\partial x_2}\{(\mu_{oo} + p/c^2)(1 - v^2/c^2)^{-1}v^2x_1x_2/r^2\}$$
$$+ \frac{\partial}{\partial x_3}\{(\mu_{oo} + p/c^2)(1 - v^2/c^2)^{-1}v^2x_1x_3/r^2\} = 0$$

Since μ_{oo}, p and v are functions of r alone and $\partial r/\partial x_1 = x_1/r$, etc., this equation is equivalent to

$$r^3\frac{d}{dr}\{(\mu_{oo} + p/c^2)(1 - v^2/c^2)^{-1}v^2/r^2\} + r\frac{dp}{dr} + 4(\mu_{oo} + p/c^2)(1 - v^2/c^2)^{-1}v^2 = 0$$

having cancelled a factor x_1/r^2. The cases i = 2,3 of equation (21.20) lead to the same result.

Introducing λ as defined, the last equation can be written

$$r^3\frac{d}{dr}(\lambda v/r) + r\frac{dp}{dr} + 4\lambda vr = 0$$

which is equivalent to the first of the stated equations.

Taking i = 4 in equation (21.20), we find

$$r^2\frac{d}{dr}\{(\mu_{oo} + p/c^2)(1 - v^2/c^2)^{-1}v/r\} + 3(\mu_{oo} + p/c^2)(1 - v^2/c^2)^{-1}v = 0$$

or

$$r\frac{d\lambda}{dr} + 3\lambda = 0$$

The last equation is easily solved to give

$$\lambda r^3 = \text{constant} \qquad \text{(i)}$$

It follows from the other equation of motion that

$$\frac{dp}{dv} = -\lambda r = -\frac{(\mu_{oo}c^2 + p)v}{c^2 - v^2}$$

This equation is also readibly integrable and leads to

$$\mu_{oo}c^2 + p = A\sqrt{(c^2 - v^2)}, \qquad \text{(ii)}$$

A being a constant of integration. Thus,

$$\lambda = \frac{Acv}{r\sqrt{(c^2 - v^2)}}$$

and it follows from (i) that

$$\frac{v}{\sqrt{(c^2 - v^2)}} = \frac{B}{r^2} \qquad \text{(iii)}$$

where B is constant. Clearly, $v \to 0$ as $r \to \infty$.

The stated boundary conditions inserted ito equations (ii) and (iii) now permit A and B to be calculated and the results stated obtained.

No.70. Since the rod is at rest in the first frame S, equations (22.19) give

$$(T_{\alpha\beta}) = (\tau_{\alpha\beta}) = \begin{bmatrix} -F/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_{\alpha 4} = 0, \quad T_{44} = -mc^2/A$$

This verifies that (T_{ij}) is a diagonal matrix with the diagonal elements stated.

If $\overline{S}$ is a parallel frame moving with the observer, the Minkowski coordinates in the two frames are related by the orthogonal transformation $\overline{x}_i = a_{ij}x_j$, where

$$(a_{ij}) = \begin{bmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{bmatrix}$$

and $\tan\alpha = iu/c$. Thus,

$$\overline{T}_{44} = a_{4i}a_{4j}T_{ij} = \sin^2\alpha(-F/A) + \cos^2\alpha(-mc^2/A)$$

$$= \frac{u^2F/c^2 - mc^2}{A(1 - u^2/c^2)}$$

Suppose $\overline{m}$ is the inertial mass per unit length of the rod as observed in $\overline{S}$. The cross-section in $\overline{S}$ will still be A, since the Fitzgerald contraction is longitudinal only. Hence, the mass density in $\overline{S}$ is $\overline{m}/A$ and, thus, $\overline{T}_{44} = -c^2\overline{m}/A$. It follows that

$$\overline{m} = \frac{m - u^2F/c^4}{1 - u^2/c^2}$$

$\overline{m}$ must be positive and this requires $F < mc^4/u^2$. Since this inequality must be valid for all possible values of u, i.e. $|u| < c$, we deduce that $F < mc^2$.

No.71. The special Lorentz transformation referred to is given by the equations

$$\overline{x}_\alpha = a_{\alpha\beta}x_\beta + b_\alpha, \quad \overline{t} = t + b_4.$$

The first of these equations corresponds to a rotation of the frame $Ox_1x_2x_3$ and a shift of origin, without relative motion; the second confirms that the clocks fixed in the two frames advance at the same rate.

With respect to this special Lorentz transformation,

$$\overline{T}_{\alpha\beta} = a_{\alpha i}a_{\beta j}T_{ij} = a_{\alpha\gamma}a_{\beta\delta}T_{\gamma\delta}$$

$$\overline{T}_{\alpha 4} = a_{\alpha i}a_{4j}T_{ij} = a_{\alpha\beta}T_{\beta 4}$$

$$\overline{T}_{44} = a_{4i}a_{4j}T_{ij} = T_{44}$$

Equations (21.14) show that these equations are equivalent to

$$\overline{g}_{\alpha\beta} = a_{\alpha\gamma}a_{\beta\delta}g_{\gamma\delta}, \quad \overline{g}_\alpha = a_{\alpha\beta}g_\beta, \quad \overline{\mu} = \mu,$$

proving that $g_{\alpha\beta}$, g_α and μ transform as 3-tensors with respect to a simple rotation of axes.

No.72. In S^o, the fluid is stationary and the components of the energy-momentum tensor are given by equations (22.19) to be

$$T^o_{\alpha\beta} = \tau^o_{\alpha\beta}, \quad T^o_{\alpha 4} = T^o_{4\alpha} = 0, \quad T^o_{44} = -c^2\mu_{oo}$$

Since T^o_{ij} is symmetric, these equations show that $\tau^o_{\alpha\beta}$ is symmetric.

Minkowski coordinates in the frames S and S^o are related by the transformation equation $x_i = a_{ij}x^o_j$, where $a_{11} = a_{44} = \cos\alpha$, $a_{41} = -a_{14} = \sin\alpha$, $a_{22} = a_{33} = 1$ and $\tan\alpha = iu/c$ (non-listed a_{ij} all vanish). Thus, the components of the energy-momentum tensor in S are given by

$$T_{11} = a_{1i}a_{1j}T^o_{ij} = T^o_{11}\cos^2\alpha + T^o_{44}\sin^2\alpha$$

$$= \frac{\tau^o_{11} + \mu_{oo}u^2}{1 - u^2/c^2}$$

$$T_{12} = a_{1i}a_{2j}T^o_{ij} = T^o_{12}\cos\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}\tau^o_{12}$$

$$T_{13} = a_{1i}a_{3j}T^o_{ij} = T^o_{13}\cos\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}\tau^o_{13}$$

$$T_{14} = a_{1i}a_{4j}T^o_{ij} = (T^o_{11} - T^o_{44})\sin\alpha\cos\alpha = \frac{\tau^o_{11} + c^2\mu_{oo}}{1 - u^2/c^2}\cdot\frac{iu}{c}$$

$$T_{21} = a_{2i}a_{1j}T^o_{ij} = T^o_{21}\cos\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}\tau^o_{21}$$

$$T_{22} = a_{2i}a_{2j}T^o_{ij} = T^o_{22} = \tau^o_{22}$$

$$T_{23} = a_{2i}a_{3j}T^o_{ij} = T^o_{23} = \tau^o_{23}$$

$$T_{24} = a_{2i}a_{4j}T^o_{ij} = T^o_{21}\sin\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}\tau^o_{21}iu/c$$

$$T_{31} = a_{3i}a_{1j}T^o_{ij} = T^o_{31}\cos\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}\tau^o_{31}$$

$$T_{32} = a_{3i}a_{2j}T^o_{ij} = T^o_{32} = \tau^o_{32}$$

$$T_{33} = a_{3i}a_{3j}T^o_{ij} = T^o_{33} = \tau^o_{33}$$

$$T_{34} = a_{3i}a_{4j}T^o_{ij} = T^o_{31}\sin\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}\tau^o_{31}iu/c$$

$$T_{44} = a_{4i}a_{4j}T^o_{ij} = T^o_{11}\sin^2\alpha + T^o_{44}\cos^2\alpha$$

$$= -\frac{\tau^o_{11}u^2/c^2 + c^2\mu_{oo}}{1 - u^2/c^2}$$

Since T_{ij} is symmetric, the expressions for T_{41}, T_{42}, T_{43} are identical with those for T_{14}, T_{24}, T_{34}.

In S, the components T_{ij} are given by equations (22.19) to be

$$T_{11} = \mu u^2 + (1 + u^2/c^2)\tau_{11}, \quad T_{12} = \tau_{12},$$

$$T_{13} = \tau_{13}, \quad T_{14} = icu(\mu + \tau_{11}/c^2)$$

$$T_{21} = \tau_{21} + u^2\tau_{12}/c^2, \quad T_{22} = \tau_{22}, \quad T_{33} = \tau_{33},$$

$$T_{24} = iu\tau_{12}/c, \quad T_{31} = \tau_{31} + u^2\tau_{13}/c^2,$$

$$T_{32} = \tau_{32}, \quad T_{33} = \tau_{33}, \quad T_{34} = iu\tau_{13}/c,$$

$$T_{44} = - c^2\mu$$

Comparing these expressions with those already found, the remaining stated equations all follow without difficulty.

EXERCISES 4

No.1. The orthogonal transformation relating the Minkowski coordinates in the frames S and $\bar{S}$ is given by equations (5.1), where $\tan\alpha = iu/c$. Thus, the corresponding transformation equations for the 4-vector $\underline{J}$ are:

$$\bar{J}_1 = J_1\cos\alpha + J_4\sin\alpha, \quad \bar{J}_2 = J_2,$$

$$\bar{J}_4 = -J_1\sin\alpha + J_4\cos\alpha, \quad \bar{J}_3 = J_3.$$

Since $\underline{J} = (j_x, j_y, j_z, ic\rho)$, $\underline{\bar{J}} = (\bar{j}_x, \bar{j}_y, \bar{j}_z, ic\bar{\rho})$, $\cos\alpha = (1 - u^2/c^2)^{-\frac{1}{2}}$, $\sin\alpha = (1 - u^2/c^2)^{-\frac{1}{2}} iu/c$, substitution now yields the equations required.

No.2. Differentiating the Maxwell equation with respect to x_i, we get

$$\mu_0 J_{i,i} = F_{ij,ji} = 0$$

since the terms of the sum are either zero (e.g. $F_{11,11} = 0$) or cancel in pairs (e.g. $F_{12,21} + F_{21,12} = 0$) due to the anti-symmetry of F_{ij}.

No.3. If $F_{ij} = \Omega_{j,i} - \Omega_{i,j}$, then

$$F_{ij,j} = \Omega_{j,ij} - \Omega_{i,jj} = (\Omega_{j,j})_{,i} - \Omega_{i,jj}$$

$$= \mu_0 J_i$$

by equations (26.12).

Also

$$F_{ij,k} + F_{jk,i} + F_{ki,j}$$

$$= \Omega_{j,ik} - \Omega_{i,jk} + \Omega_{k,ji} - \Omega_{j,ki} + \Omega_{i,kj} - \Omega_{k,ij} = 0$$

since $\Omega_{j,ik} = \Omega_{j,ki}$, etc.

No.4. (i) $F_{ij}F_{ij}$ is the sum of the squares of the elements of the 4 × 4 matrix (26.5), viz.

$$2(B_x^2 + B_y^2 + B_z^2) - 2(E_x^2 + E_y^2 + E_z^2)/c^2 = 2B^2 - 2E^2/c^2$$

Since $B = \mu_0 H$ and $c^2 = 1/\varepsilon_0\mu_0$, this is equivalent to the result stated. Also, since $F_{ij}F_{ij}$ is an invariant, the deduction indicated follows.

(ii) The summation is to be carried out over the 24 permutations of (1234). Since both F_{ij} and $\mathcal{E}_{ijk\ell}$ are anti-symmetric with respect to all pairs of indices, the 8 permutations (1234), (2134), (1243), (2143), (3412), (3421), (4312) and (4321) make the same contributions to the sum. Thus, we need only calculate the contributions of the permutations (1234), (1324), (1423) and multiply each by 8 to give

$$\mathcal{E}_{ijk\ell}F_{ij}F_{k\ell} = 8(\mathcal{E}_{1234}F_{12}F_{34} + \mathcal{E}_{1324}F_{13}F_{24} + \mathcal{E}_{1423}F_{14}F_{23})$$

$$= -8i(B_zE_z + B_yE_y + B_xE_x)/c = -8i\underline{E}\cdot\underline{B}/c$$

Since $\mathcal{E}_{ijk\ell}F_{ij}F_{k\ell}$ is a pseudoinvariant, we now deduce that $\underline{E}\cdot\underline{B}$ is also one.

No.5. If α is the angle between $\underline{E}$ and $\underline{H}$, then $|\underline{S}| = |\underline{E} \times \underline{H}| = EH\sin\alpha$. Hence

$$\underline{S}^2 = S^2 = E^2H^2\sin^2\alpha = E^2H^2 - E^2H^2\cos^2\alpha$$

$$= E^2H^2 - (\underline{E}\cdot\underline{H})^2$$

Using equation (29.16), we now find

$$U^2 - \underline{S}^2/c^2 = \tfrac{1}{4}(\varepsilon_0 E^2 + \mu_0 H^2)^2 - \varepsilon_0\mu_0[E^2H^2 - (\underline{E}\cdot\underline{H})^2]$$

$$= (\varepsilon_0 E^2 - \mu_0 H^2)^2 + (\underline{E}\cdot\underline{B})^2\varepsilon_0/\mu_0$$

But, in the previous exercise, it was shown that $\mu_0 H^2 - \varepsilon_0 E^2$ is an invariant and $\underline{E}\cdot\underline{B}$ is a pseudoinvariant. It follows that $(\underline{E}\cdot\underline{B})^2$ is an invariant and, hence, $U^2 - \underline{S}^2/c^2$ is an invariant.

No.6. Let S and $\overline{S}$ be the frames of 0 and $\overline{0}$, with the origin of $\overline{S}$ moving along the x-axis of S with velocity V. In S, $\underline{E} = (0,E,0)$ and $\underline{B} = (0,0,0)$. Thus, equations (27.3) and (27.4) show that in $\overline{S}$ the field has components

$$\overline{B}_x = \overline{B}_y = 0, \quad \overline{B}_z = -\beta VE/c^2,$$

$$\overline{E}_x = \overline{E}_z = 0, \quad \overline{E}_y = \beta E,$$

where $\beta = (1 - V^2/c^2)^{-\frac{1}{2}}$. Since $\underline{V} = (V,0,0)$, thus $\underline{V} \times \underline{\overline{E}} = (0,0,\beta VE)$. Hence $\underline{V} \times \underline{\overline{E}} = -c^2\underline{\overline{B}}$. This is a 3-vector equation and so is valid for all parallel right-handed frames moving with 0 and $\overline{0}$.

No.7. Using equation (29.5), we find

$$\mu_0 S_{ii} = F_{ik}F_{ik} - \tfrac{1}{4}\delta_{ii}F_{k\ell}F_{k\ell} = 0$$

since $\delta_{ii} = 4$.

No.8. Substitution of the components of $\underline{E}$ and $\underline{B}$ into equations (27.3) and (27.4), we obtain

$$\overline{E}_y = \beta(1 - u/c)a\sin\omega(t - x/c) = \lambda a\sin\omega(t - x/c)$$

$$\overline{B}_z = \beta(1 - u/c)\frac{a}{c}\sin\omega(t - x/c) = \lambda\frac{a}{c}\sin\omega(t - x/c)$$

all other components being zero.

Using equations (5.8), we find

$$\overline{t} - \overline{x}/c = \beta(1 + u/c)(t - x/c) = (t - x/c)/\lambda$$

Thus, $(t - x/c) = \lambda(\overline{t} - \overline{x}/c)$ and the results stated now follow.

Thus, an observer moving in the direction of propagation experiences a reduction in amplitude and frequency by a factor λ.

No.9. We can write

$$H = c\{(p_\alpha - eA_\alpha)(p_\alpha - eA_\alpha) + m_0^2c^2\}^{\frac{1}{2}} + e\phi = cR + e\phi$$

and then Hamilton's equations are

$$\dot{x}_\alpha = cR^{-1}(p_\alpha - eA_\alpha), \qquad \text{(i)}$$

$$\dot{p}_\alpha = ecR^{-1}(p_\beta - eA_\beta)A_{\beta,\alpha} - \phi_{,\alpha} \qquad \text{(ii)}$$

From (i), we deduce that

$$v^2 = \dot{x}_\alpha \dot{x}_\alpha = c^2 R^{-2}(p_\alpha - eA_\alpha)(p_\alpha - eA_\alpha) = c^2(1 - m_0^2 c^2/R^2)$$

where v is the particle's speed. Solving for R, we get

$$R = \frac{m_0 c}{\sqrt{(1 - v^2/c^2)}} \qquad \text{(iii)}$$

Putting $\dot{x}_\alpha = v_\alpha$ in (i) and eliminating p_α between equations (i) and (ii), yields

$$\frac{d}{dt}\left(\frac{R}{c}v_\alpha\right) + eA_{\alpha,\beta}\dot{x}_\beta = ev_\beta A_{\beta,\alpha} - e\phi_{,\alpha}$$

Using (iii), this can be written

$$\frac{d}{dt}\left[\frac{m_0 v_\alpha}{\sqrt{(1 - v^2/c^2)}}\right] = ev_\beta(A_{\beta,\alpha} - A_{\alpha,\beta}) - e\phi_{,\alpha}$$

$$= e(\underline{v} \times \text{curl}\underline{A} - \text{grad}\phi)_\alpha$$

$$= e(\underline{v} \times \underline{B} + \underline{E})$$

which is the x_α-component of the valid equation of motion

$$\frac{d}{dt}(m\underline{v}) = e(\underline{v} \times \underline{B} + \underline{E})$$

No.10. For the proposed solution

$$\Omega_{i,i} = iA_i k_i e^{ik_p x_p} = 0$$

provided $A_i k_i = 0$. Also

$$\Omega_{i,jj} = - A_i k_j k_j e^{ik_p x_p} = 0$$

provided $k_j k_j = 0$. Thus, equations (26.12) are satisfied with $J_i = 0$.

It is clear that the 4-vector property of Ω_i can only be ensured by requiring that A_i behave as a 4-vector and $k_p x_p$ as an invariant. Then, provided we restrict ourselves to transformations of Minkowski frames with no shift of origin, x_p will behave as a vector and, to ensure the invariance of $k_p x_p$, it will be necessary for k_p to behave as a vector.

In the case of the wave cited

$$k_1 = 2\pi\nu\cos\alpha/c, \quad k_2 = 2\pi\nu\sin\alpha/c, \quad k_3 = 0, \quad k_4 = 2i\pi\nu/c$$

and it can be verified that $k_p k_p = 0$. The Minkowski transformation relating the two frames S ans $\bar{S}$ is given by equations (5.1) (replace α by β, where $\tan\beta = iu/c$). Thus,

$$\bar{k}_1 = k_1\cos\beta + k_4\sin\beta = \frac{2\pi\nu}{c}\cdot\frac{\cos\alpha - u/c}{\sqrt{(1 - u^2/c^2)}}$$

$$\bar{k}_4 = - k_1\sin\beta + k_4\cos\beta = \frac{2\pi i\nu}{c}\cdot\frac{1 - u\cos\alpha/c}{\sqrt{(1 - u^2/c^2)}}$$

Since $\bar{k}_1 = 2\pi\bar{\nu}\cos\bar{\alpha}/c$, $\bar{k}_4 = 2i\pi\bar{\nu}/c$, we now derive the results stated immediately.

No.11. By equation (28.6), the force acting on the particle is $q\underline{v} \times \underline{B}$ and the equation of work is accordingly

$$\dot{m}c^2 = q(\underline{v} \times \underline{B})\cdot\underline{v} = 0$$

Thus, m is constant and this implies v is constant.

Resolving the equation of motion (17.1) along the axes, we obtain the equations

$$\dot{p}_x = qBv_y, \quad \dot{p}_y = -qBv_x, \quad \dot{p}_z = 0.$$

Since the particle moves in the xy-plane, the last of these equations is automatically satisfied. Putting $v_x = \dot{x}$, $p_x = m\dot{x}$, etc., the remaining equations reduce to

$$\ddot{x} = \omega\dot{y}, \quad \ddot{y} = -\omega\dot{x}.$$

These equations integrate to

$$\dot{x} = \omega y, \quad \dot{y} = -\omega x$$

assuming that the origin of the xy-plane is chosen so as to eliminate any constants of integration.

Eliminating y between the last pair of equations, we find $\ddot{x} + \omega^2 x = 0$ and it then follows that

$$x = R\sin\omega t$$

where R is a constant of integration and the other constant is suppressed by appropriate choice of the instant t = 0. We now find

$$y = R\cos\omega t$$

and the particle therefore moves in a circle of radius R.

Further,

$$v^2 = \dot{x}^2 + \dot{y}^2 = R^2\omega^2$$

and, thus, $R = v/\omega$ gives the radius of the orbit for a given speed.

This result provides the basis for the design of an instrument called the synchrotron, in which electrified particles (usually protons) circulate around an evacuated torus under the control of an intense magnetic field acting in a direction perpendicular to the plane of motion. Impulses are applied to the particles by local electric fields and these result in increases in speed which transfer them to orbits of steadily increasing radius. Eventually, the particles leave the torus by a tangential exit channel, their speed then being close to c, and the high energy beam is used in experiments relating to nuclear interactions.

In S, $\underline{E} = \underline{0}$ and $\underline{B} = (0,0,B)$. Equations (27.3) and (27.4) show that in $\bar{S}$,

$$\bar{B}_x = \bar{B}_y = 0, \quad \bar{B}_z = (1 - u^2/c^2)^{-\frac{1}{2}}B,$$
$$\bar{E}_x = \bar{E}_z = 0, \quad \bar{E}_y = -(1 - u^2/c^2)^{-\frac{1}{2}}uB.$$

To generate the required field in $\bar{S}$, we must choose u and B so that

$$B_0 = (1 - u^2/c^2)^{-\frac{1}{2}}B, \quad E_0 = -(1 - u^2/c^2)^{-\frac{1}{2}}uB.$$

Hence $u = -E_0/B_0$ and $B = \surd(B_0^2 - E_0^2/c^2)$. Clearly, it is necessary that $|B_0| > |E_0|/c$.

Using the Lorentz transformation equations (5.8), for the particle motion observed from $\bar{S}$,

$$\bar{x} = (1 - u^2/c^2)^{-\frac{1}{2}}(x - ut) = (1 - u^2/c^2)^{-\frac{1}{2}}(R\sin\omega t - ut)$$

$$\bar{t} = (1 - u^2/c^2)^{-\frac{1}{2}}(t - ux/c^2) = (1 - u^2/c^2)^{-\frac{1}{2}}(t - uR\sin\omega t/c^2)$$

The average velocity along the $\bar{x}$-axis is therefore

$$\bar{x}/\bar{t} = \frac{R\sin\omega t - ut}{t - uR\sin\omega t/c^2} \rightarrow -u \text{ as } \bar{t} \rightarrow \infty,$$

since t, $\bar{t}$ tend to infinity together. Thus, the average velocity is E_0/B_0.

No.12. Transforming from the frame S to the frame $\bar{S}$ by means of equations (27.3) and (27.4), we find

$$\bar{B}_x = 0, \quad \bar{B}_y = 0, \quad \bar{B}_z = \beta(B - uE/c^2),$$

$$\bar{E}_x = 0, \quad \bar{E}_y = \beta(E - uB), \quad \bar{E}_z = 0.$$

In $\bar{S}$, the electric field is clearly null if $u = E/B$. With this value of u, $\beta = B/\surd(B^2 - E^2/c^2)$ and $\bar{B}_z = \surd(B^2 - E^2/c^2)$. Since $\bar{B}_x = \bar{B}_y = 0$, the magnetic field is directed along the $\bar{z}$-axis.

No.13. Since the Lorentz force acting upon the particle is always perpendicular to its motion, the equation of work gives $\dot{m} = 0$ (see Ex.11 above) and the inertial mass is accordingly constant. The particle's speed must therefore have the constant value u.

The force acting upon the particle as it moves through the magnetic field is $q(\underline{v} \times \underline{B})$. If(x,y,z) are the particle's coordinates at time t after entering the field at O, then $\underline{v} = (\dot{x},\dot{y},\dot{z})$, $\underline{B} = (0,0,B)$, and the force has components $(qB\dot{y}, -qB\dot{x}, 0)$; since the z-component vanishes, the particle moves in the plane $z = 0$. The x- and y-components of the equation of motion are

$$\frac{d}{dt}(m\dot{x}) = qB\dot{y}, \quad \frac{d}{dt}(m\dot{y}) = -qB\dot{x},$$

where $m = m_0/\surd(1 - u^2/c^2)$. Defining k as stated in the question, these equations can be written

$$k\ddot{x} = u\dot{y}, \quad k\ddot{y} = -u\dot{x}.$$

Integrating under the initial conditions $x = y = 0$, $\dot{x} = u$, $\dot{y} = 0$, we calculate that

$$k\dot{x} = u(y + k), \quad k\dot{y} = -ux,$$

from which, by division, we find

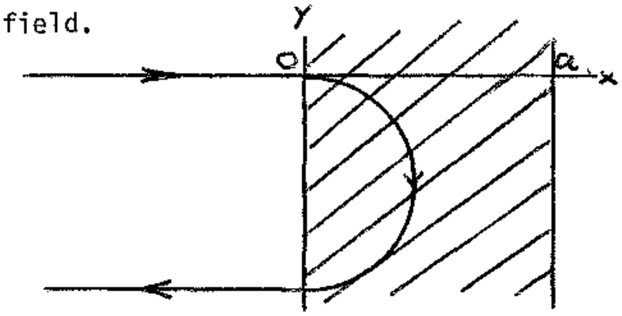
$$x\,dx + (y + k)dy = 0$$

A further integration under the stated initial conditions, leads to the equation

$$x^2 + (y + k)^2 = k^2$$

for the trajectory. This is equivalent to the result quoted.

The trajectory in the magnetic field is a circle of radius k and centre (0,-k). Thus, the particle can complete a semi-circle, as indicated in the diagram below, without leaving the field, provided $k < a$; it then finally moves parallel to the x-axis and has been turned back by the field. If $k > a$, the particle enters the region $x > a$ and is only deflected by the field.



No.14. *If the x and y axes are rotated through an angle θ in their plane to give coordinates x' and y', it is easy to verify that the field components in the new frame are:*

$$\underline{E}' = (0, AX, 0), \quad \underline{B}' = (0, 0, AX/c),$$

where $X = \sin 2\pi f(t - x'/c)$. Maxwell's equations in vacuo are satisfied by $\underline{E}'$ and $\underline{B}'$ and represent a wave of frequency f being propagated along the x'-axis with speed c. The given equations accordingly specify a wave being propagated in a direction parallel to the xy-plane and making an angle θ with the x-axis, as stated.

Transforming to the frame $\bar{S}$ by equations (27.3) and (27.4), we find that

$$\underline{\bar{B}} = \{0, 0, \beta AX(c - u\cos\theta)/c^2\}, \quad \underline{\bar{E}} = \{-AX\sin\theta, \beta AX(c\cos\theta - u)/c, 0\}.$$

Hence, if $u = c\cos\theta$, then $\beta = \operatorname{cosec}\theta$ and

$$\underline{\bar{B}} = (0, 0, AX\sin\theta/c), \quad \underline{\bar{E}} = (-AX\sin\theta, 0, 0).$$

Also, the inverse Lorentz transformation equations show that

$$t - (x\cos\theta + y\sin\theta)/c = \beta(\bar{t} + u\bar{x}/c^2) - \frac{\beta}{c}(\bar{x} + u\bar{t})\cos\theta - \frac{\bar{y}}{c}\sin\theta$$

$$= (\bar{t} - \bar{y}/c)\sin\theta$$

Thus,

$$X = \sin\{2\pi f\sin\theta(\bar{t} - \bar{y}/c)\} = \bar{X},$$

completing the calculation of the wave's components in $\bar{S}$.

The electric and magnetic fields in $\bar{S}$ take a fixed value when $\bar{t} - \bar{y}/c$ is constant. This defines a plane perpendicular to the $\bar{y}$-axis, which advances along this axis with speed c. It follows that the wave propagates parallel to the $\bar{y}$-axis in $\bar{S}$. The frequency is obviously $f\sin\theta$.

No.15. Choose an inertial frame in which the conducting medium is inztantaneously at rest at the point under consideration. In this frame, $\underline{V} = (0,0,0,ic)$ and the components J_i, F_{ij} are given by equations (24.7), (26.5) respectively (N.B. $\underline{V}$ is not the 4-velocity of charge flow). The four components of the stated equation will now be found to be

$$j_x = \sigma E_x, \quad j_y = \sigma E_y, \quad j_z = \sigma E_z, \quad ic\rho - ic\rho = 0.$$

The equation is accordingly valid in the selected frame. But, being a 4-tensor equation, this ensures its validity in all inertial frames.

No.16. Referring to equation (26.5), we find that $F_{12} = -F_{21} = B$, all other components of the field tensor vanishing. It follows that $F_{k\ell}F_{k\ell} = 2B^2$.

Substitution in equation (29.5) now gives

$$\mu_0 S_{11} = \mu_0 S_{22} = \tfrac{1}{2}B^2, \qquad \mu_0 S_{33} = \mu_0 S_{44} = -\tfrac{1}{2}B^2,$$

all other components of the energy-momentum tensor vanishing.

No.17. In a frame $\bar{S}$ with the charge at its origin, the field at $(\bar{x},\bar{y},\bar{z})$ due to the charge is determined by

$$\bar{\underline{E}} = \frac{e}{4\pi\varepsilon_0\bar{r}^3}(\bar{x},\bar{y},\bar{z}), \qquad \bar{\underline{B}} = (0,0,0),$$

where $\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$.

The Lorentz transformation equations relating coordinates and times in S and $\bar{S}$ are

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = \beta(z - vt), \quad \bar{t} = \beta(t - vz/c^2),$$

where $\beta = 1/\surd(1 - v^2/c^2)$. Thus, at $t = 0$ in S and at the point with spherical polar coordinates (r,θ,ϕ), we have

$$\bar{x} = x = r\sin\theta\cos\phi, \quad \bar{y} = y = r\sin\theta\sin\phi, \quad \bar{z} = \beta z = \beta r\cos\theta.$$

It then follows that

$$\bar{r}^2 = \beta^2 r^2(1 - v^2\sin^2\theta/c^2).$$

The transformation equations inverse to (27.4) (after exchanging the roles of the x- and z-axes) show that

$$E_x = \beta\bar{E}_x, \quad E_y = \beta\bar{E}_y, \quad E_z = \bar{E}_z.$$

Whence

$$\underline{E} = \frac{e}{4\pi\varepsilon_0 r^3}(1 - v^2/c^2)(1 - v^2\sin^2\theta/c^2)^{-3/2}(x,y,z),$$

proving that the electric field in S at $t = 0$ is radial and of the magnitude stated.

The equations inverse to (27.3) give

$$B_x = -\beta v\bar{E}_y/c^2, \quad B_y = \beta v\bar{E}_x/c^2, \quad B_z = 0.$$

Whence

$$\underline{B} = \frac{ev}{4\pi\varepsilon_0 c^2 r^3}(1 - v^2/c^2)(1 - v^2\sin^2\theta/c^2)^{-3/2}(-y,x,0)$$

$$= \frac{1}{c^2}(\underline{v} \times \underline{E}),$$

since $\underline{v} = (0,0,v)$.

No.1. The transformation law for B_{ij} can be calculated from that for A_{ij}, thus:

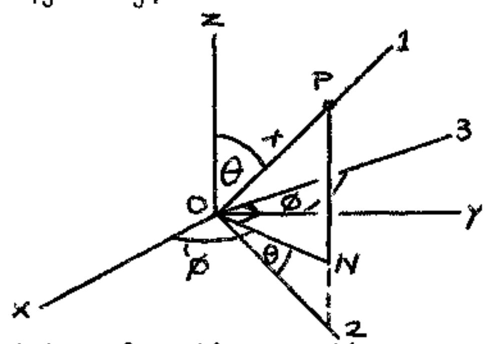
$$\bar{B}_{ij} = \bar{A}_{ji} = \frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial x^s}{\partial \bar{x}^i} A_{rs} = \frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial x^r}{\partial \bar{x}^j} B_{sr} ,$$

showing that B_{ij} transforms as a covariant tensor.

The equation $A_{ij} = A_{ji}$ may be written $A_{ij} = B_{ij}$. This is a tensor equation and hence is valid in all frames if it is valid in the x-frame.

Similarly, $A_{ij} = -A_{ji}$ is valid in all frames, if it is true in one.

No.2.



