Solution Manual

for

A Journey into Mathematics

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written by

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Solutions Chapter 1

His answer trickled through my head, like water through a sieve.

Lewis Carroll

1.1. Find a formula for $1 + \sum_{j=1}^{n} j!j$ and use mathematical induction to prove that your formula is correct.

* ~~A list of~~ The sums for $n = 1, 2, 3, 4, 5$ ~~is~~ are 2, 6, 24, 120, 720. These are factorials; better, they are 2!, 3!, 4!, 5!, 6! We have been led to the *guess*

$$S(n): \; 1 + \sum_{j=1}^{n} j!j = (n + 1)!.$$

We now use induction to prove that the guess is *always* true. The base step $S(1)$ has already been checked; ~~it is on the list.~~ $S(1) = 2$. For the inductive step, we must prove

$$S(n + 1): 1 + \sum_{j=1}^{n+1} j!j = (n + 2)!.$$

Rewrite the left side as $[1 + \sum_{j+1}^{n} j!j] + (n + 1)!(n + 1)$. By the inductive hypothesis, the bracketed term ~~on the left side~~ is $(n + 1)!$, and so the left side equals $(n + 1)! + (n + 1)!(n + 1) = (n + 1)![1 + (n + 1)] = (n + 1)!(n + 2) = (n + 2)!$. By induction, $S(n)$ is true for all $n \geq 1$.

1.2. If $r \neq 1$, prove, for all $n \geq 1$, that

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} = (r^n - 1)/(r - 1).$$

* The statement $S(1)$ is true because the left side is 1 (this is what the formula $1 + r + r^2 + r^3 + \cdots + r^{n-1}$ means when $n = 1$), while $(r - 1)/(r - 1) = 1$ as well $[r - 1 \neq 0$ because $r \neq 1]$.

Inductive Step: We state $S(n + 1)$:

$$[1 + r + r^2 + r^3 + \cdots + r^{n-1}] + r^n = (r^{n+1} - 1)/(r - 1).$$

The inductive hypothesis allows us to rewrite the left side as

$$\begin{aligned}(r^n - 1)/(r - 1) + r^n &= (r^n - 1 + (r - 1)r^n)/(r - 1) \\ &= (r^{n+1} - 1)/(r - 1).\end{aligned}$$

One can also prove the formula by showing that

$$(r - 1)(1 + r + r^2 + r^3 + \cdots + r^{n-1}) = r^n - 1.$$

The proof here does not use induction explicitly (but it uses other results needing induction in their proofs; e.g., the generalized distributive law).

$$\begin{aligned}&(r - 1)(1 + r + r^2 + r^3 + \cdots + r^{n-1}) = \\ &= r(1 + r + r^2 + r^3 + \cdots + r^{n-1}) - (1 + r + r^2 + r^3 + \cdots + r^{n-1}) \\ &= r + r^2 + r^3 + \cdots + r^n + (- 1 - r - r^2 - r^3 - \cdots - r^{n-1}) \\ &= r^n - 1.\end{aligned}$$

1.3. Show, for all $n \geq 1$, that 10^n leaves remainder 1 after dividing by 9.

* We prove, by induction on $n \geq 1$, that there is an integer q_n with $10^n = 9q_n + 1$.

Base Step: $n = 1$. Since $10 = 9 \times 1 + 1$, we may set $q_1 = 1$.

Inductive Step: We are to prove $S(n + 1)$:

$$10^{n+1} = 9q_{n+1} + 1 \text{ for some integer } q_{n+1}.$$

The inductive hypothesis gives an integer q_n with

$$10^n = 9q_n + 1.$$

Hence

$$\begin{aligned}10^{n+1} &= 10 \times 10^n = 10(9q_n + 1)\\ &= 9(10q_n) + 10\\ &= 9(10q_n) + 9 + 1\\ &= 9(10q_n + 1) + 1.\end{aligned}$$

In fact, q_n is the integer having n digits all equal to 1.

Define $q_{n+1} = 10q_n + 1$; it is an integer because q_n is.

1.4. If $a \le b$ are positive numbers, prove that $a^n \le b^n$ for all integers $n \ge 0$.

* *Base step.* $a^0 = 1 = b^0$.

Inductive step. $a^{n+1} = aa^n \le ab^n$, using the inductive hypothesis and Theorem 1.4(i), and $ab^n \le bb^n = b^{n+1}$, again using Theorem 1.4(i).

1.5. (i) Prove that $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.

* *Base step:* $1(1 + 1)(2 + 1)/6 = 1$.

Inductive step:

$$\begin{aligned}[1^2 + 2^2 + \cdots + n^2] + (n + 1)^2 &= [n(n + 1)(2n + 1)/6] + (n + 1)^2\\ &= (n + 1)(n + 2)(2n + 3)/6.\end{aligned}$$

(ii) Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

* *Base step:* When $n = 1$, both sides equal 1.

Inductive step:

$$[1^3 + 2^3 + \cdots + n^3] + (n + 1)^3 = (1 + 2 + \cdots + n)^2 + (n + 1)^3$$
$$= [n(n + 1)/2]^2 + (n + 1)^3$$
$$= (n + 1)^2(n + 2)^2/4$$
$$= (1 + 2 + \cdots + n + (n + 1))^2,$$

using Theorem 1.6.

(iii) Prove that

$$1^4 + 2^4 + \cdots + n^4 = n^5/5 + n^4/2 + n^3/3 - n/30.$$

* *Base step*:

$$1/5 + 1/2 + 1/3 - 1/30 = (6 + 15 + 10 - 1)/30 = 1.$$

Inductive step:

$$[1^4 + 2^4 + \cdots + n^4] + (n + 1)^4$$
$$= n^5/5 + n^4/2 + n^3/3 - n/30 + (n + 1)^4$$
$$= (n + 1)^5/5 + (n + 1)^4/2 + (n + 1)^3/3 - (n + 1)/30.$$

1.6. (i) Find a formula for $a_n = 1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3$, and then prove that your guess is correct using induction.

* Comparing small values of a_n with corresponding values of n^4 and n^2 ultimately leads to the guess: $a_n = 2n^4 - n^2$. The proof by induction follows.

Base step: If $n = 1$, then $a_1 = 1^3 = 1$, while $2 \times 1^4 - 1^2 = 1$.

Inductive step: We must prove that

$$a_{n+1} = 1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 + (2n+1)^3$$

$$= 2(n+1)^4 - (n+1)^2.$$

Now

$$1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 + (2n+1)^3$$

$$= [1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3] + (2n+1)^3$$ \cdots

$$= 2n^4 - n^2 + (2n+1)^3.$$

It is now a matter of high school algebra to expand this and compare it to the expanded form of $2(n+1)^4 - (n+1)^2$.

(ii) Give a second proof of part (i) based on Exercise 1.5(ii) and the following observation: If $b_{2n} = 1^3 + 2^3 + 3^3 + \ldots + (2n)^3$, then

\cdots 3x

$$b_{2n} = a_n + [2^3 + 4^3 + \ldots + (2n)^3]$$

$$= a_n + 8[1^3 + 2^3 + \ldots + n^3]$$

$$= a_n + 8b_n.$$

$*$ By Exercise 1.5(ii),

$$b_n = \tfrac{1}{4}n^2(n+1)^2 \text{ and } b_{2n} = \tfrac{1}{4}(2n)^2(2n+1)^2;$$

By the hint,

$$a_n = b_{2n} - b_n$$

$$= \tfrac{1}{4}(2n)^2(2n+1)^2 - 8[\tfrac{1}{4}n^2(n+1)^2]$$

$$= \tfrac{1}{4}[4n^2(4n^2+4n+1) - 8n^2(n^2+2n+1)]$$

$$= n^2(4n^2+4n+1) - 2n^2(n^2+2n+1)$$

$$= 4n^4 + 4n^3 + n^2 - (2n^4 + 4n^3 + 1)$$

$$= 2n^4 - n^2.$$

1.7. (i) If $n = ab$, where n, a, and b are positive integers, then either $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

* If, on the contrary, $a > \sqrt{n}$ and $b > \sqrt{n}$, then

$$n = ab > \sqrt{n}\sqrt{n} = n,$$

a contradiction.

(ii) If $n \ge 2$ is a composite integer, prove that it has a prime factor p with $p \le \sqrt{n}$.

* Since n is composite, $n = ab$, where $2 \le a$ and $2 \le b$. By (i), we may assume that $a \le \sqrt{n}$. By Theorem 1.2, either a is prime or a product of primes. In either case, there is a prime p dividing a (perhaps $p = a$), hence dividing n, with

$$p \le a \le \sqrt{n}.$$

(iii) If $f(n) = n^2 - n + 41$, use part (ii) to show that $f(10)$, $f(20)$, $f(30)$, and $f(40)$ are prime.

* It suffices to show that no prime $\le \sqrt{f(n)}$ is a divisor of $f(n)$. If $n = 10$, then $f(n) = 131$. Since $\sqrt{131} \approx 11.45$, we see that 131 is prime if it is not divisible by 2, 3, 5, 7, or 11. And one checks easily that 131 is not divisible by any of these numbers.

Similarly, $f(20) = 421$, $\sqrt{421} \approx 20.52$, and one must check whether 421 is divisible by the primes ≤ 19; $f(30) = 971$, $\sqrt{971} \approx 31.16$, and one must check whether 971 is divisible by the primes ≤ 31; $f(40) = 1601$, $\sqrt{1601} \approx 40.01$, and one must check whether 1601 is divisible by the primes ≤ 37.

1.8. Prove that $(1 + x)^n \geq 1 + nx$ if $1 + x > 0$.

* We prove the inequality by induction on $n \geq 1$. The base step $n = 1$ is obvious. For the inductive step,

$$(1 + x)^{n+1} = (1 + x)(1 + x)^n > (1 + x)(1 + nx)$$

[because $1 + x > 0$]

$$= 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x.$$

(The binomial theorem should not be used here.)

1.9. (i) Prove that $2^n > n^3$ for all $n \geq 10$.

* *Base step*: $2^{10} = 1024 > 10^3 = 1000$. (Note that $2^9 = 512 < 9^3 = 729$.)

Inductive step ($n \geq 10$ implies $n \geq 4$): The inductive hypothesis is $2^n > n^3$; multiplying both sides by 2 gives

$$2^{n+1} = 2 \times 2^n > 2n^3 = n^3 + n^3 \geq n^3 + 4n^2$$

$$\geq n^3 + 3n^2 + n^2 > n^3 + 3n^2 + 4n$$

$$\geq n^3 + 3n^2 + 3n + 1 = (n + 1)^3.$$

(ii) Prove that $2^n > n^4$ for all $n \geq 17$.

* *Base step*: $2^{17} = 131072 > 17^4 = 83521$.
(Note: $16^4 = (2^4)^4 = 2^{16}$.)

Inductive step ($n \geq 17$ implies $n \geq 7$). The inductive hypothesis is $2^n > n^4$; multiplying both sides by 2 gives

$$2^{n+1} = 2 \times 2^n > 2n^4 = n^4 + n^4 \geq n^4 + 5n^3$$

$$\geq n^4 + 4n^3 + n^3 \geq n^4 + 4n^3 + 7n^2$$

$$\geq n^4 + 4n^3 + 6n^2 + n^2 \geq n^4 + 4n^3 + 6n^2 + 5n$$

$$\geq n^4 + 4n^3 + 6n^2 + 4n + 1 = (n+1)^4.$$

1.10. Let $g_1(x), \ldots, g_n(x)$ be differentiable functions, and let $f(x) = g_1(x)\cdots g_n(x)$. Prove, for all $n \geq 2$, that the derivative

$$f'(x) = g_1'(x)g_2(x)\cdots g_n(x) + \Sigma_{i=2}^{n}\, g_1(x)\cdots g_{i-1}(x)g_i'(x)g_{i+1}(x)\cdots g_n(x).$$

$*$ *Base step* $n = 2$: the usual product rule for derivatives.

Inductive step: Define $h(x) = g_1(x)\cdots g_n(x) = f(x)/g_{n+1}(x)$. Rewrite the conclusion: $f'(x) = \Sigma_{j=1}^{n+1}\, g_j'(x)f(x)/g_j(x)$; now

$$f'(x) = (h(x)g_{n+1}(x))'$$

$$= h'(x)g_{n+1}(x) + h(x)g'_{n+1}(x)$$

$$= \Sigma_{i=1}^{n}\, [g_i'(x)h(x)/g_i(x)]g_{n+1}(x) + [f(x)/g_{n+1}(x)]g'_{n+1}(x)$$

$$= \Sigma_{j=1}^{n+1}\, g_j'(x)f(x)/g_j(x).$$

1.11. Prove that every positive integer a has a factorization $a = 3^k m$, where $k \geq 0$ and m is not a multiple of 3.

$*$ Second form of induction on $a \geq 1$.

Base step: Take $k = 0$ and $m = 1$.

Inductive step: If $a \geq 1$, then a is either not a multiple of 3 or it is. If a is not a multiple of 3, then take $k = 0$ and $m = a$. If a is a multiple of 3, then $a = 3b$. By the inductive hypothesis, $b = 3^\ell n$, where $\ell \geq 0$ and n is not a multiple of 3. Hence, the desired factorization of a is $a = 3^{\ell+1}n$.

1.12. Prove that $2^n < n!$ for all $n \geq 4$.

The proof is by induction on $n \geq 4$.
Base step. If $n = 4$, then $2^n = 2^4 = 16$, while $n! = 4! = 24$, and $16 < 24$.