The relevant transformation equations are

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \text{ and inversely } \left\{\begin{aligned} r &= \sqrt{(x^2 + y^2 + z^2)} \\ \theta &= \tan^{-1}\{\sqrt{(x^2 + y^2)}/z\} \\ \phi &= \tan^{-1}(y/x) \end{aligned}\right.$$

Since $\underline{A}$ is a contravariant vector

$$A^r = \frac{\partial r}{\partial x}A^x + \frac{\partial r}{\partial y}A^y + \frac{\partial r}{\partial z}A^z$$

$$A^\theta = \frac{\partial \theta}{\partial x}A^x + \frac{\partial \theta}{\partial y}A^y + \frac{\partial \theta}{\partial z}A^z$$

$$A^\phi = \frac{\partial \phi}{\partial x}A^x + \frac{\partial \phi}{\partial y}A^y + \frac{\partial \phi}{\partial z}A^z$$

The partial derivatives are easily calculable from the inverse transformation equations, but are best expressed in terms of the spherical polar coordinates. For example,

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (x^2+y^2)/z^2}\cdot\frac{x}{z\sqrt{(x^2+y^2)}} = \frac{xz}{r^2\sqrt{(x^2+y^2)}} = \frac{1}{r}\cos\theta\cos\phi$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1 + y^2/x^2}\cdot(-y/x^2) = -\frac{y}{x^2 + y^2} = -\frac{\sin\phi}{r\sin\theta}$$

and so on. Substitution then yields the equations

$$A^r = A^x\sin\theta\cos\phi + A^y\sin\theta\sin\phi + A^z\cos\theta$$

$$rA^\theta = A^x\cos\theta\cos\phi + A^y\cos\theta\sin\phi - A^z\sin\theta$$

$$r\sin\theta A^\phi = -A^x\sin\phi + A^y\cos\phi$$

Since (A^x, A^y, A^z) and (A^1, A^2, A^3) are components relative to rectangular Cartesian axes, we can relate them by elementary procedures. Thus,

A^x directed along Ox can be further resolved into $A^x\cos\phi$ along ON (see diagram) and $-A^x\sin\phi$ along O3. Then, $A^x\cos\phi$ along ON resolves into $A^x\cos\phi\cos\theta$ along O2 and $A^x\cos\phi\sin\theta$ along O1. Similarly, we first resolve A^y along Oy into $A^y\cos\phi$ along O3 and $A^y\sin\phi$ along ON; the latter component then resolves into $A^y\sin\phi\cos\theta$ along O2 and $A^y\sin\phi\sin\theta$ along O1. Finally, A^z along Oz resolves into $A^z\cos\theta$ along O1 and $-A^z\sin\theta$ along O2. Then, collecting together the various components along the three axes O1, O2, O3, we find that

$$A^1 = A^x\cos\phi\sin\theta + A^y\sin\phi\sin\theta + A^z\cos\theta = A^r$$

$$A^2 = A^x\cos\phi\cos\theta + A^y\sin\phi\cos\theta - A^z\sin\theta = rA^\theta$$

$$A^3 = -A^x\sin\phi + A^y\cos\phi = r\sin\theta A^\phi$$

using the earlier equations.

This exercise demonstrates that, in the notation of tensor calculus, the polar components A^r, A^θ, A^ϕ, of a contravariant vector are not identical with the Cartesian components A^1, A^2, A^3 in the Cartesian polar frame O123. However, the reader is warned that, in elementary textbooks, the Cartesian components A^1, A^2, A^3, are often referred to as the polar components of the vector.

No.3. Making use of equation (33.2), we find that

$$\bar{A}_{i,j} - \bar{A}_{j,i} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^\ell}{\partial \bar{x}^j}\frac{\partial A_k}{\partial x^\ell} - \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial x^\ell}{\partial \bar{x}^i}\frac{\partial A_k}{\partial x^\ell}$$

$$= \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^\ell}{\partial \bar{x}^j}(A_{k,\ell} - A_{\ell,k})$$

having exchanged the dummy indices k, ℓ in the second term of the right-hand member of the first equation. This proves that $A_{i,j} - A_{j,i}$ is a tensor.

If $\underline{A} = \nabla\phi$, then $A_i = \phi_{,i}$. Thus, $A_{i,j} - A_{j,i} = \phi_{,ij} - \phi_{,ji} = 0$.

No.4. Differentiating the transformation equation

$$\bar{A}_{ij} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}A_{rs}$$

partially with respect to $\bar{x}^k$, we find

$$\bar{A}_{ij,k} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k}A_{rs,t} + \frac{\partial^2 x^r}{\partial \bar{x}^i\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^j}A_{rs} + \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial^2 x^s}{\partial \bar{x}^j\partial \bar{x}^k}A_{rs}$$

Cyclically permuting the free indices i, j, k, and, at the same time, in the first term of the right-hand member cyclically permuting the dummies r, s, t, in the same way, we obtain two further equations, viz.

$$\bar{A}_{jk,i} = \frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^i}A_{st,r} + \frac{\partial^2 x^r}{\partial \bar{x}^j\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^k}A_{rs} + \frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial^2 x^s}{\partial \bar{x}^k\partial \bar{x}^i}A_{rs}$$

$$\bar{A}_{ki,j} = \frac{\partial x^t}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} A_{tr,s} + \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^j}\frac{\partial x^s}{\partial \bar{x}^i} A_{rs} + \frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} A_{rs}$$

Adding the last three equations, we find that the terms involving second order partial derivatives cancel in pairs, thus:

$$\frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^j} A_{rs} + \frac{\partial x^r}{\partial \bar{x}^j}\frac{\partial^2 x^s}{\partial \bar{x}^k \partial \bar{x}^i} A_{rs} = \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^j} A_{rs} + \frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^i} A_{sr} = 0$$

since $A_{rs} + A_{sr} = 0$. We conclude that

$$\bar{A}_{ij,k} + \bar{A}_{jk,i} + \bar{A}_{ki,j} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k}(A_{rs,t} + A_{st,r} + A_{tr,s})$$

and hence that $A_{ij,k} + A_{jk,i} + A_{ki,j}$ is a covariant tensor of rank three.

Such a tensor arises in Maxwell's equations for an electromagnetic field in the presence of a gravitational field (see equation (56.5)).

No.5. The 'chain' rule for partial differentiation requires that

$$\frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^k} = \frac{\partial \bar{x}^i}{\partial \bar{x}^k} = \delta^i_k$$

It follows from the rule for multiplication of matrices that, if P is the matrix whose ijth element is $\partial \bar{x}^i/\partial x^j$ and Q is the matrix whose jkth element is $\partial x^j/\partial \bar{x}^k$, then PQ = I, where I is the unit matrix. Taking determinants, we conclude that $|P|.|Q| = 1$, which is the result to be proved.

We have the transformation equation

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^r} A^r_s \frac{\partial x^s}{\partial \bar{x}^j}$$

If A denotes the matrix whose rsth element is A^r_s, the matrix multiplication rule shows that $\bar{A} = PAQ$. Taking determinants, we find

$$|\bar{A}| = |P|.|A|.|Q| = |A|.$$

Thus $|A^i_j|$ is an invariant.

No.6. Since the difference of two affinities is a tensor (see equation (34.9)), T^k_{ij} is a tensor.

We have

$$g_{ij;k} = \partial g_{ij}/\partial x^k - \Gamma^r_{ik} g_{rj} - \Gamma^r_{jk} g_{ir}$$

Cyclically permuting the indices i, j, k, we are led to the further equations

$$g_{jk;i} = \partial g_{jk}/\partial x^i - \Gamma^r_{ji} g_{rk} - \Gamma^r_{ki} g_{jr}$$

$$g_{ki;j} = \partial g_{ki}/\partial x^j - \Gamma^r_{kj} g_{ri} - \Gamma^r_{ij} g_{kr}$$

Adding the first and third equations and subtracting the second, we prove that

$$g_{ij;k} + g_{ki;j} - g_{jk;i} = 2[jk,i] - (\Gamma^r_{jk} + \Gamma^r_{kj})g_{ir} + 2T^r_{ki}g_{jr} + 2T^r_{ji}g_{kr}$$

Assuming that the covariant derivative of g_{ij} is identically zero, it is necessary that

$$\tfrac{1}{2}(\Gamma^r_{jk} + \Gamma^r_{kj})g_{ir} = [jk,i] + T^r_{ki}g_{jr} + T^r_{ji}g_{kr}$$

Raising the index i (as s) by multiplication through by g^{si}, we then deduce that

$$\tfrac{1}{2}(\Gamma^r_{jk} + \Gamma^r_{kj})\delta^s_r = \left\{{s \atop j\ k}\right\} + g^{si}(T^r_{ki}g_{jr} + T^r_{ji}g_{kr})$$

or

$$\tfrac{1}{2}(\Gamma^s_{jk} + \Gamma^s_{kj}) = \left\{{s \atop j\ k}\right\} + g^{si}(T^r_{ji}g_{rk} + T^r_{ki}g_{rj})$$

But $\tfrac{1}{2}\Gamma^s_{kj} = \tfrac{1}{2}\Gamma^s_{jk} - T^s_{jk}$, so that the result stated follows immediately (with some changes of indices).

The tensor T^i_{jk} can be chosen arbitrarily except that it must clearly be skew-symmetric. Then, the formula for Γ^i_{jk} gives T^i_{jk} as the skew-symmetric part of the affinity and the remaining terms constitute the symmetric part. If the affinity is to be symmetric (as in the text), then we must choose $T^i_{jk} = 0$ and the affinity becomes identical with Christoffel's symbol.

It may now be verified that, with this choice of affinity, the covariant derivative of g_{ij} does indeed vanish; for

$$\begin{aligned} g_{ij;k} &= g_{ij,k} - g_{rj}\left[\left\{{r \atop i\ k}\right\} + T^r_{ik} + g^{rt}(T^s_{it}g_{sk} + T^s_{kt}g_{si})\right] \\ &\qquad - g_{ir}\left[\left\{{r \atop j\ k}\right\} + T^r_{jk} + g^{rt}(T^s_{jt}g_{sk} + T^s_{kt}g_{sj})\right] \\ &= g_{ij,k} - [ik,j] - [jk,i] - g_{rj}T^r_{ik} - T^s_{ij}g_{sk} - T^s_{kj}g_{si} \\ &\qquad - g_{ir}T^r_{jk} - T^s_{ji}g_{sk} - T^s_{ki}g_{sj} \end{aligned}$$

which cancels to zero in view of the skew-symmetry of T^i_{jk}.

No.7. $$A_{i;j} = A_{i,j} - \Gamma^r_{ij}A_r$$

$$A_{j;i} = A_{j,i} - \Gamma^r_{ji}A_r$$

Subtracting, the terms involving the affinity cancel when $\Gamma^r_{ij} = \Gamma^r_{ji}$. Thus, $A_{i;j} - A_{j;i} = A_{i,j} - A_{j,i}$.

No.8. Taking the covariant derivative of the second rank tensor $A_{i;j}$ with respect to x^k, we get

$$\begin{aligned} A_{i;jk} &= \partial A_{i;j}/\partial x^k - \Gamma^r_{ik}A_{r;j} - \Gamma^r_{jk}A_{i;r} \\ &= \frac{\partial}{\partial x^k}\left(A_{i,j} - \Gamma^s_{ij}A_s\right) - \Gamma^r_{ik}(A_{r,j} - \Gamma^s_{rj}A_s) - \Gamma^r_{jk}A_{i;r} \\ &= A_{i,jk} - \Gamma^s_{ij}A_{s,k} - \Gamma^s_{ij,k}A_s - \Gamma^r_{ik}A_{r,j} + \Gamma^r_{ik}\Gamma^s_{rj}A_s - \Gamma^r_{jk}A_{i;r} \end{aligned}$$

Exchanging the indices j, k, we obtain

$$A_{i;kj} = A_{i,kj} - \Gamma^s_{ik}A_{s,j} - \Gamma^s_{ik,j}A_s - \Gamma^r_{ij}A_{r,k} + \Gamma^r_{ij}\Gamma^s_{rk}A_s - \Gamma^r_{kj}A_{i;r}$$

Subtracting these equations, we find

$$\begin{aligned} A_{i;jk} - A_{i;kj} &= (\Gamma^s_{ik,j} - \Gamma^s_{ij,k} + \Gamma^r_{ik}\Gamma^s_{rj} - \Gamma^r_{ij}\Gamma^s_{rk})A_s + (\Gamma^r_{kj} - \Gamma^r_{jk})A_{i;r} \\ &= B^s_{ijk}A_s + (\Gamma^r_{kj} - \Gamma^r_{jk})A_{i;r} \end{aligned}$$

which is equivalent to the result stated.

Since $\Gamma^r_{kj} - \Gamma^r_{jk}$ is a tensor, the last equation shows that $B^s_{ijk}A_s$ is a tensor for arbitrary covariant vectors A_s. It follows by the quotient rule that B^s_{ijk} is a tensor.

If B^s_{ijk} vanishes identically and $\Gamma^r_{kj} = \Gamma^r_{jk}$, then clearly $A_{i;jk} = A_{i;kj}$ and covariant differentiations commute.

A similar analysis can be applied to the contravariant vector A^i. The result is

$$A^i_{;jk} - A^i_{;kj} = -B^i_{rjk}A^r + (\Gamma^r_{kj} - \Gamma^r_{jk})A^i_{;r}$$

No.9. The calculation proceeds as in section 36. The successive equations need amendment as follows:

(36.2) $\delta A_i = \Gamma^j_{ik}A_j d\xi^k$

(36.3) $\Gamma^j_{ik} + \Gamma^j_{ik,\ell}\xi^\ell$

(36.4) $A_j + \Gamma^r_{j\ell}A_r\xi^\ell$

(36.5) $\delta A_i = \left[\Gamma^j_{ik}A_j + (A_j\Gamma^j_{ik,\ell} + \Gamma^j_{ik}\Gamma^r_{j\ell}A_r)\xi^\ell\right]d\xi^k$

(36.6) $\Delta A_i = \Gamma^j_{ik}A_j\oint_C d\xi^k + (\Gamma^r_{ik}\Gamma^j_{r\ell} + \Gamma^j_{ik,\ell})A_j\oint_C \xi^\ell d\xi^k$

(36.11) $\Delta A_i = (\Gamma^r_{ik}\Gamma^j_{r\ell} + \Gamma^j_{ik,\ell})A_j\alpha^{k\ell}$

(36.18) $\Delta A_i = (\Gamma^r_{i\ell}\Gamma^j_{rk} + \Gamma^j_{i\ell,k})A_j\alpha^{\ell k}$

(36.19) $\Delta A_i = \frac{1}{2}(\Gamma^r_{ik}\Gamma^j_{r\ell} - \Gamma^r_{i\ell}\Gamma^j_{rk} + \Gamma^j_{ik,\ell} - \Gamma^j_{i\ell,k})A_j\alpha^{k\ell}$

$= -\frac{1}{2}B^j_{ik\ell}A_j\alpha^{k\ell}$

which is equivalent to the result stated.

No.10. Let the coordinates of P' be $x^i + \Delta x^i$. Then, to the first order of small quantities using Taylor's theorem, if A^i_j denotes the field value at P, the field value at P' will be $A^i_j + A^i_{j,k}\Delta x^k$.

The field value after parallel displacement from P to P' is $A^i_j - \Gamma^i_{rk}A^r_j\Delta x^k + \Gamma^r_{jk}A^i_r\Delta x^k$. Thus, to the first order, It now follows that

$$\lim \frac{\Delta A^i_j}{\Delta t} = \lim A^i_{j;k} \frac{\Delta x^k}{\Delta t} = A^i_{j;k} \frac{dx^k}{dt}.$$

No.11. Differentiating the transformation equation for g_{jk}, viz.

$$\bar{g}_{jk} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} g_{rs}$$

partially with respect to $\bar{x}^m$, we obtain

$$\frac{\partial \bar{g}_{jk}}{\partial \bar{x}^m} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^t}{\partial \bar{x}^m} \frac{\partial g_{rs}}{\partial x^t} + \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^m} \frac{\partial x^s}{\partial \bar{x}^k} g_{rs} + \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial^2 x^s}{\partial \bar{x}^k \partial \bar{x}^m} g_{rs}$$

Cyclically permuting the indices j, k, m, we get the further equations

$$\frac{\partial \bar{g}_{km}}{\partial \bar{x}^j} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^t}{\partial \bar{x}^m} \frac{\partial g_{st}}{\partial x^r} + \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^m} g_{rs} + \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial^2 x^s}{\partial \bar{x}^m \partial \bar{x}^j} g_{rs}$$

$$\frac{\partial \bar{g}_{mj}}{\partial \bar{x}^k} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^t}{\partial \bar{x}^m} \frac{\partial g_{tr}}{\partial x^s} + \frac{\partial^2 x^r}{\partial \bar{x}^m \partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^j} g_{rs} + \frac{\partial x^r}{\partial \bar{x}^m} \frac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^k} g_{rs}$$

having, at the same time, cyclically permuted the dummies in the first term of the right-hand member.

Adding the last two equations and subtracting the original equation, we arrive at the result

$$\overline{[jk,m]} = \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial x^t}{\partial \bar{x}^m} [rs,t] + \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^m} g_{rs}$$

having appealed to the symmetry of g_{rs}. This is the transformation equation for the Christoffel symbol of the first kind.

To raise the index m in this transformation, we multiply by the transformation

$$\bar{g}^{im} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^m}{\partial x^q} g^{pq}$$

to give

$$\overline{\left\{ {i \atop j\ k} \right\}} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} \delta^t_q\, g^{pq} [rs,t] + \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} \delta^s_q\, g^{pq} g_{rs}$$

since $\frac{\partial x^t}{\partial \bar{x}^m} \frac{\partial \bar{x}^m}{\partial x^q} = \delta^t_q$. This equation immediately reduces to

$$\overline{\left\{ {i \atop j\ k} \right\}} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} \left\{ {p \atop r\ s} \right\} + \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k}$$

since $\delta^s_q g^{pq} g_{rs} = g^{ps} g_{rs} = \delta^p_r$. This shows that Christoffel's symbol of the second kind transforms like an affinity (equation (34.8)).

No.12. $A_{...} = A_{...} - \Gamma^r_{..} A_{.} - \Gamma^r_{..} A_{.} = A_{...} - \Gamma^r_{.} A_{.} - \Gamma^r_{..} A_{.}$

No.13. In the general case, when the affinity is not symmetric, the only relationship between the components of $B^i_{jk\ell}$ arises from the tensor's skew-symmetry with respect to the indices k and ℓ. If $k = \ell$, the component is zero. Thus, for each choice of values of i and j, the number of independent components is equal to the number of ways we can choose two different integers from the set (1,2,...,N), viz. $\binom{N}{2} = \frac{1}{2}N(N-1)$. The number of possible pairs (i,j) is N^2, since each can take N values. Thus, the number of independent components is $\frac{1}{2}N^3(N-1)$.

If the affinity is symmetric, the components are related by the additional equations (41.1). To count the number of independent equations for a given value of i, first consider the cases where (j,k,ℓ) are all different. Permutation of any given set of values does not lead to an independent equation (thus, $j = 1$, $k = 2$, $\ell = 3$ yields the same relationship as $j = 2$, $k = 3$, $\ell = 1$). Hence, the number of independent equations equals the number of ways of choosing three different integers from (1,2,...,N), viz. N(N-1)(N-2)/6.

Next, consider the cases where two of the indices (j,k,ℓ) are the same. For example, suppose $j = 1$ and $k = \ell = 2$. Equation (41.1) then yields $B^i_{122} + B^i_{221} + B^i_{212} = 0$. Since $B^i_{122} = 0$, this only reasserts the skew-symmetry property. Hence, none of these cases need be counted.

The cases where $j = k = \ell$ reduce to 0 = 0 and may also be ignored.

Since i can take all values from 1 to N, the total number of independent relationships (41.1) is counted to be $N^2(N-1)(N-2)/6$. Each reduces the number of independent components by one. Thus, the number of independent components is

$$\frac{1}{2}N^3(N-1) - \frac{1}{6}N^2(N-1)(N-2) = \frac{1}{3}N^2(N^2 - 1).$$

No.14. The components of $B_{ijk\ell}$ are related by equations (41.2), (41.5), (41.6), (41.7), but are otherwise independent.

Only components for which i and j are different and for which k and ℓ are different, can be non-zero. There are $\frac{1}{2}N(N-1)$ possibilities for the pair (i,j) and the same number of possibilities for the pair (k,ℓ) —the property of skew-symmetry indicates that the order of a pair should be ignored in our count. Hence, there are $\frac{1}{4}N^2(N-1)^2$ distinct non-zero components. These are related by equations (41.2) and (41.7), which we proceed to count.

Equation (41.7) makes no further restriction if (i,j) is the same pair as (k,ℓ) (e.g. $B_{1221} = B_{2112}$ already follows from equations (41.5) and (41.6)). The number of possible pairs (i,j) is $\frac{1}{2}N(N-1)$. With each of these, we can associate $\frac{1}{2}N(N-1) - 1$ pairs (k,ℓ) to give $\frac{1}{2}N(N-1)\{\frac{1}{2}N(N-1) - 1\} = \frac{1}{4}N(N^2-1)(N-2)$ effective constraints of the type (41.7). However, each constraint appears twice, e.g. $B_{1213} = B_{1312}$ also appears as $B_{1312} = B_{1213}$. Thus, the number of independent constraints is $N(N^2-1)(N-2)/8$.

In equation (41.2), if two of the indices (i,j,k,ℓ) are the same, the constraint is already included amongst the equations (41.5) - (41.7), e.g.

$$B_{1223} + B_{1232} + B_{1322} = 0 \rightarrow B_{1223} = - B_{1232}$$

$$B_{1123} + B_{1231} + B_{1312} = 0 \rightarrow B_{1231} = -B_{1312} = B_{3112}$$

If three or more indices are the same, the equation reduces to 0 = 0.

Thus, we suppose (i,j,k,ℓ) to be all different. A permutation of the values of (i,j,k,ℓ) does not lead to a distinct constraint. For example (2,4,3,1) gives the equation

$$B_{2431} + B_{2314} + B_{2143} = 0$$

which, by the other constraints, is equivalent to

$$B_{1342} + B_{1423} + B_{1234} = 0$$

and this equation follows from $i = 1$, $j = 2$, $k = 3$, $\ell = 4$. Thus, the number of essentially distinct constraints (41.2) is the number of ways in which four different numbers can be chosen from $(1,2,\ldots,N)$, i.e. $N(N-1)(N-2)(N-3)/24$.

Hence, the total number of distinct relationships (41.2) and (41.7) is

$$\frac{1}{8}N(N^2-1)(N-2) + \frac{1}{24}N(N-1)(N-2)(N-3) = \frac{1}{6}N^2(N-1)(N-2)$$

These reduce the number of independent non-zero components to

$$\frac{1}{4}N^2(N-1)^2 - \frac{1}{6}N^2(N-1)(N-2) = \frac{1}{12}N^2(N^2 - 1).$$

The case N = 4 is, of course, of most interest. There are then 20 independent components.

No.15. Differentiation of the stated identity yields

$$\frac{\partial g^{ij}}{\partial x^\ell} g_{jk} + g^{ij}\frac{\partial g_{jk}}{\partial x^\ell} = 0$$

Multiplying through by g^{mk}, we obtain

$$\frac{\partial g^{ij}}{\partial x^\ell} \delta^m_j + g^{mk}g^{ij}\frac{\partial g_{jk}}{\partial x^\ell} = 0$$

which immediately reduces to the result given.

It follows that

$$\frac{\partial g^{im}}{\partial x^\ell} + g^{ij}\begin{Bmatrix} m \\ j\ \ell \end{Bmatrix} + g^{mj}\begin{Bmatrix} i \\ j\ \ell \end{Bmatrix}$$

$$= g^{im}{}_{,\ell} + (g^{ij}g^{mk} + g^{mj}g^{ik})[j\ell,k]$$

$$= g^{im}{}_{,\ell} + \tfrac{1}{2}(g^{ij}g^{mk} + g^{mj}g^{ik})(g_{jk,\ell} + g_{\ell k,j} - g_{j\ell,k})$$

$$= g^{im}{}_{,\ell} - g^{im}{}_{,\ell} + \tfrac{1}{2}(g^{ij}g^{mk} + g^{mj}g^{ik})(g_{\ell k,j} - g_{j\ell,k})$$

$$= \tfrac{1}{2}(g^{ij}g^{mk} + g^{mj}g^{ik})g_{\ell k,j} - \tfrac{1}{2}(g^{ik}g^{mj} + g^{mk}g^{ij})g_{k\ell,j}$$

Since

$$g^{ij}{}_{;k} = g^{ij}{}_{,k} + \Gamma^i_{rk}g^{rj} + \Gamma^j_{rk}g^{ir}$$

if $\Gamma^i_{jk} = \left\{{i \atop j\ k}\right\}$, the result we have proved is seen to be equivalent to $g^{ij}{}_{;k} = 0$.

No.16. We have

$$R_{jk} = B^i_{jki} = \Gamma^i_{ji,k} - \Gamma^i_{jk,i} + \Gamma^i_{rk}\Gamma^r_{ji} - \Gamma^i_{ri}\Gamma^r_{jk}$$

If $\Gamma^i_{jk} = \left\{{i \atop j\ k}\right\}$, then $\Gamma^i_{ji} = \left\{{i \atop j\ i}\right\} = \frac{\partial}{\partial x^j}(\log\sqrt{g})$, using equation (42.5). Hence

$$R_{jk} = \frac{\partial^2}{\partial x^j \partial x^k}(\log\sqrt{g}) - \frac{\partial}{\partial x^i}\left\{{i \atop j\ k}\right\} + \left\{{i \atop r\ k}\right\}\left\{{r \atop j\ i}\right\} - \left\{{r \atop j\ k}\right\}\frac{\partial}{\partial x^r}(\log\sqrt{g})$$

Since $\left\{{i \atop j\ k}\right\}$ is symmetric in j, k, and the order of partial differentiations in the first term can be reversed and, further,

$$\left\{{i \atop r\ k}\right\}\left\{{r \atop j\ i}\right\} = \left\{{r \atop i\ k}\right\}\left\{{i \atop j\ r}\right\} = \left\{{i \atop r\ j}\right\}\left\{{r \atop k\ i}\right\}$$

(after exchanging dummies r and i), it follows that R_{jk} is symmetric.

Also

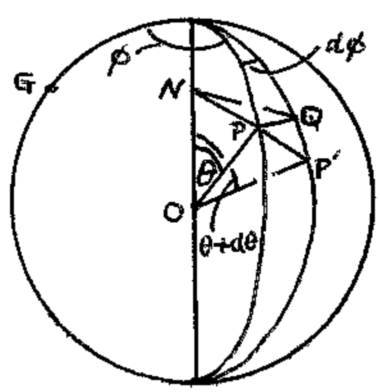
$$S_{k\ell} = B^i_{ik\ell} = \Gamma^i_{rk}\Gamma^r_{i\ell} - \Gamma^i_{r\ell}\Gamma^r_{ik} + \Gamma^i_{i\ell,k} - \Gamma^i_{ik,\ell}$$

Since $\Gamma^i_{r\ell}\Gamma^r_{ik} = \Gamma^r_{i\ell}\Gamma^i_{rk}$ (by exchange of dummies i and r) and

$$\Gamma^i_{i\ell,k} = \frac{\partial^2}{\partial x^k \partial x^\ell}(\log\sqrt{g}), \qquad \Gamma^i_{ik,\ell} = \frac{\partial^2}{\partial x^\ell \partial x^k}(\log\sqrt{g}),$$

it follows that $S_{k\ell} = 0$.

No.17.



P has colatitude θ and longitude ϕ. P' has coordinates (θ+dθ,ϕ+dϕ). PQ is a parallel of latitude with centre N. Since PN = sinθ, PQ = dϕsinθ, P'Q = dθ. To the second order of infinitesimals, we have

$$PP'^2 = P'Q^2 + PQ^2 \quad \text{or} \quad ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

With this metric, taking $x^1 = \theta$ and $x^2 = \phi$, we have

$$g_{11} = 1, \quad g_{22} = \sin^2\theta, \quad g_{12} = 0.$$

$$G^{11} = \sin^2\theta, \quad G^{22} = 1, \quad G^{12} = 0.$$

Hence

$$g^{11} = 1, \quad g^{22} = \operatorname{cosec}^2\theta, \quad g^{12} = 0.$$

We can now calculate the Christoffel symbols of the first kind:

$$[11,1] = \tfrac{1}{2}\partial g_{11}/\partial x^1 = 0, \quad [12,1] = [21,1] = \tfrac{1}{2}\partial g_{11}/\partial x^2 = 0$$

$$[22,1] = -\tfrac{1}{2}\partial g_{22}/\partial x^1 = -\sin\theta\cos\theta, \quad [11,2] = -\tfrac{1}{2}\partial g_{11}/\partial x^2 = 0,$$

$$[12,2] = [21,2] = \tfrac{1}{2}\partial g_{22}/\partial x^1 = \sin\theta\cos\theta, \quad [22,2] = \tfrac{1}{2}\partial g_{22}/\partial x^2 = 0.$$

Raising an index, we calculate that

$$\left\{ {1 \atop 1\ 1} \right\} = g^{1r}[11,r] = g^{11}[11,1] = 0$$

$$\left\{ {1 \atop 1\ 2} \right\} = \left\{ {1 \atop 2\ 1} \right\} = g^{1r}[12,r] = g^{11}[12,1] = 0$$

$$\left\{ {1 \atop 2\ 2} \right\} = g^{1r}[22,r] = g^{11}[22,1] = -\sin\theta\cos\theta$$

$$\left\{ {2 \atop 1\ 1} \right\} = g^{2r}[11,r] = g^{22}[11,2] = 0$$

$$\left\{ {2 \atop 1\ 2} \right\} = \left\{ {2 \atop 2\ 1} \right\} = g^{2r}[11,r] = g^{22}[11,2] = 0$$

$$\left\{ {2 \atop 2\ 2} \right\} = g^{2r}[22,r] = g^{22}[22,2] = 0$$

If $B_{ijk\ell}$ is not to vanish, the pair (i,j) must be different and the pair (k,ℓ) must be different. B_{1212} and the components obtained by exchanging the first two and/or the last two indices are therefore the only possible non-vanishing ones. We have

$$\begin{aligned} B_{1212} &= g_{1r}B^r_{212} = g_{11}B^1_{212} \\ &= g_{11}\left[\left\{ {1 \atop r\ 1} \right\}\left\{ {r \atop 2\ 2} \right\} - \left\{ {1 \atop r\ 2} \right\}\left\{ {r \atop 2\ 1} \right\} + \frac{\partial}{\partial x^1}\left\{ {1 \atop 2\ 2} \right\} - \frac{\partial}{\partial x^2}\left\{ {1 \atop 2\ 1} \right\}\right] \\ &= \sin\theta\cos\theta\cot\theta - \frac{\partial}{\partial\theta}(\sin\theta\cos\theta) \\ &= \sin^2\theta \end{aligned}$$

The other three non-zero components follow immediately due to the skew-symmetry.

For the Ricci tensor,

$$R_{11} = B^r_{11r} = g^{rs}B_{s11r} = g^{22}B_{2112} = -1$$

$$R_{12} = R_{21} = B^r_{12r} = g^{rs}B_{s12r} = g^{12}B_{2121} = 0$$

$$R_{22} = B^r_{22r} = g^{rs}B_{s22r} = g^{11}B_{1221} = -\sin^2\theta$$

It now follows that

No.18. The metric for cylindrical polar coordinates (ρ,ϕ,z) is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

and, hence,

$$g_{11} = 1, \quad g_{22} = \rho^2, \quad g_{33} = 1$$

the other components of the metric tensor vanishing. The determinant $g = g_{11}g_{22}g_{33} = \rho^2$ and the cofactors of its elements are

$$G^{11} = g_{22}g_{33} = \rho^2, \quad G^{22} = g_{33}g_{11} = 1, \quad G^{33} = g_{11}g_{22} = \rho^2,$$

the remaining cofactors being zero. Thus,

$$g^{11} = 1/g_{11} = 1, \quad g^{22} = 1/g_{22} = 1/\rho^2, \quad g^{33} = 1/g_{33} = 1,$$

the remaing components of the contravariant metric tensor vanishing.

We can now substitute into equation (42.9) to show that

$$\nabla^2 V = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial V}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial V}{\partial z}\right)\right]$$
$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In terms of spherical polar coordinates (r,θ,ϕ), the metric takes the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

giving

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta,$$
$$g = r^4\sin^2\theta,$$
$$g^{11} = 1, \quad g^{22} = 1/r^2, \quad g^{33} = 1/(r^2\sin^2\theta),$$

other components vanishing.

Thus

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial\phi}\right)\right]$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

No.19. Substituting for the affinity into equation (36.21), we find

$$B^i_{jk\ell} = (\delta^i_r\phi_{,k} + \delta^i_k\psi_{,r})(\delta^r_j\phi_{,\ell} + \delta^r_\ell\psi_{,j}) - (\delta^i_r\phi_{,\ell} + \delta^i_\ell\psi_{,r})(\delta^r_j\phi_{,k} + \delta^r_k\psi_{,j})$$
$$+ \frac{\partial}{\partial x^k}\left(\delta^i_j\phi_{,\ell} + \delta^i_\ell\psi_{,j}\right) - \frac{\partial}{\partial x^\ell}\left(\delta^i_j\phi_{,k} + \delta^i_k\psi_{,j}\right)$$
$$= \delta^i_j\phi_{,k}\phi_{,\ell} + \delta^i_k\psi_{,j}\psi_{,\ell} + \delta^i_k\phi_{,\ell}\psi_{,j} + \delta^i_\ell\phi_{,k}\psi_{,j} - \delta^i_j\phi_{,\ell}\phi_{,k} - \delta^i_\ell\psi_{,j}\psi_{,k}$$
$$- \delta^i_\ell\phi_{,k}\psi_{,j} - \delta^i_k\phi_{,\ell}\psi_{,j} + \delta^i_j\phi_{,\ell k} + \delta^i_\ell\psi_{,jk} - \delta^i_j\phi_{,k\ell} - \delta^i_k\psi_{,j\ell}$$

If $\psi = -\log(a_i x^i)$, then

$$\psi_{,j} = -\frac{a_j}{a_i x^i}, \qquad \psi_{,jk} = \frac{a_j a_k}{(a_i x^i)^2}.$$

Hence

$$\psi_{,j}\psi_{,k} - \psi_{,jk} = 0$$

from which it follows that $B^i_{jk\ell} = 0$ and, hence, $R_{jk} = 0$.

No.20. Putting $x^1 = r$, $x^2 = \theta$, the non-zero components of the metric tensor are

$$g_{11} = -\frac{a^2}{(r^2 - a^2)^2}, \qquad g_{22} = \frac{r^2}{r^2 - a^2}.$$

Substitution into equation (43.19) shows that the geodesics have equations

$$\frac{d}{ds}\left[-\frac{a^2}{(r^2 - a^2)^2}\frac{dr}{ds}\right] - \frac{2a^2 r}{(r^2 - a^2)^3}\left(\frac{dr}{ds}\right)^2 + \frac{a^2 r}{(r^2 - a^2)^2}\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left[\frac{r^2}{r^2 - a^2}\frac{d\theta}{ds}\right] = 0 \qquad \text{(i)}$$

We reject the first of these equations in favour of the known first integral (43.11), viz.

$$-\frac{a^2}{(r^2 - a^2)^2}\left(\frac{dr}{ds}\right)^2 + \frac{r^2}{r^2 - a^2}\left(\frac{d\theta}{ds}\right)^2 = 1 \qquad \text{(ii)}$$

It follows from (i) that

$$\frac{r^2}{r^2 - a^2}\frac{d\theta}{ds} = \text{constant} = 1/c \qquad \text{(iii)}$$

Eliminating ds between equations (ii) and (iii), we arrive at the equation

$$a^2\left(\frac{dr}{d\theta}\right)^2 + a^2 r^2 = (1 - c^2)r^4 = k^2 r^4,$$

writing $k^2 = 1 - c^2$.