Inductive step. We must prove the statement for $n + 1$, namely, $2^{n+1} < (n + 1)!$. Multiplying both sides of the nth statement $2^n > n!$ by 2, we have

$$2^{n+1} = 2 \times 2^n < 2 \times n!.$$

But since $4 \leq n$, we surely have $2 < n + 1$, so that

$$2 \times n! < (n + 1)n! = (n + 1)!,$$

as desired. \ldots

1.13. If the Fibonacci sequence is $F_0, F_1, F_2, \ldots$, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$, prove that $F_n < 2^n$ for all $n \geq 0$.

* The proof is by the second form of induction.
Base step: $F_0 = 0 < 1 = 2^0$ and $F_1 = 1 < 2^1 = 2$.
(There are two base steps because we will have to use two predecessors for the inductive step.)

Inductive step: If $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2} < 2^{n-1} + 2^{n-2} < 2^{n-1} + 2^{n-1} = 2 \times 2^{n-1} = 2^n.$$

Notice that the second form is the appropriate induction here, for we are using both $S(n - 2)$ and $S(n - 1)$ to prove $S(n)$.

1.14. For every acute angle θ, i.e., $0° < \theta < 90°$, prove that

$$\sin\theta + \cot\theta + \sec\theta \geq 3.$$

* That θ is an acute angle implies that the numbers $\sin\theta$, $\cot\theta$, and $\sec\theta$ are all positive. By Theorem 1.11,

$$[\tfrac{1}{3}(\sin\theta + \cot\theta + \sec\theta)]^3 \geq \sin\theta\cot\theta\sec\theta.$$

Now

$$\sin\theta\cot\theta\sec\theta = \sin\theta(\cos\theta/\sin\theta)(1/\cos\theta) = 1.$$

Therefore,

$$[\tfrac{1}{3}(\sin\theta + \cot\theta + \sec\theta)]^3 \geq 1,$$

$$\tfrac{1}{3}(\sin\theta + \cot\theta + \sec\theta) \geq 1$$

and

$$\sin\theta + \cot\theta + \sec\theta \geq 3.$$

1.15. Prove that if $a_1, a_2, \cdots, a_n$ are positive numbers, then

$$(a_1 + a_2 + \cdots + a_n)(1/a_1 + 1/a_2 + \cdots + 1/a_n) \geq n^2.$$

* By Theorem 1.11, $[(a_1 + a_2 + \cdots + a_n)/n]^n \geq a_1a_2\cdots a_n$ and $[1/a_1 + 1/a_2 + \cdots + 1/a_n)/n]^n \geq 1/a_1a_2\cdots a_n$. Therefore,

$$[(a_1 + a_2 + \cdots + a_n)/n]^n[1/a_1 + 1/a_2 + \cdots + 1/a_n)/n]^n \geq$$

$$a_1a_2\cdots a_n/a_1a_2\cdots a_n = 1.$$

Taking nth roots, we have

$$[(a_1 + a_2 + \cdots + a_n)/n][1/a_1 + 1/a_2 + \cdots + 1/a_n)/n] \geq 1,$$

and so $(a_1 + a_2 + \cdots + a_n)(1/a_1 + 1/a_2 + \cdots + 1/a_n) \geq n^2$.

1.16. For every $n \geq 2$, prove that there are n consecutive composite numbers.

* If $2 \leq a \leq n + 1$, then a is a divisor of $(n + 1)!$; say, $(n + 1)! = da$ for some integer d. It follows that $(n + 1)! + a = (d + 1)a$, and so $(n + 1)! + a$ is composite for all a between 2 and $n + 1$.

1.17. Show, for all r with $0 \le r \le n$, that

$$\binom{n}{r} = \binom{n}{n-r}.$$

∗ By Theorem 1.19, both $\binom{n}{r}$ and $\binom{n}{n-r}$ are equal to $n!/r!(n-r)!$.

1.18. Show, for every n, that the sum of the binomial coefficients is 2^n:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

∗ Use Corollary 1.20 with $x = 1$.

1.19. (i) Show, for every n, that the "alternating sum" of the binomial coefficients is zero:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n} = 0.$$

∗ Use Corollary 1.20 with $x = -1$.

(ii) Use part (i) to prove, for a given n, that the sum of all the binomial coefficients $\binom{n}{r}$ with r even is equal to the sum of all those $\binom{n}{r}$ with r odd.

∗ In the alternating sum, all the "even" terms have one sign, and all the "odd" terms have the opposite sign. Since the alternating sum is 0, transposing gives the result.

1.20. What is the coefficient of x^{16} in $(1 + x)^{20}$?

∗ Pascal's formula gives $\binom{20}{16} = 4845$.

1.21. How many ways are there to choose 4 colors from a palette containing 20 different paints?

∗ The answer is "20 choose 4" = $\binom{20}{4}$ = 4845. One can calculate with Pascal's formula, or one can use the symmetry in Exercise 1.17 and the calculation done in Exercise 1.20.

1.22. Prove that a set X with n elements has exactly 2^n subsets.

∗ There are many proofs of this. We offer three: a proof with binomial coefficients, an algebraic proof, and a combinatorial one.

Binomial coefficients. By Exercise 1.18,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

On the other hand, $\binom{n}{r}$ is just the number of subsets of X having exactly r elements. Thus, $\binom{n}{0}$ is the number of subsets with $r = 0$ elements (there is only one, the empty set $\varnothing$), $\binom{n}{1}$ is the number of subsets with exactly 1 element (there are n such), $\binom{n}{2}$ is the number of subsets with exactly 2 elements, and so forth. Every subset S of X is counted, for S must have r elements for $0 \le r \le n$.

\ldots

Algebraic. Let $X = \{a_1, a_2, \cdots, a_n\}$. We may describe each subset S of X by an n-tuple $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$, where $\varepsilon_i = 0$ if a_i is not in S and $\varepsilon_i = 1$ if a_i is in S (after all, a set is determined by the elements comprising it). But there are exactly 2^n such n-tuples, for there are two choices for each coordinate.

Combinatorial. Induction on $n \ge 1$ (taking base step $n = 0$ is also fine; the only set with 0 elements is $\varnothing$, which has exactly one subset, itself). If X has just one element, then there are two subsets: $\varnothing$ and X. For the inductive step, assume that X has $n + 1$ elements, of which one is colored red and n are

\varnothing

colored blue. There are two types of subsets S: those that are solid blue; those that contain the red. By induction, there are 2^n solid blue subsets. There are as many subsets containing the red as there are solid blue subsets: each such subset arises by adjoining the red element to a solid blue subset (even the singleton subset consisting of the red element alone arises in this way, by adjoining the red element to $\varnothing$). Hence, there are $2^n + 2^n = 2^{n+1}$ subsets.

1.23. A weekly lottery asks you to select 5 numbers between 1 and 45. At the week's end, 5 such numbers are drawn at random, and you win the jackpot if all your numbers, in some order, match the drawn numbers. How many selections of 5 numbers are there?

* The answer is "45 choose 5", which is $\binom{45}{5} = 1{,}221{,}759$. The odds against your winning are more than a million to one.

1.24. Assume that term-by-term differentiation of power series is valid: if

$$f(x) = \Sigma_{k\geq 0}\, a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

then the power series for its derivative $f'(x)$ is

$$f'(x) = \Sigma_{k\geq 1}\, k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots.$$

(i) Prove that $f(0) = a_0$.

* $f(0) = c_0$, for all the other terms are 0. (If one wants to be fussy--this is the wrong time for analytic fussiness--then the partial sums of the series form the constant sequence $c_0, c_0, c_0, (\cdots)$ \ldots

(ii) Prove, for all $n \geq 0$, that the nth derivative

$$f^{(n)}(x) = \Sigma_{k\geq n}\, k(k-1)(k-2)\cdots(k-n+1)a_k x^{k-n}.$$

[$f^{(0)}(x)$ is defined to be $f(x)$]. Conclude that $a_n = f^{(n)}(0)/n!$ for all $n \geq 0$.

* One prove this by induction on $n \geq 0$. The base step is obvious because $f^{(0)}(x)$ is defined to be $f(x)$. For the inductive step,

prime

$$f^{(n+1)}(x) = [f^{(n)}(x)]'$$

$$= [\Sigma_{k \geq n}\, k(k-1)\cdots(k-n+1)a_k x^{k-n}]'$$

$$= \Sigma_{k \geq n}\, k(k-1)\cdots(k-n+1)(k-n)a_k x^{k-n-1},$$

by term-by-term differentiation. Notice that the constant term in the expansion of $f^{(n)}(x)$ involves a_n, so that this term is not present in the expansion of $f^{(n+1)}(x)$; thus, the summation is really for $k \geq n+1$.

for

1.25. (Leibniz) Prove that if f and g are C^∞-functions, then

$$(fg)^{(n)} = \Sigma_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

* Base step. If $n = 0$, then the left side is $(fg)^{(0)} = fg$, while the right side is $\binom{0}{0} f^{(0)} g^{(0)} = \binom{0}{0} fg = fg$, because $\binom{0}{0} = 1$.
Inductive step.

prime

$$(fg)^{(n+1)} = [(fg)^{(n)}]'$$

$$= [\Sigma_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}]'$$

$$= \Sigma_{k=0}^{n} \binom{n}{k} [f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)}]$$

$$= \Sigma_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \Sigma_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)}$$

$$= \binom{n}{0} f' g^{(n)} + [\binom{n}{1} + \binom{n}{0}] f^{(2)} g^{(n-1)} + [\binom{n}{2} +$$

$$\binom{n}{1}]\, f^{(3)}g^{(n-2)} + [\binom{n}{3} + \binom{n}{2}]\, f^{(4)}g^{(n-3)} + \cdots + \binom{n}{n} f^{(n)}g'.$$

Now add, using Lemma 1.18.

1.26. Prove, for all $n \geq 0$ and for all $r \geq 0$, that

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}.$$

* We use induction on $r \geq 0$.

Base step. If $r = 0$, then the left side is $\binom{n}{0} = 1$ while the right side is $\binom{n+1}{0} = 1$.

Inductive step.

$$[\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r}] + \binom{n+r+1}{r+1}$$

$$= \binom{n+r+1}{r} + \binom{n+r+1}{r+1}$$

$$= \frac{(n+r+1)!}{r!(n+1)!} + \frac{(n+r+1)!}{(r+1)!n!}$$

$$= \frac{(n+r+1)!}{r!n!} \quad \frac{1}{n+1} + \frac{1}{r+1}$$

$$= \frac{(n+r+1)!}{r!n!} \quad \frac{n+r+2}{(n+1)(r+1)}$$

$$= \binom{n+r+2}{r+1}.$$

Remark. A proof by induction on n is more complicated.

Chapter 2

2.1. (i) In Figure 2.11a, QPSR is a rectangle and YZSR is a parallelogram. Show that ΔQYR and ΔPZS are congruent.

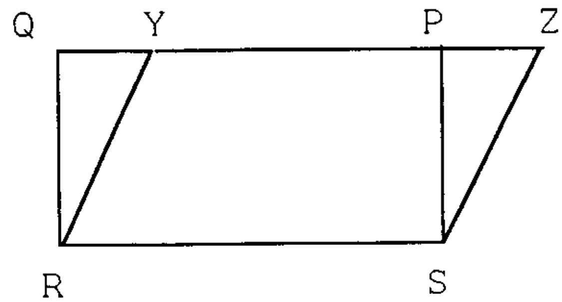


Figure 2.11a

* ΔQZR and ΔPYS are congruent by "side-angle-side": $|QR| = |PS|$ (opposite sides of the rectangle), $|RY| = |SZ|$ (opposite sides of the parallelogram), and the included angles are right angles.

(ii) Show that one can construct the parallelogram YZSR from the rectangle QPSR by cutting off ΔPZS and pasting it in position ΔQYR; conclude that the parallelogram has the same area as the rectangle.

* area(ZYSR) = area(PSRQ) - area(ΔQYR) + area(ΔPZS). By (i), however, area(ΔQYR) = area(ΔPZS).

(iii) Prove that the parallelogram YZSR has the same area as the rectangle QPSR when Y is not between Q and P (see Figure 2.11b.)

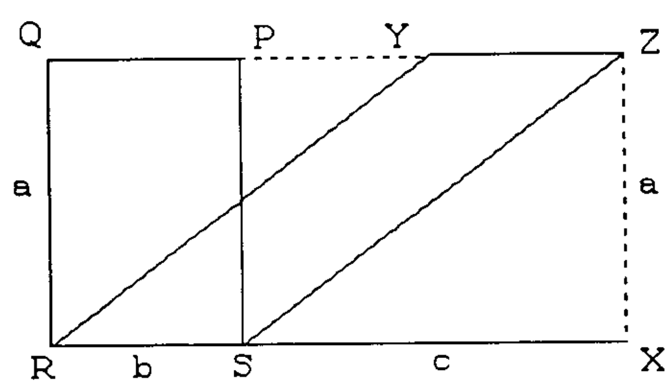


Figure 2.11b

* Note first that $\Delta QYR \approx \Delta XSZ$, by "side-side-side": $|QR| = a = |XZ|$; $|RY| = |ZS|$; $c = |XS| = |XR| - |SR| = |QZ| - |YZ| = |QY|$. (One could, instead, use the fact that two right triangles having the same hypotenuse and one leg must be congruent.) Therefore, both triangles have the same area, namely, $\frac{1}{2}ac$. We now write the area of the rectangle QZXR in two ways:

$$a(b + c) = 2(\tfrac{1}{2}ac) + p,$$

where p is the area of the parallelogram YZSR. It follows that $ab = p$, as desired.