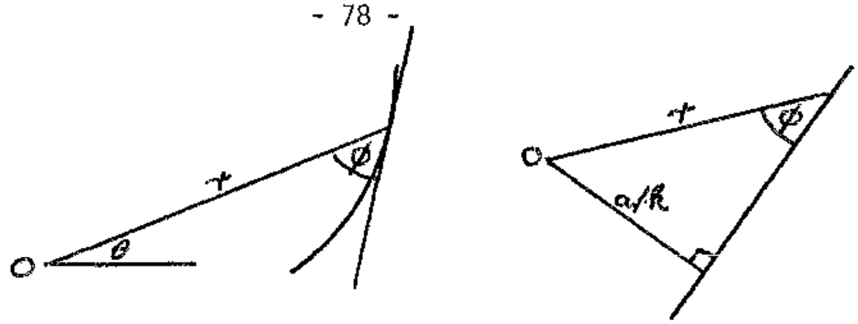
For a null-geodesic, we must replace s by the parameter λ and the right-hand member of equation (ii) by 0. Then, equation (ii) yields immediately

$$a^2\left(\frac{dr}{d\theta}\right)^2 + a^2 r^2 = r^4,$$

i.e. $k^2 = 1$.

Mapping the space on to a Euclidean plane by treating (r,θ) as polar coordinates, if ϕ is the angle between the tangent to a geodesic and the radius vector r, it is a well-known result from elementary differential geometry that $\tan\phi = r\,d\theta/dr$. Hence, along a geodesic,

$$a^2 r^2 \cot^2\phi + a^2 r^2 = k^2 r^4$$



perpendicular distance from O is a/k. It follows that the geodesics are mapped by straight lines. In the case of null-geodesics, k = 1 and the straight lines are tangents to the circle r = a.

No.21. Note that $g_{12} = 0$. Thus, $g = g_{11}g_{22}$ and $g^{11} = 1/g_{11}$, $g^{22} = 1/g_{22}$, $g^{12} = 0$.

The only non-zero components of $B_{ijk\ell}$ are B_{1212}, B_{2112}, B_{2121}, B_{1221}, and these only differ in sign.

For the Ricci tensor

$$R_{12} = B^{i}{}_{12i} = g^{ir}B_{r12i} = g^{12}B_{2121} = 0$$

$$R_{11} = B^{i}{}_{11i} = g^{ir}B_{r11i} = g^{22}B_{2112} = B_{1221}/g_{22}$$

$$R_{22} = B^{i}{}_{22i} = g^{ir}B_{r22i} = g^{11}B_{1221} = B_{1221}/g_{11}.$$

It now follows that

$$R = g^{ij}R_{ij} = g^{11}R_{11} + g^{22}R_{22} = 2B_{1221}/(g_{11}g_{22})$$

The tensor equation $R_{ij} = \frac{1}{2}Rg_{ij}$ is now immediately verifiable, component by component. It follows from (42.20) that Einstein's tensor vanishes in such a space. However, the coordinate frame can always be chosen (in an infinity of ways) in any $\mathcal{R}_2$ so that the metric takes the above form*, with $g_{12} = 0$. We conclude that Einstein's tensor is identically zero in any ${}_2$. A space for which the Einstein tensor vanishes is called an Einstein Space. Thus, every $\mathcal{R}_2$ is an Einstein space.

* If $g_{12} \neq 0$, transform to fresh coordinates $\overline{x}^i$. Then

$$\overline{g}_{12} = \frac{\partial x^i}{\partial \overline{x}^1}\frac{\partial x^j}{\partial \overline{x}^2} g_{ij}$$

Thus, the condition $\overline{g}_{12} = 0$ requires that the two functions $x^1(\overline{x}^1,\overline{x}^2)$, $x^2(\overline{x}^1,\overline{x}^2)$ satisfy one partial differential equation, which can be arranged in an infinity of ways.

No.22. (i) $A^{ij}{}_{;i} = A^{ij}{}_{,i} + \Gamma^i_{ki}A^{kj} + \Gamma^j_{ki}A^{ik}$

If $\Gamma^i_{jk} = \left\{ {i \atop j\ k} \right\}$, then $\Gamma^i_{ki} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g})$ (see equation (42.5)). Then

$$A^{ij}{}_{;i} = \frac{1}{\sqrt{g}}\left[\sqrt{g}A^{ij}{}_{,i} + A^{ij}\frac{\partial}{\partial x^i}(\sqrt{g})\right] + \left\{ {j \atop i\ k} \right\}A^{ik}$$

$$\text{(ii)}\ X^{ij}{}_{;i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}X^{ij}) + \left\{ {j \atop i\ k} \right\} X^{ik} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}X^{ij})$$

since $\left\{ {j \atop i\ k} \right\}$ is symmetric and X^{ik} is skew-symmetric in i, k. Thus

$$X^{ij}{}_{;ij} = \frac{\partial}{\partial x^j}(X^{ij}{}_{;i}) + \left\{ {i \atop r\ j} \right\} X^{rj}{}_{;i} + \left\{ {j \atop r\ j} \right\} X^{ir}{}_{;i} - \left\{ {r \atop i\ j} \right\} X^{ij}{}_{;r}$$

$$= \frac{\partial}{\partial x^j}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}X^{ij})\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}(\sqrt{g}).\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}X^{ir})$$

since the affinity is symmetric and $X^{ij}{}_{;k}$ is skew-symmetric. But

$$\frac{\partial}{\partial x^j}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}X^{ij})\right) = \frac{1}{\sqrt{g}}\frac{\partial^2}{\partial x^i \partial x^j}(\sqrt{g}X^{ij}) + \frac{\partial}{\partial x^j}\left(\frac{1}{\sqrt{g}}\right).\frac{\partial}{\partial x^i}(\sqrt{g}X^{ij})$$

$$= \frac{\partial}{\partial x^j}\left(\frac{1}{\sqrt{g}}\right).\frac{\partial}{\partial x^i}(\sqrt{g}X^{ij})$$

$$= -\frac{1}{2}g^{-3/2}\frac{\partial g}{\partial x^j}.\frac{\partial}{\partial x^i}(\sqrt{g}X^{ij})$$

Further

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}(\sqrt{g}).\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}X^{ir}) = \frac{1}{2}g^{-3/2}\frac{\partial g}{\partial x^r}.\frac{\partial}{\partial x^i}(\sqrt{g}X^{ir})$$

It now follows that $X^{ij}{}_{;ij} = 0$.

For any tensor A^{ij}, the tensor $B^{ij} = A^{ij} - A^{ji}$ is skew-symmetric. Hence, $B^{ij}{}_{;ij} = 0$, from which it follows that

$$A^{ij}{}_{;ij} = A^{ji}{}_{;ij} = A^{ij}{}_{;ji}$$

after exchange of dummies.

No.23. Writing $dx^k/dt = \dot{x}^k$, Euler's condition that the integral

$$\int_A^B F(x^k, \dot{x}^k)dt$$

should be stationary with respect to variation of the curve joining A and B, is

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) - \frac{\partial F}{\partial x^k} = 0$$

Thus, if $F = \sqrt{(g_{ij}\dot{x}^i\dot{x}^j)}$, noting that

$$\frac{\partial}{\partial \dot{x}^k}(g_{ij}\dot{x}^i\dot{x}^j) = 2g_{jk}\dot{x}^j$$

we are led to the equation

$$\frac{d}{dt}\left(\frac{1}{F}g_{jk}\dot{x}^j\right) - \frac{1}{2F}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = 0, \quad k = 1,2,\ldots,N. \qquad \text{(i)}$$

We use this equation to change the independent variable in (i) from t to s to yield

$$F\frac{d}{ds}\left(g_{jk}\frac{dx^j}{ds}\right) - \frac{1}{2}F\frac{\partial g^{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

After cancelling through by F, this reduces to equation (43.19).

Thus, a geodesic can be defined to be a curve joining two points, whose length is stationary with respect to small variations. For example, on the surface of a sphere (an $\mathcal{R}_2$), the great circle joining two points is either the minimum distance between the two points (the minor arc) or the maximum distance between the two points (the major arc, which together with the minor arc forms a complete great circle) - in either case it is stationary with respect to small variations and so constitutes a geodesic on the surface.

No.24. Γ^{i*}_{jk} is certainly an affinity, since it is the sum of an affinity and a tensor.

$$\Gamma^{i*}_{kj} = \Gamma^i_{kj} + \delta^i_k A_j + \delta^i_j A_k = \Gamma^i_{jk} + \delta^i_j A_k + \delta^i_k A_j = \Gamma^{i*}_{jk}$$

Using equation (36.21), we have

$$\begin{aligned}
B^{i*}_{jk\ell} &= (\Gamma^i_{rk} + \delta^i_r A_k + \delta^i_k A_r)(\Gamma^r_{j\ell} + \delta^r_j A_\ell + \delta^r_\ell A_j) \\
&\quad - (\Gamma^i_{r\ell} + \delta^i_r A_\ell + \delta^i_\ell A_r)(\Gamma^r_{jk} + \delta^r_j A_k + \delta^r_k A_j) \\
&\quad + \Gamma^i_{j\ell,k} + \delta^i_j A_{\ell,k} + \delta^i_\ell A_{j,k} - \Gamma^i_{jk,\ell} - \delta^i_j A_{k,\ell} - \delta^i_k A_{j,\ell} \\
&= B^i_{jk\ell} + \delta^i_k(A_j A_\ell + \Gamma^r_{j\ell}A_r - A_{j,\ell}) - \delta^i_\ell(A_j A_k + \Gamma^r_{jk}A_r - A_{j,k}) \\
&\quad + \delta^i_j(A_{\ell,k} - A_{k,\ell})
\end{aligned}$$

after simplification using the characteristic property of the Kronecker delta and cancellation of terms.

Since

$$A_{j;k} = A_{j,k} - \Gamma^r_{jk}A_r$$

we can write

$$A_j A_k + \Gamma^r_{jk}A_r - A_{j,k} = A_j A_k - A_{j;k} = A_{jk}$$

and then

$$B^{i*}_{jk\ell} = B^i_{jk\ell} + \delta^i_k A_{j\ell} - \delta^i_\ell A_{jk} + \delta^i_j(A_{\ell,k} - A_{k,\ell})$$

Also

$$A_{\ell,k} - A_{k,\ell} = A_{\ell;k} - A_{k;\ell} = A_{k\ell} - A_{\ell k}$$

Hence, finally,

$$B^{i*}_{jk\ell} = B^i_{jk\ell} + \delta^i_k A_{j\ell} - \delta^i_\ell A_{jk} + \delta^i_j(A_{k\ell} - A_{\ell k})$$

Suppose $A = \phi$. Then $A = \phi\,\phi - \phi$ and A is seen to

Exchanging j, ℓ, we have

$$B^{i*}_{\ell ij} = B^{i}_{\ell ij} + (N-1)A_{\ell j}$$

Since $A_{j\ell}$ is symmetric, it follows that

$$B^{i*}_{ji\ell} - B^{i*}_{\ell ij} = B^{i}_{ji\ell} - B^{i}_{\ell ij}$$

No.25. Transforming from the x-frame to the $\bar{x}$-frame, we have

$$\bar{\Gamma}^{i}_{jk} = \frac{\partial\bar{x}^{i}}{\partial x^{r}}\frac{\partial x^{s}}{\partial\bar{x}^{j}}\frac{\partial x^{t}}{\partial\bar{x}^{k}}\Gamma^{r}_{st} + \frac{\partial\bar{x}^{i}}{\partial x^{r}}\frac{\partial^2 x^{r}}{\partial\bar{x}^{j}\partial\bar{x}^{k}}$$

Transforming from the $\bar{x}$-frame to the $\bar{\bar{x}}$-frame, we have

$$\bar{\bar{\Gamma}}^{\ell}_{mn} = \frac{\partial\bar{\bar{x}}^{\ell}}{\partial\bar{x}^{i}}\frac{\partial\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}}\frac{\partial\bar{x}^{k}}{\partial\bar{\bar{x}}^{n}}\bar{\Gamma}^{i}_{jk} + \frac{\partial\bar{\bar{x}}^{\ell}}{\partial\bar{x}^{r}}\frac{\partial^2\bar{x}^{r}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}}$$

It now follows from this pair of equations that

$$\bar{\bar{\Gamma}}^{\ell}_{mn} = \frac{\partial\bar{\bar{x}}^{\ell}}{\partial\bar{x}^{i}}\frac{\partial\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}}\frac{\partial\bar{x}^{k}}{\partial\bar{\bar{x}}^{n}}\left(\frac{\partial\bar{x}^{i}}{\partial x^{r}}\frac{\partial x^{s}}{\partial\bar{x}^{j}}\frac{\partial x^{t}}{\partial\bar{x}^{k}}\Gamma^{r}_{st} + \frac{\partial\bar{x}^{i}}{\partial x^{r}}\frac{\partial^2 x^{r}}{\partial\bar{x}^{j}\partial\bar{x}^{k}}\right) + \frac{\partial\bar{\bar{x}}^{\ell}}{\partial\bar{x}^{r}}\frac{\partial^2\bar{x}^{r}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}}.$$

But

$$\frac{\partial\bar{\bar{x}}^{i}}{\partial\bar{x}^{j}}\frac{\partial\bar{x}^{j}}{\partial x^{k}} = \frac{\partial\bar{\bar{x}}^{i}}{\partial x^{k}}$$

Hence

$$\bar{\bar{\Gamma}}^{\ell}_{mn} = \frac{\partial\bar{\bar{x}}^{\ell}}{\partial x^{r}}\frac{\partial x^{s}}{\partial\bar{\bar{x}}^{m}}\frac{\partial x^{t}}{\partial\bar{\bar{x}}^{n}}\Gamma^{r}_{st} + \frac{\partial\bar{\bar{x}}^{\ell}}{\partial x^{r}}\frac{\partial\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}}\frac{\partial\bar{x}^{k}}{\partial\bar{\bar{x}}^{n}}\frac{\partial^2 x^{r}}{\partial\bar{x}^{j}\partial\bar{x}^{k}} + \frac{\partial\bar{\bar{x}}^{\ell}}{\partial\bar{x}^{r}}\frac{\partial^2\bar{x}^{r}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}} \qquad (i)$$

Now differentiate the identity

$$\frac{\partial x^{r}}{\partial\bar{x}^{j}}\frac{\partial\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}} = \frac{\partial x^{r}}{\partial\bar{\bar{x}}^{m}}$$

partially with respect to x^{n} to give

$$\frac{\partial^2 x^{r}}{\partial\bar{x}^{j}\partial\bar{x}^{k}}\frac{\partial\bar{x}^{k}}{\partial\bar{\bar{x}}^{n}}\frac{\partial\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}} + \frac{\partial x^{r}}{\partial\bar{x}^{j}}\frac{\partial^2\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}} = \frac{\partial^2 x^{r}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}}$$

Multiplying through by $\partial\bar{\bar{x}}^{\ell}/\partial x^{r}$, this leads to

$$\frac{\partial\bar{\bar{x}}^{\ell}}{\partial x^{r}}\frac{\partial\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}}\frac{\partial\bar{x}^{k}}{\partial\bar{\bar{x}}^{n}}\frac{\partial^2 x^{r}}{\partial\bar{x}^{j}\partial\bar{x}^{k}} + \frac{\partial\bar{\bar{x}}^{\ell}}{\partial\bar{x}^{j}}\frac{\partial^2\bar{x}^{j}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}} = \frac{\partial\bar{\bar{x}}^{\ell}}{\partial x^{r}}\frac{\partial^2 x^{r}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}}$$

This permits us to reduce equation (i) to the form

$$\bar{\bar{\Gamma}}^{\ell}_{mn} = \frac{\partial\bar{\bar{x}}^{\ell}}{\partial x^{r}}\frac{\partial x^{s}}{\partial\bar{\bar{x}}^{m}}\frac{\partial x^{t}}{\partial\bar{\bar{x}}^{n}}\Gamma^{r}_{st} + \frac{\partial\bar{\bar{x}}^{\ell}}{\partial x^{r}}\frac{\partial^2 x^{r}}{\partial\bar{\bar{x}}^{m}\partial\bar{\bar{x}}^{n}}$$

We have now proved that the 'product' of two affinity transformations is also an infinity transformation.

we must put $\bar{\bar{x}}^i = x^i$ in the last equation, which thereupon reduces to the product transformation

$$\bar{\bar{\Gamma}}^{\ell}_{mn} = \delta^{\ell}_{r}\delta^{s}_{m}\delta^{t}_{n}\Gamma^{r}_{st} + 0 = \Gamma^{\ell}_{mn},$$

which is the identity (or unit) transformation. This shows that the product of inverses is the identity. Thus, the transformations satisfy all the requirements for a group.

No.26. We have $(\nabla\phi)^2 = \nabla\phi\cdot\nabla\phi = g^{ij}\phi_{,i}\phi_{,j}$ by equations (38.4) and (38.5). Whence

$$\frac{\partial}{\partial x^k}(\nabla\phi)^2 = g^{ij}{}_{,k}\phi_{,i}\phi_{,j} + g^{ij}\phi_{,ik}\phi_{,j} + g^{ij}\phi_{,i}\phi_{,jk} = \phi_{,i}(g^{ij}{}_{,k}\phi_{,j} + 2g^{ij}\phi_{,jk}) \quad \text{(i)}$$

after exchanging the dummies (i,j) in the middle term of the right-hand member. Choosing the frame to be geodesic at the point under consideration, then $g^{ij}{}_{,k} = 0$ and equation (i) reduces to

$$\frac{\partial}{\partial x^k}(\nabla\phi)^2 = 2g^{ij}\phi_{,i}\phi_{,jk} = 2g^{ij}\phi_{,i}\phi_{;jk}$$

since $\phi_{,jk} = (\phi_{,j})_{,k} = (\phi_{;j})_{;k} = \phi_{;jk}$ in a geodesic frame. This proves the result in this frame. However, the result to be established is a tensor equation and, if valid in one frame, is true in all.

Alternatively, we can continue to work in a general frame and proceed as follows: We have

$$\phi_{;jk} = (\phi_{,j})_{;k} = \phi_{,jk} - \Gamma^{r}_{jk}\phi_{,r} = \phi_{,jk} - g^{rs}[jk,s]\phi_{,r}$$

so that

$$g^{ij}\phi_{,i}\phi_{;jk} = g^{ij}\phi_{,i}\phi_{,jk} - g^{ij}g^{rs}[jk,s]\phi_{,r}\phi_{,i} \quad \text{(ii)}$$

Differentiating the identity

$$g^{ij}g_{js} = \delta^{i}_{s}$$

with respect to x^k gives

$$g^{ij}{}_{,k}g_{js} + g^{ij}g_{js,k} = 0$$

Multiplying through by g^{rs}, this becomes

$$g^{ij}{}_{,k}\delta^{r}_{j} + g^{ij}g^{rs}g_{js,k} = 0$$

or

$$g^{ir}{}_{,k} = -g^{ij}g^{rs}g_{js,k} = -g^{ij}g^{rs}([jk,s] + [sk,j])$$

after using the identity (42.3). Thus

$$g^{ir}{}_{,k}\phi_{,i}\phi_{,r} = -g^{ij}g^{rs}[jk,s]\phi_{,i}\phi_{,r} - g^{ij}g^{rs}[sk,j]\phi_{,i}\phi_{,r} = -2g^{ij}g^{rs}[jk,s]\phi_{,i}\phi_{,r}$$

after exchanging the dummies (i,r) and the dummies (j,s) in the second term of the right-hand member.

Equation (ii) can now be written in the form

No.27. We have $\overline{g}_{ij} = e^{\sigma}g_{ij}$, so that $\overline{g} = e^{N\sigma}g$, $\overline{G}^{ij} = e^{(N-1)\sigma}G^{ij}$ and $\overline{g}^{ij} = e^{-\sigma}g^{ij}$.

It then follows that

$$\overline{[jk,r]} = \tfrac{1}{2}(\overline{g}_{jr,k} + \overline{g}_{kr,j} - \overline{g}_{jk,r})$$

$$= \tfrac{1}{2}e^{\sigma}(g_{jr,k} + g_{kr,j} - g_{jk,r} + g_{jr}\sigma_{,k} + g_{kr}\sigma_{,j} - g_{jk}\sigma_{,r})$$

$$= \tfrac{1}{2}e^{\sigma}([jk,r] + g_{jr}\sigma_{,k} + g_{kr}\sigma_{,j} - g_{jk}\sigma_{,r})$$

and thus

$$\overline{\Gamma}^i_{jk} = \overline{g}^{ir}\overline{[jk,r]} = \tfrac{1}{2}g^{ir}([jk,r] + g_{jr}\sigma_{,k} + g_{kr}\sigma_{,j} - g_{jk}\sigma_{,r})$$

$$= \Gamma^i_{jk} + \tfrac{1}{2}(\delta^i_j\sigma_{,k} + \delta^i_k\sigma_{,j} - g^{ir}g_{jk}\sigma_{,r})$$

$$= \Gamma^i_{jk} + A^i_{jk}$$

Since A^i_{jk} is the difference of two affinities defined in the same x-frame, it is a tensor; it is symmetric since the affinities are symmetric.

Using equation (36.21), we calculate that

$$\overline{B}^i_{jk\ell} = (\Gamma^i_{rk} + A^i_{rk})(\Gamma^r_{j\ell} + A^r_{j\ell}) - (\Gamma^i_{r\ell} + A^i_{r\ell})(\Gamma^r_{jk} + A^r_{jk})$$

$$+ \Gamma^i_{j\ell,k} + A^i_{j\ell,k} - \Gamma^i_{jk,\ell} - A^i_{jk,\ell}$$

$$= B^i_{jk\ell} + \Gamma^i_{rk}A^r_{j\ell} + \Gamma^r_{j\ell}A^i_{rk} - \Gamma^i_{r\ell}A^r_{jk} - \Gamma^r_{jk}A^i_{r\ell} + A^i_{rk}A^r_{j\ell}$$

$$- A^i_{r\ell}A^r_{jk} + A^i_{j\ell,k} - A^i_{jk,\ell}$$

$$= B^i_{jk\ell} + A^i_{j\ell;k} - A^i_{jk;\ell} + A^i_{rk}A^r_{j\ell} - A^i_{r\ell}A^r_{jk}$$

a pair of terms $\Gamma^r_{k\ell}A^i_{jr} - \Gamma^r_{\ell k}A^i_{jr}$ cancelling by virtue of the symmetry of the affinity.

Contracting, we find

$$\overline{R}_{jk} = \overline{B}^i_{jki} = R_{jk} + A^i_{ji;k} - A^i_{jk;i} + A^i_{rk}A^r_{ji} - A^i_{ri}A^r_{jk}$$

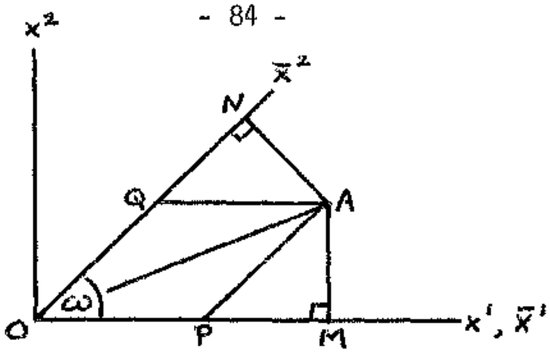
Finally

$$A^i_{ij} = \tfrac{1}{2}(\delta^i_j\sigma_{,i} + \delta^i_i\sigma_{,j} - g_{ij}g^{ir}\sigma_{,r})$$

$$= \tfrac{1}{2}(\sigma_{,j} + N\sigma_{,j} - \delta^r_j\sigma_{,r}) = \tfrac{1}{2}N\sigma_{,j}$$

No.28. Let $O\overline{x}^1\overline{x}^2$ be the oblique axes and Ox^1x^2 rectangular axes as indicated in the figure. If A has coordinates x^i, $\overline{x}^i$ in the two frames, then

$$\overline{x}^1 = OP = OM - PM = x^1 - x^2\cot\omega$$



Suppose $\underline{A}$ has components (A_1,A_2) in the rectangular frame. Choose the displacement vector $\underline{OA}$ such that the point A has coordinates (A_1,A_2) in the x-frame. The contravariant components of $\underline{A}$ transform like the differentials of the coordinates, viz.

$$\bar{A}^1 = A^1 - A^2\cot\omega = A_1 - A_2\cot\omega = OP$$

$$\bar{A}^2 = A^2\operatorname{cosec}\omega = A_2\operatorname{cosec}\omega = OQ$$

recalling that the covariant and contravariant components are identical in the rectangular frame. These equations show that the contravariant components of $\underline{A}$ in the $\bar{x}$-frame are obtained by projecting the displacement vector $\underline{OA}$ on to the axes, by parallels to the axes.

The inverse transformation equations for the coordinate differentials are

$$dx^1 = d\bar{x}^1 + d\bar{x}^2\cos\omega, \qquad dx^2 = d\bar{x}^2\sin\omega$$

Hence, the metric is

$$ds^2 = (dx^1)^2 + (dx^2)^2 = (d\bar{x}^1)^2 + 2d\bar{x}^1 d\bar{x}^2\cos\omega + (d\bar{x}^2)^2$$

and the components of the metric tensor in the oblique frame are $\bar{g}_{11} = \bar{g}_{22} = 1$, $\bar{g}_{12} = \cos\omega$.

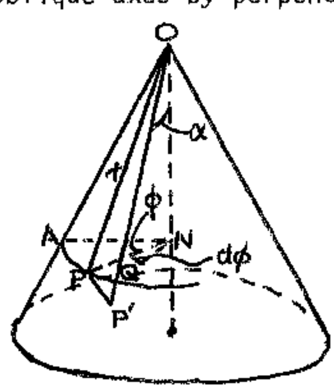
We can now calculate the covariant components of $\underline{A}$ in the oblique frame thus:

$$\bar{A}_1 = \bar{g}_{1j}\bar{A}^j = \bar{A}^1 + \cos\omega\bar{A}^2 = OP + OQ\cos\omega = OM$$

$$\bar{A}_2 = \bar{g}_{2j}\bar{A}^j = \cos\omega\bar{A}^1 + \bar{A}^2 = OP\cos\omega + OQ = ON$$

These equations show that the covariant components can be found by projecting $\underline{OA}$ on to the oblique axes by perpendiculars from A.

No.29.



P is any point on the conical surface. PN is the perpendicular from P to the axis ON. φ is defined to be the angle between the plane OPN

$$ds^2 = PP'^2 = dr^2 + r^2\sin^2\alpha d\phi^2$$

With $x^1 = r$, $x^2 = \phi$, the metric tensor has components $g_{11} = 1$, $g_{12} = 0$, $g_{22} = r^2\sin^2\alpha$. It follows that the equations (43.19) for the geodesics take the form

$$\frac{d^2r}{ds^2} - r\sin^2\alpha\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left(r^2\sin^2\alpha\frac{d\phi}{ds}\right) = 0$$

We discard the first equation and replace it by the metric as a first integral. The second equation shows that

$$ds = cr^2d\phi$$

where c is a constant. Elimination of ds between this equation and the metric yields the equation

$$rd\phi\sqrt{(c^2r^2 - \sin^2\alpha)} = dr$$

or

$$d\phi\sin\alpha = \frac{dr}{r\sqrt{(b^2r^2 - 1)}}$$

where $b = c\ \mathrm{cosec}\alpha$. Putting $r = 1/u$, this equation is seen to be equivalent to

$$d\phi\sin\alpha = -\frac{du}{\sqrt{(b^2 - u^2)}}$$

This integrates to

$$\phi\sin\alpha = \cos^{-1}(u/b) + \beta$$

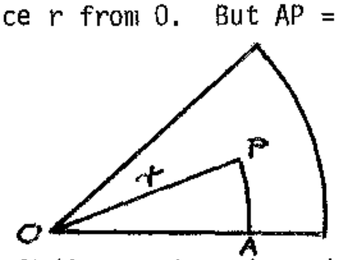
where β is a constant. It follows that

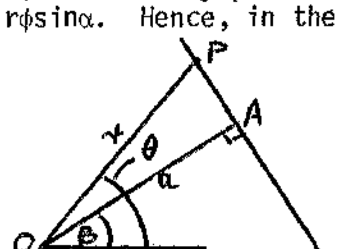
$$u = b\cos(\phi\sin\alpha - \beta)$$

or
$$r = a\sec(\phi\sin\alpha - \beta),$$

where $a = 1/b$.

If we cut the cone along the generator OA and open it out into a plane, the arc AP will continue to be circular, since every point on it has the same distance r from O. But $AP = AN.\phi = r\phi\sin\alpha$. Hence, in the plane,





the angle $AOP = AP/OP = \phi\sin\alpha$; i.e. $(r,\phi\sin\alpha)$ are polar coordinates in the plane. But geodesics in the plane are straight lines and any straight line AP has polar equation $r = OP = a\sec(\theta - \beta)$. Thus, in terms of (r,ϕ), the geodesics in the plane have equations $r = a\sec(\phi\sin\alpha - \beta)$. But these straight lines are curves of minimum distance joining two points and remain so when the plane is rolled back into the cone, i.e. they remain geodesics. This explains why $r = a\sec(\phi\sin\alpha - \beta)$ along a geodesic on the cone.

components of the Christoffel symbols of the first kind must have at least two indices the same and we calculate that

$$[QQ,P] = -\tfrac{1}{2}\partial g_{QQ}/\partial x_P = -\tfrac{1}{2}e^{\lambda}\lambda_P$$

$$[QQ,Q] = \tfrac{1}{2}\partial g_{QQ}/\partial x_Q = \tfrac{1}{2}e^{\lambda}\lambda_Q$$

$$[PQ,P] = \tfrac{1}{2}\partial g_{PP}/\partial x_Q = \tfrac{1}{2}e^{\lambda}\lambda_Q$$

provided $P \neq Q$. (N.B. Repetition of a capital letter index does not indicate summation.) The restriction $P \neq Q$ can now be removed by combining the first and second results into the formula

$$[QQ,P] = e^{\lambda}(\delta^P_Q - \tfrac{1}{2})\lambda_P.$$

The third result is clearly valid when $P = Q$.

The non-zero symbols of the second kind now follow immediately, viz.

$$\left\{ {P \atop Q\ Q} \right\} = g^{PP}[QQ,P] = (\delta^P_Q - \tfrac{1}{2})\lambda_P$$

$$\left\{ {P \atop P\ Q} \right\} = g^{PP}[PQ,P] = \tfrac{1}{2}\lambda_Q$$

Thus, summing with respect to i and r,

$$\left\{ {i \atop r\ P} \right\}\left\{ {r \atop P\ i} \right\} = \sum_{i=1}^{N} \left\{ {i \atop i\ P} \right\}\left\{ {i \atop P\ i} \right\} + \sum_{i \neq r} \left\{ {i \atop r\ P} \right\}\left\{ {r \atop P\ i} \right\}$$

where we have separated out the terms in which $i = r$ from those in which $i \neq r$. Then

$$\sum_{i=1}^{N} \left\{ {i \atop i\ P} \right\}\left\{ {i \atop P\ i} \right\} = \sum_{i=1}^{N} \tfrac{1}{4}\lambda_P^2 = \tfrac{1}{4}N\lambda_P^2$$

If $i \neq r$, for a non-zero contribution to the sum we must have either $i = P$ or $r = P$. If $i = P$, the net contribution is

$$\sum_{r \neq P} \left\{ {P \atop r\ P} \right\}\left\{ {r \atop P\ P} \right\} = -\tfrac{1}{4}\sum_{r \neq P} \lambda_r^2 = -\tfrac{1}{4}\lambda_r\lambda_r + \tfrac{1}{4}\lambda_P^2$$

If $r = P$, the net contribution is the same. Hence

$$\left\{ {i \atop r\ P} \right\}\left\{ {r \atop P\ i} \right\} = \tfrac{1}{4}N\lambda_P^2 - \tfrac{1}{2}\lambda_r\lambda_r + \tfrac{1}{2}\lambda_P^2 = \tfrac{1}{4}(N+2)\lambda_P^2 - \tfrac{1}{2}\lambda_r\lambda_r.$$

Now

$$R = g^{jk}R_{jk} = e^{-\lambda}\sum_{P=1}^{N} R_{PP}$$

and

$$R_{PP} = B^i_{PPi} = \left\{ {i \atop r\ P} \right\}\left\{ {r \atop P\ i} \right\} - \left\{ {i \atop r\ i} \right\}\left\{ {r \atop P\ P} \right\} + \frac{\partial}{\partial x^P}\left\{ {i \atop P\ i} \right\} - \frac{\partial}{\partial x^i}\left\{ {i \atop P\ P} \right\}$$

We calculate that

$$\left\{ {i \atop r\ i} \right\} = \sum_{i=1}^{N} \tfrac{1}{2}\lambda_r = \tfrac{1}{2}N\lambda_r$$

$$\frac{\partial}{\partial x^p}\left\{{i \atop p\ i}\right\} = \tfrac{1}{2}N\lambda_{pp}$$

$$\frac{\partial}{\partial x^i}\left\{{i \atop p\ p}\right\} = \sum_{i\neq p} (-\tfrac{1}{2}\lambda_{ii}) + \tfrac{1}{2}\lambda_{pp} = -\tfrac{1}{2}\lambda_{ii} + \lambda_{pp}$$

Substitution of these results shows that

$$R_{pp} = \tfrac{1}{4}(N+2)\lambda_p^2 - \tfrac{1}{2}\lambda_r\lambda_r + \tfrac{1}{4}N\lambda_r\lambda_r - \tfrac{1}{2}N\lambda_p^2 + \tfrac{1}{2}N\lambda_{pp} + \tfrac{1}{2}\lambda_{ii} - \lambda_{pp}$$

$$= -\tfrac{1}{4}(N-2)\lambda_p^2 + \tfrac{1}{4}(N-2)\lambda_r\lambda_r + \tfrac{1}{2}\lambda_{rr} + \tfrac{1}{2}(N-2)\lambda_{pp}$$

Thus

$$R = e^{-\lambda}\sum_{p=1}^{N} R_{pp} = e^{-\lambda}\left[-\tfrac{1}{4}(N-2)\lambda_r\lambda_r + \tfrac{1}{4}N(N-2)\lambda_r\lambda_r + \tfrac{1}{2}N\lambda_{rr} + \tfrac{1}{2}(N-2)\lambda_{rr}\right]$$

$$= (N-1)e^{-\lambda}\left[\tfrac{1}{4}(N-2)\lambda_r\lambda_r + \lambda_{rr}\right]$$

No.31. The non-zero components of the metric affinity for the sphere have been calculated in Ex.17 above. They are

$$\Gamma^1_{22} = -\sin\theta\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$$

Referring to equation (33.7), it follows that a parallel displacement of A_i from (θ,ϕ) to $(\theta+d\theta,\phi+d\phi)$ leads to increments

$$\delta A_1 = \Gamma^i_{1j}A_i dx^j = \cot\theta A_2 d\phi,$$

$$\delta A_2 = \Gamma^i_{2j}A_i dx^j = \cot\theta A_2 d\theta - \sin\theta\cos\theta A_1 d\phi.$$

Along the parallel of latitude $\theta = \alpha$, $d\theta = 0$ and the equations reduce to

$$\delta A_1 = A_2\cot\alpha d\phi, \quad \delta A_2 = -A_1\sin\alpha\cos\alpha d\phi$$

Thus, variation of A_i along this parallel is governed by the differential equations

$$\frac{dA_1}{d\phi} = A_2\cot\alpha, \quad \frac{dA_2}{d\phi} = -A_1\sin\alpha\cos\alpha$$

Eliminating A_2, we arrive at the equation

$$\frac{d^2A_1}{d\phi^2} + \cos^2\alpha A_1 = 0$$

whose general solution is

$$A_1 = P\cos(\phi\cos\alpha) + Q\sin(\phi\cos\alpha)$$

P and Q being integration constants determined by the initial conditions $A_1 = X$, $A_2 = Y$ at $\phi = 0$ (we choose the zero meridian through the initial point.

A_2 can now be found, thus:

$$A_2 = \frac{dA_1}{d\phi}\tan\alpha = \left\{-P\sin(\phi\cos\alpha) + Q\cos(\phi\cos\alpha)\right\}\sin\alpha$$

Substitution for P and Q now leads to the results stated.

If A is the magnitude of A_i, then

$$A^2 = g^{ij}A_iA_j = A_1^2 + \mathrm{cosec}^2\theta A_2^2$$

Thus, initially $A^2 = X^2 + Y^2\mathrm{cosec}^2\alpha$. Finally

$$\begin{aligned} A^2 &= \{X\cos(\phi\cos\alpha) + Y\mathrm{cosec}\alpha\sin(\phi\cos\alpha)\}^2 \\ &\quad + \{-X\sin\alpha\sin(\phi\cos\alpha) + Y\cos(\phi\cos\alpha)\}^2\mathrm{cosec}^2\alpha \\ &= X^2 + Y^2\mathrm{cosec}^2\alpha \end{aligned}$$

and A is unchanged.

No.32. We have $g_{11} = \lambda$, $g_{22} = r^2$, $g_{33} = r^2\sin^2\theta$, all other components vanishing. Thus, equation (43.19) for the geodesics yields the equations

$$\frac{d}{ds}\left(\lambda\frac{dr}{ds}\right) - \tfrac{1}{2}\lambda'\left(\frac{dr}{ds}\right)^2 - r\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \qquad \text{(i)}$$

$$\frac{d}{ds}\left(r^2\sin^2\theta\frac{d\phi}{ds}\right) = 0 \qquad \text{(ii)}$$

where $\lambda' = d\lambda/dr$. The metric also provides a first integral

$$\lambda\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 1 \qquad \text{(iii)}$$

If $\theta = \frac{1}{2}\pi$, $d\theta/ds = 0$ at $s = 0$, equation (i) shows that $d^2\theta/ds^2 = 0$ at this point also. Further, by differentiating this equation with respect to s once, twice, etc. times, we can show that derivatives of θ of all orders vanish at the initial point. It follows by Maclaurin's theorem that

$$\theta(s) = \theta(0) + s\theta'(0) + \frac{1}{2!}s^2\theta''(0) + \ldots = \tfrac{1}{2}\pi$$

i.e. along the geodesic θ takes the value $\frac{1}{2}\pi$ constantly.

Thus, putting $\theta = \frac{1}{2}\pi$ in equations (ii) and (iii) we are led to the equations

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0, \qquad \lambda\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 = 1.$$

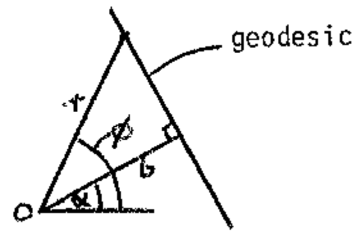
It follows that $d\phi/ds = b/r^2$, where b is constant. Then, eliminating ds from the second equation, we conclude that

$$\lambda\left(\frac{dr}{d\phi}\right)^2 + r^2 = r^4/b^2$$

Putting $r = b\sec\psi$, the last equation reduces to

$$\lambda\left(\frac{d\psi}{d\phi}\right)^2 = 1 \quad \text{or} \quad \frac{d\phi}{d\psi} = \lambda^{\frac{1}{2}}$$

If $\lambda = 1$, then $\phi = \psi$ + constant, implying that $r = b\sec(\phi-\alpha)$ (α constant). In this case, the $\mathcal{R}_3$ is Euclidean and (r,θ,ϕ) are spherical polar coordinates. The equation $r = b\sec(\phi-\alpha)$ gives the geodesics in the plane $\theta = \frac{1}{2}\pi$, in terms of polar coordinates (r,ϕ) in this plane. It is clear from the diagram that these are straight lines, as expected.



No.33. By trigonometry we can eliminate the parameters θ, ϕ, ψ, to give

$$(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = R^2$$

which is the equation of a hypersphere of radius R.

Differentiating, we find

$$dy^1 = -R\sin\theta d\theta$$

$$dy^2 = R\cos\theta\cos\phi d\theta - R\sin\theta\sin\phi d\phi$$

$$dy^3 = R\cos\theta\sin\phi\cos\psi d\theta + R\sin\theta\cos\phi\cos\psi d\phi - R\sin\theta\sin\phi\sin\psi d\psi$$

$$dy^4 = R\cos\theta\sin\phi\sin\psi d\theta + R\sin\theta\cos\phi\sin\psi d\phi + R\sin\theta\sin\phi\cos\psi d\psi$$

Whence

$$ds^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 + (dy^4)^2$$

$$= R^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2)$$

We now calculate that

$$g_{11} = R^2, \quad g_{22} = R^2\sin^2\theta, \quad g_{33} = R^2\sin^2\theta\sin^2\phi$$

$$g^{11} = 1/R^2, \quad g^{22} = 1/(R^2\sin^2\theta), \quad g^{33} = 1/(R^2\sin^2\theta\sin^2\phi)$$

$$\left\{{1 \atop 2\ 2}\right\} = -\sin\theta\cos\theta, \quad \left\{{1 \atop 3\ 3}\right\} = -\sin\theta\cos\theta\sin^2\phi, \quad \left\{{2 \atop 3\ 3}\right\} = -\sin\phi\cos\phi$$

$$\left\{{2 \atop 1\ 2}\right\} = \cot\theta, \quad \left\{{3 \atop 1\ 3}\right\} = \cot\theta, \quad \left\{{3 \atop 2\ 3}\right\} = \cot\phi$$

all other distinct components vanishing.

The component $B_{ijk\ell}$ of the curvature tensor vanishes if $i = j$ or $k = \ell$. Of the remaining components, we calculate that

$$B_{1212} = g_{11}B^1_{212} = g_{11}\left[\left\{{1 \atop r\ 1}\right\}\left\{{r \atop 2\ 2}\right\} - \left\{{1 \atop r\ 2}\right\}\left\{{r \atop 2\ 1}\right\} + \frac{\partial}{\partial x^1}\left\{{1 \atop 2\ 2}\right\} - \frac{\partial}{\partial x^2}\left\{{1 \atop 2\ 1}\right\}\right]$$

$$= R^2\left[-\left\{{1 \atop 2\ 2}\right\}\left\{{2 \atop 1\ 2}\right\} + \frac{\partial}{\partial\theta}\left\{{1 \atop 2\ 2}\right\}\right]$$

$$= R^2\sin^2\theta(-\cos^2\theta\sin^2\phi + \cos^2\phi + \sin^2\phi - \cos^2\phi)$$

$$= R^2\sin^4\theta\sin^2\phi$$

$$B_{3131} = g_{33}B^3_{131} = g_{33}\left[\begin{Bmatrix}3\\r\;3\end{Bmatrix}\begin{Bmatrix}r\\1\;1\end{Bmatrix} - \begin{Bmatrix}3\\r\;1\end{Bmatrix}\begin{Bmatrix}r\\1\;3\end{Bmatrix} + \frac{\partial}{\partial x^3}\begin{Bmatrix}3\\1\;1\end{Bmatrix} - \frac{\partial}{\partial x^1}\begin{Bmatrix}3\\1\;3\end{Bmatrix}\right]$$

$$= R^2\sin^2\theta\sin^2\phi(-\cot^2\theta + \operatorname{cosec}^2\theta) = R^2\sin^2\theta\sin^2\phi$$

The remaing 'mixed' components B_{1213}, B_{1223}, B_{1323}, vanish, e.g.

$$B_{1223} = g_{11}B^1_{223} = g_{11}\left[\begin{Bmatrix}1\\r\;2\end{Bmatrix}\begin{Bmatrix}r\\2\;3\end{Bmatrix} - \begin{Bmatrix}1\\r\;3\end{Bmatrix}\begin{Bmatrix}r\\2\;2\end{Bmatrix} + \frac{\partial}{\partial x^2}\begin{Bmatrix}1\\2\;3\end{Bmatrix} - \frac{\partial}{\partial x^3}\begin{Bmatrix}1\\2\;2\end{Bmatrix}\right]$$

$$= 0$$

If $i = j$ or $k = \ell$, then $g_{ik}g_{j\ell} - g_{i\ell}g_{jk}$ is obviously zero and the equation $B_{ijk\ell} = K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})$ is validated. If (i,j) is a different pair from (k,ℓ), each of the terms $g_{ik}g_{j\ell}$, $g_{i\ell}g_{jk}$ must contain a zero factor and, again, the equation reduces to $0 = 0$ and is validated.

The tensor $g_{ik}g_{j\ell} - g_{i\ell}g_{jk}$ is clearly skew-symmetric with respect to (i,j) and also with respect to (k,ℓ). Hence, it only remains to validate the equation for B_{1212}, B_{2323}, B_{3131}. We have

$$B_{1212} = K(g_{11}g_{22} - g_{12}g_{21}) = KR^4\sin^2\theta$$

which is true with $K = 1/R^2$. Further,

$$B_{2323} = K(g_{22}g_{33} - g_{23}g_{32}) = R^2\sin^4\theta\sin^2\phi$$

$$B_{3131} = K(g_{33}g_{11} - g_{13}g_{31}) = R^2\sin^2\theta\sin^2\phi$$

showing the equation is valid in all the other cases with $K = 1/R^2$.

No.34. Putting $g_{11} = g_{22} = \operatorname{sech}^2 y$, $g_{12} = 0$ in equations (43.19), we find the geodesics are determined by

$$\frac{d}{ds}\left(\operatorname{sech}^2 y\frac{dx}{ds}\right) = 0,$$

$$\frac{d}{ds}\left(\operatorname{sech}^2 y\frac{dy}{ds}\right) + \operatorname{sech}^2 y \tanh y\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right] = 0$$

The first of these equations integrates to yield

$$\frac{dx}{ds} = A\cosh^2 y,$$

where A is constant. The metric supplies a first integral

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = \cosh^2 y$$

Eliminating ds between the last two equations, we find

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{1}{A^2}\operatorname{sech}^2 y$$

where $\alpha^2 = \frac{1}{A^2} - 1$. Thus

$$x = \int \frac{dz}{\sqrt{(\alpha^2 - z^2)}} = \sin^{-1}(z/\alpha) + \text{constant}$$

or

$$\sinh y = z = \alpha \sin(x + \beta)$$

This is the equation of the geodesics.

The metric for the spherical surface is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Now

$$dy = \tan\tfrac{1}{2}\theta(-\text{cosec}^2\tfrac{1}{2}\theta)\tfrac{1}{2}d\theta = -\text{cosec}\theta d\theta$$

Also

$$e^y = \cot\tfrac{1}{2}\theta, \quad e^{-y} = \tan\tfrac{1}{2}\theta$$

so that

$$\cosh y = \tfrac{1}{2}(\tan\tfrac{1}{2}\theta + \cot\tfrac{1}{2}\theta) = \text{cosec}\theta$$

Hence

$$\text{sech} y \, dy = -d\theta$$

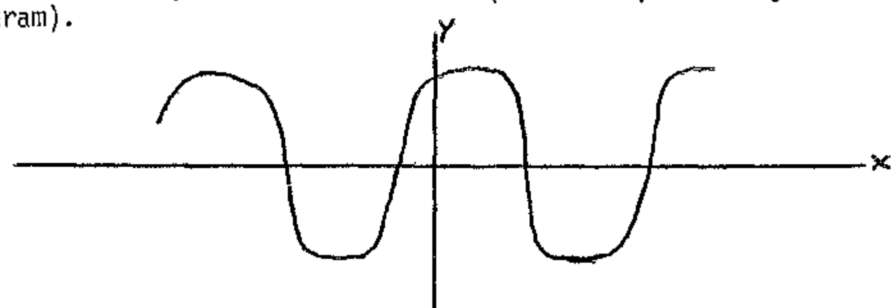
In terms of coordinates (x,y), the metric can now be expressed in the form

$$ds^2 = \text{sech}^2 y(dx^2 + dy^2)$$

Since the great circles on the sphere are geodesics, in terms of the coordinates (x,y) they must have equations

$$\sinh y = \alpha \sin(x + \beta)$$

If, as in Mercator's projection, the coordinates (x,y) are plotted as rectangular Cartesians, this equation shows that the great circles will appear as sine-waves, somewhat distorted (flattened) in the y-direction (see diagram).



No.35. $$B_{ijk\ell} = g_{is}B^s_{jk\ell} = g_{is}\left[\left\{{s \atop r\ k}\right\}\left\{{r \atop j\ \ell}\right\} - \left\{{s \atop r\ \ell}\right\}\left\{{r \atop j\ k}\right\} + \frac{\partial}{\partial x^k}\left\{{s \atop j\ \ell}\right\} - \frac{\partial}{\partial x^\ell}\left\{{s \atop j\ k}\right\}\right]$$

$$= [rk,i]\left\{{r \atop j\ \ell}\right\} - [r\ell,i]\left\{{r \atop j\ k}\right\} + g_{is}\frac{\partial}{\partial x^k}\left\{{s \atop j\ \ell}\right\} - g_{is}\frac{\partial}{\partial x^\ell}\left\{{s \atop j\ k}\right\}$$

But

$$g_{is}\frac{\partial}{\partial x^k}\left\{{s \atop j\ \ell}\right\} = \frac{\partial}{\partial x^k}\left[g_{is}\left\{{s \atop j\ \ell}\right\}\right] - g_{is,k}\left\{{s \atop j\ \ell}\right\} = \frac{\partial}{\partial x^k}[j\ell,i] - g_{is,k}\left\{{s \atop j\ \ell}\right\}$$

Hence,

$$B_{ijk\ell} = [sk,i]\left\{{s \atop j\ \ell}\right\} - [s\ell,i]\left\{{s \atop j\ k}\right\} - g_{is,k}\left\{{s \atop j\ \ell}\right\} + g_{is,\ell}\left\{{s \atop j\ k}\right\}$$

$$+ \frac{\partial}{\partial x^k}[j\ell,i] - \frac{\partial}{\partial x^\ell}[jk,i]$$

$$= [sk,i]\left\{ {s \atop j\ \ell} \right\} - [s\ell,i]\left\{ {s \atop j\ k} \right\} - ([ik,s] + [sk,i])\left\{ {s \atop j\ \ell} \right\}$$

$$+ ([i\ell,s] + [s\ell,i])\left\{ {s \atop j\ k} \right\} + \frac{\partial}{\partial x^k}[j\ell,i] - \frac{\partial}{\partial x^\ell}[jk,i]$$

$$= - [ik,s]\left\{ {s \atop j\ \ell} \right\} + [i\ell,s]\left\{ {s \atop j\ k} \right\} + \frac{\partial}{\partial x^k}[j\ell,i] - \frac{\partial}{\partial x^\ell}[jk,i]$$

having used the identity (42.3).

Since

$$[i\ell,s]\left\{ {s \atop j\ k} \right\} = g_{sr}\left\{ {r \atop i\ \ell} \right\}\left\{ {s \atop j\ k} \right\}$$

$$\frac{\partial}{\partial x^k}[j\ell,i] = \tfrac{1}{2}(g_{ij,k\ell} + g_{i\ell,jk} - g_{j\ell,ik})$$

the last identity is equivalent to

$$B_{ijk\ell} = \tfrac{1}{2}(g_{i\ell,jk} + g_{jk,i\ell} - g_{ik,j\ell} - g_{j\ell,ik})$$

$$+ g_{sr}\left\{ {r \atop i\ \ell} \right\}\left\{ {s \atop j\ k} \right\} - g_{sr}\left\{ {r \atop i\ k} \right\}\left\{ {s \atop j\ \ell} \right\}$$

The identities (41.5) - (41.7) will be found to follow directly from this result.

No.36. The components of the metric tensor are $g_{11} = g_{22} = 0$, $g_{12} = \phi$. Thus, $g = -\phi^2$, $G^{11} = G^{22} = 0$, $G^{12} = -\phi$; hence, $g^{11} = g^{22} = 0$, $g^{12} = 1/\phi$.

The christoffel symbols of the first kind are

$$[11,1] = 0, \quad [12,1] = 0, \quad [22,1] = \phi_y$$

$$[11,2] = \phi_x, \quad [12,2] = 0, \quad [22,2] = 0$$

The symbols of the second kind now follow thus:

$$\left\{ {1 \atop 1\ 1} \right\} = g^{12}[11,2] = \phi_x/\phi$$

$$\left\{ {1 \atop 1\ 2} \right\} = g^{12}[12,2] = 0$$

$$\left\{ {1 \atop 2\ 2} \right\} = g^{12}[22,2] = 0$$

$$\left\{ {2 \atop 1\ 1} \right\} = g^{21}[11,1] = 0$$

$$\left\{ {2 \atop 1\ 2} \right\} = g^{21}[12,1] = 0$$

$$\left\{ {2 \atop 2\ 2} \right\} = g^{21}[22,1] = \phi_y/\phi$$

Then

$$B_{1212} = g_{12}B^2_{212} = g_{12}\left[\left\{ {2 \atop r\ 1} \right\}\left\{ {r \atop 2\ 2} \right\} - \left\{ {2 \atop r\ 2} \right\}\left\{ {r \atop 2\ 1} \right\} + \frac{\partial}{\partial x}\left\{ {2 \atop 2\ 2} \right\} - \frac{\partial}{\partial x}\left\{ {2 \atop 2\ 1} \right\}\right]$$

$$= \phi\frac{\partial}{\partial x}(\phi_y/\phi) = \phi_{xy} - \phi_x\phi_y/\phi$$

the remaining 12 components of the curvature tensor vanishing.

The space is flat provided all components of the curvature tensor vanish. This is the case if $B_{1212} = 0$, i.e. if $\phi\phi_{xy} = \phi_x\phi_y$.

If $\phi = e^\psi$, then $\phi_x = \psi_x e^\psi$, $\phi_y = \psi_y e^\psi$, $\phi_{xy} = \psi_{xy}e^\psi + \psi_x\psi_y e^\psi$ and the condition for a flat space reduces to $\psi_{xy} = 0$. For this to be true, ψ_x must be a function of x alone and, hence,

$$\psi = f(x) + g(y)$$

In this case,

$$\phi = e^{f+g} = e^f e^g = F(x)\,G(y)$$

i.e. ϕ is a product of a function of x and a function of y. Thus

$$ds^2 = 2FGdxdy$$

Now choose (ξ,η) such that

$$\frac{d\xi}{dx} = F, \quad \frac{d\eta}{dy} = G$$

Then

$$ds^2 = 2\frac{d\xi}{dx}dx\frac{d\eta}{dy}dy = 2d\xi d\eta$$

No.37. It is being assumed the affinity is symmetric so that (see Ex.8 above)

$$A_{i;jk} - A_{i;kj} = A_r B^r_{ijk}$$

Cyclically permuting (i,j,k) we obtain the additional equations

$$A_{j;ki} - A_{j;ik} = A_r B^r_{jki}$$

$$A_{k;ij} - A_{k;ji} = A_r B^r_{kij}$$

Adding the first two equations and subtracting the third, we find that

$$A_{i;jk} - (A_{i;k} + A_{k;i})_{;j} + (A_{j;k} + A_{k;j})_{;i} - A_{j;ik}$$

$$= A_r(B^r_{ijk} + B^r_{jki} + B^r_{kij})$$

which reduces to

$$A_{i;jk} - A_{j;ik} = A_r(B^r_{ijk} + B^r_{jki} - B^r_{kij})$$

in view of the given condition.

But

$$A_{i;jk} - A_{j;ik} = (A_{i;j} - A_{j;i})_{;k} = 2A_{i;jk}$$

again using the given condition. Also, it follows from the identity (41.1) that

$$B^r_{ijk} + B^r_{jki} - B^r_{kij} = -2B^r_{kij}$$

No.38. The Cauchy-Riemann conditions which have to be satisfied by the real and imaginary parts of an analytic function are

$$\frac{\partial\phi}{\partial u} = \frac{\partial\psi}{\partial v}, \qquad \frac{\partial\phi}{\partial v} = -\frac{\partial\psi}{\partial u}$$

Differentiating the transformation equations $x = \phi$, $y = \psi$ we arrive at the differential transformation

$$dx = \frac{\partial\phi}{\partial u}du + \frac{\partial\phi}{\partial v}dv$$

$$dy = \frac{\partial\psi}{\partial u}du + \frac{\partial\psi}{\partial v}dv = -\frac{\partial\phi}{\partial v}du + \frac{\partial\phi}{\partial u}dv$$

Squaring and adding, we obtain the metric

$$ds^2 = dx^2 + dy^2 = \left[\left(\frac{\partial\phi}{\partial u}\right)^2 + \left(\frac{\partial\phi}{\partial v}\right)^2\right](du^2 + dv^2)$$

In the special case $\phi + i\psi = 1/w = (u-iv)/(u^2+v^2)$, we have

$$\phi = \frac{u}{u^2 + v^2}, \quad \psi = -\frac{v}{u^2 + v^2}, \quad \frac{\partial\phi}{\partial u} = \frac{v^2 - u^2}{(u^2 + v^2)^2}, \quad \frac{\partial\phi}{\partial v} = -\frac{2uv}{(u^2 + v^2)^2}$$

and so

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2}$$

$$ds^2 = dx^2 + dy^2 = \frac{du^2 + dv^2}{(u^2 + v^2)^2}$$

It follows that the transformation from coordinates (u,v) to coordinates (x,y) may be regarded as a transformation from curvilinear coordinates in a plane to rectangular Cartesian coordinates. In terms of the Cartesians (x,y), the geodesics form a family of straight lines with equation

$$ax + by = 1,$$

(a,b) being parameters. Thus, in terms of u and v, the equation for the family of geodesics must take the form

$$au - bv = u^2 + v^2 \qquad \text{(i)}$$

If, instead, we use the transformation equations $u = r\cos\theta$, $v = r\sin\theta$, then

$$du = dr\cos\theta - r\sin\theta d\theta, \qquad dv = dr\sin\theta + r\cos\theta d\theta$$

and the metric reduces to

$$ds^2 = r^{-4}dr^2 + r^{-2}d\theta^2$$

The equations governing the geodesics now follow by substitution in equation (43.19); they are

$$\frac{d}{ds}\left(r^{-4}\frac{dr}{ds}\right) + 2r^{-5}\left(\frac{dr}{ds}\right)^2 + r^{-3}\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = r^4$$

The second equation shows that $d\theta/ds = Ar^2$, where A is cinstant. Eliminating ds, we conclude that, along a geodesic

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = 1/A^2 = c^2$$

Thus, $d\theta/dr = (c^2 - r^2)^{-\frac{1}{2}}$ and integration leads to

$$\theta = \sin^{-1}(r/c) + \alpha$$

where α is constant. Hence

$$r = c\sin(\theta-\alpha) = c\sin\theta\cos\alpha - c\cos\theta\sin\alpha = -b\sin\theta + a\cos\theta$$

where $a = -c\sin\alpha$, $b = -c\cos\alpha$. Multiplying through by r, and using the transformation equations, we once again derive equation (i).

No.39. The repeated index summation convention is suspended in the solution to this exercise.

The transformation equations are

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r = \sqrt{(x^2 + y^2)}, \quad \theta = \tan^{-1}(y/x)$$

From these, it is easy to verify that

$$\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial\theta}{\partial x} = -\frac{1}{r}\sin\theta, \quad \frac{\partial\theta}{\partial y} = \frac{1}{r}\cos\theta$$

Also, $A_{xy} = \tan\theta + \cot\theta = 2\operatorname{cosec}2\theta$. Since there is no distinction between covariant and contravariant components in a rectangular Cartesian frame, we conclude that $A^{xx} = A^{yy} = 0$, $A^{xy} = A^{yx} = 2\operatorname{cosec}2\theta$.

The transformation equations for a second rank contravariant tensor now show that

$$A^{rr} = \left(\frac{\partial r}{\partial x}\right)^2 A^{xx} + 2\left(\frac{\partial r}{\partial x}\right)\left(\frac{\partial r}{\partial y}\right)A^{xy} + \left(\frac{\partial r}{\partial y}\right)^2 A^{yy} = 2$$

$$A^{r\theta} = \frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x}A^{xx} + \frac{\partial r}{\partial x}\frac{\partial\theta}{\partial y}A^{xy} + \frac{\partial r}{\partial y}\frac{\partial\theta}{\partial x}A^{yx} + \frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y}A^{yy} = \frac{2}{r}\cot2\theta$$

$$A^{\theta\theta} = \left(\frac{\partial\theta}{\partial x}\right)^2 A^{xx} + 2\left(\frac{\partial\theta}{\partial x}\right)\left(\frac{\partial\theta}{\partial y}\right)A^{xy} + \left(\frac{\partial\theta}{\partial y}\right)^2 A^{yy} = -\frac{2}{r^2}$$

That $A^{rr} + r^2 A^{\theta\theta} = 0$, now follows immediately.

No.40. Again, we suspend the summation convention.

$$ds^2 = dx^2 + dy^2 + dz^2 = (v^2+2)du + 2uvdudv + (u^2+2)dv$$

This shows that the covariant components of the metric tensor on the surface are $g_{uu} = v^2+2$, $g_{uv} = uv$, $g_{vv} = u^2+2$. The contravariant components follow from these, viz.

$$g^{uu} = (u^2+2)/g, \quad g^{uv} = -uv/g, \quad g^{vv} = (v^2+2)/g$$

where $g = (u^2+2)(v^2+2) - u^2v^2 = 2u^2 + 2v^2 + 4$.

We can now raise the indices of the covariant tensor A_{ij} thus:

$$A^{uu} = \sum g^{ur}g^{us}A_{rs}$$

$$= \frac{1}{g^2}\left[(u^2+2)^2A_{uu} - (u^2+2)uvA_{uv} - uv(u^2+2)A_{vu} + u^2v^2A_{vv}\right]$$

$$= \frac{1}{g^2}u^2(u^2 + v^2 + 2)^2 = \tfrac{1}{4}u^2 = \tfrac{1}{4}A_{uu}$$

$$A^{uv} = \sum g^{ur}g^{vs}A_{rs}$$

$$= \frac{1}{g^2}\left[-uv(u^2+2)A_{uu} + (u^2+2)(v^2+2)A_{uv} + u^2v^2A_{vu} - uv(v^2+2)A_{vv}\right]$$

$$= -\frac{1}{g^2}uv(u^2 + v^2 + 2)^2 = -\tfrac{1}{4}uv = \tfrac{1}{4}A_{uv}$$

$$A^{vv} = \sum g^{vr}g^{vs}A_{rs}$$

$$= \frac{1}{g^2}\left[u^2v^2A_{uu} - uv(v^2+2)A_{uv} - uv(v^2+2)A_{vu} + (v^2+2)^2A_{vv}\right]$$

$$= \frac{1}{g^2}v^2(u^2 + v^2 + 2)^2 = \tfrac{1}{4}v^2 = \tfrac{1}{4}A_{vv}$$

No.41. Differentiating the transformation equations partially with respect to x (keeping y constant), we obtain

$$1 = a \sinh u \cos v \frac{\partial u}{\partial x} - a \cosh u \sin v \frac{\partial v}{\partial x}$$

$$0 = a \cosh u \sin v \frac{\partial u}{\partial x} + a \sinh u \cos v \frac{\partial v}{\partial x}$$

Solving for $\partial u/\partial x$ and $\partial v/\partial x$, we calculate that

$$\frac{\partial u}{\partial x} = \frac{\sinh u \cos v}{a(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)}$$

$$\frac{\partial v}{\partial x} = -\frac{\cosh u \sin v}{a(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)}$$

Since

$$\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v = \sinh^2 u(1 - \sin^2 v) + (1 + \sinh^2 u)\sin^2 v$$

$$= \sinh^2 u + \sin^2 v = \tfrac{1}{2}(\cosh 2u - \cos 2v)$$

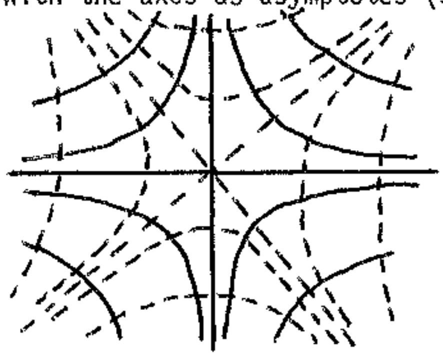
these results can be written

$$\frac{\partial u}{\partial x} = 2\sinh u \cos v/D \qquad \frac{\partial v}{\partial x} = -2\cosh u \sin v/D$$

$$A_x = \frac{\partial u}{\partial x}A_u + \frac{\partial v}{\partial x}A_v$$

$$= 2(A_u \sinh u \cos v - A_v \cosh u \sin v)/D.$$

No.42. The curves u = constant are rectangular hyperbolae with asymptotes $y = \pm x$. The curves v = constant are orthogonal to the first family and are rectangular hyperbolae with the axes as asymptotes (see diagram).



Differentiating the transformation equations, we find that the differentials are related thus:

$$du = xdx - ydy, \quad dv = ydx + xdy.$$

Solving for dx and dy, we get

$$(x^2 + y^2)dx = xdu + ydv, \quad (x^2 + y^2)dy = xdv - ydu$$

Hence,

$$ds^2 = dx^2 + dy^2 = (x^2 + y^2)^{-1}(du^2 + dv^2)$$

But

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 4(u^2 + v^2)$$

so that

$$ds^2 = \tfrac{1}{2}(u^2 + v^2)^{-\frac{1}{2}}(du^2 + dv^2).$$

Differentiating the transformation equations partially with respect to u (with v constant) gives

$$1 = x\frac{\partial x}{\partial u} - y\frac{\partial y}{\partial u}, \quad 0 = y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u}$$

Solving for $\partial x/\partial u$ and $\partial y/\partial u$, we calculate that

$$\frac{\partial x}{\partial u} = \frac{x}{x^2 + y^2}, \quad \frac{\partial y}{\partial u} = -\frac{y}{x^2 + y^2}$$

Thus, a transformation equation for the covariant vector A_i is

$$A_u = \frac{\partial x}{\partial u}A_x + \frac{\partial y}{\partial u}A_y = \frac{xA_x - yA_y}{x^2 + y^2}$$

Similarly, differentiating the coordinate transformation equations partially with respect to v and solving for $\partial x/\partial v$, $\partial y/\partial v$, we find that

$$\frac{\partial x}{\partial v} = \frac{y}{x^2 + y^2}, \quad \frac{\partial y}{\partial v} = \frac{x}{x^2 + y^2}$$

$$A_v = \frac{\partial x}{\partial v}A_x + \frac{\partial y}{\partial v}A_y = \frac{yA_x + xA_y}{x^2 + y^2}$$

No.43. Differentiating the transformation equations

$$dx = udu + vdv, \quad dy = vdu + udv$$

Hence,

$$ds^2 = dx^2 + dy^2 = (u^2+v^2)du^2 + 4uvdudv + (u^2+v^2)dv^2$$

gives the metric in terms of curvilinear coordinates.

The components of the metric tensor in the uv-frame are now found to be

$$g_{uu} = g_{vv} = u^2 + v^2, \quad g_{uv} = 2uv,$$

$$g^{uu} = g^{vv} = (u^2 + v^2)/g, \quad g^{uv} = -2uv/g$$

where $g = (u^2 - v^2)^2$.