2.2. Show that the area of the trapezoid is $\frac{1}{2}(a + b)h$.

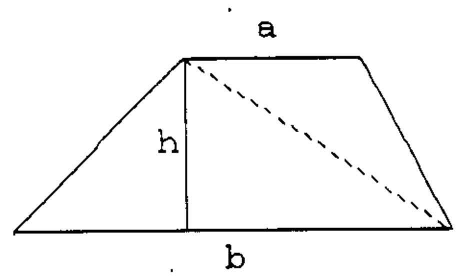


Figure 2.12a

* Dissect the trapezoid into two triangles using the dashed diagonal. The area A of the trapezoid is the sum of the areas of the two triangles:

$$A = \tfrac{1}{2}ah + \tfrac{1}{2}bh = \tfrac{1}{2}(a + b)h.$$

2.3. Assume, in Figure 2.13, that PA, EH, and QR are parallel.

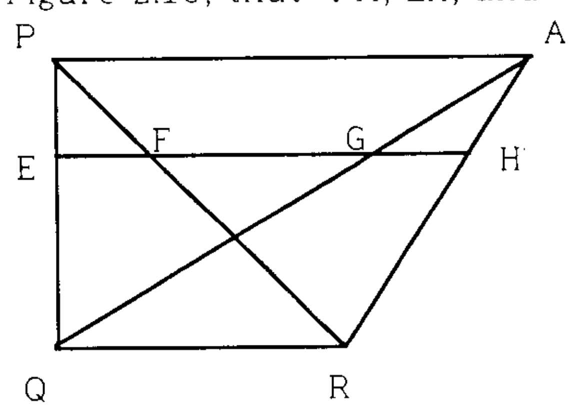


Figure 2.13

If P and Q are points, let us denote the length of the line segment PQ by $|PQ|$. Prove that $|EF| = |GH|$, and conclude that Cavalieri's Principle applies to ΔPQR and ΔAQR. (Hint. Let ℓ and ℓ' be parallel lines, and let t and t' be transversals. If ℓ'' is parallel to ℓ (and to ℓ'), then ℓ'' divides the transversals proportionally. In Figure 2.13, $|PE|/|PQ| = |AH|/|AR|$.)

* Since ΔPEF and ΔPQR are similar, $|PE|/|PQ| = |EF|/|QR|$, and since ΔAGH and ΔAQR are similar, $|AH|/|AR| = |GH|/|QR|$. By the hint, $|PE|/|PQ| = |AH|/|AR|$. Therefore, $|EF|/|QR| = |GH|/|QR|$, and so $|EF| = |GH|$.

2.4. Let a and b be integers.
(i) If a is even, then ab is even for every integer b.

* Since a is even, $a = 2m$, and so $ab = (2m)b = 2(mb)$ is even.

(ii) If both a and b are odd, then ab is odd while $a + b$ is even.

* Since both a and b are odd, $a = 2m + 1$ and $b = 2n + 1$ for some integers m and n. Thus,

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1, \end{aligned}$$

while

$$a + b = 2m + 1 + 2n + 1 = 2(m + n + 1).$$

(iii) If neither a nor b is a multiple of 3, prove that ab is not a multiple of 3.

* Dividing a by 3 leaves remainder either 1 or 2, so that $a = 3q + 1$ or $a = 3q + 2$ for some integer q; similarly, b also has one of these forms. There are thus four cases to check (all are easy). For example, $ab = (3q + 2)(3t + 2) = 9qt + 6q + 6t + 4 =$

$3(3qt + 2q + 2t + 1) + 1$; that is, dividing ab by 3 leaves remainder 1 (and not 0), so that ab is not a multiple of 3. One deals with the other three cases in the same way.

~~Note: Congruences are done in the next course.~~

2.5. If $r = p/q$ is a nonzero rational number, show that $r + \sqrt{2}$ and $r\sqrt{2}$ are irrational numbers. Conclude that there are infinitely many irrational numbers.

* Suppose, on the contrary, that $r + \sqrt{2}$ is rational; that is, there is a rational number r' with $r' = r + \sqrt{2}$. Thus, $\sqrt{2} = r' - r$ is also rational (for the difference of two rationals is rational), contradicting Theorem 2.5. A similar argument, using the (easily proved) fact that the quotient of two rationals is also rational, shows that $r\sqrt{2}$ is irrational (when $r \neq 0$).

2.6. Use the Pythagorean theorem to prove that if $\underline{a}$ is the side length of a cube and $|AB|$ is the length of a diagonal joining opposite corners, then $x^2 = 3a^2$.

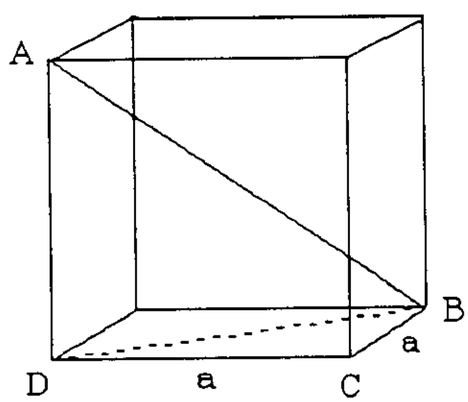


Figure 2.14a

* In the cube drawn in Figure 2.14a, all side lengths are equal to a, and we seek the length $|AB|$. Applying the Pythagorean theorem to $\triangle BCD$ (lying on the floor), we see that $|DB|^2 = 2a^2$. Now apply the Pythagorean theorem to $\triangle ABD$ to see that $|AB|^2 = 2a^2 + a^2 = 3a^2$.

2.7. Prove that $\sqrt{3}$ is irrational.

* Assume, on the contrary, that $\sqrt{3}$ is rational; that is,

$$\sqrt{3} = p/q,$$

where both p and q are positive integers (with $q \neq 0$). By (ii), $p = 3^k m$ and $q = 3^\ell n$, where $k, \ell \geq 0$ and both m and n are not a multiple of 3. If $k \geq \ell$, then $p/q = 3^{k-\ell} m/n$. We may thus replace p by $3^{k-\ell} m$ and q by the number n (which is not a multiple of 3); a similar replacement can be made if $k < \ell$. Therefore, we may assume that at least one of p and q is not a multiple of 3.

Squaring both sides, $3 = p^2/q^2$, and cross multiplying gives

$$3q^2 = p^2.$$

Now p must be a multiple of 3, otherwise p^2 would not be a multiple of 3, contradicting $p^2 = 3q^2$ and Exercise 1.11. Hence, $p = 3r$ for some integer r. Substituting gives $3q^2 = (3r)^2 = 9r^2$, so that

$$q^2 = 3r^2.$$

It follows, as above, that q is a multiple of 3, contradicting our assumption that at least one of p and q is not a multiple of 3.

2.8. (i) Prove that an integer $m \geq 2$ is a perfect square if and only if each of its prime factors occurs an even number of times.

* Let $m = p^e q^f \cdots$ be the factorization into primes, where we have collected all like primes together. If all the exponents $e, f, \ldots$ are even, say, $e = 2e'$, $f = 2f'$, ..., then $m = p^{2e'} q^{2f'} \cdots$. If we define $a = p^{e'} q^{f'} \cdots$, then a is an integer and $a^2 = m$. Thus, m is a perfect square.

Conversely, assume that $m = a^2$. If the prime factorization of a is $p^e q^f \cdots$, then $m = p^{2e} q^{2f} \cdots$, and so every prime factor of m occurs an even number of times.

(ii) Prove that if m is a positive integer for which $\sqrt{m}$ is rational, then m is a perfect square. Conclude that $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ are irrational.

* If $\sqrt{m} = a/b$, then $m = a^2/b^2$ and $mb^2 = a^2$. Now every prime factor occurs an even number of times in both mb^2 (= a^2) and in b^2, so that every prime factor occurs an even number of times in m. By (i), m is a perfect square, and so $\sqrt{m}$ is an integer.

(iii) If n is a positive integer, show that $n^3 + n^2$ is a perfect square if and only if $n + 1$ is a perfect square.

* If $n^3 + n^2$ is a perfect square, then there is an integer a with $a^2 = n^3 + n^2 = n^2(n + 1)$. Hence $n + 1 = (a/n)^2$, $\sqrt{n + 1}$ is rational, and, by (ii), $n + 1$ is a perfect square.

2.9. Let p be a prime number, and consider the number

$$N = 1 + 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p.$$

\ldots

Prove that none of the prime numbers 2, 3, 5, 7, 11, $\cdots$, p used in the definition of N is a divisor of N.

* If a and N are positive numbers, then we may divide N by a obtaining the quotient q and the remainder r (where $0 \le r < a$). We write this as the equation

$$N = qa + r.$$

Of course, a is a divisor of N precisely when $r = 0$. Now N is an odd number, so that dividing N by 2 leaves remainder 1; that is, $N = 2q + 1$, where $q = 3 \times 5 \times 7 \times \cdots \times p$. Similarly, dividing N by 3 also leaves remainder 1 ($q = 2 \times 5 \times 7 \times \cdots \times p$); indeed, dividing N by any of the numbers $2, 3, 5, 7, \ldots, p$ leaves remainder 1 and not 0.

2.10. Use Exercise 2.9 to prove that there are infinitely many prime numbers.

* Suppose there were only a finite number of primes. One could then write a list of them, say, $2, 3, 5, \ldots, p$. Since finitely many numbers can be multiplied together, the number $N = 1 + 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p$ is defined. By Exercise 2.9, none of the primes $2, 3, 5, \ldots, p$ is a divisor of N, and this contradicts Theorem 1.2, for every positive integer ≥ 2 is either a prime or a product of primes.

2.11. (i) If $p = 11$, the number N in Exercise 2.9 is 2311. Show that 2311 is prime.

* $\sqrt{2311} < 48$, and so one need check only whether primes up to and including 47 divide 2311.

(ii) If $p = 13$, the number N is 30031. Show that 30031 is not prime.

* $30031 = 59 \times 509$.

(iii) If $p = 17$, show that 19 is a divisor of $N = 510511$.

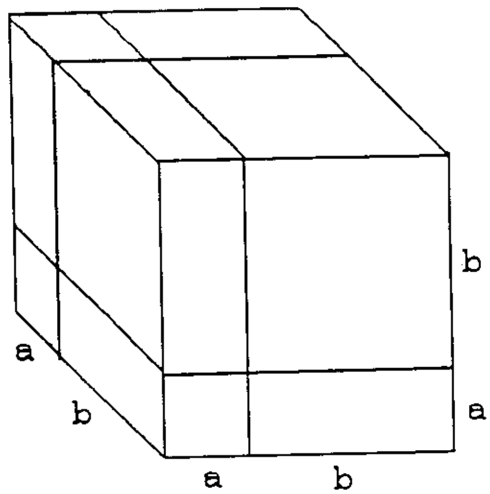
* $510511 = 19 \times 26869$.

2.12. A mad architect has designed the symmetric building shown in Figure 2.15. Find the area of the building's front (not counting the two circular windows (of radius 2) and the semicircular entrance way), given the dimensions in the figure.

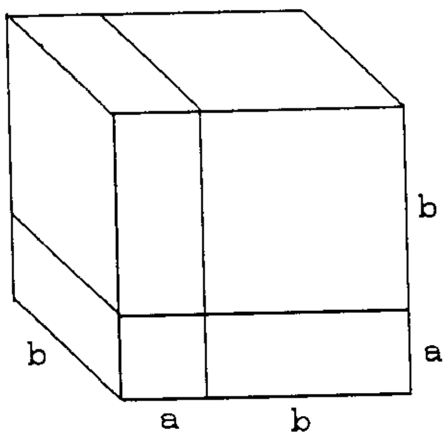
* The area is $2040 - (75 + 175 + 80) = 1710$.

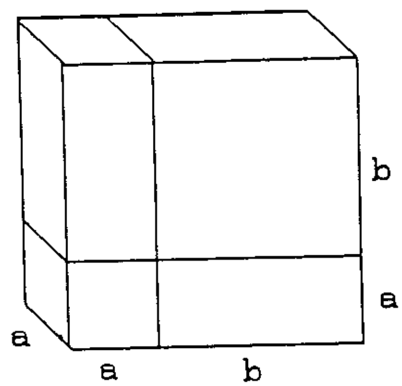
2.13. Use a dissection of a cube having side lengths $a + b$ to prove

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$



Consider the front slab of the cube having sides of length $a + b$: it consists of 4 boxes having volumes a^3, a^2b, a^2b, and ab^2.





The back slab also consists of 4 boxes; they have volumes a^2b, ab^2, ab^2 and b^3. The volume of the whole cube is $(a + b)^3$ as well as the sum of the 8 boxes.

2.14. Give another proof of the Pythagorean theorem (attributed to U.S. President James A. Garfield). Take a vertical line CC' of length $a + b$, and construct two replicas of $\triangle ABC$ as in Figure 2.28.