We can now raise the index of the covariant vector A_i, thus:

$$A^u = g^{uu}A_u + g^{uv}A_v = (u - v)^2$$

$$A^v = g^{vu}A_u + g^{vv}A_v = (u - v)^2$$

The Christoffel symbols of the first kind are

$$[11,1] = u, \quad [12,1] = v, \quad [22,1] = u$$

$$[11,2] = v, \quad [12,2] = u, \quad [22,2] = v$$

leading to the following formulae for symbols of the second kind

$$\left\{ {1 \atop 1\ 1} \right\} = g^{11}[11,1] + g^{12}[11,2] = u/(u^2 - v^2)$$

$$\left\{ {1 \atop 1\ 2} \right\} = -v/(u^2 - v^2), \quad \left\{ {1 \atop 2\ 2} \right\} = u/(u^2 - v^2),$$

$$\left\{ {2 \atop 1\ 1} \right\} = -v/(u^2 - v^2), \quad \left\{ {2 \atop 1\ 2} \right\} = u/(u^2 - v^2),$$

$$\left\{ {2 \atop 2\ 2} \right\} = -v/(u^2 - v^2).$$

The divergence of $\underline{A}$ is given by

$$A^i{}_{;i} = A^i{}_{,i} + \Gamma^i_{ri}A^r$$

$$= A^1{}_{,1} + A^2{}_{,2} + \left\{ {1 \atop 1\ 1} \right\}A^1 + \left\{ {1 \atop 2\ 1} \right\}A^2 + \left\{ {2 \atop 1\ 2} \right\}A^1 + \left\{ {2 \atop 2\ 2} \right\}A^2$$

$$= 2(u-v) - 2(u-v) + \frac{u}{u^2-v^2}(u-v)^2 - \frac{v}{u^2-v^2}(u-v)^2$$

$$+ \frac{u}{u^2-v^2}(u-v)^2 - \frac{v}{u^2-v^2}(u-v)^2$$

$$= 2(u-v)^2/(u+v)$$

$$\delta A^u = -\left\{{1 \atop 1\;1}\right\}A^u du - \left\{{1 \atop 1\;2}\right\}A^u dv - \left\{{1 \atop 2\;1}\right\}A^v du - \left\{{1 \atop 2\;2}\right\}A^v dv$$

$$\delta A^v = -\left\{{2 \atop 1\;1}\right\}A^u du - \left\{{2 \atop 1\;2}\right\}A^u dv - \left\{{2 \atop 2\;1}\right\}A^v du - \left\{{2 \atop 2\;2}\right\}A^v dv$$

Along the curve $u = 0$, we have $du = 0$ and these reduce to

$$\delta A^u = -\left\{{1 \atop 1\;2}\right\}A^u dv - \left\{{1 \atop 2\;2}\right\}A^v dv = -\frac{1}{v}A^u dv$$

$$\delta A^v = -\left\{{2 \atop 1\;2}\right\}A^u dv - \left\{{2 \atop 2\;2}\right\}A^v dv = -\frac{1}{v}A^v dv$$

Thus, variations in A^u and A^v during parallel displacement along $u = 0$ are governed by the differential equations

$$\frac{dA^u}{A^u} = -\frac{dv}{v}, \qquad \frac{dA^v}{A^v} = -\frac{dv}{v}$$

The variables are separable in both cases. Integrating from the initial conditions $A^u = A^v = 1$ at $u = 0$, $v = 1$, we deduce that

$$A^u = A^v = 1/v$$

Thus, at the point $u = 0$, $v = 2$, the components of the parallel displaced vector are given by $A^u = A^v = \frac{1}{2}$.

No.44. In the usual notation:

$$g_{11} = 1, \quad g_{22} = x^2, \quad g_{12} = 0,$$

$$g^{11} = 1, \quad g^{22} = 1/x^2, \quad g^{12} = 0,$$

$$[11,1] = [12,1] = [11,2] = [22,2] = 0,$$

$$[12,2] = -[22,1] = x$$

$$\left\{{1 \atop 1\;1}\right\} = \left\{{1 \atop 1\;2}\right\} = \left\{{2 \atop 1\;1}\right\} = \left\{{2 \atop 2\;2}\right\} = 0,$$

$$\left\{{2 \atop 1\;2}\right\} = 1/x, \quad \left\{{1 \atop 2\;2}\right\} = -x.$$

The contravariant components of the vector field are given by

$$A^x = g^{11}A_x = x\cos 2y, \quad A^y = g^{22}A_y = -\sin 2y.$$

The divergence of the vector field is given by

$$A^i{}_{;i} = A^i{}_{,i} + \Gamma^i_{ji}A^j$$

$$= A^x{}_{,x} + A^y{}_{,y} + \left\{{2 \atop 1\;2}\right\}A^x$$

$$= \cos 2y - 2\cos 2y + \cos 2y = 0$$

No.45. We have

$$= \tfrac{1}{2}g^{rp}{}_{,j}(g_{tp,s} + g_{sp,t} - g_{st,p}) + \tfrac{1}{2}g^{rp}(g_{tp,sj} + g_{sp,tj} - g_{st,pj})$$

Differentiating with respect to x^i and ignoring terms containing first order derivatives of the metric tensor (since the frame is geodesic), we get

$$\frac{\partial^2}{\partial x^i \partial x^j}\left\{{r \atop s\ t}\right\} = \tfrac{1}{2}g^{rp}(g_{tp,ijs} + g_{sp,ijt} - g_{st,ijp})$$

as stated.

Now

$$R^i_j = g^{ik}R_{jk} = g^{ik}\left[\left\{{s \atop r\ k}\right\}\left\{{r \atop j\ s}\right\} - \left\{{s \atop r\ s}\right\}\left\{{r \atop j\ k}\right\} + \frac{\partial}{\partial x^k}\left\{{s \atop j\ s}\right\} - \frac{\partial}{\partial x^s}\left\{{s \atop j\ k}\right\}\right] \quad (i)$$

$$= g^{ik}\left[\frac{\partial}{\partial x^k}\left\{{s \atop j\ s}\right\} - \frac{\partial}{\partial x^s}\left\{{s \atop j\ k}\right\}\right]$$

for a geodesic frame. (N.B. Although the Christoffel symbols vanish at the point, their derivatives do not.)

Also, by differentiating (i) with respect to x^i and <u>then</u> ignoring terms containing first order derivatives of the metric tensor, <u>we</u> calculate that

$$R^i_{j,i} = g^{ik}\left[\frac{\partial^2}{\partial x^i \partial x^k}\left\{{s \atop j\ s}\right\} - \frac{\partial^2}{\partial x^i \partial x^s}\left\{{s \atop j\ k}\right\}\right]$$

But, in a geodesic frame, $R^i_{j,i} = R^i_{j;i}$ so that we have proved the second result.

Putting j = i in (i), we get

$$R = g^{ik}\left[\left\{{s \atop r\ k}\right\}\left\{{r \atop i\ s}\right\} - \left\{{s \atop r\ s}\right\}\left\{{r \atop i\ k}\right\} + \frac{\partial}{\partial x^k}\left\{{s \atop i\ s}\right\} - \frac{\partial}{\partial x^s}\left\{{s \atop i\ k}\right\}\right]$$

Differentiating with respect to x^j and neglecting terms containing first order derivatives of the metric tensor, we find

$$R_{,j} = g^{ik}\left[\frac{\partial^2}{\partial x^j \partial x^k}\left\{{s \atop i\ s}\right\} - \frac{\partial^2}{\partial x^j \partial x^s}\left\{{s \atop i\ k}\right\}\right]$$

which is the third result.

We now make use of the first identity to show that

$$R^i_{j;i} = \tfrac{1}{2}g^{ik}g^{sp}(g_{jp,iks} + g_{sp,ikj} - g_{js,ikp} - g_{jp,isk} - g_{kp,isj} + g_{jk,isp})$$

$$= \tfrac{1}{2}g^{ik}g^{sp}(g_{sp,ikj} - g_{js,ikp} - g_{kp,isj} + g_{jk,isp})$$

$$= \tfrac{1}{2}g^{ik}g^{sp}(g_{sp,ikj} - g_{kp,isj})$$

after exchanging the dummies (k,s) and (i,p) in the second term inside the bracket. Further,

$$R_{,j} = \tfrac{1}{2}g^{ik}g^{sp}(g_{ip,jks} + g_{sp,jki} - g_{is,jkp} - g_{ip,jsk} - g_{kp,jsi} + g_{ik,jsp})$$

after exchanging the dummies (i,p) and (s,k) in the second and fourth terms in the bracket.

It now follows that, in the geodesic frame, $R^i_{j;i} = \frac{1}{2}R_{,j}$. Since this is a tensor equation, we can conclude that it is valid in every frame.

No.46. In the usual notation:

$$g_{11} = y^2, \quad g_{22} = 1, \quad g_{12} = 0,$$

$$g^{11} = 1/y^2, \quad g^{22} = 1, \quad g^{12} = 0,$$

$$[11,1] = [22,1] = [12,2] = [22,2] = 0$$

$$[12,1] = -[11,2] = y$$

$$\left\{{1 \atop 1\ 1}\right\} = \left\{{1 \atop 2\ 2}\right\} = \left\{{2 \atop 1\ 2}\right\} = \left\{{2 \atop 2\ 2}\right\} = 0$$

$$\left\{{1 \atop 1\ 2}\right\} = 1/y, \quad \left\{{2 \atop 1\ 1}\right\} = -y$$

Thus,

$$\delta A_x = \delta A_1 = \Gamma^i_{1j}A_i dx^j = \left\{{1 \atop 1\ 2}\right\}A_1 dx^2 + \left\{{2 \atop 1\ 1}\right\}A_2 dx^1 = -yA_y dx + \frac{1}{y}A_x dy$$

$$\delta A_y = \delta A_2 = \Gamma^i_{2j}A_i dx^j = \left\{{1 \atop 2\ 1}\right\}A_1 dx^1 = \frac{1}{y}A_x dx$$

For parallel transfer along the curve $y = y(x)$, A_x and A_y are accordingly governed by the differential equations

$$\frac{dA_x}{dx} = \frac{1}{y}A_x\frac{dy}{dx} - yA_y, \quad \frac{dA_y}{dx} = \frac{1}{y}A_x$$

In particular, if $y = \sec x$,

$$\frac{dA_x}{dx} = A_x \tan x - A_y \sec x, \quad \frac{dA_y}{dx} = A_x \cos x \qquad \text{(i)}$$

Thus,

$$\cos x \frac{dA_x}{dx} = A_x \sin x - A_y$$

Eliminating A_y thus:

$$\frac{d}{dx}\left(\cos x \frac{dA_x}{dx}\right) = \frac{d}{dx}(A_x \sin x) - A_x \cos x$$

or

$$\cos x \frac{d^2A_x}{dx^2} - \sin x \frac{dA_x}{dx} = \sin x \frac{dA_x}{dx}$$

Putting $dA_x/dx = u$, this equation may be written

$$\frac{du}{u} = 2\tan x \, dx$$

which integrates immediately to give

where P is constant. Hence,

$$A_x = P\tan x + Q$$

whence, using the first of equations (i),

$$A_y = -P\cos x + Q\sin x$$

Initially, $A_x = 0$, $A_y = 1$ at the point (0,1). Thus, $P = -1$, $Q = 0$ and so

$$A_x = -\tan x, \quad A_y = \cos x.$$

At the point $(\pi/3,2)$, these equations show that $A_x = -\sqrt{3}$, $A_y = \frac{1}{2}$.

No.47. The metric for the sphere is

$$ds^2 = d\theta^2 + \cos^2\theta d\phi^2$$

and equation (43.19) shows that the great circle geodesics are determined by the equations

$$\frac{d^2\theta}{ds^2} + \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \quad \frac{d}{ds}\left(\cos^2\theta\frac{d\phi}{ds}\right) = 0.$$

The second equation integrates to give $ds = a\cos^2\theta d\phi$, where a is constant. Eliminating ds between this equation and the metric, we find

$$a^2\cos^4\theta d\phi^2 = d\theta^2 + \cos^2\theta d\phi^2$$

or

$$\frac{d\phi}{d\theta} = \frac{1}{\sqrt{(a^2\cos^4\theta - \cos^2\theta)}} = \frac{\sec^2\theta}{\sqrt{(b^2 - \tan^2\theta)}}$$

where $b^2 = a^2 - 1$. Integrating this equation, we calculate that

$$\phi = \sin^{-1}\left(\frac{1}{b}\tan\theta\right) - \beta$$

where β is constant. Thus

$$\tan\theta = b\sin(\phi + \beta)$$

which is the equation stated if we choose α so that $\tan\alpha = b$.

Since the point $\theta = 0$, $\phi = -\beta$ lies on the great circle, $-\beta$ can be identified as the longitude of the point where the great circle intersects the equator. Also, since the point $\theta = \alpha$, $\phi = \frac{1}{2}\pi-\beta$, also lies on the great circle, α can be identified as the latitude of the most northerly (or southerly) point on the great circle.

No.48. We calculate that

$$g_{11} = g_{22} = \text{sech}^2 y, \quad g_{12} = 0,$$

$$g^{11} = g^{22} = \cosh^2 y, \quad g^{12} = 0,$$

$$[11,2] = \text{sech}^2 y \tanh y = -[12,1] = -[22,2]$$

$$\left\{ {2 \atop 1\ 1} \right\} = \tanh y = -\left\{ {1 \atop 1\ 2} \right\} = -\left\{ {2 \atop 2\ 2} \right\}$$

$$\delta A_1 = \left\{ {i \atop 1\ j} \right\} A_i dx^j = \left\{ {2 \atop 1\ 1} \right\} A_2\, dx^1 + \left\{ {1 \atop 1\ 2} \right\} A_1\, dx^2$$

$$= A_2 \tanh y\, dx - A_1 \tanh y\, dy$$

$$\delta A_2 = \left\{ {i \atop 2\ j} \right\} A_i dx^j = \left\{ {1 \atop 2\ 1} \right\} A_1\, dx^1 + \left\{ {2 \atop 2\ 2} \right\} A_2\, dx^2$$

$$= -A_1 \tanh y\, dx - A_2 \tanh y\, dy$$

Along the line $y = b$, we have $dy = 0$ and the equations reduce to

$$\delta A_1 = A_2 \tanh b\, dx, \qquad \delta A_2 = -A_1 \tanh b\, dx$$

i.e. parallel displacement along this line is governed by the differential equations

$$\frac{dA_1}{dx} = A_2 \tanh b, \qquad \frac{dA_2}{dx} = -A_1 \tanh b \qquad \text{(i)}$$

Eliminating A_2, we find that A_1 satisfies the equation

$$\frac{d^2 A_1}{dx^2} + A_1 \tanh^2 b = 0,$$

whose general solution is

$$A_1 = X\cos(x\tanh b) + Y\sin(x\tanh b),$$

X and Y being constants of integration. Substituting in the first of equations (i), we then get

$$A_2 = -X\sin(x\tanh b) + Y\cos(x\tanh b)$$

At $x = 0$, we see that $A_1 = X$, $A_2 = Y$, so that (X,Y) are correctly identified as the initial components of the vector. The components of the vector in its final position at $x = a$ are now found to be as stated.

No.49. In the $\bar{x}$-frame, we can differentiate A^i covariantly thus:

$$\bar{A}^i{}_{;j} = \frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{\Gamma}^i_{kj} \bar{A}^k$$

Substituting from the tensor transformation equations

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^r} A^r, \qquad \bar{A}^i{}_{;j} = \frac{\partial \bar{x}}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^j} A^r{}_{;s},$$

into the first equation, we get

$$\frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^j} A^r{}_{;s} = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial A^r}{\partial x^s} + \frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s} \frac{\partial x^s}{\partial \bar{x}^j} A^r + \bar{\Gamma}^i_{kj} \frac{\partial \bar{x}^k}{\partial x^r} A^r$$

We next put

$$A^r{}_{;s} = \frac{\partial A^r}{\partial x^s} + \Gamma^r_{ts} A^t$$

and, after cancelling a pair of identical terms, arrive at the equation

$$\frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^j} \Gamma^r_{ts} A^t = \frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s} \frac{\partial x^s}{\partial \bar{x}^j} A^r + \bar{\Gamma}^i_{kj} \frac{\partial \bar{x}^k}{\partial x^r} A^r$$

$$\frac{\partial\bar{x}^i}{\partial x^t}\frac{\partial x^s}{\partial\bar{x}^j}\Gamma^t_{rs} = \frac{\partial^2\bar{x}^i}{\partial x^r\partial x^s}\frac{\partial x^s}{\partial\bar{x}^j} + \bar{\Gamma}^i_{kj}\frac{\partial\bar{x}^k}{\partial x^r}$$

Multiplying through by $\partial x^r/\partial\bar{x}^\ell$ and rearranging, we arrive at the transformation equation

$$\bar{\Gamma}^i_{\ell j} = \frac{\partial\bar{x}^i}{\partial x^t}\frac{\partial x^r}{\partial\bar{x}^\ell}\frac{\partial x^s}{\partial\bar{x}^j}\Gamma^t_{rs} - \frac{\partial^2\bar{x}^i}{\partial x^r\partial x^s}\frac{\partial x^r}{\partial\bar{x}^\ell}\frac{\partial x^s}{\partial\bar{x}^j} \qquad \text{(i)}$$

which is equivalent to the one given in the exercise.

If we differentiate the identity

$$\frac{\partial\bar{x}^i}{\partial x^r}\frac{\partial x^r}{\partial\bar{x}} = \delta^i_\ell$$

partially with respect to $\bar{x}^j$ (keeping the remaining $\bar{x}$s constant), we find that

$$\frac{\partial^2\bar{x}^i}{\partial x^r\partial x^s}\frac{\partial x^s}{\partial\bar{x}^j}\frac{\partial x^r}{\partial\bar{x}^\ell} + \frac{\partial\bar{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial\bar{x}^\ell\partial\bar{x}^j} = 0$$

Thus

$$-\frac{\partial^2\bar{x}^i}{\partial x^r\partial x^s}\frac{\partial x^r}{\partial\bar{x}^\ell}\frac{\partial x^s}{\partial\bar{x}^j} = \frac{\partial\bar{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial\bar{x}^\ell\partial\bar{x}^j}$$

and equation (i) is seen to be equivalent to equation (34.8).

No.50. We have

$$\begin{aligned}\bar{\Gamma}^i &= \bar{g}^{jk}\bar{\Gamma}^i_{jk}\\ &= \frac{\partial\bar{x}^j}{\partial x^r}\frac{\partial\bar{x}^k}{\partial x^s} g^{rs}\left[\frac{\partial\bar{x}^i}{\partial x^u}\frac{\partial x^v}{\partial\bar{x}^j}\frac{\partial x^w}{\partial\bar{x}^k}\Gamma^u_{vw} - \frac{\partial^2\bar{x}^i}{\partial x^u\partial x^v}\frac{\partial x^u}{\partial\bar{x}^j}\frac{\partial x^v}{\partial\bar{x}^k}\right]\\ &= g^{rs}\frac{\partial\bar{x}^i}{\partial x^u}\delta^v_r\delta^w_s\Gamma^u_{vw} - g^{rs}\frac{\partial^2\bar{x}^i}{\partial x^u\partial x^v}\delta^u_r\delta^v_s\\ &= g^{rs}\frac{\partial\bar{x}^i}{\partial x^u}\Gamma^u_{rs} - g^{rs}\frac{\partial^2\bar{x}^i}{\partial x^r\partial x^s}\\ &= \frac{\partial\bar{x}^i}{\partial x^u}\Gamma^u - g^{rs}\frac{\partial^2\bar{x}^i}{\partial x^r\partial x^s}\end{aligned}$$

Suppose g^{rs} and Γ^u are given over the space and $\bar{\Gamma}^i$ is to vanish everywhere; then we need

$$\frac{\partial\bar{x}^i}{\partial x^u}\Gamma^u = g^{rs}\frac{\partial^2\bar{x}^i}{\partial x^r\partial x^s} \qquad \text{(i)}$$

This is a set of N second order partial differential equations for the N unknown functions $\bar{x}^i(x)$; in general, these will have an infinity of solutions.

Note that

$$\nabla^2 V = \text{div grad}V = \left(g^{jk}\frac{\partial V}{\partial x^k}\right)_{;j} = g^{jk}\left(\frac{\partial V}{\partial x^k}\right)_{;j}$$

since the covariant derivative of the metric tensor vanishes. Hence

$$\nabla^2 \bar{x}^i = g^{jk}\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} - \Gamma^r \frac{\partial \bar{x}^i}{\partial x^r}$$

Equation (i) can now be written $\nabla^2 \bar{x}^i = 0$.

EXERCISES 6

No.1. At the point P, the y-frame is geodesic and the metric is given by

$$ds^2 = dy^k dy^k$$

(Note: Summation with respect to the index k is understood.) Now $dy^k = (\partial y^k/\partial x^i)dx^i$, so that at P

$$ds^2 = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j}dx^i dx^j,$$

showing that, in the x-frame, $g_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j}$ at this point.

Then, using equation (47.2),

$$T^i_j = g_{jk}T^{ik} = \frac{\partial y^p}{\partial x^j}\frac{\partial y^p}{\partial x^k}\frac{\partial x^i}{\partial y^r}\frac{\partial x^k}{\partial y^s}T^{(o)}_{rs}$$

$$= \delta^p_s \frac{\partial y^p}{\partial x^j}\frac{\partial x^i}{\partial y^r}T^{(o)}_{rs} = \frac{\partial y^s}{\partial x^j}\frac{\partial x^i}{\partial y^r}T^{(o)}_{rs}$$

in agreement with equation (47.3).

This shows that our definitions of T^{ij}, T^i_j, T_{ij} in the x-frame are consistent with the rules for raising and lowering indices.

No.2. $x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = t,$

$$g_{11} = g_{22} = 1, \quad g_{33} = e^{2\theta}, \quad g_{44} = -e^{2\phi}$$

$$g^{11} = g^{22} = 1, \quad g^{33} = e^{-2\theta}, \quad g^{44} = -e^{-2\phi}$$

The only distinct non-zero components of the Christoffel symbol of the second kind prove to be

$$\left\{ {3 \atop 3\ 3} \right\} = \theta', \quad \left\{ {3 \atop 4\ 4} \right\} = e^{2\phi-2\theta}\phi', \quad \left\{ {4 \atop 3\ 4} \right\} = \phi'$$

It now follows that the only distinct non-zero components of the curvature tensor $B^i_{jk\ell}$ are

$$B^3_{434} = e^{2\phi-2\theta}(\phi'' - \theta'\phi' + \phi'^2), \quad B^4_{334} = \phi'' - \theta'\phi' + \phi'^2$$

Hence, $B^i_{jk\ell} = 0$ provided $\phi'' - \theta'\phi' + \phi'^2 = 0$.

If $\phi = -\theta$, then for flat space-time we need

$$\phi'' + 2\phi'^2 = 0$$

or

$$\frac{d\phi'}{\phi'^2} = -2dz$$

Integration yields

$$-1/\phi' = -2z + \text{constant}$$

or

$$\frac{d\phi}{dz} = \frac{1}{2(z + \alpha)}$$

$$\phi \;=\; \tfrac{1}{2}\log(z + \alpha) + \text{constant}$$

Taking the constant in the form $\frac{1}{2}\log b$, we obtain the stated result.

No.3. Interpreting (r,ϕ,z) as cylindrical polar coordinates, the given metric has axial symmetry about the z-axis and can represent the gravitational field in the empty space surrounding a static distribution of matter possessing this type of symmetry. The proof that r and z can be defined in such a manner that $g_{33}g_{44} = -r^2$(as indicated in the metric), will be found on page 311 of Relativity: The General Theory by J L Synge (North Holland, 1971).

The derivation of the equations proceeds as below:

$$g_{11} = g_{22} = e^{\lambda}, \qquad g_{33} = r^2e^{-\rho}, \qquad g_{44} = -e^{\rho}$$
$$g^{11} = g^{22} = e^{-\lambda}, \qquad g^{33} = r^{-2}e^{\rho}, \qquad g^{44} = -e^{-\rho}$$

and the only distinct non-zero components of the Christoffel symbol of the second kind are listed thus:

$$\left\{ {1 \atop 1\ 1} \right\} = \tfrac{1}{2}\lambda_1, \quad \left\{ {2 \atop 1\ 1} \right\} = -\tfrac{1}{2}\lambda_2, \quad \left\{ {1 \atop 1\ 2} \right\} = \tfrac{1}{2}\lambda_2, \quad \left\{ {1 \atop 2\ 2} \right\} = -\tfrac{1}{2}\lambda_1,$$
$$\left\{ {2 \atop 2\ 2} \right\} = \tfrac{1}{2}\lambda_2, \quad \left\{ {2 \atop 1\ 2} \right\} = \tfrac{1}{2}\lambda_1, \quad \left\{ {1 \atop 3\ 3} \right\} = \tfrac{1}{2}(r^2\rho_1 - 2r)e^{-\rho-\lambda}$$
$$\left\{ {2 \atop 3\ 3} \right\} = \tfrac{1}{2}r^2\rho_2e^{-\rho-\lambda}, \quad \left\{ {3 \atop 1\ 3} \right\} = \frac{1}{r} - \tfrac{1}{2}\rho_1, \quad \left\{ {3 \atop 2\ 3} \right\} = -\tfrac{1}{2}\rho_2,$$
$$\left\{ {1 \atop 4\ 4} \right\} = \tfrac{1}{2}\rho_1e^{\rho-\lambda}, \quad \left\{ {2 \atop 4\ 4} \right\} = \tfrac{1}{2}\rho_2e^{\rho-\lambda}, \quad \left\{ {4 \atop 1\ 4} \right\} = \tfrac{1}{2}\rho_1, \quad \left\{ {4 \atop 2\ 4} \right\} = \tfrac{1}{2}\rho_2.$$

The non-zero components of the Ricci tensor now follow:

$$R_{11} = \left\{ {i \atop r\ 1} \right\}\left\{ {r \atop 1\ i} \right\} - \left\{ {i \atop r\ i} \right\}\left\{ {r \atop 1\ 1} \right\} + \frac{\partial}{\partial x^1}\left\{ {i \atop 1\ i} \right\} - \frac{\partial}{\partial x^i}\left\{ {i \atop 1\ 1} \right\}$$
$$= \tfrac{1}{2}\lambda_{11} + \tfrac{1}{2}\lambda_{22} - \frac{1}{2r}\lambda_1 + \tfrac{1}{2}\rho_1^2 - \frac{1}{r}\rho_1$$

$$R_{22} = \left\{ {i \atop r\ 2} \right\}\left\{ {r \atop 2\ i} \right\} - \left\{ {i \atop r\ i} \right\}\left\{ {r \atop 2\ 2} \right\} + \frac{\partial}{\partial x^2}\left\{ {i \atop 2\ i} \right\} - \frac{\partial}{\partial x^i}\left\{ {i \atop 2\ 2} \right\}$$
$$= \tfrac{1}{2}\lambda_{11} + \tfrac{1}{2}\lambda_{22} + \frac{1}{2r}\lambda_1 + \tfrac{1}{2}\rho_2^2$$

$$R_{33} = \left\{ {i \atop r\ 3} \right\}\left\{ {r \atop 3\ i} \right\} - \left\{ {i \atop r\ i} \right\}\left\{ {r \atop 3\ 3} \right\} + \frac{\partial}{\partial x^3}\left\{ {i \atop 3\ i} \right\} - \frac{\partial}{\partial x^i}\left\{ {i \atop 3\ 3} \right\}$$
$$= -\tfrac{1}{2}(r^2\rho_{11} + r^2\rho_{22} + r\rho_1)e^{-\rho-\lambda}$$

$$R_{44} = \left\{ {i \atop r\ 4} \right\}\left\{ {r \atop 4\ i} \right\} - \left\{ {i \atop r\ i} \right\}\left\{ {r \atop 4\ 4} \right\} + \frac{\partial}{\partial x^4}\left\{ {i \atop 4\ i} \right\} - \frac{\partial}{\partial x^i}\left\{ {i \atop 4\ 4} \right\}$$
$$= -\tfrac{1}{2}(\rho_{11} + \rho_{22} + \frac{1}{r}\rho_1)e^{\rho-\lambda}$$

$$R_{12} = \left\{ {i \atop r\ 2} \right\}\left\{ {r \atop 1\ i} \right\} - \left\{ {i \atop r\ i} \right\}\left\{ {r \atop 1\ 2} \right\} + \frac{\partial}{\partial x^2}\left\{ {i \atop 1\ i} \right\} - \frac{\partial}{\partial x^i}\left\{ {i \atop 1\ 2} \right\}$$
$$= \tfrac{1}{2}\rho_1\rho_2 - \frac{1}{2r}(\rho_2 + \lambda_2)$$

equivalent, we derive four equations, viz.

$$\lambda_{11} + \lambda_{22} - \frac{1}{r}\lambda_1 + \rho_1^2 - \frac{2}{r}\rho_1 = 0 \qquad \text{(i)}$$

$$\lambda_{11} + \lambda_{22} + \frac{1}{r}\lambda_1 + \rho_2^2 = 0 \qquad \text{(ii)}$$

$$\rho_{11} + \rho_{22} + \frac{1}{r}\rho_1 = 0 \qquad \text{(iii)}$$

$$\lambda_2 + \rho_2 = r\rho_1\rho_2 \qquad \text{(iv)}$$

The last two equations are included in the list given in the exercise. The remaining pair of equations in the list are reached by the operations (ii) - (i) and (i) + (ii) + 2(iii).

Note that, by putting $\nu = \lambda + \rho$, the stated equations can be reduced to the form

$$\nu_1 = \tfrac{1}{2}r(\rho_1^2 - \rho_2^2), \quad \nu_2 = r\rho_1\rho_2,$$

$$\nabla^2\rho + \frac{1}{r}\rho_1 = 0, \quad \nabla^2\nu + \tfrac{1}{2}(\rho_1^2 + \rho_2^2) = 0$$

for the unknown functions ρ and ν. (Note: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}$.) Then, differentiating the first pair of equations, we get

$$\nu_{11} = \tfrac{1}{2}(\rho_1^2 - \rho_2^2) + r(\rho_1\rho_{11} - \rho_2\rho_{12})$$

$$\nu_{22} = r(\rho_2\rho_{12} + \rho_1\rho_{22})$$

Addition now gives

$$\nabla^2\nu = \tfrac{1}{2}(\rho_1^2 - \rho_2^2) + r\rho_1\nabla^2\rho = -\tfrac{1}{2}(\rho_1^2 + \rho_2^2)$$

by appeal to the third equation. This shows that the fourth equation is derivable from the other three and so may be discarded.

Comments on the solution of these equations will be found on pages 312 - 317 of the book by Synge referred to earlier.

No.4. With $g_{11} = g_{22} = g_{33} = -g_{44} = e^{2kx}$, the equations (43.19) for geodesics through space-time take the form:

$$\frac{d}{ds}\left(e^{2kx}\frac{dx}{ds}\right) - ke^{2kx}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 - \left(\frac{dt}{ds}\right)^2\right] = 0$$

$$\frac{d}{ds}\left(e^{2kx}\frac{dy}{ds}\right) = \frac{d}{ds}\left(e^{2kx}\frac{dz}{ds}\right) = \frac{d}{ds}\left(e^{2kx}\frac{dt}{ds}\right) = 0$$

These determine the motion of a freely falling body in the gravitational field.

Integration of the last equation yields $dt/ds = Ae^{-2kx}$, where A is constant.

From the metric, we deduce that

$$e^{2kx}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 1) = (ds/dt)^2 = \frac{1}{A^2}e^{4kx}$$

If, when $x = 0$, then $v = V$, we find that $1/A^2 = V^2 - 1$. Thus

$$1 - v^2 = (1 - V^2)e^{2kx}$$

Note: For a photon, $ds = 0$ and $v = 1$. Thus, the velocity of light is unity in the units being used. Hence, v and V are less than unity for a particle.

No.5. As in the previous exercise, we obtain a geodesic equation

$$\frac{d}{ds}\left(\alpha\frac{dt}{ds}\right) = 0$$

which integrates to give

$$\frac{dt}{ds} = \frac{A}{\alpha}.$$

The metric now leads immediately to the equation

$$\alpha^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - c^2\alpha = (ds/dt)^2 = \alpha^2/A^2$$

or

$$v^2 = c^2\alpha^{-1} + \frac{1}{A^2} = - c^2kx + \text{constant}$$

If $v = V$ at $x = 0$, the constant is V^2 and so

$$V^2 - v^2 = c^2kx.$$

No.6. The given metric describes a static gravitational field directed along the z-axis and uniform over any plane parallel to the xy-plane. The field could be generated by a distribution of matter whose density was uniform over any such plane and which therefore extended to infinity over such a plane, e.g. an infinite uniform plane plate.

To calculate the field in empty space outside the matter distribution, we require $R_{jk} = 0$.

With the usual notation and listing only non-zero components,

$$g_{11} = g_{22} = g_{33} = e^{\alpha}, \quad g_{44} = -e^{\beta}$$

$$g^{11} = g^{22} = g^{33} = e^{-\alpha}, \quad g^{44} = -e^{-\beta}$$

$$\left\{{1 \atop 1\ 3}\right\} = -\left\{{3 \atop 1\ 1}\right\} = \left\{{2 \atop 2\ 3}\right\} = -\left\{{3 \atop 2\ 2}\right\} = \left\{{3 \atop 3\ 3}\right\} = \tfrac{1}{2}\alpha'$$

$$\left\{{4 \atop 3\ 4}\right\} = \tfrac{1}{2}\beta', \quad \left\{{3 \atop 4\ 4}\right\} = \tfrac{1}{2}\beta' e^{\beta-\alpha}$$

Hence

$$R_{11} = R_{22} = \tfrac{1}{2}\alpha'' + \tfrac{1}{4}\alpha'^2 + \tfrac{1}{4}\alpha'\beta',$$

$$R_{33} = \alpha'' + \tfrac{1}{2}\beta'' - \tfrac{1}{4}\alpha'\beta' + \tfrac{1}{4}\beta'^2$$

$$R_{44} = -(\tfrac{1}{2}\beta'' + \tfrac{1}{4}\alpha'\beta' + \tfrac{1}{4}\beta'^2)e^{\beta-\alpha}$$

leading to the Einstein equations stated.