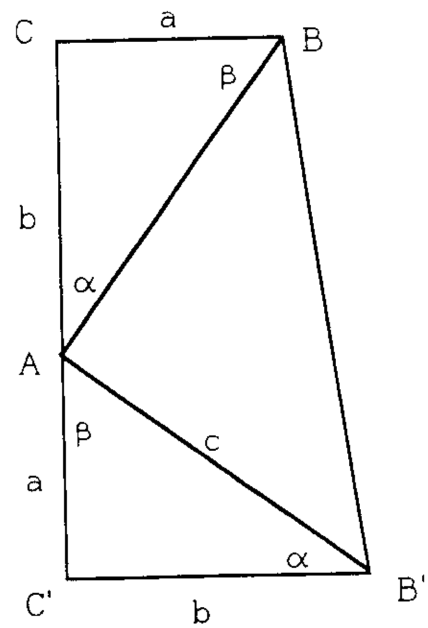


Figure 2.28

* Construct Figure 2.28 by drawing two copies of the original right triangle: $\triangle ABC$ and $\triangle AB'C'$. Notice that $\measuredangle BAB' = 90°$, for the sum of the three angles at A is $180°$, and the other two angles at A are the acute angles of $\triangle ABC$. Thus, $CBB'C'$ is a trapezoid whose area T can be computed in two ways. On the one hand, Exercise 2.2 gives $T = \frac{1}{2}(a + b)\times(a + b)$; on the other hand, T is the sum of the areas of the three triangles:

$$T = \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}c^2.$$

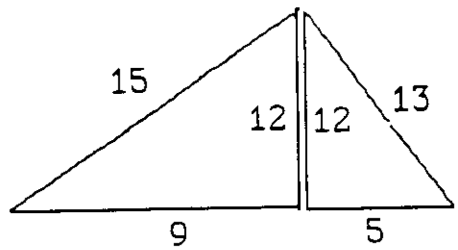
Therefore,

$$\tfrac{1}{2}(a + b)^2 = ab + \tfrac{1}{2}c^2,$$

and high school algebra gives $a^2 + b^2 = c^2$.

2.15 (i) In a triangle with sides of lengths 13, 14, and 15, what is the length of the altitude drawn to the side of length 14?

* A "slick" way is to see that one can construct a triangle with sides of lengths 13, 14, and 15 by pasting together a right triangle with sides 5, 12, 13 and a right triangle of sides 9, 12, 15, as in the figure below.



It follows that the length of the altitude drawn to the side of length 14 is 12.

Here is a less tricky solution. The altitude divides the side of length 14 into two pieces of lengths a and b, so that $a + b = 14$; it also divides the triangle into two right triangles. The Pythagorean theorem gives the equations

$$h^2 = 15^2 - a^2$$

$$h^2 = 13^2 - b^2.$$

Hence, $15^2 - a^2 - (13^2 - b^2) = 0$, so that

$$a^2 - b^2 = 15^2 - 13^2 = 225 - 169 = 56;$$

that is,

$$(a - b)(a + b) = 56.$$

But $a + b = 14$, so that $a - b = 56/14 = 4$. We solve the system:

$$a + b = 14$$

$$a - b = \ 4;$$

$a = 9$ and $b = 5$. Finally, $h^2 = 15^2 - a^2 = 225 - 81 = 144$, and so $h = 12$.

(ii) Find the area of this triangle.

* The area is $\frac{1}{2} \times 12 \times 14 = 84$.

2.16. Given a right triangle with perpendicular sides a and b, find the side s of the inscribed square.

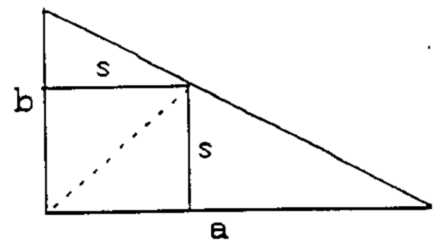


Figure 2.29a

* The area of the given triangle is $A = \frac{1}{2}ab$. A diagonal of the square divides the given triangle into two triangles, as in Figure 2.29a. Thus

$$A = \tfrac{1}{2}as + \tfrac{1}{2}bs = \tfrac{1}{2}(a + b)s.$$

From $\frac{1}{2}ab = \frac{1}{2}(a + b)s$, we obtain $s = ab/(a + b)$.

2.17. Given a right triangle with legs a and b, find the radius r of the inscribed circle.

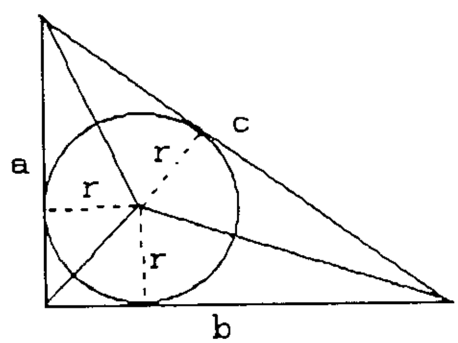


Figure 2.30a

* Draw line segments from the vertices of the triangle to the center of the circle. The triangle of area $\frac{1}{2}ab$ is thus divided into three triangles. It follows that

$$\tfrac{1}{2}ab = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr,$$

where c is the length of the hypotenuse. Thus

$$ab = (a + b + c)r,$$

and

$$r = ab/(a + b + c), \qquad \text{where } c^2 = a^2 + b^2.$$

2.18. The length of the perimeter of a right triangle is 60 units, and the length of the altitude perpendicular to the hypotenuse is 12 units. How long are the sides?

* If a and b are the lengths of the perpendicular sides and c is the length of the hypotenuse, then

$$a + b + c = 60 \quad \text{and} \quad a^2 + b^2 = c^2.$$

The area of the triangle can be computed in two ways: it is $\frac{1}{2}ab$ as well as $\frac{1}{2} \times 12c$. Therefore,

$$ab = 12c.$$

It follows that

$$(60 - c)^2 = (a + b)^2 = a^2 + b^2 + 2ab$$

$$= c^2 + 2ab = c^2 + 24c,$$

or

$$3600 - 120c = 24c.$$

Thus, $144c = 3600$, and hence $c = 25$. It follows that

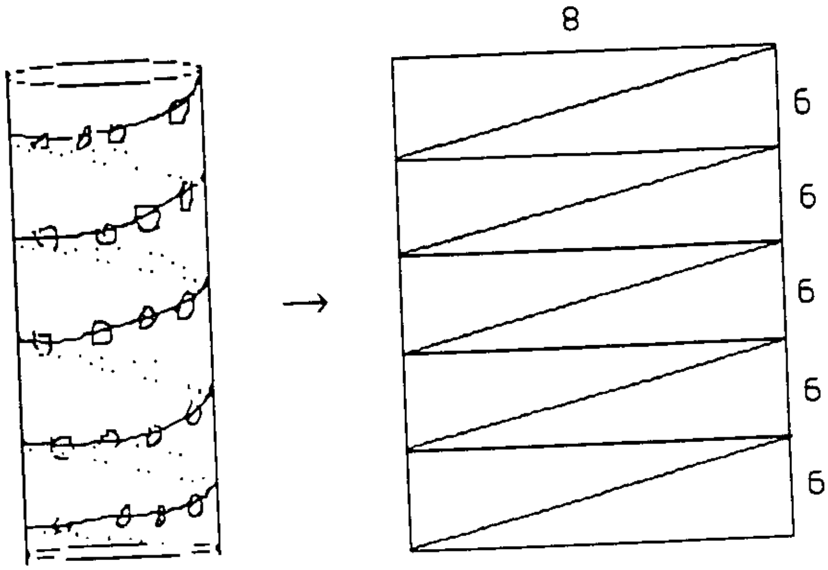
$$a + b = 35 \quad \text{and} \quad ab = 300.$$

Setting $b = 300/a$ gives $a + 300/a = 35$; we rewrite this as $a^2 - 35a + 300 = 0$. The quadratic formula now gives

$$a = 15 \quad \text{and} \quad b = 20.$$

2.19. A cylindrical column has height 30 feet and circumference 8 feet. A garland is wound evenly, in spiral fashion, 5 times around the column, reaching from top to bottom. Show that the garland must be at least 50 feet long.

*Think of the cylinder as a 30 by 8 rectangle which has been rolled up. If we unroll the cylinder, we see five rectangles, each 6 by 8, as in the figure.



The portion of the garland in any of these small rectangles must go, in some way, from the upper right corner to the lower left corner. The shortest such path is the straight line; that is, the diagonal, which has length 10, by the Pythagorean theorem. The garland is thus of length at least $5 \times 10 = 50$ feet.

2.20. (i) Show that an equilateral triangle Δ with side lengths a has altitudes of height $a\sqrt{3}/2$.

*The Pythagorean theorem gives $a^2 = (a/2)^2 + h^2$, and this gives $h = a\sqrt{3}/2$.

(ii) Show that the area of Δ is $a^2\sqrt{3}/4$.

* area $= \frac{1}{2} \times a \times a\sqrt{3}/2 = a^2\sqrt{3}/4$.

2.21. Find the area and circumference of the circumscribed circle of a regular hexagon whose sides have length s.

* The radius of the circumscribed circle is s, and so $A_s = \pi s^2$ and $L_s = 2\pi s$.

2.22. Find the area and circumference of the circumscribed circle of an equilateral triangle whose side has length s.

* Let r be the radius of the circumscribed circle and let h be the height of the equilateral triangle. By Exercise 2.20,

$$h = s\sqrt{3}/2.$$

Moreover,

$$r = \tfrac{2}{3}h = s\sqrt{3}/3.$$

Thus

$$A_r = \pi r^2 = \tfrac{1}{3}\pi s^2 \quad \text{and} \quad L_r = 2\pi r = 2\pi s\sqrt{3}/3.$$

2.23. Is there a Pythagorean triple (a, b, c) with $a = 1$ or 2?

* No. In a Pythagorean triple, $c^2 - b^2 = a^2$, so that $(c + b)(c - b) = a^2$. If $a = 1$, then $a^2 = 1$; since there is only one factorization of 1 into positive integers, so that $c + b = 1 = c - b$, forcing $b = 0$. If $a = 2$, a similar contradiction arises from $(c + b)(c - b) = 4$. Either $c + b = 2 = c - b$ or $c + b = 4$ and $c - b = 1$; in the first case, $b = 0$, and in the second case, $c = 5/2$.

2.24. (i) Verify that (781, 2460, 2581) is a Pythagorean triple.

* One can verify that $781^2 + 2460^2 = 2581^2$:

$$609961 + 6051600 = 6661561,$$

or one can check that $781 = q^2 - p^2$, $2460 = 2qp$, and $2581 = q^2 + p^2$, where $q = 41$ and $p = 30$.

(ii) Verify that (3 993, 7 972 024, 7 972 025) is a Pythagorean triple.

* A hand calculator with an 8 or 9 digit display is inadequate for this problem.

$$(3993)^2 = 15944049$$

$$(7972024)^2 = 63553166656576$$

$$(7972024)^2 = 63553182600625.$$

Here is an alternative way. Note that $c = b + 1$. If this is a Pythagorean triple, then

$$a^2 + b^2 = (b + 1)^2 = b^2 + 2b + 1,$$

and $b = \frac{1}{2}(a^2 - 1)$. And this does, in fact, hold here, for $\frac{1}{2}(3993^2 - 1) = 7\,972\,024$.

(iii) Is (169568, 1139826, 1152370) a Pythagorean triple?

* Yes., for one can find $q = 813$ and $q = 701$.

2.25. Show that no two of the Pythagorean triples in Eq. (2) are similar.

* In each case, show that $c/c' \neq a/a'$ and $c/c' \neq a/b'$. For example, (5, 12, 13) and (8, 15, 17), for $13/17 \neq 5/8$ and $13/17 \neq 5/15$ (as one checks with cross multiplication).

2.26. Prove that if $b = 4\binom{m+1}{2}$ for some $m \geq 2$, then there is a Pythagorean triple (a, b, c) with $c = b + 1$. Conversely, prove that if there is a Pythagorean triple (a, b, c) with $c = b + 1$, then there is an integer $m \geq 2$ with $b = 4\binom{m+1}{2}$.

* If $b = 4\binom{m+1}{2} = 2m(m + 1) = 2m^2 + 2m$, it is only a question of finding a with

$$a^2 + b^2 = (b + 1)^2 = b^2 + 2b + 1.$$

Thus, $a^2 = 2b + 1 = 2m^2 + 4m + 1 = (2m + 1)^2$, and $a = 2m + 1$. Hence, $(2m + 1, b, b + 1)$ is a Pythagorean triple.

If $a^2 + b^2 = (b + 1)^2 = b^2 + 2b + 1$, then $a^2 = 2b + 1$. There are two consequences: a is odd, say, $a = 2m + 1$; $b = \frac{1}{2}(a^2 - 1) = \frac{1}{2}(a + 1)(a - 1)$. Hence,

$$b = \tfrac{1}{2}(a + 1)(a - 1)$$

$$= \tfrac{1}{2}(2m + 2)2m = 4\binom{m+1}{2}.$$

2.27. Let Δ be a right triangle with legs a and b and hypotenuse c. In Exercise 2.17, one found that the radius r of the inscribed circle is given by $r = ab/(a + b + c)$ [r is called the ***inradius*** of Δ].

If (a, b, c) is a Pythagorean triple, prove that the inradius of the corresponding right triangle is an integer.

*

$$r = ab/(a + b + c)$$

$$= 2qp(q^2 - p^2)/[2qp + q^2 - p^2 + q^2 + p^2]$$

$$= 2qp(q + p)(q - p)/2q(p + q)$$

$$= p(q - p).$$

2.28. Let Δ be a right triangle with side lengths a, b, and c all integers. Prove that the height h of the altitude to the hypotenuse is a rational number. (Hint. What is area(Δ)?)

* Area(Δ) = $\frac{1}{2}ab = \frac{1}{2}hc$, and so $h = ab/c$ is a rational number.

2.29. Find the values of g and h for $t = \frac{2}{3}, \frac{3}{4}, 3/5$, and $4/5$.

*

$t =$	$\frac{2}{3}$	$\frac{3}{4}$	3/5	4/5
$g =$	5/13	7/25	8/17	9/41
$h =$	12/13	24/25	15/17	40/41

make all fractions same size

2.30. Find q and p in Theorem 2.12 for each of the following Pythagorean triples.

(i) (7, 24, 25).

* $q = 5$, $p = 3$.

(ii) (129396, 261547, 291805).

* $q = 526$, $p = 123$.

2.31. Show that the same number can occur as a leg in two non-similar Pythagorean triples.

* (6, 8, 10) and (8, 15, 17) are Pythagorean triples that are not similar.

2.32. Show that there are distinct Pythagorean triples (a, b, c) and (α, β, c) having the same c.

* If (a, b, c) and (x, y, z) are Pythagorean triples with $c \neq z$, then (az, bz, cz) and (cx, cy, cz) are Pythagorean triples with the same hypotenuse. Thus, (3, 4, 5) and (5, 12, 13) give distinct Pythagorean triples (39, 52, 65) and (25, 60, 65).

2.33. Show that every integer $n \geq 3$ occurs as a leg of some Pythagorean triple.

* If n is even, say, $n = 2k$ $(k \geq 2)$, then $(n, k^2 - 1, k^2 + 1)$ is a Pythagorean triple because

$$\begin{aligned}(2k)^2 + (k^2 - 1)^2 &= 4k^2 + k^4 - 2k^2 + 1 \\ &= k^4 + 2k^2 + 1 = (k^2 + 1)^2;\end{aligned}$$

a similar calculations shows that if n is odd, say, $n = 2k + 1$ $(k \geq 1)$, then $(n, 2k(k + 1), 2k^2 + 2k + 1)$ is a Pythagorean triple.

2.34. Use Heron's formula to solve Exercise 2.15(i).

* The sides have lengths 13, 14, and 15, so that the semiperimeter $s = \frac{1}{2}(13 + 14 + 15) = 21$. If A is the area, then $A^2 = 21 \times 8 \times 7 \times 6 = 7056$, and so $A = 84$.

2.35. Let $P = (1, 1)$, $Q = (2, 3)$, and $R = (-2, 2)$ be three points in the plane.
(i) Show that P, Q, and R are not collinear.

* It suffices to show that the slopes of the lines PQ and QR are different. The first slope is $\frac{1}{2}$, and the second slope is 4.

(ii) Find area(ΔPQR).

\approx

* The distance formula ~~gives:~~

$$|PQ| = \sqrt{5} \approx 2.24; \quad |QP| = \sqrt{17} \approx 4.12; \quad |PR| = \sqrt{10} \approx 3.16.$$

Thus, the semiperimeter is $s \approx 4.76$, and so Heron's formula gives:

$$(\text{area})^2 = 2.24 \times (4.76 - 2.24) \times (4.76 - 4.12) \times (4.76 - 3.16)$$

$$= 4.76 \times 2.52 \times 0.64 \times 1.6 \approx 12.28,$$

and area(ΔPQR) ≈ 3.50.

2.36. To find the distance between two points A and C on opposite sides of a river, a distance of 100 feet is paced off on one side of the river from A to a point B with AB perpendicular to BC. The angle $\measuredangle CAB$ is measured as 60°. Find the distance $|AC|$.

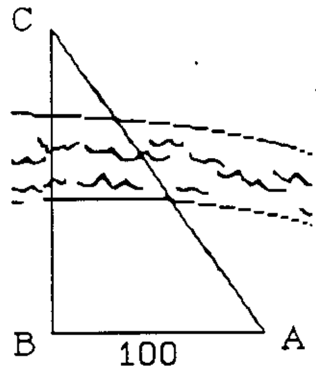


Figure 2.40

* $|AC| = |AB|\cos 60^\circ = 100 \times ½ = 50$.

2.37. A pole 30 feet tall casts a shadow 50 feet long. Find the approximate angle of elevation of the sun.

* If α is the angle of elevation, then $\cos\alpha = 30/50 = .6$, and so $\alpha \approx 53.1^\circ$.

approx

2.38. New Orleans, Louisiana is due south of Madison, Wisconsin. If the latitude of New Orleans is 30°N, the latitude of Madison is 43°N, and the radius of the earth is 4000 miles, what is the distance between the two cities?

* Since the radius of the earth is 4000, the circumference of the earth is 8000π. Now 13° (43 - 30 = 13) corresponds to 13/360 of the circumference, and the distance between the cities (on the surface of the earth) is $13/360 \times 8000\pi \approx 907.6$ miles.

approx

2.39. Assuming the fact that the arc and the chord subtended by a small central angle in a circle are approximately the same length, compute the diameter of the sun using the facts that the distance from the earth to the sun is about 93,000,000 miles and that the sun as seen from the earth subtends an angle of 0.53 degrees.

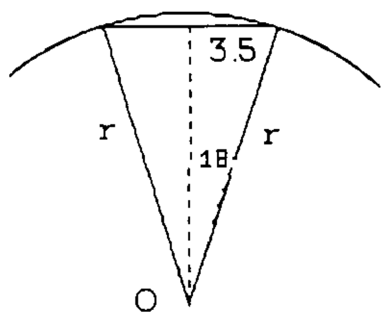
* The arclength in miles is approximately approx

$$.53/360 \times 93{,}000{,}000\pi \approx 430{,}000.$$

Using the assumed fact, this is the sun's diameter.

2.40. If a regular decagon whose sides are 7 inches long is inscribed in a circle, what is the radius of the circle?

* Connecting each vertex of the decagon to the center of the circle dissects the decagon into isosceles triangles, with equal sides r (the radius of the circle), the other side of length 7, and with the angle at the center 36°. The perpendicular bisector



thus splits this triangle into two right triangles, each with hypotenuse r, angle 18°, and opposite leg 3.5. Hence, cos 18° $= 3.5/r$, and so $r = 3.5/\cos 18° \approx 3.68$.

2.41. Let α be an angle and let $A = (\cos\alpha, \sin\alpha)$. Show that if m is the slope of the line OA (where O denotes the origin), then $m = \tan\alpha$.

* The slope of the line joining points (a, b) and (c, d) is $m = (b - d)/(a - c)$. Here, $m = (\cos\alpha - 0)/(\sin\alpha - 0) = \tan\alpha$.

2.42. Let (g, h) be a point in the first quadrant lying on the arc of the unit circle, and let the line ℓ joining (g, h) and $(-1, 0)$ have slope t. Prove that if the line L joining (h, g) and $(-1, 0)$ has slope T, then $T = (1 - t)/(1 + t)$.

* By Eqs. (2), and (3), we have $g = 2t/(1 + t^2)$ and $h = (1 - t^2)/(1 + t^2)$; by Eq. (1), we have $T = g/(h + 1)$. Substituting,

$$T = 2t/(1 + t^2)/[(1 - t^2)/(1 + t^2) + 1].$$

After simplifying, the term on the right is $(1 - t)/(1 + t)$.

2.43. Prove that $1/(\csc\theta - \cot\theta) - 1/(\csc\theta + \cot\theta) = 2\cot\theta$.

* Rewrite in terms of $\sin\theta$ and $\cos\theta$; then use Theorem 2.14; after simplifying, each side is $(1 - t^2)/t$.

2.44. Prove that $\cot^4\theta + \cot^2\theta = \csc^4\theta - \csc^2\theta$.

* Rewrite in terms of $\sin\theta$ and $\cos\theta$; then use Theorem 2.14; after simplifying, each side is

$$(1 - 2t^4 + t^8)/16t^4.$$

2.45. Of all the triangles with a given perimeter p and with one side of a given length a, prove that the triangle having the maximal area is isosceles.

* If $p = 2s$, then Heron's formula give

$$(s - b)(s - c) = A^2/s(s - a),$$

where A is the area of the triangle, and so the arithmetic mean-geometric mean inequality (Theorem 1.11) gives

$$(\tfrac{1}{2}[s - b + s - c])^2 \geq A^2/s(s - a),$$

with equality if and only if $s - b = s - c$; i.e., if and only if $b = c$. Thus, the triangle with largest area is isosceles.

2.46. Prove, for every triangle Δ, that

$$(p/4)^2 \geq \mathrm{area}(\Delta),$$

where p is the perimeter of Δ. Can there be equality?

* Let $p = a + b + c = 2s$. By Heron's formula,

$$s(s - a)(s - b)(s - c) = A^2,$$

where $A = \text{area}(\Delta)$, and the arithmetic mean-geometric mean inequality gives

$$(\tfrac{1}{4}[s + s - a + s - b + s - c])^4 \geq A^2.$$

But $s + s - a + s - b + s - c = 4s - (a + b + c) = 4s - 2s = 2s = p$, so that $(p/4)^4 \geq A^2$. Taking square roots, $(p/4)^2 \geq A$.

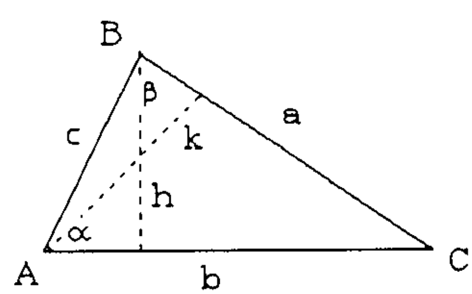
The inequality is always strict, for it can occur only when $s = s - a = s - b = s - c$. But $s = s - a$ gives $a = 0$, which is not a length of a side of a triangle.

2.47. Prove the Law of Sines: In a triangle ΔABC with angles α, β, and γ opposite sides of lengths a, b, and c, respectively,

$$\sin\alpha/a = \sin\beta/b = \sin\gamma/c.$$

* Since $\text{area}(\Delta ABC) = \tfrac{1}{2}bh = \tfrac{1}{2}bc\sin\alpha$, we have

$$\sin\alpha/a = 2\text{area}(\Delta ABC)/abc.$$



Since $\text{area}(\Delta ABC) = \tfrac{1}{2}ak = \tfrac{1}{2}ac\sin\beta$, we have

$$\sin\beta/b = 2\text{area}(\Delta ABC)/abc.$$

Therefore, $\sin\alpha/a = \sin\beta/b$. The hint is essentially the whole proof!

2.48. For any two angles α and β, prove

$$\cos[\tfrac{1}{2}(\alpha + \beta)]\cos[\tfrac{1}{2}(\alpha - \beta)] = \tfrac{1}{2}(\cos\alpha + \cos\beta).$$

* $\cos[\frac{1}{2}(\alpha + \beta)]\cos[\frac{1}{2}(\alpha - \beta)]$

$= (\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta - \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta)(\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta + \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta)$

$= \cos^2\frac{1}{2}\alpha\cos^2\frac{1}{2}\beta - \sin^2\frac{1}{2}\alpha\sin^2\frac{1}{2}\beta$

$= \cos^2\frac{1}{2}\alpha\cos^2\frac{1}{2}\beta - (1 - \cos^2\frac{1}{2}\alpha)(1 - \cos^2\frac{1}{2}\beta)$

$= \cos^2\frac{1}{2}\alpha\cos^2\frac{1}{2}\beta - (1 - \cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\beta + \cos^2\frac{1}{2}\alpha\cos^2\frac{1}{2}\beta)$

$= -1 + \cos^2\frac{1}{2}\alpha + \cos^2\frac{1}{2}\beta$

$= -1 + \frac{1}{2}(\cos\alpha + 1) + \frac{1}{2}(\cos\beta + 1)$

$= \frac{1}{2}\cos\alpha + \frac{1}{2}\cos\beta.$

2.49. Prove that $\cos 20^\circ\cos 40^\circ\cos 80^\circ = 1/8$.

* Using Exercise 2.48,

$$\cos 40^\circ\cos 80^\circ = \tfrac{1}{2}(\cos 120^\circ + \cos 40^\circ)$$

[$\frac{1}{2}(\alpha + \beta) = 80$ and $\frac{1}{2}(\alpha - \beta) = 40$ give

$$\alpha + \beta = 160,\ \alpha - \beta = 80,$$

so that

$$\alpha = 120 \text{ and } \beta = 40].$$

Hence,

$$\cos 20^\circ\cos 40^\circ\cos 80^\circ = \cos 20^\circ\tfrac{1}{2}[(\cos 120^\circ + \cos 40^\circ)]$$

$$= -\tfrac{1}{4}\cos 20^\circ + \tfrac{1}{2}(\cos 20^\circ\cos 40^\circ).$$

Using Exercise 2.48 again,

$$\cos 20°\cos 40° = ½(\cos 60° + \cos 20°).$$

$$[½(\alpha + \beta) = 40,\ ½(\alpha - \beta) = 20$$

give

$$\alpha + \beta = 80,\ \alpha - \beta = 40,$$

so that

$$\alpha = 60 \text{ and } \beta = 20].$$

Hence,

$$\cos 20°\cos 40°\cos 80° = -¼\cos 20° + ½(½(\cos 60° + \cos 20°)$$

$$= -¼\cos 20° + ½(½(\cos 60° + ½\cos 20°)$$

$$= ¼\cos 60° = 1/8.$$

2.50. (i) Prove that

$$\cos x \cos 2x \cos 4x \cdots \cos 2^n x = \sin 2^{n+1}x/2^{n+1}\sin x.$$

* The proof is by induction on $n \geq 0$, and the idea is to multiply the left side by $\sin x$.
Base step $n = 0$: The double angle formula $\sin 2x = 2\sin x\cos x$ gives $\cos x = \sin 2x/2\sin x$.
Inductive step:

$$\cos x \cos 2x \cos 4x \cdots \cos 2^n x\cos 2^{n+1}x$$

$$= \sin 2^{n+1}x\cos 2^{n+1}x/2^{n+1}\sin x$$

$$= ½\sin 2^{n+2}x/2^{n+1}\sin x$$

$$= \sin 2^{n+2}x/2^{n+2}\sin x.$$

(ii) Use part (i) to give a second proof of Exercise 2.49.