Eliminating α'' and β'' between the Einstein equations, we deduce the equation $\alpha'(\tfrac{1}{2}\alpha' + \beta') = 0$.

and integration gives $\beta' = 2/(z + A)$, with A constant. Hence, $\beta = 2\log(z+A)$ + constant and

$$e^{\beta} = C(z + A)^2.$$

If $\frac{1}{2}\alpha' + \beta' = 0$, then eliminating α' from any one of the Einstein equations, we arrive at the equation

$$\beta'' - \tfrac{1}{2}\beta'^2 = 0,$$

showing that the equations are consistent. Two integrations now lead to the equations

$$\beta' = 2/(k - z), \qquad \beta = -2\log(k - z) + \text{constant}$$

Thus, $e^{\beta} = B(k - z)^{-2}$. It then follows that $\alpha = -2\beta$ + const. $= 4\log(k - z)$ + constant and so

$$e^{\alpha} = A(k - z)^4.$$

No.7. The non-zero Christoffel symbols are listed below:

$$\left\{{1 \atop 1\ 1}\right\} = \tfrac{1}{2}\alpha', \quad \left\{{1 \atop 2\ 2}\right\} = -re^{-\alpha}, \quad \left\{{1 \atop 3\ 3}\right\} = -\tfrac{1}{2}e^{\beta-\alpha}\beta',$$

$$\left\{{1 \atop 4\ 4}\right\} = \tfrac{1}{2}e^{\gamma-\alpha}\gamma', \quad \left\{{2 \atop 1\ 2}\right\} = \frac{1}{r}, \quad \left\{{3 \atop 1\ 3}\right\} = \tfrac{1}{2}\beta', \quad \left\{{4 \atop 1\ 4}\right\} = \tfrac{1}{2}\gamma'$$

It follows that the non-zero components of the Ricci tensor are:

$$R_{11} = \tfrac{1}{2}\beta'' + \tfrac{1}{2}\gamma'' + \tfrac{1}{4}\beta'^2 + \tfrac{1}{4}\gamma'^2 - \tfrac{1}{4}\alpha'\beta' - \tfrac{1}{4}\alpha'\gamma' - \frac{1}{2r}\alpha'$$

$$R_{22} = \tfrac{1}{2}re^{-\alpha}(\beta' + \gamma' - \alpha')$$

$$R_{33} = \tfrac{1}{2}e^{\beta-\alpha}(\beta'' + \tfrac{1}{2}\beta'^2 - \tfrac{1}{2}\alpha'\beta' + \tfrac{1}{2}\beta'\gamma' + \frac{1}{r}\beta')$$

$$R_{44} = -\tfrac{1}{2}e^{\gamma-\alpha}(\gamma'' + \tfrac{1}{2}\gamma'^2 - \tfrac{1}{2}\alpha'\gamma' + \tfrac{1}{2}\beta'\gamma' + \frac{1}{r}\gamma')$$

and these lead to the stated Einstein equations in vacuo.

Eliminating α' from the Einstein equations by appeal to the second equation, these reduce to

$$\beta'' + \gamma'' - \frac{1}{r}\beta' - \frac{1}{r}\gamma' - \beta'\gamma' = 0, \quad \beta'' + \frac{1}{r}\beta' = 0, \quad \gamma'' + \frac{1}{r}\gamma' = 0 \qquad \text{(i)}$$

The last pair of equations are easily integrated to yield $\beta = -\lambda\log r$ + const., $\gamma = -\mu\log r$ + const. Thus

$$e^{\beta} = Br^{-\lambda}, \qquad e^{\gamma} = Cr^{-\mu}.$$

Substitution for β and γ in the first of the equations (i) shows that $2(\lambda + \mu) = \lambda\mu$.

Finally,

$$\alpha = \beta + \gamma + \text{const.} = -(\lambda + \mu)\log r + \text{const.}$$

and so

$$e^{\alpha} = Ar^{-(\lambda+\mu)}$$

No.8. Geodesics through the space-time are governed by the equations

There are clearly solutions for which $y = z = 0$ identically and, hence, for which a falling particle moves along the x-axis. Putting $y = z = 0$ in the metric, we are led to the first integral

$$\left(\frac{ds}{dt}\right)^2 = \dot{x}^2 - x^2$$

where $\dot{x} = dx/dt$. But, from the third of equations (i), we deduce that $ds/dt = Ax^2$, where A is constant. Hence

$$\dot{x}^2 = x^2 + A^2x^4$$

Initially, $\dot{x} = 0$ at $x = 1$. Hence, $A^2 = -1$ (N.B. ds is always purely imaginary for a real particle). Thus

$$\dot{x}^2 = x^2 - x^4$$

Putting $x = \operatorname{sech}u$, this gives

$$\operatorname{sech}^2u \tanh^2u \, \dot{u}^2 = \operatorname{sech}^2u - \operatorname{sech}^4u = \operatorname{sech}^2u \tanh^2u$$

i.e. $\dot{u} = 1$. We conclude that $u = t + \alpha$ and so $x = \operatorname{sech}(t + \alpha)$. Since $x = 1$ at $t = 0$, we see that $\alpha = 0$ so that $x = \operatorname{sech}t$.

For a photon, the equations of a null-geodesic must be used, viz.

$$\left.\begin{aligned} \frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = \frac{d}{d\lambda}\left(x^2\frac{dt}{d\lambda}\right) &= 0, \\ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= x^2. \end{aligned}\right\} \qquad \text{(ii)}$$

Initially $x = 1$ and the motion is along the y-axis, so that $\dot{x} = \dot{z} = 0$. Substituting these values in the last of equations (ii), we find that $\dot{y}^2 = 1$; hence $\dot{y} = 1$ initially.

Integration of the first two equations (ii) shows that

$$dy/d\lambda = \text{const.}, \quad dz/d\lambda = \text{const.}, \quad x^2dt/d\lambda = \text{constant} \quad \text{(iii)}$$

But, initially $dz/d\lambda = \dot{z}(dt/d\lambda) = 0$; hence $dz/d\lambda = 0$ throughout the motion, showing that $z = 0$, i.e. the photon moves in the xy-plane. Dividing the first and third of equations (iii), we get $x^2dt/dy = \text{const.}$ Thus, $\dot{y} = x^2$, since initially $x = 1$, $\dot{y} = 1$. The last of equations (ii) now yields

$$\dot{x}^2 + x^4 = x^2,$$

whose solution is $x = \operatorname{sech}t$ as before. Thus $\dot{y} = x^2 = \operatorname{sech}^2t$ and so $y = \tanh t$, since $y = 0$ at $t = 0$.

It now follows that the photon moves in a circle since

$$x^2 + y^2 = \operatorname{sech}^2t + \tanh^2t = 1$$

No.9. The equations governing null-geodesics in the universe are

$$\frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$A^{-1}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 - Ac^2 = 0$$

The first two equations can be satisfied by taking θ and ϕ to be constant, confirming that radial motion of a photon is possible. The equations then reduce to the pair

$$\frac{d}{d\lambda}\left(A\frac{dt}{d\lambda}\right) = 0, \qquad \left(\frac{dr}{dt}\right)^2 = A^2c^2$$

Thus

$$\frac{dt}{dr} = \frac{R^2/c}{R^2 - r^2}$$

which, after integration, shows that

$$t = \frac{R}{2c}\log\left(\frac{R+r}{R-r}\right) + \text{constant}$$

Since $r = 0$ at $t = 0$, the constant vanishes. Solving for r, we find that

$$r = R\tanh(ct/R)$$

As $t \to +\infty$, $\tanh(ct/R) \to +1$ and so $r \to R$.

Putting $r = \frac{1}{2}R$ in the equation for t, we get $t = \frac{R}{2c}\log 3$.

No.10. Equations for the geodesics are:

$$\frac{d}{ds}\left(r^2\frac{dr}{ds}\right) - r\left(\frac{dr}{ds}\right)^2 - r\left(\frac{d\theta}{ds}\right)^2 - \tfrac{1}{2}\left(\frac{dz}{ds}\right)^2 + \tfrac{1}{2}\left(\frac{dt}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = \frac{d}{ds}\left(r\frac{dz}{ds}\right) = \frac{d}{ds}\left(-r\frac{dt}{ds}\right) = 0$$

Integrating the last three equations, we deduce that

$$\frac{d\theta}{ds} = \frac{A}{r^2}, \qquad \frac{dz}{ds} = \frac{B}{r}, \qquad \frac{dt}{ds} = \frac{C}{r}, \tag{i}$$

where A, B, C, are constants. Dividing the last two equations, we see that $\dot{z} = B/C$ and, since $\dot{z} = 0$ initially, this means $B = 0$. Thus $z =$ constant $= 0$ throughout the motion, which lies in the quasi-plane $z = 0$.

Putting $z = 0$ in the metric, we have the first integral

$$r^2\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 - r\left(\frac{dt}{ds}\right)^2 = 1$$

Using equations (i), this leads to

$$C^2\dot{r}^2 + \frac{A^2}{r^2} - \frac{C^2}{r} = 1 \tag{ii}$$

Dividing the first and last of equations (i), we find

$$\dot{\theta} = \frac{A}{Cr} \tag{iii}$$

Hence, by the initial conditions, $A/C = \sqrt{3}/2$. Also, by substituting initial values in (ii), we calculate that $A^2 - C^2 = 1$. We can now deduce

The last equation proves that $\dot{r}$ can only be real if $1 \leqslant r \leqslant 3$ and the particle's trajectory accordingly lies between the circles $r = 1$ and $r = 3$. $\dot{r}$ can only change sign when $r = 1$ or $r = 3$; it follows that r increases to the value 3 and then decreases to the value 1 alternately.

Dividing (iv) by the square of equation (iii), we derive an equation for the trajectory in the form

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{3}(3 - r)(r - 1)$$

i.e.

$$\theta = \int \frac{3dr}{\sqrt{\{1 - (r-2)^2\}}} = 3\sin^{-1}(r-2) + \text{const.}$$

Since $\theta = 0$ when $r = 1$, the constant is $3\pi/2$. The polar equation of the trajectory is therefore

$$r = 2 - \cos(\theta/3).$$

For a photon, the equations (i) are replaced by

$$\frac{d\theta}{d\lambda} = \frac{A}{r^2}, \quad \frac{dz}{d\lambda} = \frac{B}{r}, \quad \frac{dt}{d\lambda} = \frac{C}{r}$$

and the first integral by

$$r^2\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 + r\left(\frac{dz}{d\lambda}\right)^2 - r\left(\frac{dt}{d\lambda}\right)^2 = 0$$

If initially $z = 0$, $\dot{z} = 0$, then $B = 0$ and $z = 0$ throughout the motion. Thus,

$$\frac{C^2}{r^2}\left(\frac{dr}{d\theta}\right)^2 + \frac{A^2}{r^2} - \frac{C^2}{r} = 0$$

Initially, $dr/d\theta = \dot{r}(dt/d\theta) = 0$ and $r = 1$. Hence, $A^2/C^2 = 1$ and we deduce that

$$\left(\frac{dr}{d\theta}\right)^2 = r - 1$$

or

$$\theta = \int \frac{dr}{\sqrt{(r - 1)}} = 2\sqrt{(r - 1)} + \text{const.}$$

Since $\theta = 0$ when $r = 1$, the constant of integration vanishes and we deduce that $r = 1 + \frac{1}{4}\theta^2$.

No.11. Along a geodesic

$$\frac{d}{ds}\left(e^{2ct/R}\frac{dx}{ds}\right) = \frac{d}{ds}\left(e^{2ct/R}\frac{dy}{ds}\right) = \frac{d}{ds}\left(e^{2ct/R}\frac{dz}{ds}\right) = 0$$

which integrate to give

$$\frac{dx}{ds} = Ae^{-2ct/R}, \quad \frac{dy}{ds} = Be^{-2ct/R}, \quad \frac{dz}{ds} = Ce^{-2ct/R}$$

It follows immediately that

$$dy/dx = B/A, \quad dz/dx = C/A$$

so that

For the motion of a photon, the argument can be repeated with s replaced by λ.

If a particle is projected from the origin along the x-axis, it will continue to move along the x-axis so that $y = z = 0$. Thus, the metric first integral for the particle's motion reduces to

$$e^{2ct/R} - c^2\left(\frac{dt}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2 = \frac{1}{A^2}e^{4ct/R} \qquad \text{(i)}$$

At $t = 0$, $\dot{x} = V$ and, thus, $1/A^2 = 1 - c^2/V^2$. (i) now reduces to

$$\dot{x} = \frac{cVe^{-2ct/R}}{\surd(c^2 - V^2 + V^2e^{-2ct/R})}$$

Integrating (change the variable to $e^{-2ct/R}$), we find that

$$Vx = R\{c - \surd(c^2 - V^2 + V^2e^{-2ct/R})\},$$

since $x = 0$ at $t = 0$.

For a photon moving along the x-axis, equation (i) is replaced by

$$e^{2ct/R} - c^2\left(\frac{dt}{dx}\right)^2 = 0$$

or

$$\dot{x} = -ce^{-ct/R}$$

for motion in the negative sense. Thus,

$$x = Re^{-ct/R} + \text{const.}$$

If $x = X$ at $t = 0$, the integration constant is $X - R$ and

$$x = X - R(1 - e^{-ct/R}). \qquad \text{(ii)}$$

Then $x = 0$ when

$$e^{-ct/R} = 1 - X/R$$

or

$$t = -\frac{R}{c}\log(1 - X/R).$$

Equation (ii) shows that x decreases steadily towards the value $X - R$ as $t \to +\infty$. Hence, if $X > R$, the photon never reaches 0. The exponential time factor present in the metric, implies that the universe expands at a steadily increasing rate and this increasingly retards the photon's motion in the xyz-frame, ultimately ($t = +\infty$) bringing it to rest. (Note however the comments at the end of section 66 regarding the unreal nature of this expansion.)

No.12. Equations for geodesics through the space-time are

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left(r^2\sin^2\theta\frac{d\phi}{ds}\right) = \frac{d}{ds}\left(\frac{r}{r+2}\,\frac{dt}{ds}\right) = 0$$

These, together with the metric, determine the motion of a freely falling identically, showing that the trajectories lying in this quasi-plane are possible. The initial conditions $\theta = \frac{1}{2}\pi$, $\dot{\theta} = 0$, indicate that the trajectory to be calculated is of this type.

Integrating the other two equations, we get

$$\frac{d\phi}{ds} = \frac{A}{r^2}, \qquad \frac{dt}{ds} = B(r+2)/r \qquad \text{(i)}$$

Also, the metric yields the equation

$$\left(\frac{r}{r+1}\right)^2\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - \frac{r}{r+2}\left(\frac{dt}{ds}\right)^2 = 1$$

Eliminating $d\phi/ds$ and dt/ds between these equations, we find that

$$\left(\frac{r}{r+1}\right)^2\left(\frac{dr}{ds}\right)^2 + \frac{A^2}{r^2} - B^2\frac{r+2}{r} = 1$$

or

$$\frac{A^2}{r^2(r+1)^2}\left(\frac{dr}{d\phi}\right)^2 + \frac{A^2}{r^2} - B^2\frac{r+2}{r} = 1 \qquad \text{(ii)}$$

Dividing the equations (i), we have $\dot{\phi} = \frac{A}{Br(r+2)}$. Substituting initial values, we then find $A/B = \sqrt{(3/2)}$. Also, putting initial values in (ii), we get $A^2 - 3B^2 = 1$. Hence, $A^2 = -1$, $B^2 = -2/3$. (ii) now leads to the equation

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{1}{3}(r+1)^2(3-r)(r-1) \qquad \text{(iii)}$$

Putting $u = 1/(r+1)$, this equation reduces to;

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{3}(2u-1)(4u-1)$$

or

$$\phi = \sqrt{\frac{3}{8}}\int\frac{du}{\sqrt{\{1/64 - (u-3/8)^2\}}} = \sqrt{\frac{3}{8}}\sin^{-1}(8u-3) + \text{const.}$$

Initially $u = \frac{1}{2}$, $\phi = 0$, so that the constant of integration is $-\sqrt{(3/8)}(\pi/2)$. We now deduce that

$$8u - 3 = \sin(a\phi + \tfrac{1}{2}\pi) = \cos(a\phi)$$

or

$$r = \frac{5 - \cos(a\phi)}{3 + \cos(a\phi)}.$$

Since $\cos(a\phi)$ oscillates between the values ± 1, r oscillates between the values 1 and 3 (this also follows directly from (iii)). As $a\phi$ increases by 2π, r makes one complete oscillation. Thus, two successive contacts of the particle with the circle $r = 1$ are made as ϕ increases by $2\pi/a$.

No.13. Equations for geodesics through de Sitter space-time are:

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left(r^2\sin^2\theta\frac{d\phi}{ds}\right) = \frac{d}{ds}\left(Ac^2\frac{dt}{ds}\right) = 0$$

identifying trajectories in the plane $\theta = \frac{1}{2}\pi$. the remaining equations then integrate to give

$$\frac{d\phi}{ds} = \frac{\alpha}{r^2}, \qquad \frac{dt}{ds} = \frac{\beta}{A}, \text{ where } A = 1 - r^2/R^2, \qquad \text{(i)}$$

and α, β, are constants. The metric contributes the first integral

$$A^{-1}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - Ac^2\left(\frac{dt}{ds}\right)^2 = 1$$

Eliminating dt and ds between these three equations, we arrive at the equation

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{r^4}{\alpha^2} - r^2\right)A + \frac{\beta^2c^2}{\alpha^2}r^4.$$

Also, by division of the equations (i), we get $\dot{\phi} = \alpha A/(\beta r^2)$.

Substituting the initial values in the last two equations (N.B. $dr/d\phi = \dot{r}/\dot{\phi} = 0$, initially), we calculate that $\alpha^2 = -\frac{1}{4}R^2$, $\beta^2 = -3/2c^2$. It now follows that

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^2}{R^4}(r^2 + R^2)(4r^2 - R^2),$$

determining the polar equation of the trajectory in the plane $\theta = \frac{1}{2}\pi$.

Putting $r = u^{-\frac{1}{2}}$, the last equation reduces to

$$\left(\frac{du}{d\phi}\right)^2 = \frac{4}{R^4}(1 + R^2u)(4 - R^2u)$$

from which it follows that

$$\phi = \tfrac{1}{2}R^2\int\left[\left(\frac{5}{2}\right)^2 - \left(R^2u - \frac{3}{2}\right)^2\right]^{-\frac{1}{2}}du$$

$$= \tfrac{1}{2}\sin^{-1}\left[\frac{1}{5}(2R^2u - 3)\right] + \text{const.}$$

Initially, $\phi = 0$ when $u = 4/R^2$; hence, the constant of integration is $-\frac{1}{4}\pi$. We now deduce that

$$\frac{1}{5}(2R^2u - 3) = \sin(2\phi + \tfrac{1}{2}\pi) = \cos 2\phi,$$

which is equivalent to the result stated.

No.14. We have geodesic equations

$$\frac{d}{ds}\left(z\frac{dx}{ds}\right) = \frac{d}{ds}\left(z\frac{dy}{ds}\right) = \frac{d}{ds}\left(z\frac{dt}{ds}\right) = 0$$

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 - \left(\frac{dt}{ds}\right)^2 = \frac{1}{z}$$

Integrating, we find

$$\frac{dx}{ds} = \frac{A}{z}, \qquad \frac{dy}{ds} = \frac{B}{z}, \qquad \frac{dt}{ds} = \frac{C}{z}.$$

Since $\dot{y} = 0$ initially, B must vanish and so $y = 0$ throughout the motion, which lies in the xz-plane. Also, dividing the equations for x and t, we conclude that $\dot{x} = A/C = v$ throughout the motion.

Putting in the initial values $z = 1$, $\dot{z} = 0$, we find $1/C^2 = v^2 - 1$. Hence

$$\dot{z}^2 = (1 - v^2)(1 - z).$$

Since $v < 1$, we must have $z < 1$ for $\dot{z}$ to be real; i.e. z must decrease from its initial value. Thus

$$-\frac{dz}{\sqrt{(1 - z)}} = \sqrt{(1 - v^2)}dt$$

which integrates to give

$$2\sqrt{(1 - z)} = \sqrt{(1 - v^2)}t \text{ or } z = 1 - \tfrac{1}{4}(1 - v^2)t^2$$

the integration constant being zero. It now appears that $z = 0$ when $t = 2/\sqrt{(1 - v^2)}$.

Since $\dot{x} = v$, $x = vt$. Thus

$$z = 1 - \frac{1}{4v^2}(1 - v^2)x^2$$

which is the equation of a parabola in the xz-plane.

No.15. This result follows immediately from equations (47.14) and (47.16).

No.16. From the components of the energy-momentum tensor stated, we calculate that $T = 3p - c^2\mu$, $T_{11} = pe^{\alpha}$, $T_{22} = pr^2$, $T_{33} = pr^2\sin^2\theta$, $T_{44} = c^4\mu e^{\beta}$.

The non-zero components of the Ricci tensor are listed at (51.18). It now follows that Einstein's equation (47.15) requires that

$$\tfrac{1}{2}\beta'' + \tfrac{1}{4}\beta'^2 - \tfrac{1}{4}\alpha'\beta' - \frac{1}{r}\alpha' = \tfrac{1}{2}\kappa e^{\alpha}(p - c^2\mu) \qquad \text{(i)}$$

$$e^{-\alpha}(\tfrac{1}{2}r\beta' - \tfrac{1}{2}r\alpha' + 1) - 1 = \tfrac{1}{2}\kappa r^2(p - c^2\mu) \qquad \text{(ii)}$$

$$c^2e^{\beta-\alpha}(-\tfrac{1}{2}\beta'' - \tfrac{1}{4}\beta'^2 + \tfrac{1}{4}\alpha'\beta' - \frac{1}{r}\beta') = -\tfrac{1}{2}\kappa c^2e^{\beta}(3p + c^2\mu) \qquad \text{(iii)}$$

Equation (i) is equivalent to the third of the stated equations.

From equation (iii), we deduce

$$-\tfrac{1}{2}\beta'' - \tfrac{1}{4}\beta'^2 + \tfrac{1}{4}\alpha'\beta' - \frac{1}{r}\beta' = -\tfrac{1}{2}\kappa e^{\alpha}(3p + c^2\mu) \qquad \text{(iv)}$$

Adding (i) and (iv), we get

$$\frac{1}{r}(\beta' + \alpha') = \kappa e^{\alpha}(p + c^2\mu)$$

Equation (ii) is equivalent to

$$\frac{1}{r}(\beta' - \alpha') = \frac{2}{r^2}(e^{\alpha} - 1) + \kappa e^{\alpha}(p - c^2\mu)$$

Hence, adding and subtracting the last two equations, we obtain the remaining two equations asked for.

Integrating the first of the stated equations, we find

$$r(1 - e^{-\alpha}) = \frac{1}{3}\kappa c^2\mu r^3 + \text{const.}$$

Thus,

$$e^{-\alpha} = 1 - qr^2$$

where $q = \kappa c^2\mu/3$. This condition on A is clearly necessary if we are to avoid a singularity at $r = 0$.

Eliminating p between the second and third of the stated equations, we get

$$\beta'' + \tfrac{1}{2}\beta'^2 - \tfrac{1}{2}\alpha'\beta' - \frac{2}{r}\alpha' - \frac{1}{r}\beta' + \frac{1}{r^2}(e^{\alpha} - 1) = -\kappa c^2\mu e^{\alpha}$$

or

$$e^{-\alpha}(\beta'' + \tfrac{1}{2}\beta'^2 - \tfrac{1}{2}\alpha'\beta' - \frac{2}{r}\alpha' - \frac{1}{r}\beta') + \frac{1}{r^2}(1 - e^{-\alpha}) = -\kappa c^2\mu$$

Substituting $e^{-\alpha} = 1 - qr^2$, $\alpha' e^{-\alpha} = 2qr$, the last equation reduces to

$$(1 - qr^2)(\beta'' + \tfrac{1}{2}\beta'^2) = \frac{1}{r}\beta'$$

This is a Bernoulli type equation for β'. We solve it by first substituting $\beta' = 1/y$ — this leads to

$$\frac{dy}{dr} + \frac{y}{r(1-qr^2)} = \tfrac{1}{2},$$

which is a first order linear equation for y. The integrating factor is found to be $r/\sqrt{(1-qr^2)}$; thus

$$\frac{d}{dr}\left[\frac{ry}{\sqrt{(1-qr^2)}}\right] = \frac{r}{2\sqrt{(1-qr^2)}}$$

This integrates to yield

$$\frac{ry}{\sqrt{(1-qr^2)}} = -\frac{1}{2q}\sqrt{(1-qr^2)} + \text{const.}$$

so that

$$\frac{1}{\beta'} = y = \frac{1}{2qr}\left[B\sqrt{(1-qr^2)} - (1-qr^2)\right]$$

where B is constant.

We can now substitute into the second of the listed equations in the exercise to obtain an expression for p, thus:

$$\kappa p = \frac{1}{r}\beta' e^{-\alpha} - \frac{1}{r^2}(1 - e^{-\alpha})$$

$$= \frac{3q\sqrt{(1-qr^2)} - qB}{B - \sqrt{(1-qr^2)}}$$

If $p = 0$ at $r = a$, then $B = 3\sqrt{(1-qa^2)}$. Thus, finally

$$\kappa p = \frac{3q\{\sqrt{(1-qr^2)} - \sqrt{(1-qa^2)}\}}{3\sqrt{(1-qa^2)} - \sqrt{(1-qr^2)}}$$

Since $q = \kappa c^2\mu/3$, this is equivalent to the result stated.

Clearly, as r decreases from its surface value a, the pressure increases towards a maximum value at the centre where

$$p_{max} = c^2\mu\,\frac{1 - \sqrt{(1-qa^2)}}{3\sqrt{(1-qa^2)} - 1}$$

If qa^2 is small (i.e. density not too great or radius too large) since $\kappa = 8\pi G/c^4$ (equation (49.9)), G being the Newtonian constant of gravitation. This is the classical result for the pressure at the centre of a uniform sphere of gravitating liquid.

No.17. For space-time with the metric (52.10), null geodesics for which θ and ϕ are constant are governed by the equation

$$\frac{1}{1 - 2m/r}\left(\frac{dr}{d\lambda}\right)^2 - c^2(1 - 2m/r)\left(\frac{dt}{d\lambda}\right)^2 = 0.$$

Thus,

$$\dot{r} = \pm c(1 - 2m/r)$$

Inside the Schwarzschild sphere, $r < 2m$ and for a photon moving away from O therefore

$$\dot{r} = c(2m/r - 1)$$

or

$$\frac{rdr}{2m - r} = cdt$$

Integration leads to

$$-r - 2m\log(2m - r) = ct + \text{const.}$$

If $r = 0$ at $t = 0$, the constant of integration is $-2m\log 2m$ and so

$$ct = 2m\log\left(\frac{2m}{2m - r}\right) - r$$

Then, as $r \to 2m$, $t \to +\infty$ and so the photon takes an infinite coordinate time to reach the sphere $r = 2m$.

For a photon moving towards O,

$$\dot{r} = -c(2m/r - 1)$$

and the integral takes the form

$$r + 2m\log(2m - r) = ct + \text{const.}$$

If $r = R$ when $t = 0$, the integration constant is $R + 2m\log(2m - R)$. Hence

$$ct = 2m\log\left(\frac{2m - r}{2m - R}\right) + r - R$$

If $t = T$ when the photon arrives at O,

$$cT = 2m\log\left(\frac{2m}{2m - R}\right) - R = -R - 2m\log(1 - R/2m)$$

No.18. To evaluate the integral in equation (57.3), we put $r = R\cos^2\theta$, so that $dr = -2R\cos\theta\sin\theta d\theta$. When $r = R$, then $\theta = 0$, and the integral reduces thus:

$$\int_0^\theta \frac{R^{3/2}\cos^3\theta . 2R\cos\theta\sin\theta d\theta}{(R\cos^2\theta - 2m)R^{1/2}\sin\theta} = 2R\int_0^\theta \frac{\cos^4\theta \, d\theta}{\cos^2\theta - 2m/R}$$

$$= 2R\int_0^\theta \left[\cos^2\theta + \alpha^2 + \frac{\alpha^4}{\cos^2\theta - \alpha^2}\right]d\theta \qquad \text{(i)}$$

where $\alpha^2 = 2m/R$.

$$\int_0^\theta \cos^2\theta d\theta \;=\; \tfrac{1}{2}\int_0^\theta (1 + \cos 2\theta)d\theta \;=\; \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta \;=\; \tfrac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$\int_0^\theta \alpha^2 d\theta \;=\; \alpha^2\theta$$

$$\int_0^\theta \frac{\alpha^4}{\cos^2\theta - \alpha^2}d\theta \;=\; \int_0^\theta \frac{\alpha^4\sec^2\theta}{1 - \alpha^2 - \alpha^2\tan^2\theta}d\theta \;=\; \int_0^\theta \frac{\alpha^4 d(\tan\theta)}{1 - \alpha^2 - \alpha^2\tan^2\theta}$$

$$= \frac{\alpha^3}{2\sqrt{(1 - \alpha^2)}}\log\left[\frac{\sqrt{(1-\alpha^2)} + \alpha\tan\theta}{\sqrt{(1-\alpha^2)} - \alpha\tan\theta}\right]$$

The expression (i) is therefore equal to

$$R(\theta + \sin\theta\cos\theta) + 4m\theta + \frac{2m\alpha}{\sqrt{(1 - \alpha^2)}}\log\left[\frac{\sqrt{(1-\alpha^2)} + \alpha\tan\theta}{\sqrt{(1-\alpha^2)} - \alpha\tan\theta}\right]$$

$$= (R + 4m)\cos^{-1}\sqrt{(r/R)} + \sqrt{r(R - r)} + \frac{2m}{\sqrt{(R/2m - 1)}}\log\left\{\frac{1 + \gamma}{1 - \gamma}\right\}$$

Multiplying through by $(R/2m - 1)^{\frac{1}{2}}$, equation (57.3) now yields the expression for ct stated.

As $r \to 2m$ (approaching the Schwarzschild sphere), we see that $\gamma \to 1$ and the log term $\to +\infty$.

No.19. Equation (57.4) can be rewritten in the form

$$c\frac{d\tau}{dr} \;=\; -\frac{1}{\sqrt{(2m)}}\left\{\frac{Rr}{R - r}\right\}^{\frac{1}{2}}$$

for the case r decreasing. Hence,

$$c\tau \;=\; -\frac{R}{2m}\int\left\{\frac{r}{R - r}\right\}^{\frac{1}{2}}dr \;=\; -\frac{R}{2m}\int\frac{r\,dr}{\sqrt{(Rr - r^2)}}$$

$$= \frac{R}{2m}\int\frac{\frac{1}{2}R - r}{\sqrt{(Rr - r^2)}}dr \;-\; \frac{R}{2m}\int\frac{\frac{1}{2}R}{\sqrt{(Rr - r^2)}}dr$$

$$= \sqrt{(R^3/2m)}\left[\sqrt{(\rho - \rho^2)} + \tfrac{1}{2}\cos^{-1}(2\rho - 1)\right] + \text{const.}$$

where $\rho = r/R$. If $\tau = 0$ when $r = R$ (or $\rho = 1$), the constant vanishes and we have obtained equation (57.5).

In the special case when the particle commences its fall from the Schwarzschild sphere, we take $R = 2m$ and then

$$c\tau \;=\; 2m\left[\sqrt{(\rho - \rho^2)} + \tfrac{1}{2}\cos^{-1}(2\rho - 1)\right]$$

where $\rho = r/2m$. This shows that $\rho = 0$ when

$$c\tau \;=\; m\cos^{-1}(-1) \;=\; \pi m$$

i.e. $\tau = \pi m/c$ as stated.

Referring to equation (52.14), $m = GM/c^2$ and the time of fall is therefore $\pi GM/c^3$, where M is the mass of the black hole. In the case of a black hole of solar mass, $GM = 1.33 \times 10^{20}$ and $c = 3 \times 10^8$ in SI units; these data lead to a time of fall into the hole of 16×10^{-6} secs.

$$g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \quad g_{44} = -c^2(1 - 2m/r), \quad g_{14} = g_{41} = c.$$

Thus, $g = -c^2 r^4 \sin^2\theta$ and so

$$g^{11} = 1 - 2m/r, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2}\operatorname{cosec}^2\theta, \quad g^{14} = g^{41} = \frac{1}{c}$$

the remaining components of the contravariant metric tensor vanishing.

We can now list the non-zero components of the Christoffel symbol as below:

$$\left\{\begin{matrix}1\\2\ 2\end{matrix}\right\} = 2m - r, \quad \left\{\begin{matrix}4\\2\ 2\end{matrix}\right\} = -\frac{r}{c}, \quad \left\{\begin{matrix}1\\3\ 3\end{matrix}\right\} = (2m - r)\sin^2\theta,$$

$$\left\{\begin{matrix}4\\3\ 3\end{matrix}\right\} = -\frac{r}{c}\sin^2\theta, \quad \left\{\begin{matrix}2\\3\ 3\end{matrix}\right\} = -\sin\theta\cos\theta, \quad \left\{\begin{matrix}1\\4\ 4\end{matrix}\right\} = \frac{mc^2}{r^3}(r - 2m)$$

$$\left\{\begin{matrix}4\\4\ 4\end{matrix}\right\} = \frac{mc}{r^2}, \quad \left\{\begin{matrix}2\\1\ 2\end{matrix}\right\} = \left\{\begin{matrix}3\\1\ 3\end{matrix}\right\} = \frac{1}{r}, \quad \left\{\begin{matrix}3\\2\ 3\end{matrix}\right\} = \cot\theta, \quad \left\{\begin{matrix}1\\1\ 4\end{matrix}\right\} = -\frac{mc}{r^2}.$$

The 10 distinct components of R_{jk} can now be checked through to vanish.

No.21. Differentiating the transformation equations, we derive the differential transformation

$$du = \frac{r}{8m^2}\left(\frac{r}{2m} - 1\right)^{-\frac{1}{2}} e^{r/4m}\cosh(ct/4m)dr + \frac{c}{4m}\left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{r/4m}\sinh(ct/4m)dt$$

$$dv = \frac{r}{8m^2}\left(\frac{r}{2m} - 1\right)^{-\frac{1}{2}} e^{r/4m}\sinh(ct/4m)dr + \frac{c}{4m}\left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{r/4m}\cosh(ct/4m)dt$$

Squaring these equations and subtracting, we find that

$$du^2 - dv^2 = \frac{r^2}{64m^4}\left(\frac{r}{2m} - 1\right)^{-1} e^{r/2m}dr^2 - \frac{c^2}{16m^2}\left(\frac{r}{2m} - 1\right)e^{r/2m}dt^2$$

showing that

$$\left(1 - \frac{2m}{r}\right)^{-1}dr^2 - c^2\left(1 - \frac{2m}{r}\right)dt^2 = \frac{32m^3}{r}e^{-r/2m}(du^2 - dv^2)$$

which proves that the metric transforms as stated.