* Exercise 2.49 follows from the case $n = 2$:

$$\cos x \cos 2x \cos 4x = \sin 2^3x/2^3\sin x = \sin 8x/8\sin x;$$

set $x = 20^\circ$ and obtain

$$\cos 20^\circ \cos 40^\circ \cos 80^\circ = \sin 160^\circ / 8 \sin 20^\circ.$$

The result follows, for $\sin 20^\circ = \sin(180^\circ - 20^\circ)$.

2.51. Prove that

$$2\cos(n + 1)x = (2\cos x)(2\cos nx) - 2\cos(n - 1)x.$$

* We prove $2\cos(n + 1)x + 2\cos(n - 1)x = (2\cos x)(2\cos nx)$ by using Exercise 2.50.

$$2\cos(n + 1)x + 2\cos(n - 1)x = 2[\cos(n + 1)x + \cos(n - 1)x]$$

$$= 2[2\cos([n + 1 + n - 1]/2)\cos([n + 1 - n + 1]/2)$$

$$= 4\cos nx \cos x = (2\cos x)(2 \cos nx).$$

2.52. Prove the half angle formula for tangent:

$$\tan(\theta/2) = \sin \theta/(1 + \cos \theta).$$

* In Figure 2.41, we see that $t = \tan(\theta/2)$ is the slope of the line joining $(-1, 0)$ to $(\cos \theta, \sin \theta)$. But Eq. (1) gives

$$\tan(\theta/2) = t = h/(g + 1) = \sin \theta/(\cos \theta + 1).$$

2.53. Find the indefinite integral $\int \sin \theta d\theta/(2 + \cos \theta)$.

* After substituting using Theorem 2.14, the integral is

$$\int 4tdt/(1 + t^2)(3 + t^2).$$

The partial fraction decomposition is

$$4t/(1 + t^2)(3 + t^2) = 2t/(1 + t^2) - 2t/(3 + t^2),$$

so that the indefinite integral is

$$\ln|1 + t^2| - \ln|3 + t^2| = \ln|(1 + t^2)/(3 + t^2)|.$$

By Eq. (4), $t = \sin\theta/(1 + \cos\theta)$. Anything more is cosmetic.

2.54. Find the indefinite integral

$$\int[\sin\theta - \cos\theta]d\theta/[\sin\theta + \cos\theta].$$

* After substituting using Theorem 2.14, the integral is

$$\int 2(-1 + 2t + t^2)dt/(1 + 2t - t^2)(1 + t^2).$$

The partial fraction decomposition is

$$2(-1 + 2t + t^2)/(1 + 2t - t^2)(1 + t^2) =$$

$$(2t - 2)/(1 + 2t - t^2) + 2t/(1 + t^2),$$

so that the indefinite integral is

$$-\ln|1 + 2t - t^2| + \ln|1 + t^2| = \ln|(1 + t^2)/(1 + 2t - t^2)|.$$

By Eq. (4) and some manipulation, the indefinite integral is $-\ln|\cos\theta + \sin\theta|$.

2.55. Find the indefinite integral $\int(\sqrt{x} - 1)dx/(\sqrt{x} + 1)$.

* The substitution $\sqrt{x} = 1/t$ rewrites the integral as

$$\int -2(1 - t)dt/(1 + t)t^3.$$

The partial fraction decomposition is

$$-2(1 - t)/(1 + t)t^3 = -4/t + 4/t^2 - 2/t^3 + 4/(1 + t).$$

Therefore,

$$\int -2(1 - t)dt/(1 + t)t^3 = 4\ln|1 + 1/t| -4/t + 1/t^2.$$

Recalling that $1/t = \sqrt{x}$, the indefinite integral is

$$4\ln(\sqrt{x} + 1) - 4\sqrt{x} + x.$$

2.56. Reduce the indefinite integral $\int x^n \sqrt{x^2 + 1}\, dx$, for any integer $n \geq 0$, to an indefinite integral of a rational function.

* The appropriate substitution is:

$$x = (1 + t^2)/(1 - t^2),$$

so that $y = 2t/(1 - t^2)$, and $dx = 4tdt/(1 - t^2)^2$ (for we are in the case of a hyperbola). The integral is rewritten as

$$\int 8t^2(1 + t^2)^n dt/(1 - t^2)^{n+3}.$$

The drudgery of partial fractions is not recommended.

2.57. (i) Show that the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$ can be parametrized by $x = a\cos\vartheta$ and $y = b\sin\vartheta$.

* $(a\cos\vartheta)^2/a^2 + (b\sin\vartheta)^2/b^2 = \cos^2\vartheta + \sin^2\vartheta = 1$.

(ii) Show that if $a > b$, then the arclength of this curve is given by the integral

$$2\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\cos^2\vartheta}\, d\vartheta.$$

* If a curve is parametrized by $x = x(\vartheta)$ and $y = y(\vartheta)$, then its arclength is given by $\int \sqrt{(dx/d\vartheta)^2 + (dy/d\vartheta)^2}\, d\vartheta$. Here, the expression under the radical is:

$$a^2\sin^2\vartheta + b^2\cos^2\vartheta = a^2 - (a^2 - b^2)\cos^2\vartheta.$$

(iii) Show that the $\tan\vartheta/2$ substitution rewrites this as an elliptic integral of the form $\int R(t, \sqrt{f(t)})dt$, where $R(u, v)$ is a rational function of two variables and $f(t)$ is a quartic polynomial.

* The substitution gives $\cos\vartheta = (1 - t^2)/(1 + t^2)$ and $d\vartheta = 2dt/(1 + t^2)$. The integral becomes

$$\int\sqrt{a^2 + (a^2 - b^2)\cos^2\vartheta}\, d\vartheta = \int\sqrt{f(t)}\, dt/(1 + t^2)^2,$$

where $f(t) = b^2t^4 + 2(2a^2 - b^2)t^2 + b^2$.

Chapter 3

3.1. (i) For $n \geq 1$, define $k_n = 1 - (1/10)^n$; show that $k_* \nearrow 1$.

* Clearly, $k_1 < k_2 < k_3 < \cdots < 1$; moreover,

$$1 - k_n = 1 - [1 - (1/10)^n] = (1/10)^n < \tfrac{1}{2}^n \text{ for all } n.$$

(ii) For $n \geq 1$, define $\ell_n = 1 - 2(1/10)^n$; show that $\ell_* \nearrow 1$.

* Clearly, $\ell_1 < \ell_2 < \ell_3 < \cdots < 1$; moreover,

$$1 - \ell_n = 1 - [1 - 2(1/10)^n] = 2(1/10)^n < \tfrac{1}{2}^n \text{ for all } n.$$

3.2. Show that 1.1, 1.01, 1.001, ... approximates 1 from above.

* Define $K_n = 1 + (1/10)^n$ for all $n \geq 1$. Clearly, $K_1 > K_2 > K_3 > \cdots > 1$ for all n; moreover

$$K_n - 1 = (1/10)^n < \tfrac{1}{2}^n \text{ for all } n.$$

3.3. Use Theorem 3.1 to prove (again) that the sequence .9, .99, .999, ... approximates 1 from below.

* The sequence in question is $k_n = 1 - (1/10)^n$, which clearly is increasing. Now

$$1 - k_1 = 1 - .9 = .1 < \tfrac{1}{2} = .5.$$

We must also check, for all $n \geq 1$, that

$$1 - (1 - (1/10)^{n+1}) < \tfrac{1}{2}(1 - [1 - (1/10)^n]);$$

that is,

$$(1/10)^{n+1} < \tfrac{1}{2}(1/10)^n.$$

This last inequality follows from Exercise 1.4, for $1/10 < \tfrac{1}{2}$.

\ldots

3.4. Modify the proof of Theorem 3.3 to show that $\frac{1}{3}$ is equal to $.3333\cdots$ (an unending string of 3's).

* Let $B = .3333\cdots$ (an unending string of 3's). It is plain that $B \le \frac{1}{3}$ (because $3B \le 1$), so that either $B < \frac{1}{3}$ or $B = \frac{1}{3}$. Our strategy is to eliminate the first possibility.

\ldots The sequence here is $k_n = \frac{1}{3}(1 - (1/10)^n)$; that is, $k_1 = .3$, $k_2 = .33$, $k_3 = .333, \cdots$. We show that this increasing sequence approximates $\frac{1}{3}$ from below. For all $n \ge 1$,

$$\tfrac{1}{3} - k_n = \tfrac{1}{3} - \tfrac{1}{3}[1 - (1/10)^n] = \tfrac{1}{3}(1/10)^n < (1/10)^n < \tfrac{1}{2}^n.$$

If $B < \frac{1}{3}$, then the Getting Close Principle says that there is some k_n with $B < k_n$; that is, there is some $n \ge 1$ so that a string of only n 3's is larger than $B = .33333\cdots$, a never-ending string of 3's. This contradicts the criterion for determining inequality between two numbers which are given as decimal expansions. We have eliminated the possibility $B < \frac{1}{3}$, and the only remaining option is $B = \frac{1}{3}$. \ldots

3.5 (i). Show that if M is a positive number and $k_* \nearrow A$, then $Mk_* \nearrow MA$.

* By hypothesis, $k_n < A$ and $A - k_n < \frac{1}{2}^n A$ for all $n \ge 1$. It follows from Theorem 1.4 that $Mk_1 < Mk_2 < Mk_3 < \cdots < MA$ and $M(A - k_n) = MA - Mk_n < \frac{1}{2}^n MA$ for all $n \ge 1$.

(ii). Show that if M is a positive number and $K_* \searrow A$, then $MK_* \searrow MA$.

* By hypothesis, $K_n > A$ and $K_n - A < \frac{1}{2}^n A$ for all $n \ge 1$. It follows from Theorem 1.4 that $MK_1 > MK_2 > MK_3 > \cdots > MA$ and $M(K_n - A) = MK_n - MA < \frac{1}{2}^n MA$ for all $n \ge 1$.

(iii) Use part # (i) of this exercise, together with the fact (proved in Theorem 3.4) that the increasing sequence .9, .99, .999, ... approximates 1 from below, to give a second solution to Exercise 3.4.

* In (i), take $k_n = 1 - (1/10)^n$, $A = 1$, and $M = \frac{1}{3}$.

3.6. Prove that if $a_* \nearrow A$ and $b_* \nearrow B$, then $a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$, ... approximates $A + B$ from below.

* It is easy to see, using Theorem 1.4, that $a_1 + b_1 < a_2 + b_2 < a_3 + b_3 < \ldots < A + B$. For all $n \geq 1$, we have $A - a_n < \frac{1}{2}^n A$ and
\cdots $B - b_n < \frac{1}{2}^n B$. By Theorem 1.4,

$$A + B - (a_n + b_n) = A - a_n + B - b_n < \tfrac{1}{2}^n A + \tfrac{1}{2}^n B = \tfrac{1}{2}^n (A + B).$$

3.7. Assume that $k_* \nearrow A$. \ldots
(i) Show that $k_2, k_3, k_4, \cdots$ approximates A from below.

\cdots \cdots
* Clearly, $k_1 < k_2 < k_3 < \ldots < A$ implies $k_2 < k_3 < \ldots < A$. Also, we know that $A - k_n < \frac{1}{2}^n A$ for all $n \geq 1$. Define $\ell_n = k_{n+1}$; that is, $\ell_1 = k_2$, $\ell_2 = k_3$, etc. We must show that $A - \ell_n < \frac{1}{2}^n A$ for all $n \geq 1$. But $A - \ell_n = A - k_{n+1} < \frac{1}{2}^{n+1} A < \frac{1}{2}^n A$.

\ldots
(ii) For every $m \geq 1$, show that $k_{m+1}, k_{m+2}, k_{m+3}, \cdot\cdot$ approximates A from below.

* This is essentially the same argument as in (ii). Define $a_n = k_{n+m}$; that is, $a_1 = k_{m+1}$, $a_2 = k_{m+2}$, etc. Clearly, $a_1 < a_2 < a_3 <$
\cdots $\ldots < A$; that is, $k_{m+1} < k_{m+2} < k_{m+3} < \cdots < A$; moreover, $a_n = k_{n+m} < A$ and $A - a_n = A - k_{n+m} < \frac{1}{2}^{n+m} A < \frac{1}{2}^n A$.

3.8. Show that $k_1 < k_2 < k_3 < \cdots < 1$ does not approximate 1 from below, where $k_n = 1 - \frac{2}{3}^n$ for all $n \geq 1$.

* If $n = 1$, then $1 - k_1 = \frac{2}{3} > \frac{1}{2}$. Since one needs the inequality $1 - k_n < \frac{1}{2}^n$ for all $n \geq 1$, this suffices. However, one sees that $1 - k_n = \frac{2}{3}^n > \frac{1}{2}^n$ for all $n \geq 1$.

3.9. A pizzeria charges \$2.50 for a 10" pizza and \$5.00 for a 15" pizza (a 10" pizza is circular with diameter 10 inches). Should four hungry students order four 10" pizzas or two 15" pizzas?

* The area of a 10" pizza is 25π, so that four 10" pizzas give 100π square inches of pizza. The area of a 15" pizza is $(7.5)^2\pi = 56.25\pi$ square inches. Thus, two 15" pizzas having 112.5π square inches are more filling.

3.10. Complete the proof of Theorem 3.8 by showing that $A'/A > r'^2/r^2$ leads to a contradiction.

* If $A'/A > r'^2/r^2$, there is some number M' with $M'/A = r'^2/r^2$, so that $A'/A > M'/A$. Multiplying both sides by A gives $A' > M'$.

We have seen that $\text{area}(P'_*) \nearrow \text{area}(D')$, where P'_n is an inscribed regular 2^{n+1}-gon. Since $M' < A'$, the Getting Close Principle says that there is some inscribed polygon P'_ℓ with $M' < \text{area}(P'_\ell)$. Let P_ℓ be the corresponding polygon in D. By Lemma 3.7,

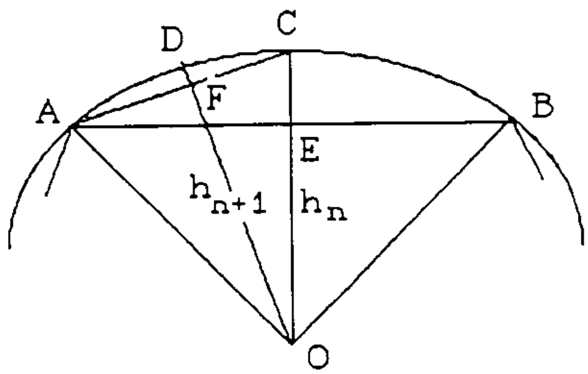
$$\text{area}(P'_\ell)/\text{area}(P_\ell) = r'^2/r^2.$$

But $M'/A = r'^2/r^2$, so that $\text{area}(P'_\ell)/\text{area}(P_\ell) = M'/A$. Hence,

$$\text{area}(P'_\ell)/M' = \text{area}(P_\ell)/A.$$

This is a contradiction, for the right side is smaller than 1 (because P_ℓ is inside the disk D, hence has smaller area) while the left side is greater than 1 [for P'_ℓ was chosen so that $M' < \text{area}(P'_\ell)$].

3.11. The inscribed polygon P_n consists of 2^{n+1} congruent isosceles triangles, each with height h_n. Prove that $h_* \nearrow r$.



cdots

Figure 3.10

* Clearly, $h_1 < h_2 < h_3 < \ldots < r$, the last inequality because h_n is a leg of a right triangle whose hypotenuse is r.

Second, we must show that $r - h_1 < \frac{1}{2}r$. We can find h_1 explicitly. Let us assume, in Figure 3.10, that $n = 1$; thus, AB is a side of P_1. Now $\triangle OAB$ is an isosceles right triangle with equal sides of length r. Thus, $|AB|^2 = 2r^2$, and $|AB| = \sqrt{2}r$; hence, $|EB| = \sqrt{2}r/2$. But $\triangle OEB$ is also isosceles, and so $h_1 = \sqrt{2}r/2$. Therefore, $r - h_1 = (1 - \sqrt{2}/2)r \approx .293r < .5r = \frac{1}{2}r$.

We describe the desired inequality $r - h_{n+1} < \frac{1}{2}(r - h_n)$ in geometric language. $\triangle AOB$ is one of the isosceles triangles that dissects P_n, OC is the perpendicular bisector of AB, and $|OE| = h_n$; the perpendicular bisector of AC is OD, and $|OF| = h_{n+1}$. We must show that $r - |OF| < \frac{1}{2}(r - |OE|)$. Now $|OF| = r\cos\theta$, where $\theta = \measuredangle DOC$, and $|OE| = r\cos 2\theta$, where $2\theta = \measuredangle AOC$, so we must prove that

$$r(1 - \cos\theta) < \tfrac{1}{2}r(1 - \cos 2\theta).$$

By the double angle formula, $\cos 2\theta = 2\cos^2\theta - 1$,

$$\begin{aligned}
\tfrac{1}{2}(1 - \cos 2\theta) &= \tfrac{1}{2}(1 - 2\cos^2\theta + 1) \\
&= \tfrac{1}{2}(2 - 2\cos^2\theta) \\
&= 1 - \cos^2\theta \\
&= (1 + \cos\theta)(1 - \cos\theta) \\
&> 1 - \cos\theta, \quad \text{because } 1 + \cos\theta > 1.
\end{aligned}$$

3.12. For all $n \geq 1$, let P_n be the regular 2^n-gon inscribed in a disk D of radius 1 (as constructed in the text). Prove, for all $n \geq 1$, that there are inequalities

$$\text{area}(P_n) < \pi < \text{area}(P_n) + (\tfrac{1}{2})^n \pi.$$

$*$ Theorem 3.6 gives $\text{area}(P_*) \nearrow \text{area}(D) = \pi$. Therefore, for all $n \geq 1$,

$$0 < \pi - \text{area}(P_n) < (\tfrac{1}{2})^n \pi.$$

Adding $\text{area}(P_n)$ to all sides gives

$$\text{area}(P_n) < \pi < \text{area}(P_n) + (\tfrac{1}{2})^n \pi.$$

3.13. Let P_n be as in Exercise 3.12, and let s_n be the length of a side of P_n.
(i) Prove that $s_1 = \sqrt{2}$.

$*$ In Figure 3.10, ΔOAB is an isosceles right triangle with equal sides of length 1. The Pythagorean theorem gives $s_1 = \sqrt{2}$.

(ii) Prove, for all $n \geq 1$ that

$$s_{n+1} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

$*$ In Figure 3.10, $s_n = |AB|$ and $s_{n+1} = |AC|$. We begin by computing the altitude $|OE| = h_n$; the Pythagorean theorem applied to ΔOEB gives

$$h_n^2 = |OE| = 1 - \tfrac{1}{4} s_n^2.$$

Similarly,

$$h_{n+1}^2 = |OF| = 1 - \tfrac{1}{4} s_{n+1}^2.$$

Now

$$|EC| = 1 - |OE| = 1 - h_n = 1 - \sqrt{1 - \tfrac{1}{4} s_n^2}.$$

The Pythagorean theorem applied to ΔACE gives

$$s_{n+1}^2 = |CE|^2 + |AE|^2$$

$$= (1 - h_n)^2 + (\tfrac{1}{2} s_n)^2$$

$$= 1 - 2h_n + h_n^2 + \tfrac{1}{4} s_n^2$$

$$= 1 - 2\sqrt{1 - \tfrac{1}{4} s_n^2} + 1 - \tfrac{1}{4} s_n^2 + \tfrac{1}{4} s_n^2$$

$$= 2 - 2\sqrt{1 - \tfrac{1}{4} s_n^2}.$$

The result follows from this if we rewrite the last term

$$2\sqrt{1 - \tfrac{1}{4} s_n^2} = \sqrt{4}\sqrt{1 - \tfrac{1}{4} s_n^2} = \sqrt{4 - s_n^2}.$$

3.14. (i) Prove that area$(P_1) = 2$.

* By Exercise 3.12(i), each side of P_1 has length $s_1 = \sqrt{2}$; it follows that area$(P_1) = 2$.

(ii) Prove, for all $n \geq 1$ that area$(P_{n+1}) = 2^n s_n$.

* In Figure 3.10, we compute area(ΔOAC) by letting OC be the base and AE be the altitude. Notice that $|OC| = 1$, for OC is a radius, and that $|AC| = \tfrac{1}{2} s_n$. Therefore,

$$\text{area}(\Delta OAC) = \tfrac{1}{2}|OC||AE| = \tfrac{1}{2}(\tfrac{1}{2} s_n) = \tfrac{1}{4} s_n;$$

since P_{n+1} is dissected into 2^{n+2} triangles congruent to ΔOAC, we have

$$\text{area}(P_n) = 2^{n+2}\tfrac{1}{4} s_n = 2^n s_n.$$

3.15. Use the previous three exercises to estimate π.
(i) Show that $2 < \pi < 4$.

* A disk D of radius 1 (and diameter 2) can be inscribed in a square Q with side lengths 2, and so

$$\pi = \text{area}(D) < \text{area}(Q) = 2^2 = 4.$$

By Exercise 3.12, a lower bound is $\text{area}(P_1) = 2$.

(ii) Show that $2.828427125 < \pi < 3.828427125$. (Hint. Use Exercise 3.12 with the estimate $\pi < 4$ in the upper bound.)

* When $n = 2$, the inequality in Exercise 3.12 becomes

$$\text{area}(P_2) < \pi < \text{area}(P_2) + (\tfrac{1}{2})^2\pi.$$

Now $\text{area}(P_2) = 2s_1 = 2\sqrt{2} \approx 2.828427125$, by Exercise 3.13(i), so that

$$2.828427125 < \pi < 2.828427125 + (\tfrac{1}{2})^2\pi.$$

Since $\pi < 4$, we have $(\tfrac{1}{2})^2\pi < 4/4 = 1$, and so

$$2.828427125 < \pi < 3.828427125.$$

(iii) Show that $s_2 \approx 0.765366864$, $\text{area}(P_3) \approx 3.06147456$, and $3.0611467456 < \pi < 3.5611467456$.

* When $n = 3$, the inequality in Exercise 3.12 becomes

$$\text{area}(P_3) < \pi < \text{area}(P_3) + (\tfrac{1}{2})^3\pi.$$

By Exercise 3.14(ii), $\text{area}(P_3) = 4s_2$, and by Exercise 3.13(ii),

$$s_2 = \sqrt{2 - \sqrt{4 - s_1^2}}.$$

\approx

Since $s_1 = \sqrt{2}$, we have $s_2 \approx 0.765366864$ and

$$\text{area}(P_3) = 4 \times 0.765366864 \approx 3.06147456.$$

Hence,

$$3.06147456 < \pi < 3.06147456 + (\tfrac{1}{2})^3\pi.$$

Since $(\frac{1}{2})^3\pi < (\frac{1}{2})^3 4 = \frac{1}{2} = .5$, we have

$$3.06147456 < \pi < 3.56147456.$$

approx

(iv) Show that $s_3 \approx 0.390180643$, $\text{area}(P_4) \approx 3.121445144$, and $3.121445144 < \pi < 3.371445144$.

* When $n = 4$, the inequality in Exercise 3.12 becomes

$$\text{area}(P_4) < \pi < \text{area}(P_4) + (\tfrac{1}{2})^4\pi.$$

By Exercise 3.14(ii), $\text{area}(P_4) = 8s_3$, and by Exercise 3.13(ii),

$$s_3 = \sqrt{2 - \sqrt{4 - s_2^2}}.$$

approx

Since $s_2 = .765366864$, we have $s_3 \approx 0.390180643$ and

$$\text{area}(P_4) = 8 \times 0.390180643 \approx 3.121445144.$$

Hence,

$$3.121445144 < \pi < 3.121445144 + (\tfrac{1}{2})^4\pi.$$

Since $(\frac{1}{2})^4\pi < (\frac{1}{2})^4 4 = \frac{1}{4} = .25$, we have

$$3.121445144 < \pi < 3.371445144.$$

(v) Repeat this procedure until you can estimate π well enough to see that its first digits are 3.14.

* We keep iterating this process until we reach an inequality of the form $3.140 < \pi < 3.149$.