If we square and subtract the original transformation equations, we obtain the result

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right)e^{r/2m}$$

This is an inverse transformation equation determining r in terms of u and v.

For radial motion of a photon, we put $ds = d\theta = d\phi = 0$ in the new metric to give

$$\frac{32m^3}{r}e^{-r/2m}(du^2 - dv^2) = 0$$

Hence, $du = \pm dv$, which integrates to $u = \pm v +$ constant.

No.22. From the first transformation equation, we conclude that

$$r = \left(\frac{3a}{2}\right)^{2/3}(u - v)^{2/3} = \mu(u - v)^{2/3}.$$

Differentiating the transformation equations, we get

$$du = dv + (r^{\frac{1}{2}}/a)dr$$

$$dv = dt + \frac{ar^{\frac{1}{2}}}{r - a^2}\,dr$$

It follows that

$$dr = \frac{a}{\sqrt{r}}(du - dv), \qquad dt = \frac{rdv - a^2du}{r - a^2}$$

Hence

$$\frac{1}{1 - 2m/r}\,dr^2 - (1 - 2m/r)dt^2$$

$$= \frac{r}{r - a^2}\cdot\frac{a^2}{r}(du - dv)^2 - \frac{(rdv - a^2du)^2}{r(r - a^2)}$$

$$= \frac{a^2}{r}du^2 - dv^2$$

$$= \frac{a^2}{\mu}(u - v)^{-2/3}du^2 - dv^2 = \frac{4}{9}\mu^2(u - v)^{-2/3}du^2 - dv^2$$

which completes the transformation of the Schwarzschild metric.

Note: Units have been chosen so that $c = 1$. If this is not done, it is necessary to replace t by ct in the second transformation equation.

No.23. As shown in Ex.13 above, any trajectory which lies initially in the plane $\theta = \frac{1}{2}\pi$, remains in this plane. Hence, we shall put $\theta = \frac{1}{2}\pi$, $d\theta = 0$, $ds = 0$, in the metric to obtain an equation governing the motion of the photon, viz.

$$(1 - 2m/r)^{-1}dr^2 + r^2d\phi^2 - c^2(1 - 2m/r)dt^2 = 0 \qquad \text{(i)}$$

Substituting initial values for r and $\dot{\phi}$, we calculate that $\dot{r} = \pm 2\sqrt{7}c$ initially.

We also have the geodesic equations

$$\frac{d}{d\lambda}\left(r^2\frac{d\phi}{d\lambda}\right) = \frac{d}{d\lambda}\left[(1 - 2m/r)\frac{dt}{d\lambda}\right] = 0$$

which together imply that

$$\frac{r^2}{1 - 2m/r}\frac{d\phi}{dt} = \text{constant} = A \qquad \text{(ii)}$$

Eliminating dt between equations (i) and (ii), we deduce that, for a photon,

$$\left(\frac{dr}{d\phi}\right)^2 = r(2m - r) + \frac{c^2}{A^2}r^4$$

Initially $r = m$ and $dr/d\phi = \dot{r}/\dot{\phi} = -\frac{2}{3}\sqrt{\frac{7}{3}}m$; it follows that $c^2/A^2 = 1/(27m^2)$ and so

$$\left(\frac{dr}{d\phi}\right)^2 = r(2m - r) + \frac{r^4}{27m^2}$$

Putting $r = 1/u$, this equation transforms to

$$\left(\frac{du}{d\phi}\right)^2$$

$$\phi = 3\sqrt{3}m\int\frac{du}{(3mu - 1)\sqrt{(6mu + 1)}}$$

(Note: $d\phi/du = -\frac{1}{u}\frac{d\phi}{dr} = -r^2\dot{\phi}/\dot{r} > 0$ initially.) Changing the variable of integration by putting $6mu + 1 = v^2$, we find

$$\phi = 2\sqrt{3}\int\frac{dv}{v^2 - 3} = \log\left(\frac{v - \sqrt{3}}{v + \sqrt{3}}\right) + \text{const.}$$

Since initially $\phi = 0$, $v = \sqrt{7}$, the integration constant is $-\log\left(\frac{\sqrt{7} - \sqrt{3}}{\sqrt{7} + \sqrt{3}}\right) = \log\left(\frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}}\right) = \log\{\frac{1}{2}(5 + \sqrt{21})\} = \alpha$. Solving for v, we find that

$$v = \sqrt{3}\coth\tfrac{1}{2}(\alpha - \phi)$$

It now follows that

$$\frac{6m}{r} = 6mu = v^2 - 1 = 3\coth^2\tfrac{1}{2}(\alpha - \phi) - 1.$$

As ϕ increases from 0 to α, $\coth\frac{1}{2}(\alpha - \phi)$ increases from $\coth\frac{1}{2}\alpha$ to $+\infty$ and r, accordingly, decreases from m to 0, i.e. the photon falls into the black hole.

No.24. For photons moving in the plane $\theta = \frac{1}{2}\pi$, we have null-geodesic equations

$$\frac{d}{d\lambda}\left(\frac{r}{r - 2m}\frac{dr}{d\lambda}\right) + \frac{m}{(r - 2m)^2}\left(\frac{dr}{d\lambda}\right)^2 - r\left(\frac{d\phi}{d\lambda}\right)^2 + \frac{mc^2}{r^2}\left(\frac{dt}{d\lambda}\right)^2 = 0 \quad \text{(i)}$$

$$\frac{d}{d\lambda}\left(r^2\frac{d\phi}{d\lambda}\right) = \frac{d}{d\lambda}\left[(1 - 2m/r)\frac{dt}{d\lambda}\right] = 0 \quad \text{(ii)}$$

together with the first integral

$$\frac{r}{r - 2m}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 - c^2(1 - 2m/r)\left(\frac{dt}{d\lambda}\right)^2 = 0 \quad \text{(iii)}$$

If r = constant is to satisfy these equations, equations (i) and (ii) require that along such a trajectory

$$\dot{\phi}^2 = \frac{mc^2}{r^3}, \qquad \dot{\phi}^2 = \frac{c^2}{r^3}(r - 2m)$$

Hence, r must satisfy the equation

$$\frac{mc^2}{r^3} = \frac{c^2}{r^3}(r - 2m)$$

i.e. $r = 3m$.

(Note: In the special case of a trajectory r = constant, (i) cannot be derived from (ii) and (iii) and so (i) cannot be ignored.)

With $r = 3m$, we have $\dot{\phi} = c/(3\sqrt{3}m)$. Hence ϕ increases by 2π in coordinate time $6\sqrt{3}\pi m/c$.

For general motion of a photon in the plane, equations (ii) show that $\dot{\phi} = A(r - 2m)/r^3$ where A is constant. Thus, eliminating $\dot{\phi}$ from equation (iii), we get

$$\left(\frac{dr}{dt}\right)^2 = \frac{c^2}{r^2}(r - 2m)^2 - \frac{A^2}{r^5}(r - 2m)^3 \quad \text{(iv)}$$

Suppose the photon is deflected at some point on its trajectory

$$\dot{\phi}^2 = \frac{c^2}{27m^2} - \frac{\delta^2}{3m}$$

immediately afterwards. Hence $A^2 = 27m^2c^2 - 243m^3\delta^2$. During the motion after the disturbance, suppose $r = (3 + \varepsilon)m$, where ε is initially small. Substituting in (iv) and ignoring terms $O(\varepsilon^3)$, we find that

$$m^2\dot{\varepsilon}^2 = \frac{c^2}{27}\varepsilon^2 + m\delta^2(1 + \tfrac{4}{3}\varepsilon - \tfrac{1}{3}\varepsilon^2)$$

Initially, $\dot{\varepsilon} = \delta$ and ε increases. This causes $\dot{\varepsilon}$ to increase due to the presence of the dominant term $c^2\varepsilon^2/27$ in the last equation and thus ε becomes large. This means that the orbit $r = 3m$ is unstable.

No.25. If r is constant and $\theta = \frac{1}{2}\pi$, the geodesic equations for the Schwarzschild metric reduce to

$$-r\left(\frac{d\phi}{ds}\right)^2 + \frac{mc^2}{r^2}\left(\frac{dt}{ds}\right)^2 = 0,$$

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = \frac{d}{ds}\left[(1 - 2m/r)\frac{dt}{ds}\right] = 0$$

with a first integral

$$r^2\left(\frac{d\phi}{ds}\right)^2 - c^2(1 - 2m/r)\left(\frac{dt}{ds}\right)^2 = 1$$

We deduce that

$$\dot{\phi}^2 = mc^2/r^3, \quad d\phi/ds = A/r^2, \quad dt/ds = \frac{Br}{r - 2m},$$

where A and B are constants. Since $mc^2 = GM$, the first equation shows that the relationship between $\dot{\phi}$ and r is the same as for classical theory. The other two equations give $\dot{\phi} = A(r - 2m)/(Br^3)$; hence $B^2/A^2 = (r - 2m)^2/(mc^2r^3)$.

Substituting in the first integral, we find that

$$\frac{A^2}{r^2} - \frac{B^2c^2r}{r - 2m} = 1$$

Hence

$$A^2 = -\frac{mr^2}{r - 3m}, \qquad B^2 = -\frac{(r - 2m)^2}{c^2r(r - 3m)}. \qquad \text{(i)}$$

But A and B must both be purely imaginary, since ds has this property for any trajectory of a real particle. We conclude that A^2 and B^2 must be negative and so $r > 3m$.

If τ is time measured by a standard clock moving with the body, then $ds = ic\,d\tau$ and so

$$\frac{d\phi}{d\tau} = \frac{icA}{r^2} = \pm\frac{c}{r}(r/m - 3)^{-\frac{1}{2}}$$

It follows that ϕ increases by 2π in a time

$$\frac{2\pi r}{c}(r/m - 3)^{\frac{1}{2}}$$

as measured on this clock.

For an observer who is stationary at some point on the orbit, if τ' is time as measured on his standard clock, then the metric shows that (see equation (55.1)). Since $\dot{\phi} = c\surd(m/r^3)$, we deduce that

$$\frac{d\phi}{d\tau'} = \frac{c}{r}(r/m - 2)^{-\frac{1}{2}}.$$

As measured by the observer's clock, therefore, the orbital period is as stated.

Equations governing the general motion of a falling body in the plane $\theta = \frac{1}{2}\pi$ are

$$\frac{d\phi}{ds} = \frac{A}{r^2}, \qquad \frac{dt}{ds} = \frac{Br}{r - 2m},$$

$$(1 - 2m/r)^{-1}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - c^2(1 - 2m/r)\left(\frac{dt}{ds}\right)^2 = 1.$$

Putting $ds = icd\tau$ and eliminating ϕ and t, we arrive at the equation

$$\left(\frac{dr}{d\tau}\right)^2 = -\frac{2mc^2A^2}{r^3} + \frac{c^2A^2}{r^2} + \frac{2mc^2}{r} - c^2 - c^4B^2.$$

Differentiating with respect to τ, we get

$$\frac{d^2r}{d\tau^2} = \frac{6mc^2A^2}{r^4} - \frac{2c^2A^2}{r^3} - \frac{2mc^2}{r^2} \qquad \text{(ii)}$$

Now suppose that the body is orbiting in the circle $r = R(>3m)$, when it is slightly disturbed. The constants A and B will then be perturbed from the values given by equations (i); let

$$A^2 = -\frac{mR^2 + \delta}{R - 3m},$$

where δ is small. On the new orbit, r will be governed by equation (ii). Suppose $r = R(1 + \varepsilon)$, where ε is initially small. Then, to the first order in ε,

$$R\frac{d^2\varepsilon}{d\tau^2} = \frac{6mc^2A^2}{R^4}(1 - 4\varepsilon) - \frac{2c^2A^2}{R^3}(1 - 3\varepsilon) - \frac{2mc^2}{R^2}(1 - 2\varepsilon)$$

$$= \frac{2c^2\delta}{R^4} - \frac{2mc^2R^2(R - 6m) + 6c^2\delta(R - 4m)}{R^4(R - 3m)}\varepsilon$$

Provided $R > 6m$, for sufficiently small δ(with either sign), this equation will take the form

$$\frac{d^2\varepsilon}{d\tau^2} + \omega^2\varepsilon = 2c^2\delta/R^5$$

where $\omega^2 > 0$, showing that ε oscillates about the value $2c^2\delta/R^5$. In this case, the new orbit will stay close to the circle $r = R$ and the circular orbit is stable. If, however, $R < 6m$, the equation will take the form

$$\frac{d^2\varepsilon}{d\tau^2} - \omega^2\varepsilon = 2c^2\delta/R^5$$

and ε will increase in magnitude so that the new orbit diverges widely from the circle $r = R$, which is accordingly unstable.

No.26. The equation governing the motion of the signal is

For the inwards motion, we choose the negative sign and integrate to yield the result

$$ct = R - r + 2m \log\left(\frac{R - 2m}{r - 2m}\right)$$

choosing t = 0 initially.

For the return journey, we choose the positive sign and integrate to give the same time for this phase of the motion.

Thus, the total coordinate time elapsing between transmission and reception is

$$\frac{2}{c}\left[R - r + 2m \log\left(\frac{R - 2m}{r - 2m}\right)\right]$$

But, as explained in the previous exercise, a standard clock stationary at (R,θ,ϕ) will indicate a time lapse which is smaller than that for an adjacent coordinate clock by a factor $\sqrt{(1 - 2m/R)}$. Multiplication of the last result by this factor gives the stated result.

If $d\ell$ is the distance between the points (r,θ,ϕ) and $(r+dr,\theta,\phi)$, then the metric shows that

$$d\ell^2 = (1 - 2m/r)^{-1}dr^2.$$

Hence, the total distance between the points (r,θ,ϕ) and (R,θ,ϕ) is given by

$$\ell = \int_r^R (1 - 2m/r)^{-\frac{1}{2}}dr = \Big|\sqrt{(r^2 - 2mr)} + m\log\left[r - m + \sqrt{(r^2 - 2mr)}\right]\Big|_r^R$$

$$= \sqrt{(R^2 - 2mR)} - \sqrt{(r^2 - 2mr)} + m\log\frac{R - m + \sqrt{(R^2 - 2mR)}}{r - m + \sqrt{(r^2 - 2mr)}}$$

$$= \sqrt{(R^2 - 2mR)} - \sqrt{(r^2 - 2mr)} + 2m\log\frac{\sqrt{R} + \sqrt{(R - 2m)}}{\sqrt{r} + \sqrt{(r - 2m)}}$$

According to classical theory, the time a wireless signal would need to cover the distance 2ℓ is $2\ell/c$ and this is the result given.

To the first order in m, the relativistic time is

$$\frac{2}{c}(R - r - m + \frac{mr}{R} + 2m\log\frac{R}{r})$$

and the classical time is

$$\frac{2}{c}(R - r + m\log\frac{R}{r}).$$

Subtracting the classical time from the relaticistic, we obtain the difference stated.

No.27. As explained in section 55, the period (and therefore the frequency) of the radiation, measured in coordinate time, will be the same at the points of transmission and reception. But, at coordinate distance r from the centre, an interval dt of coordinate time is related to an interval $d\tau$ of time measured on a standard clock by the equation $d\tau = \sqrt{(1 - 2m/r)}dt$. Thus, the radiation frequency ν measured by a standard clock and the corresponding frequency ω measured by a coordinate clock are related by the equation

Since ω is unchanged at the radius R, we have similarly

$$\nu - \delta\nu = (1 - 2m/R)^{-\frac{1}{2}}\omega$$

Hence

$$\nu - \delta\nu = \sqrt{\frac{1 - 2m/r}{1 - 2m/R}}\,\nu = \left[1 - m\left(\frac{1}{r} - \frac{1}{R}\right)\right]\nu$$

to the first order in m. This gives the result stated.

No.28. Differentiating the transformation equation, we get

$$dr = \left(1 - \frac{m^2}{4r'^2}\right)dr'$$

Also

$$1 - \frac{2m}{r} = \left(\frac{1 - m/2r'}{1 + m/2r'}\right)^2$$

Substitution in the Schwarzschild metric now leads to the stated result.

No.29. Geodesic equations for the given space-time metric are as follows:

$$\frac{d}{ds}\left(\frac{dx}{ds} + at\frac{dt}{ds}\right) = \frac{d^2y}{ds^2} = \frac{d^2z}{ds^2} = 0$$

Integrating these equations, we derive the equations

$$\frac{dx}{ds} + at\frac{dt}{ds} = \text{const.}, \quad \frac{dy}{ds} = \text{const.}, \quad \frac{dz}{ds} = \text{const.}$$

The metric supplies a first integral, viz.

$$\left(\frac{dx}{ds} + at\frac{dt}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 - c^2\left(\frac{dt}{ds}\right)^2 = 1$$

Hence

$$dt/ds = \text{const.}$$

It now follows that

$$\frac{dx}{dt} = -at + B, \quad \frac{dy}{dt} = D, \quad \frac{dz}{dt} = F,$$

where B, D, F, are constants. The stated equations now follow immediately by integration with respect to t.

Thus, a freely falling particle experiences uniform acceleration parallel to the x-axis in the frame.

Putting $x = x' - \frac{1}{2}at^2$, $dx = dx' - atdt$, in the metric, it reduces to the Minkowski form $ds^2 = dx'^2 + dy^2 + dz^2 - c^2dt^2$. Thus, in the x'yzt-frame, a freely falling particle moves uniformly, i.e.

$$x' = A + Bt, \quad y = C + Dt, \quad z = E + Ft.$$

Substituting $x' = x + \frac{1}{2}at^2$, we recover the original equations of motion.

No.30. We have

$$dy^k = \frac{\partial y^k}{\partial x^i}dx^i$$

Substituting in the Minkowski metric of I, viz. $ds^2 = dy^k dy^k$, we calculate that

$$ds^2 = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j}\, dx^i dx^j$$

Thus, for S,

$$g_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j}$$

Along P's world-line, $dx^1 = dx^2 = dx^3 = 0$. Hence

$$dy^i = \frac{\partial y^i}{\partial x^4} dx^4$$

But, at the initial instant under consideration, since P is stationary in I, $dy^1 = dy^2 = dy^3 = 0$. It follows that $\partial y^i/\partial x^4 = 0$ for $i = 1,2,3$. But

$$g_{i4} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^4} = \frac{\partial y^4}{\partial x^i}\frac{\partial y^4}{\partial x^4}$$

In particular

$$g_{44} = \left(\frac{\partial y^4}{\partial x^4}\right)^2 .$$

Thus, $\partial y^4/\partial x^i = g_{i4}/\sqrt{(g_{44})}$ at P at the instant under consideration.

Since the points P and P' are observed at the same instant in S, $dx^4 = 0$ and so

$$dy^\alpha = \frac{\partial y^\alpha}{\partial x^i}\, dx^i = \frac{\partial y^\alpha}{\partial x^\lambda}\, dx^\lambda$$

where Greek indices take values 1, 2, 3. The standard rod at rest in I now measures the distance $d\ell$ between the positions occupied by P and P' at the time x^4 in S and, since geometry is Euclidean in I, we find

$$d\ell^2 = dy^\alpha dy^\alpha = \frac{\partial y^\alpha \partial y^\alpha}{\partial x^\lambda \partial x^\mu}\, dx^\lambda dx^\mu = \gamma_{\lambda\mu} dx^\lambda dx^\mu$$

where

$$\gamma_{\lambda\mu} = \frac{\partial y^\alpha \partial y^\alpha}{\partial x^\lambda \partial x^\mu} = \frac{\partial y^k}{\partial x^\lambda}\frac{\partial y^k}{\partial x^\mu} - \frac{\partial y^4}{\partial x^\lambda}\frac{\partial y^4}{\partial x^\mu} = g_{\lambda\mu} - g_{\lambda 4} g_{\mu 4}/g_{44},$$

using previous results.

We note that $\gamma_{\lambda\mu}$ is the metric tensor of the $\mathcal{R}_3$ which is the section of space-time by the hypersurface x_4 = constant, it being understood that distances between neighbouring points of the $\mathcal{R}_3$ are measured by a standard measuring rod which is in free fall and which is instantaneously stationary at the points.

No.31. For the frame I, $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$. But

$$dx = dr\cos(\theta+\omega t) - r\sin(\theta+\omega t)(d\theta + \omega dt)$$

$$dy = dr\sin(\theta+\omega t) + r\cos(\theta+\omega t)(d\theta + \omega dt)$$

so that

$$ds^2 = dr^2 + r^2(d\theta + \omega dt)^2 + dz^2 - c^2dt^2$$

Thus

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{44} = -(c^2 - \omega^2 r^2), \quad g_{24} = \omega r^2,$$

other components vanishing.

Using the result of the previous exercise,

$$\gamma_{11} = 1, \quad \gamma_{22} = r^2 + \frac{\omega^2 r^4}{c^2 - \omega^2 r^2} = \frac{c^2 r^2}{c^2 - \omega^2 r^2},$$

the other components vanishing. Thus

$$d\ell^2 = dr^2 + \frac{r^2 d\theta^2}{1 - \omega^2 r^2/c^2}$$

is the metric for geometry in the rotating plane.

For space geodesics in the plane, we have

$$\frac{d}{d\ell}\left\{\frac{r^2}{1 - \omega^2 r^2/c^2}\frac{d\theta}{d\ell}\right\} = 0.$$

Thus,

$$\frac{r^2}{1 - \omega^2 r^2/c^2}\frac{d\theta}{d\ell} = \text{constant} = 1/b$$

Substitution in the metric leads to

$$\frac{d\theta}{dr} = \frac{1 - \omega^2 r^2/c^2}{r\sqrt{\{(b^2 + \omega^2/c^2)r^2 - 1\}}}$$

Putting $b^2 + \omega^2/c^2 = 1/a^2$ and $c/\omega = r_1$, we have

$$\theta = \int\frac{1 - r^2/r_1^2}{r\sqrt{(r^2/a^2 - 1)}}\,dr = \int\frac{a\,dr}{r\sqrt{(r^2 - a^2)}} - \frac{a}{r_1^2}\int\frac{r\,dr}{\sqrt{(r^2 - a^2)}}$$

The first integral is evaluated by the change of variable $r = 1/u$ and the second integral is a standard form. Thus,

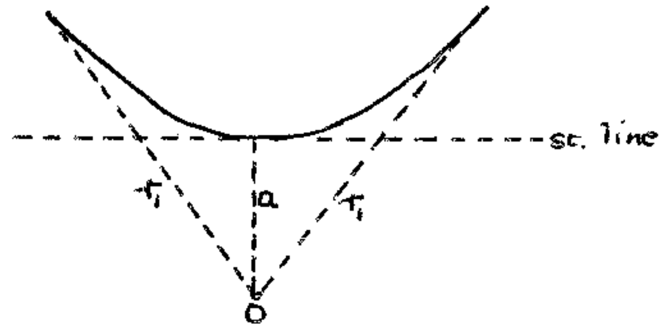
$$\theta = \text{const.} - \sin^{-1}(a/r) - \frac{a}{r_1^2}\sqrt{(r^2 - a^2)}.$$

Clearly $1/a^2 = b^2 + \omega^2/c^2 \geqslant \omega^2/c^2$; thus, $|a| \leqslant c/\omega = r_1$. For real values of θ, we need $r \geqslant |a|$. Hence $|a|$ is the closest distance of approach of the geodesic to the origin. Since $\omega r_1 = c$, points on the disc at a greater distance than r_1 from O will have speeds as measured in I which are greater than c; the frame R cannot be extended to such points and no geodesics for R lie in this region; hence, for a geodesic to be possible, we need $|a| < r_1$ and then, on the geodesic, $|a| < r < r_1$.

The shapes of the geodesics are independent of the constant in the equation and so, taking it equal to $\frac{1}{2}\pi$, we can write

$$\theta = \cos^{-1}(a/r) - \frac{a}{r_1^2}\sqrt{(r^2 - a^2)}$$

If $\omega = 0$, then $r_1 = \infty$ and $r = a\sec\theta$, which (as expected) is a straight line distant a from O. Thus, we can regard $-\frac{a}{r_1^2}\sqrt{(r^2 - a^2)}$ as a correction term to allow for the rotation. The magnitude of this term steadily increases as r increases from a and distorts the straight line as indicated below:



No.32. Note: In the left-hand member of the first equation in the exercise, $g_{i\alpha}$ should read $\gamma_{i\alpha}$.

Along the world-line of a freely falling particle, the geodesic equation (43.5) is valid. At any point where the particle is stationary in the frame, $dx^i/ds = 0$ for $i = 1, 2, 3$. Hence, at such a point,

$$\frac{d^2x^i}{ds^2} + \left\{ {i \atop 4\ 4} \right\}\left(\frac{dx^4}{ds}\right)^2 = 0 \qquad \text{(i)}$$

Instead of the arc length parameter s, we can employ x^4. Then

$$\frac{dx^i}{ds} = \frac{dx^i}{dx^4}.\frac{dx^4}{ds}, \quad \frac{d^2x^i}{ds^2} = \frac{d^2x^i}{(dx^4)^2}.\left(\frac{dx^4}{ds}\right)^2 + \frac{dx^i}{ds}.\frac{d^2x^4}{ds^2}.$$

Hence, at a point where the particle is stationary, we shall have

$$\frac{d^2x^\alpha}{ds^2} = \frac{d^2x^\alpha}{(dx^4)^2}.\left(\frac{dx^4}{ds}\right)^2,$$

where $\alpha = 1,2,3$. Also, from the first integral (43.6), we find that at such a point $g_{44}(dx^4/ds)^2 = 1$. Hence

$$\frac{d^2x^\alpha}{ds^2} = \frac{1}{g_{44}}\frac{d^2x^\alpha}{(dx^4)^2}. \qquad \text{(ii)}$$

Equations (i) and (ii) now yield the result

$$\frac{d^2x^\alpha}{(dx^4)^2} = -g^{\alpha j}[44,j] = -\frac{1}{2}g^{\alpha j}\left(2\frac{\partial g_{j4}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^j}\right)$$

Taking $\gamma_{ij} = g_{ij} - g_{i4}g_{j4}/g_{44}$ as in Ex.30, we have

$$\gamma_{i\alpha}\frac{d^2x^\alpha}{(dx^4)^2} = -\frac{1}{2}\gamma_{i\alpha}g^{\alpha j}\left(2\frac{\partial g_{j4}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^j}\right) = -\frac{1}{2}\gamma_{ik}g^{kj}\left(2\frac{\partial g_{j4}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^j}\right)$$

since $\gamma_{i4} = 0$. Now

$$\gamma_{ik}g^{kj} = g_{ik}g^{kj} - g_{i4}g_{k4}g^{kj}/g_{44} = \delta_i^j - g_{i4}\delta_4^j/g_{44}$$

Hence,

$$\gamma_{i\alpha}\frac{d^2x^\alpha}{(dx^4)^2} = -\frac{1}{2}(\delta_i^j - g_{i4}\delta_4^j/g_{44})\left(2\frac{\partial g_{j4}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^j}\right)$$
$$= -\frac{\partial g_{i4}}{\partial x^4} + \frac{1}{2}\frac{\partial g_{44}}{\partial x^i} + \frac{1}{2}\frac{g_{i4}}{g_{44}}\frac{\partial g_{44}}{\partial x^4}$$

which is the result stated (after correction).

be treated as an invariant parameter and, thus, dx^α/dx^4, $d^2x^\alpha/(dx^4)^2$ are contravariant 3-vectors (velocity and acceleration respectively). Lowering an index, the covariant acceleration vector is

$$a_\alpha = \gamma_{\alpha\beta}\frac{d^2x^\beta}{(dx^4)^2} = \frac{1}{2}\left\{\frac{\partial g_{44}}{\partial x^\alpha} + \frac{g_{\alpha 4}}{g_{44}}\frac{\partial g_{44}}{\partial x^4}\right\} - \frac{\partial g_{\alpha 4}}{\partial x^4}$$

$$= -\frac{\partial U}{\partial x^\alpha} - (c^2 + 2U)^{\frac{1}{2}}\frac{\partial \gamma_\alpha}{\partial x^4}$$

as may be verified by substituting for U and γ_α.

For the space-time metric of Ex.31, we have $g_{44} = -(c^2 - r^2\omega^2)$, $g_{14} = \omega r^2$. Thus

$$U = -\tfrac{1}{2}(g_{44} + c^2) = -\tfrac{1}{2}r^2\omega^2, \quad \gamma_1 = \frac{\omega r^2}{\sqrt{(c^2 - \omega^2 r^2)}}, \quad \gamma_2 = \gamma_3 = 0.$$

Hence

$$a_1 = \omega^2 r, \qquad a_2 = a_3 = 0,$$

i.e. the initial acceleration of a particle placed at a point of the rotating plane is radial and of magnitude $\omega^2 r$ — this is the acceleration arising from the centrifugal force to be expected.

No.33. Null geodesic equations in the plane $\theta = \frac{1}{2}\pi$ are

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = \frac{d}{ds}\left(A\frac{dt}{ds}\right) = 0$$

$$A^{-1}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - Ac^2\left(\frac{dt}{ds}\right)^2 = 0$$

Integrating the first two equations and dividing, we calculate that $d\phi/dt = CA/r^2$, where C is constant. It now follows from the third equation that

$$A^{-1}\left(\frac{dr}{d\phi}\right)^2 + r^2 - \frac{c^2r^4}{C^2A} = 0$$

or

$$\frac{dr}{d\phi} = \frac{r}{a}\sqrt{(r^2 - a^2)}$$

where $\frac{1}{a^2} = \frac{c^2}{C^2} + \frac{1}{R^2}$. Thus $|a| < R$.

Putting $r = 1/u$, the last equation for null geodesics reduces to

$$\frac{du}{d\phi} = -\sqrt{(a^{-2} - u^2)}.$$

Hence

$$\phi = \cos^{-1}(au) + \text{const.}$$

or

$$r = a\sec(\phi + \alpha) \qquad \alpha = \text{const.}$$

This is the polar equation for a straight line distant a from the origin.

No.34. Null geodesics in the plane $\theta = \frac{1}{2}\pi$ are governed by equations

$$\frac{d}{ds}\left\{r^2\frac{d\phi}{ds}\right\} = \frac{d^2t}{ds^2} = 0$$

$$\frac{1}{1-\lambda r^2}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - c^2\left(\frac{dt}{ds}\right)^2 = 0$$

As in the previous exercise, we now derive the equations $d\phi/dt = C/r^2$ and

$$\frac{1}{1-\lambda r^2}\left(\frac{dr}{d\phi}\right)^2 + r^2 - \frac{c^2}{C^2}r^4 = 0$$

Putting $c^2/C^2 = \mu$, this reduces to the equation stated.

With $r^2 = 1/v$, the equation reduces to

$$\left(\frac{dv}{d\phi}\right)^2 = 4(v-\lambda)(\mu - v)$$

showing immediately that v oscillates between the values λ and μ; i.e. r oscillates between the values $\lambda^{-\frac{1}{2}}$ and $\mu^{-\frac{1}{2}}$. Integration of the last equation leads to the result

$$\phi = \frac{1}{2}\sin^{-1}\left(\frac{2v-\lambda-\mu}{\lambda-\mu}\right) + \text{const.}$$

Hence

$$\frac{2}{r^2} = (\lambda-\mu)\sin(2\phi+\alpha) + \lambda + \mu$$

The shape of the null geodesics is not affected by the constant (only their orientation in the plane). Thus, take $\alpha = \frac{1}{2}\pi$ so that

$$\frac{2}{r^2} = (\lambda-\mu)\cos 2\phi + \lambda + \mu = (\lambda-\mu)(\cos^2\phi - \sin^2\phi) + \lambda + \mu$$

Introducing Cartesian coordinates to replace the polars, we put $r\cos\phi = x$, $r\sin\phi = y$ and $r^2 = x^2 + y^2$. Hence

$$2 = (\lambda-\mu)(x^2-y^2) + (\lambda+\mu)(x^2+y^2) = 2\lambda x^2 + 2\mu y^2$$

or

$$\lambda x^2 + \mu y^2 = 1.$$

We have

$$r^2\dot{\phi} = C = c/\sqrt{\mu}$$

This has the form of Kepler's second law, i.e. it shows that the radius vector sweeps out an area at the constant rate $c/2\sqrt{\mu}$. The semi-axes of the above ellipse are $1/\sqrt{\lambda}$ and $1/\sqrt{\mu}$, so that its area is $\pi/\sqrt{(\lambda\mu)}$. The time taken for the radius r to sweep out this area is

$$\frac{\pi}{\sqrt{(\lambda\mu)}} \div \frac{c}{2\sqrt{\mu}} = \frac{2\pi}{c\sqrt{\lambda}}.$$

No.35. We derive geodesic equations

$$\frac{d}{ds}\left(\alpha^2\frac{dy}{ds}\right) = \frac{d}{ds}\left(\alpha^2\frac{dz}{ds}\right) = \frac{d}{ds}\left(\alpha\frac{dt}{ds}\right) = 0$$
$$\alpha^2\left(\frac{dx}{ds}\right)^2 + \alpha^2\left(\frac{dy}{ds}\right)^2 + \alpha^2\left(\frac{dz}{ds}\right)^2 - k\alpha\left(\frac{dt}{ds}\right)^2 = 1$$

Integrating the first three equations, we find $dy/ds = A/\alpha^2$, $dz/ds = B/\alpha^2$, $dt/ds = C/\alpha$, where A, B, C, are constants.

The first integral can be expressed in the form

$$(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - k\alpha = \left(\frac{ds}{dt}\right)^2 = \frac{\alpha^2}{C^2}$$

i.e.

$$v^2 - \frac{k}{\alpha} = \frac{1}{C^2} = \text{constant}$$

No.36. As explained in section 56, the form taken by equation (26.13) in a gravitational field is

$$F_{ij} = \Omega_{j;i} - \Omega_{i;j}$$

But, as proved in No.7 of Exercises 5, this reduces to

$$F_{ij} = \Omega_{j,i} - \Omega_{i,j}$$

Hence, equation (26.13) remains valid in a gravitational field.

Assuming $\Omega_i = (0,0,0,\chi)$, where $\chi = \chi(r)$, the only non-zero components of the covariant field tensor are $F_{14} = -F_{41} = d\chi/dr$. Since $g^{11} = 1/a$, $g^{22} = 1/r^2$, $g^{33} = 1/(r^2\sin^2\theta)$, $g^{44} = -1/(bc^2)$, the other components vanishing, we can raise indices and show that the only non-zero components of the contravariant field tensor are $F^{14} = -F^{41} = -\frac{1}{abc^2}\frac{d\chi}{dr}$.

With $J^i = 0$, the first three equations (56.4) reduce to 0 = 0 and the fourth to

$$\frac{d}{dr}\left[\frac{r^2}{\sqrt{(ab)}}\frac{d\chi}{dr}\right] = 0$$

Hence

$$\frac{d\chi}{dr} = \frac{q}{4\pi\varepsilon_0 r^2}\sqrt{(ab)}$$

where q is a constant (and can be interpreted as the charge at O in SI units). All equations (56.5) reduce to 0 = 0.

Equation (56.6) determines the energy-momentum tensor S^i_j. We find that

$$-S^1_1 = S^2_2 = S^3_3 = -S^4_4 = \chi'^2/(2\mu_0 abc^2),$$

where $\chi' = d\chi/dr$; all other components vanish. Note that $S = S^i_i = 0$. Lowering indices, we get

$$S_{11} = -\chi'^2/(2\mu_0 bc^2), \quad S_{22} = r^2\chi'^2/(2\mu_0 abc^2)$$

$$S_{33} = r^2\chi'^2\sin^2\theta/(2\mu_0 abc^2), \quad S_{44} = \chi'^2/(2\mu_0 a),$$

the remaining components vanishing.

Putting $a = e^\alpha$, $b = e^\beta$, and referring to equations (51.18), we can write down the Einstein equations (47.15) thus:

$$\tfrac{1}{2}\beta'' + \tfrac{1}{4}\beta'^2 - \tfrac{1}{4}\alpha'\beta' - \frac{1}{r}\alpha' = \frac{\kappa}{2\mu_0 bc^2}\chi'^2 = ke^\alpha/r^4 \qquad \text{(i)}$$

$$e^{-\alpha}(\tfrac{1}{2}r\beta' - \tfrac{1}{2}r\alpha' + 1) - 1 = -\frac{\kappa}{2\mu_0 abc^2}r^2\chi'^2 = -k/r^2 \qquad \text{(ii)}$$

$$-\tfrac{1}{2}\beta'' - \tfrac{1}{4}\beta'^2 + \tfrac{1}{4}\alpha'\beta' - \frac{1}{r}\beta' = -ke^\alpha/r^4, \qquad \text{(iii)}$$

where $k = Gq^2/(4\pi\varepsilon_0 c^4)$ (we have used the results $\mu_0\varepsilon_0 = 1/c^2$ and $\kappa = 8\pi G/c^4$).

$$e^{\beta}(r\beta' + 1) = 1 - \frac{k}{r^2}$$

Putting $b = e^{\beta}$ again, this takes the form

$$\frac{d}{dr}(rb) = 1 - \frac{k}{r^2}$$

Hence

$$b = 1 + \frac{k}{r^2} + \frac{A}{r},$$

where A is constant. In the case $k = 0$, we know $A = -2m$. Thus

$$b = \frac{1}{a} = 1 - \frac{2m}{r} + \frac{k}{r^2}.$$

It can now be checked that equations (i) and (iii) are satisfied by this solution.

No.37. Referring to equation (48.6), we have the approximation

$$\Gamma^i_{jk} = \tfrac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i})$$

But, $g^{jk} = \delta^{jk}$ + first order terms in the h_{ij}. Hence

$$\Gamma^i = g^{jk}\Gamma^i_{jk} = \tfrac{1}{2}(2h_{ij,j} - h_{jj,i})$$

The condition from No.50, Ex.5 for a harmonic frame now takes the form

$$\delta^{jk}\frac{\partial^2 \overline{x}^i}{\partial x^j \partial x^k} = \frac{\partial \overline{x}^i}{\partial x^j}(h_{jr,r} - \tfrac{1}{2}h_{rr,j})$$

or

$$\overline{x}^i_{,jj} = \overline{x}^i_{,j}(h_{jr,r} - \tfrac{1}{2}h_{rr,j})$$

Putting $\overline{x}^i = x^i + \xi^i$, this takes the form

$$\xi^i_{,jj} = (\delta^i_j + \xi^i_{,j})(h_{jr,r} - \tfrac{1}{2}h_{rr,j})$$

$$= h_{ir,r} - \tfrac{1}{2}h_{rr,i}$$

since the ξ^i are small with the h_{ij}.

We have the transformation equation

$$g_{ij} = \frac{\partial \overline{x}^r}{\partial x^i}\frac{\partial \overline{x}^s}{\partial x^j}\overline{g}_{rs}$$

or

$$\delta_{ij} + h_{ij} = (\delta^r_i + \xi^r_{,i})(\delta^s_j + \xi^s_{,j})(\delta_{rs} + \overline{h}_{rs})$$

$$= \delta_{ij} + \xi^j_{,i} + \xi^i_{,j} + \overline{h}_{ij}$$

to the first order of small quantities. Thus

$$\overline{h}_{ij} = h_{ij} - \xi^j_{,i} - \xi^i_{,j}$$

$$\bar{h}_{ii} = h_{ii} - 2\xi^i_{,i}$$

Hence $\bar{h}_{ii} = 0$ provided $\xi^i_{,i} = \frac{1}{2}h_{ii}$.

If the x-frame is harmonic before transformation, then h_{ij} satisfies equation (58.2). Hence, the condition for the frame to remain harmonic after transformation is then

$$\xi^i_{,jj} = h_{ij,j} - \tfrac{1}{2}h_{jj,i} = 0.$$

No.38. To apply the formula (58.25), we need to calculate the second moments of a uniform sphere with respect to the rectangular axes Oxyz.

We have

$$I = \mu\int x^2 dV = \mu\int y^2 dV = \mu\int z^2 dV = \mu\iiint r^2\cos^2\theta . r^2\sin\theta dr d\theta d\phi$$

having introduced spherical polar coordinates. Thus

$$I = 2\pi\mu\int_0^\pi \cos^2\theta\sin\theta d\theta \times \int_0^a r^4 dr = 4\pi\mu a^5/15 = Ma^2/5$$

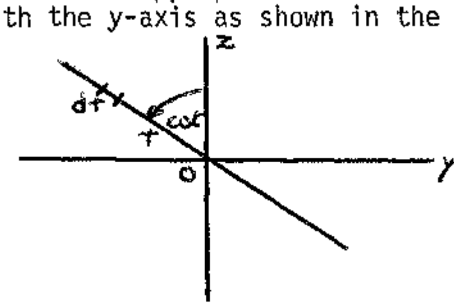
since $M = 4\pi a^3\mu/3$. The product moments $\mu\int xy dV$, etc. all vanish on account of the symmetry with respect to the axes.

Thus

$$h'_{11} = h'_{22} = h'_{33} = \frac{2GM}{5c^4 r}\frac{d^2}{dt^2}(a^2) = \frac{4GM}{5c^4 r}(\dot{a}^2 + a\ddot{a})$$

$$h'_{23} = h'_{31} = h'_{12} = 0.$$

No.39. Rename the frame Oxyz for convenience and suppose the rod rotates in the yz-plane. At the appropriate retarded time, suppose the rod makes an angle ωt with the y-axis as shown in the figure.



Then we have

$$\int x^2 dV = \int xz dV = \int xy dV = 0$$

$$\mu\int y^2 dV = \frac{M}{2a}\int_{-a}^{a} r^2\sin^2\omega t\, dr = \frac{1}{3}Ma^2\sin^2\omega t = \frac{1}{6}Ma^2(1 - \cos2\omega t)$$

$$\mu\int z^2 dV = \frac{M}{2a}\int_{-a}^{a} r^2\cos^2\omega t\, dr = \frac{1}{3}Ma^2\cos^2\omega t = \frac{1}{6}Ma^2(1 + \cos2\omega t)$$

$$\mu\int yz dV = -\frac{M}{2a}\int_{-a}^{a} r^2\sin\omega t\cos\omega t\, dr = -\frac{1}{3}Ma^2\sin\omega t\cos\omega t = -\frac{1}{6}Ma^2\sin2\omega t$$

Hence

$$h'_{33} = \frac{GMa^2}{3c^4r}\frac{d^2}{dt^2}(1 + \cos 2\omega t) = -h'_{22}$$

$$h'_{23} = h'_{32} = -\frac{GMa^2}{3c^4r}\frac{d^2}{dt^2}(\sin 2\omega t) = A\sin 2\omega t$$

the remaining components vanishing.

No.40. In empty space, the Einstein equation takes the form

$$R^i_j - \tfrac{1}{2}\delta^i_j R - \Lambda\delta^i_j = 0$$

Contracting by putting $j = i$, we get $R - 2R - 4\Lambda = 0$, i.e. $R = -4\Lambda$. Substituting for R in the covariant Einstein equation

$$R_{ij} - \tfrac{1}{2}g_{ij}R - \Lambda g_{ij} = 0$$

we arrive at the equation stated.

In the case of spherical symmetry, we can use equations (51.18) to get the Einstein equations in empty space surrounding a spherical body, in the form

$$\tfrac{1}{2}\beta'' + \tfrac{1}{4}\beta'^2 - \tfrac{1}{4}\alpha'\beta' - \frac{1}{r}\alpha' = -\Lambda e^{\alpha} \qquad \text{(i)}$$

$$e^{-\alpha}(\tfrac{1}{2}r\beta' - \tfrac{1}{2}r\alpha' + 1) - 1 = -\Lambda r^2 \qquad \text{(ii)}$$

$$-\tfrac{1}{2}\beta'' - \tfrac{1}{4}\beta'^2 + \tfrac{1}{4}\alpha'\beta' - \frac{1}{r}\beta' = -\Lambda e^{\alpha} \qquad \text{(iii)}$$

Adding (i) and (iii), we prove $\alpha = -\beta$ as in the case $\Lambda = 0$. Thus (ii) reduces to

$$e^{\beta}(r\beta' + 1) = 1 - \Lambda r^2 \quad \text{or} \quad rb' + b = 1 - \Lambda r^2$$

i.e

$$\frac{d}{dr}(rb) = 1 - \Lambda r^2$$

Integrating, we find

$$rb = r - \frac{1}{3}\Lambda r^3 + \text{const.}$$

Thus

$$b = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2$$

since the constant is known to be $-2m$ when $\Lambda = 0$.

From equation (48.17), we calculate that the Newtonian potential U is given by

$$U = \tfrac{1}{2}c^2(g_{44} - 1)$$

where $-c^2g_{44}dt^2$ is the fourth term in the metric. In our case, $g_{44} = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2$ and so

$$U = -\frac{mc^2}{r} - \frac{1}{6}\Lambda c^2 r^2$$

This potential corresponds to a radial gravitational field whose inward intensity is

$$\frac{dU}{dr} = \frac{mc^2}{r^2} - \frac{1}{3}\Lambda c^2 r$$

$mc^2/r^2 = GM/r^2$ is the Newtonian attraction due to a sphere of mass M. $-\frac{1}{3}\Lambda c^2 r$ represents an additional repulsive force, which increases with the distance r from the body.

No.1. We have to substitute $\sigma = r/(1 + \frac{1}{4}kr^2)$ in equation (61.8). Differentiation gives

$$d\sigma = \frac{1 - \frac{1}{4}kr^2}{(1 + \frac{1}{4}kr^2)^2}\, dr$$

Also

$$1 - k\sigma^2 = \left(\frac{1 - \frac{1}{4}kr^2}{1 + \frac{1}{4}kr^2}\right)^2$$

Hence

$$\frac{d\sigma^2}{1 - k\sigma^2} = \frac{dr^2}{(1 + \frac{1}{4}kr^2)^2}$$

and

$$ds^2 = \frac{S^2}{(1 + \frac{1}{4}kr^2)^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} - c^2dt^2$$

No.2. We have

$$T^{4i}{}_{;i} = \frac{\partial T^{4i}}{\partial x^i} + \left\{{4 \atop k\ i}\right\}T^{ki} + \left\{{i \atop k\ i}\right\}T^{4k}$$

With $x^1 = \sigma$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$, the non-zero components of T^{ij} are given at equation (65.4). Thus

$$\frac{\partial T^{4i}}{\partial x^i} = \dot{\mu}$$

μ being a function of the cosmical time t alone. Also

$$\left\{{4 \atop k\ i}\right\}T^{ki} = \left\{{4 \atop 1\ 1}\right\}T^{11} + \left\{{4 \atop 2\ 2}\right\}T^{22} + \left\{{4 \atop 3\ 3}\right\}T^{33} + \left\{{4 \atop 4\ 4}\right\}T^{44}$$

$$= \frac{1}{c^2}S\dot{S}\left(\frac{1}{1 - k\sigma^2}T^{11} + \sigma^2T^{22} + \sigma^2\sin^2\theta T^{33}\right)$$

$$= 3p\dot{S}/(c^2S)$$

$$\left\{{i \atop k\ i}\right\}T^{4k} = \left\{{i \atop 4\ i}\right\}T^{44} = \frac{1}{\sqrt{(-g)}}\frac{\partial}{\partial t}\{\sqrt{(-g)}\}T^{44} = 3\mu\dot{S}/S$$

Hence, the conservation equation $T^{4i}{}_{;i} = 0$ takes the form

$$\dot{\mu} + \frac{3p\dot{S}}{c^2S} + \frac{3\mu\dot{S}}{S} = 0$$

or

$$S^3\dot{\mu} + 3\mu S^2\dot{S} + \frac{3}{c^2}pS^2\dot{S} = 0$$

or

$$\frac{d}{dt}(\mu S^3) + \frac{3}{c^2}pS^2\dot{S} = 0$$

If $p = 0$, then μS^3 = constant.

No.3. Note: 'exp(2HT)' should read 'exp(2Ht)' in the first metric.

Differentiating the transformation equations, we find

$$dt = dT - \frac{Hr}{c^2}(1 - H^2r^2/c^2)^{-1}dr$$

Whence

$$A^2e^{2Ht}d\sigma^2 - c^2dt^2$$

$$= e^{2H(t-T)}\{(1-H^2r^2/c^2)^{-3}dr^2 - 2Hr(1-H^2r^2/c^2)^{-2}drdT + H^2r^2(1-H^2r^2/c^2)^{-1}dT^2\}$$

$$- c^2dT^2 + 2Hr(1-H^2r^2/c^2)^{-1}drdT - \frac{H^2r^2}{c^2}(1-H^2r^2/c^2)^{-2}dr^2$$

But

$$e^{2H(t-T)} = 1 - H^2r^2/c^2$$

by the second transformation equation. Hence

$$A^2e^{2Ht}d\sigma^2 - c^2dt^2 = (1-H^2r^2/c^2)^{-1}dr^2 - c^2(1-H^2r^2/c^2)dT^2$$

Also

$$A^2e^{2Ht}\sigma^2 = \frac{r^2}{1 - H^2r^2/c^2}e^{2H(t-T)} = r^2$$

This completes the transformation of the metric.

No.4. The equation of radial motion of a photon follows immediately from the metric, viz.

$$A^2e^{2Ht}d\sigma^2 - c^2dt^2 = 0$$

For inwards motion, this reduces to

$$\frac{d\sigma}{dt} = -\frac{c}{A}e^{-Ht}$$

Integration under the initial conditions $\sigma = \sigma_0$, $t = t_0$, shows that

$$\sigma_0 - \sigma = \frac{c}{AH}(e^{-Ht_0} - e^{-Ht})$$

Thus, when $\sigma = 0$,

$$e^{-Ht_0} - e^{-Ht} = AH\sigma_0/c$$

or $$t = -\frac{1}{H}\log(e^{-Ht_0} - AH\sigma_0/c) = t_0 - \frac{1}{H}\log\left(1 - \frac{AH\sigma_0}{c}e^{Ht_0}\right)$$

Since the time taken to reach the origin is $t - t_0$, this agrees with the result stated.

If $\frac{HA\sigma_0}{c}e^{Ht_0} > 1$, the logarithm is imaginary and the photon can never arrive at O. The proper distance of the point (σ_0,θ,ϕ) from O at time t_0, is

$$d = \int_0^{\sigma_0} Ae^{Ht_0}d\sigma = A\sigma_0e^{Ht_0}$$

Thus, if $Hd/c > 1$ or $d > c/H$, the photon cannot reach O.

No.5. We make use of equations (64.6), (63.5), (64.7), viz.

$$d_L = \sigma S_0^2/S_1, \quad z = \frac{S_0}{S_1} - 1,$$

If $k = 1$, the last equation gives

$$\sin^{-1}\sigma = c\int_{t_1}^{t_0}\frac{dS}{S\dot{S}} = \int_{t_1}^{t_0}\frac{dS}{\sqrt{S(D - S)}}$$

$$= \sin^{-1}\{(2S_0 - D)/D\} - \sin^{-1}\{(2S_1 - D)/D\} \qquad \text{(i)}$$

having used equation (67.8) for a Friedmann universe. We can now calculate σ in the form

$$\sigma = \frac{2}{D^2}\{(2S_0 - D)\sqrt{S_1(D - S_1)} - (2S_1 - D)\sqrt{S_0(D - S_0)}\}$$

$$= 2\{(2\alpha - 1)\sqrt{\beta(1 - \beta)} - (2\beta - 1)\sqrt{\alpha(1 - \alpha)}\}$$

where $\alpha = S_0/D$, $\beta = S_1/D$. Hence

$$d_L = \frac{2D\alpha^2}{\beta}\{(2\alpha - 1)\sqrt{\beta(1 - \beta)} - (2\beta - 1)\sqrt{\alpha(1 - \alpha)}\}$$

$$= 2D\alpha^2\{(2\alpha - 1)\sqrt{1/\beta - 1} - (2 - 1/\beta)\sqrt{\alpha(1 - \alpha)}\} \qquad \text{(ii)}$$

Now, using equations (67.17), (67.20), we have

$$S_0 = \frac{c}{H_0\sqrt{(2q_0 - 1)}}, \qquad D = \frac{2cq_0}{H_0(2q_0 - 1)^{3/2}}.$$

Whence

$$\alpha = (2q_0 - 1)/2q_0, \qquad \frac{1}{\beta} = \frac{1}{\alpha}(1 + z).$$

Substitution in (ii) now yields the stated result.

If $k = -1$, we use equation (67.14) and equation (i) is replaced by

$$\sinh^{-1}\sigma = \cosh^{-1}\{(2S_0 + D)/D\} - \cosh^{-1}\{(2S_1 + D)/D\}$$

leading to the equation

$$\sigma = 2\{(2\beta + 1)\sqrt{\alpha(\alpha + 1)} - (2\alpha + 1)\sqrt{\beta(\beta + 1)}\}$$

For this case,

$$D = \frac{2cq_0}{H_0(1 - 2q_0)^{3/2}}, \quad \alpha = (1 - 2q_0)/2q_0, \quad \frac{1}{\beta} = \frac{1}{\alpha}(1 + z)$$

so that the expression for d_L works out as for the previous case.

If $k = 0$, we use equation (67.12) and equation (i) is replaced by

$$\sigma = 2\{\sqrt{(S_0/D)} - \sqrt{(S_1/D)}\}$$

Hence

$$d_L = \frac{2S_0^2}{S_1}\{\sqrt{(S_0/D)} - \sqrt{(S_1/D)}\} = \frac{2S_0^{3/2}}{\sqrt{D}}\frac{S_0}{S_1}\left(\sqrt{\frac{S_0}{S_1}} - 1\right)$$

$$= \frac{2c}{H_0}\sqrt{(z + 1)}\{\sqrt{(z + 1)} - 1\}$$

using equation (67.21). This is the stated result with $q_0 = \frac{1}{2}$.

If z is small, we can use the binomial theorem to approximate to d_L thus:

$$d_L = \frac{c}{\dots}[q_0 z + (q_0 - 1)\{q_0 z - \tfrac{1}{2}q_0^2 z^2 + O(z^3)\}]$$

$$= \frac{c}{H_0}\{z - \tfrac{1}{2}(q_0 - 1)z^2 + O(z^3)\}$$

Then, referring to equation (64.5) for the apparent luminosity ℓ of a galaxy, we deduce that

$$\ell = \frac{L}{4\pi d_L^2} = \frac{LH_0^2}{4\pi c^2 z^2}\{1 - \tfrac{1}{2}(q_0 - 1)z + O(z^2)\}^{-2}$$

$$= \frac{LH_0^2}{4\pi c^2 z^2}\{1 + (q_0 - 1)z + O(z^2)\}$$

in agreement with equation (64.14).

No.6. Retaining the cosmical constant term and taking $p = 0$, equation (65.9) gives

$$2S\ddot{S} + \dot{S}^2 + kc^2 - c^2\Lambda S^2 = 0$$

This can be written

$$\frac{d}{dt}(S\dot{S}^2) = -kc^2\dot{S} + c^2\Lambda S^2\dot{S}$$

Integration leads to the result

$$S\dot{S}^2 = -kc^2 S + \frac{1}{3}c^2\Lambda S^3 + \text{constant}$$

or

$$S\dot{S}^2 = c^2(D - kS + \frac{1}{3}\Lambda S^3)$$

where D is constant.

Substitution for $\dot{S}^2$ from the last equation into equation (65.10) shows that $\kappa c^2\mu S^3 = 3D$.

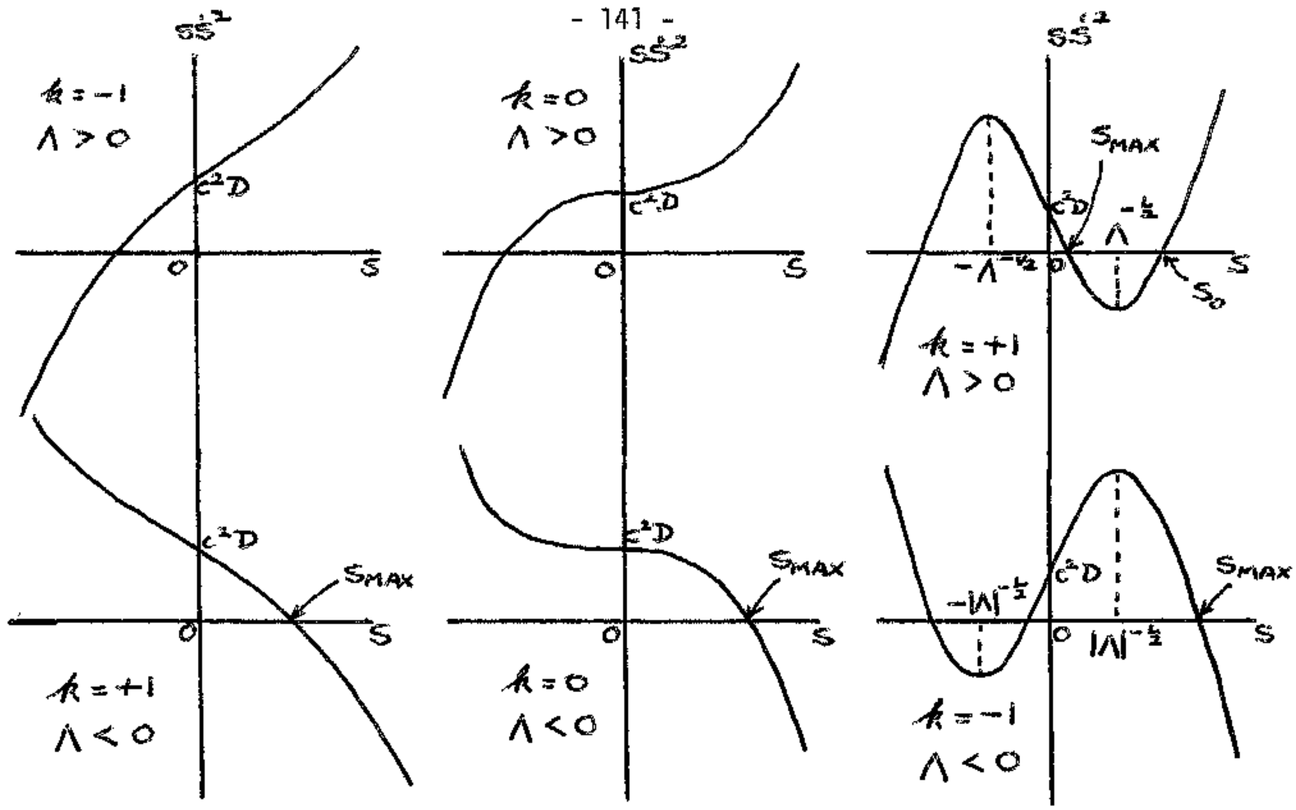
If $k = 0$, $D = 0$ (i.e. $\mu = 0$), then

$$\dot{S}^2 = \frac{1}{3}c^2\Lambda S^2 \quad \text{or} \quad \dot{S} = c\surd(\Lambda/3)S.$$

Integration now yields $S = Ae^{Ht}$, with $H = c\surd(\Lambda/3)$. This is equation (66.6) for the de Sitter universe.

No.7. $S\dot{S}^2$ is a cubic in S, which is stationary where $S = \pm\surd(k/\Lambda)$. Thus, if $k = -1$ and $\Lambda > 0$, or if $k = +1$ and $\Lambda < 0$, the cubic has no real stationary values and $S\dot{S}^2$ increases monotonically with S if $\Lambda > 0$, and decreases monotonically as S increases for $\Lambda < 0$. If $k = 0$, $S\dot{S}^2$ increases monotonically with S for $\Lambda > 0$ and decreases monotonically as S increases for $\Lambda < 0$; in these cases, $S\dot{S}^2$ is stationary at $S = 0$ (point of inflexion). If $k = +1$ and $\Lambda > 0$, or if $k = -1$ and $\Lambda < 0$, $S\dot{S}^2$ has a real maximum and a real minimum; its minimum value is $c^2(D - 2/3\surd|\Lambda|)$ and, in sketching the graphs, we shall assume this to be negative, since $|\Lambda|$ is certainly exceedingly small. The graphs of $S\dot{S}^2$ are now easily sketched as below:

S can be assumed positive and $\dot{S}$ to be initially positive. Then, S will increase steadily until $\dot{S}$ changes sign, which it can only do by passing through the value zero at $S = S_{MAX}$. Referring to the graphs, we note that this is only possible in the cases (a) $k = +1$, (b) $k = 0$, $\Lambda < 0$, (c) $k = -1$, $\Lambda < 0$. In case (a), if $\Lambda < 0$, then S_{MAX} certainly exists; if, however, $\Lambda > 0$, the S_{MAX} only exists if the minimum of $S\dot{S}^2$ is negative (as shown) — the condition for this is $2/3\surd\Lambda > D$, i.e. $\Lambda < 4/(9D^2)$. Hence, if $k = +1$, we simply require $\Lambda < 4/(9D^2)$ for S_{MAX} to exist (if $\Lambda = 4/(9D^2)$, S takes an infinite time to reach the value S_{MAX} (see Ex.8,



In the cases (a) $k = +1$, $\Lambda > 4/(9D^2)$, (b) $k = 0$, $\Lambda > 0$, (c) $k = -1$, $\Lambda > 0$, S_{MAX} does not exist and S will increase indefinitely.

If $\Lambda = 0$, the evolution of the Friedmann model has already been discussed in section 67, and we have shown that S increases indefinitely if $k = 0$ or -1, but achieves a maximum if $k = +1$. These cases have been included in the statement of Ex.7.

If $k = +1$ and $\Lambda > 0$, if $S > S_0$ (see graph above), then $S\dot{S}^2$ is positive and there is a possible motion. Then, if S is initially decreasing, S will continue to decrease until $S = S_0$, when S must change sign and S will thereafter increase to $+\infty$.

No.8. If $k = +1$ and $\Lambda = 4/(9D^2)$, then

$$S\dot{S}^2 = \frac{c^2}{27D^2}(3D - 2S)^2(S + 3D),$$

showing that $S = 0$ when $S = 3D/2 = 1/\sqrt{\Lambda}$. It follows that, if S initially increases from zero, it will do so monotonically until S achieves its stationary value $1/\sqrt{\Lambda}$. A further analysis proves that S takes an infinite time to arrive at this value. Thus, putting $S = 3D/(u^2 - 1)$, where u decreases from $+\infty$ to $\sqrt{3}$ as S increases from 0 to $1/\sqrt{\Lambda}$, we find that

$$\frac{du}{dt} = -\frac{c}{3\sqrt{3}D}(u^2 - 1)(u^2 - 3)$$

so that

$$t = \frac{3\sqrt{3}D}{2c}\int\left(\frac{1}{u^2 - 1} - \frac{1}{u^2 - 3}\right)du$$

$$= \frac{3D}{4c}\left\{\sqrt{3}\log\frac{u - 1}{u + 1} - \log\frac{u - \sqrt{3}}{u + \sqrt{3}}\right\},$$

the constant of integration vanishing if we take $t = 0$ when $S = 0$ (i.e. $u = +\infty$). Now clearly, as $u \to \sqrt{3}$, $t \to +\infty$.

made positive and S will commence to increase — thereafter $\dot{S}$ cannot vanish and $S \to +\infty$. If $S < 3D/2$, then $\dot{S} < 0$ and S will commence to decrease — thereafter $\dot{S}$ cannot vanish and $S \to 0$.

No.9. Eliminating U between equations (68.7), (68.8), we find that

$$S\ddot{S} + \dot{S}^2 + kc^2 = \tfrac{2}{3}c^2\Lambda S^2$$

Multiplying through by $2S\dot{S}$, this gives

$$\frac{d}{dt}(S^2\dot{S}^2) = (-\,2kc^2S + \tfrac{4}{3}c^2\Lambda S^3)\dot{S}$$

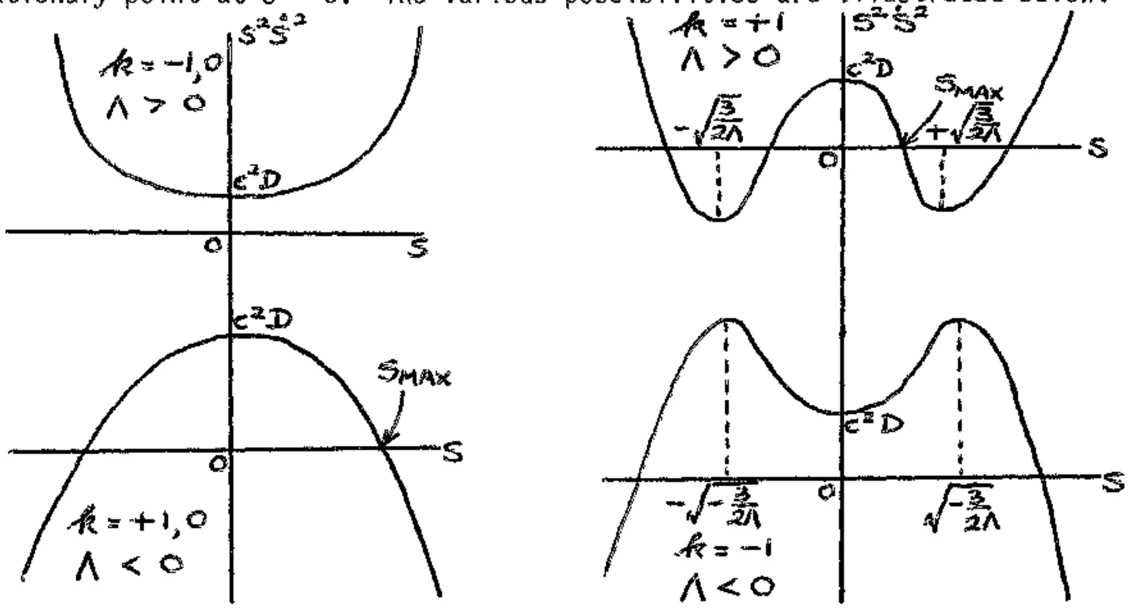
which possesses the integral

$$S^2\dot{S}^2 = -\,kc^2S^2 + \tfrac{1}{3}c^2\Lambda S^4 + \text{const.}$$

verifying the stated result.

Substituting for $\dot{S}^2$ into equation (68.8), this shows that $3D = \kappa US^4$.

No.10. $S^2\dot{S}^2$ equals a quartic in S, which is stationary where $S = \pm\sqrt{\frac{3k}{2\Lambda}}$ or $S = 0$. Unless k and Λ have the same sign, therefore, there is only one stationary point at $S = 0$. The various possibilities are illustrated below:



If $k = +1$, $\Lambda > 0$, the minimum value of $S^2\dot{S}^2$ equals $c^2(D - 3/(4\Lambda))$ and S_{MAX} only exists provided $D < 3/(4\Lambda)$, i.e. $\Lambda < 3/(4D)$. We conclude that S_{MAX} exists in the cases (a) $k = +1$, $\Lambda < 3/(4D)$, (b) $k = 0$, $\Lambda < 0$, (c) $k = -1$, $\Lambda < 0$; in these cases S increases to S_{MAX} and thereafter decreases. In all other cases, S increases indefinitely.

No.11. If $k = 1$, $\Lambda = 3/(4D)$, the equation of Ex.9 can be written

$$S^2\dot{S}^2 = \frac{c^2}{4D}(2D - S^2)^2$$

Clearly, S can increase from 0 to a maximum $\sqrt{(2D)}$. We have

dt 2√D S

Whence

$$t \;=\; -\frac{\surd D}{c}\log(2D - S^2) + \text{const.}$$

If $S = 0$ at $t = 0$, the constant $= \frac{\surd D}{c}\log 2D$. Hence

$$t \;=\; \frac{\surd D}{c}\log\frac{2D}{2D - S^2}$$

or

$$S^2 \;=\; 2D\{1 - \exp(-ct/\surd D)\}.$$

This shows that the universe takes an infinite time to reach the static state $S = \surd(2D)$.

If the universe is disturbed slightly from such a static state, if $S > \surd(2D)$, then S has been increased and $\dot{S} > 0$; S now increases indefinitely since S cannot pass through the value zero. If $S < \surd(2D)$, then S has been decreased and $\dot{S} < 0$; S now decreases towards 0 and $\dot{S}$ cannot change sign. We conclude that the static universe is unstable.