We recall that $\text{area}(P_n) = 2^{n-1}s_{n-1}$, and we define the nth upper bound $UB_n = \text{area}(P_n) + 4/2^n$.

n	s_{n-1}	2^{n-1}	$\text{area}(P_n) = 2^{n-1}s_{n-1}$	$\text{area}(P_n) + 4/2^n$
2	1.414213562	2	2.828427125	3.828427125
3	0.765366864	4	3.061467456	3.561467456
4	.390180643	8	3.121445144	3.371445144
5	.196034280	16	3.136548480	3.261548480
6	.098135347	32	3.140331104	3.202831104
7	.049082455	64	3.141277120	3.172527120
8	.024543073	128	3.141513344	3.157138344
9	.012271763	256	3.141571328	3.149383828

3.16. Let P_n be as in Exercise 3.12. Show that

$$\text{area}(P_n) = 2^n\sin(360/2^n).$$

* In Figure 3.10, $\text{area}(P_n) = 2^{n+1}\text{area}(\triangle OAB)$. If $\alpha_n = \measuredangle OAB$, then $\alpha_n = 360/2^{n+1}$. Since the disk has radius 1, the altitude $h_n = \cos \frac{1}{2}\alpha_n$ and the base $s_n = |AB| = 2|AE| = 2\sin \frac{1}{2}\alpha_n$. Therefore,

$$\text{area}(\triangle OAB) = \tfrac{1}{2}h_n s_n = \tfrac{1}{2}\cos \alpha_n 2\sin \alpha_n = \tfrac{1}{2}\sin 2\alpha_n.$$

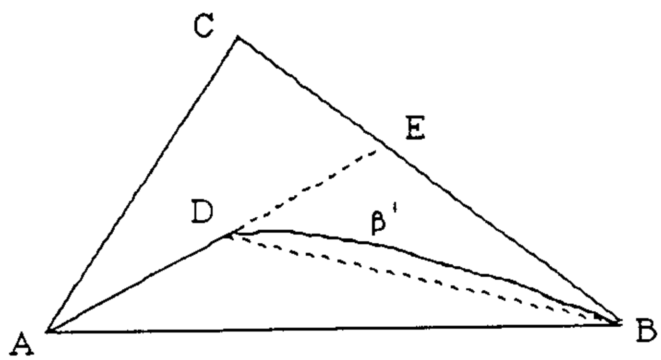
Since $2\alpha_n = 2\times 360/2^{n+1} = 360/2^n$, we conclude that

$$\text{area}(P_n) = 2^{n+1}\times\tfrac{1}{2}\sin 2\alpha_n = 2^n\sin 2\alpha_n = 2^n\sin(360/2^n).$$

3.17. Let $\triangle ABC$ be a triangle in the plane, as in Figure 3.17, and let β be a concave polygonal path from A to B lying wholly inside it. If β has $n \geq 2$ edges, prove that

$$|AC| + |CB| > \text{length } \beta.$$

* The proof is by induction on $n \geq 2$, the base step $n = 2$ being done in the text. For the inductive step, let β be an $(n+1)$-edged path from A to B; its first edge is AD and the remaining n edges form a path β' from D to B, as in the figure.



Extend AD, and let it meet CB in the point E. Since the path β is concave, it is entirely inside the triangle $\triangle AEB$. Let us compute.

$$
\begin{aligned}
|AC| + |CB| &> |AE| + |EB| \\
&= |AD| + |DE| + |EB| \\
&> |AD| + \mathrm{length}(\beta') = \mathrm{length}(\beta).
\end{aligned}
$$

The first inequality holds by the base step, for $AE + EB$ is a concave 2-edged curve inside $\triangle ABC$; the second inequality holds by the inductive hypothesis applied to $\triangle DEB$.

3.18. Show, in Figure 3.21, that

$$|AC| + |CD| + |DB| > |AE| + |EB|.$$

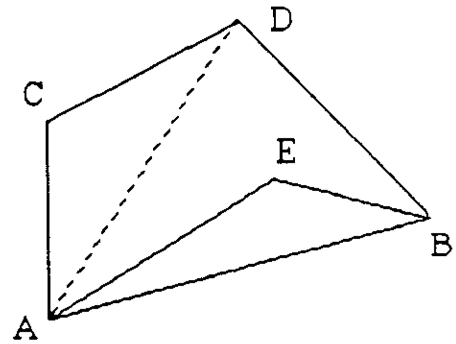


Figure 3.21a

* Draw an edge from A to D. As Figure 3.21 is drawn, the path from A to E to B lies inside the region bounded by AB and the path from A to D to B. By Exercise 3.21, $|AD| + |DB| > |AE| + |EB|$. But $|AC| + CD| > |AD|$, and this gives the result.

A more general interpretation involves any concave 2-edged path inside a region bounded by AB and a concave 3-edged path from A to B. One can show that the inside path lies either in $\triangle ADB$ or $\triangle ACB$, but this involves taking the definition of concavity more seriously than is being done in the text.

3.19. True or false: the arclength of the curve $y = x^2$ between 0 and 1 is greater than the arclength of the curve $y = x^3$ between 0 and 1. No fair integrating.

* False. Both are concave curves from $A = (0, 0)$ to $B = (1, 1)$, and since $x^3 \leq x^2 \leq 1$ for all x in $[0, 1]$, the quadratic curve is inside the region bounded by the line AB and the cubic curve. The principle of concavity now applies.

3.20. The front wheel of a tricycle is 3 feet in diameter, while its two rear wheels are 2 feet in diameter. If, on a straight road, the front wheel makes 64 revolutions, how many revolutions do the rear wheels make?

* The tricycle travels $64 \times 3\pi$ feet, so that the number of revolutions made by the rear wheels is $64 \times 3\pi/2\pi = 96$.

\ldots

3.21. Let $a_1, a_2, a_3, \ldots$ be a constant sequence; that is, there is some number c with $a_n = c$ for all n. Prove that $a_n \to c$.

* Let $\varepsilon > 0$. If we choose $\ell = 1$, then

$$|a_n - c| = |c - c| = 0 < \varepsilon$$

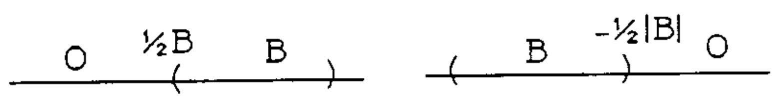
for all $n \geq 1$.

3.22. Prove that $|x - B| < \varepsilon$ if and only if $B - \varepsilon < x < B + \varepsilon$; that is, x lies in the open interval $(B - \varepsilon, B + \varepsilon)$.

* Interpret $|x - B|$ as the distance from x to B. Now both $B - \varepsilon$ and $B + \varepsilon$ have distance ε to B. If x lies in the open interval $(B - \varepsilon, B + \varepsilon)$, then the distance from x to $B < \varepsilon$, and if it lies outside this interval, its distance to $B > \varepsilon$.

3.23. (i) If $b_n \to B \neq 0$ and all $b_n \neq 0$, prove that there is an integer ℓ and some number N with $1/|b_n| < N$ for all $n \geq \ell$.

* Let $\varepsilon = |B|/2$; note that $\varepsilon > 0$ because $B \neq 0$. Since $b_n \to B$, there is an integer ℓ with $|b_n - B| < |B|/2$ for all $n \geq \ell$. Since $|b_n - B|$ is the distance between b_n and B, we are saying that for all $n \geq \ell$, b_n lies in the open interval $(B - |B|/2, B + |B|/2)$.



If $B > 0$, every point in this interval lies to the right of $\frac{1}{2}B$, and so its distance to 0 is larger than $\frac{1}{2}B$; that is, $|b_n| = |b_n - 0| > \frac{1}{2}B$. If $B < 0$, every point b_n is to the left of $-\frac{1}{2}|B|$, and so its distance to 0 is also greater than $\frac{1}{2}|B|$. By Theorem 1.4(iii), we have

$$1/|b_n| < 2/|B| \text{ for all } n \geq \ell.$$

Define $N = 2/|B|$.

(ii) Assume that $a_n \to A$ and $b_n \to B$. Prove that if $B \neq 0$ and all $b_n \neq 0$, then $a_n/b_n \to A/B$.

* Given $\varepsilon > 0$, we are going to tinker a bit. First, let's see what we need.

$$
\begin{aligned}
|a_n/b_n - A/B| &= |(Ba_n - Ab_n)/b_nB| \\
&= |(Ba_n - AB + AB - Ab_n)/b_nB| \\
&\le |(Ba_n - AB)/b_nB| + |(AB - Ab_n)/b_nB| \\
&= |a_n - A|/|b_n| + |b_n - B|\cdot|A|/|b_nB|.
\end{aligned}
$$

Choose ℓ and N as in (i), so $1/|b_n| < N$ for all $n \ge \ell$.

There is ℓ' with $|a_n - A| < \varepsilon/2N$ for all $n \ge \ell'$ (since $a_n \to A$), and there is ℓ'' with $|B - b_n| < \varepsilon|B|/2|A|N$ for all $n \ge \ell''$ (since $b_n \to B$). If we let λ denote the largest of ℓ, ℓ', and ℓ'', then all inequalities hold simultaneously when $n \ge \lambda$:

$$
\begin{aligned}
|a_n/b_n - A/B| &\le |a_n - A|/|b_n| + |b_n - B|\cdot|A|/|b_nB| \\
&< |a_n - A|N + |A||B - b_n|N/|B| \\
&< \varepsilon N/2N + \varepsilon|B||A|N/2N|A||B| \\
&= \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

(iii) Let $f(x)$ and $g(x)$ be polynomials. If $g(b) \neq 0$, prove that the rational function $f(x)/g(x)$ is continuous at b.

* Let $a_n \to b$, and assume that $f(a_n)/g(a_n)$ are all defined; that is, $g(a_n) \neq 0$ for all n (we are told that $g(b) \neq 0$, so that $f(b)/g(b)$ is defined). By Theorem 3.24, we have $f(a_n) \to f(b)$ and $g(a_n) \to g(b)$, so part (ii) gives $f(a_n)/g(a_n) \to f(b)/g(b)$.

3.24. Prove that the sequence (a_n) given by $a_n = (-1)^n$ does not converge.

* If, on the contrary, $a_n \to L$, then for every $\varepsilon > 0$, there is ℓ with $|L - a_n| < \varepsilon$ for all $n \ge \ell$. In particular, this would be true for $\varepsilon = \frac{1}{2}$. Thus, for all $n \ge \ell$, a_n lies in $(L - \frac{1}{2}, L + \frac{1}{2})$, the open interval with center L and radius $\frac{1}{2}$. Since this interval has length 1, it cannot contain both numbers -1 and 1 (the distance between them is 2). This is a contradiction, for a_n and a_{n+1} have different signs, and so both cannot be in $(L - \frac{1}{2}, L + \frac{1}{2})$.

3.25. Prove that a convergent sequence has only one limit: if $a_n \to A$ and $a_n \to L$, then $A = L$.

* If $A \ne L$, set $\varepsilon = \frac{1}{2}|L - A| > 0$ (ε is half the distance from A to L). Notice that the two open intervals of length 2ε with midpoints L and A, respectively, do not overlap. Since $a_n \to A$, there is ℓ with $|a_n - A| < \varepsilon$ for all $n \ge \ell$. This says, for all $n \ge \ell$, that a_n lies in the open interval with midpoint A; similarly, $a_n \to L$ says there is ℓ' with a_n lying in the open interval with midpoint L for all $n \ge \ell'$. If $n \ge$ larger of ℓ and ℓ', then a_n lies in both nonoverlapping intervals, and this is a contradiction. Therefore, $A = L$.

\ldots

3.26. (i) Let $a_1, a_2, a_3, \cdots$ be a sequence with $a_n \to L$, and consider the new sequence $a_2, a_3, a_4, \cdots$ obtained by eliminating the first term. Show that the new sequence also converges to L.

* Define a sequence b_n by $b_n = a_{n+1}$; thus, $b_1 = a_2$, $b_2 = a_3$, etc. Given $\varepsilon > 0$, there exists ℓ with $|a_n - L| < \varepsilon$ for all $n \ge \ell$. Therefore, $|b_n - L| = |a_{n+1} - L| < \varepsilon$ for all $n \ge \ell$.

(ii) Show that if (a_n) converges to L, then every subsequence of (a_n) also converges to L.

\ldots

* If $b_1, b_2, b_3, \cdots$ is a subsequence of (a_n), then each $b_n = a_m$ with $m \ge n$. Given $\varepsilon > 0$, there is ℓ with $|a_m - L| < \varepsilon$ for all $m \ge \ell$. If $n \ge \ell$, then $b_n = a_m$ for $m \ge n \ge \ell$, so that $|b_n - L| = |a_m - L| < \varepsilon$.

(iii) Use part (ii) to give another proof of Exercise 3.28.

* The subsequence $a_{2n-1} \to -1$, while $a_{2n} \to 1$.

3.27. (Sandwich theorem). Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $a_n \to L$, $b_n \to L$, and $a_n \le c_n \le b_n$ for all $n \ge 1$. Prove that $c_n \to L$.

* Given $\varepsilon > 0$. Since $a_n \to L$ and $b_n \to L$, we have $a_n - b_n \to L - L = 0$. There is thus ℓ with $|b_n - a_n| < \varepsilon/3$ for all $n \ge \ell$ (more tinkering afoot). Now

$$b_n - a_n = (b_n - c_n) + (c_n - a_n) > b_n - c_n,$$

so that $|b_n - c_n| < \varepsilon/3$ for all $n \ge \ell$. Now

$$|L - c_n| = |L - c_n + b_n - b_n + a_n - a_n|$$

$$\le |L - a_n| + |b_n - c_n| + |a_n - b_n|.$$

Since $a_n \to L$, there is ℓ' with $|L - a_n| < \varepsilon/3$ for all $n \ge \ell'$. Hence, if $m \ge$ larger of ℓ and ℓ', then for all $n \ge m$,

$$|L - c_n| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

3.28. (i) Show that $1 - \frac{2}{3}^n \to 1$.

* Since $0 < \frac{2}{3} < 1$, Theorem 3.19 shows that $\frac{2}{3}^n \to 0$. It follows that $1 - \frac{2}{3}^n \to 1 - 0 = 1$.

(ii) Show that $(n - 1)/n \to 1$.

* $(n - 1)/n = 1 - 1/n$. Now $1/n \to 0$, as in the text, and so the result follows from Theorem 3.23.

3.29. Prove that $a_n \to A$ if and only if $A - a_n \to 0$.

* If $a_n \to A$, then for each $\varepsilon > 0$, there is ℓ with $|A - a_n| < \varepsilon$ for all $n \ge \ell$. Hence, $|(A - a_n) - 0| < \varepsilon$ for all $n \ge \ell$, and so

$A - a_n \to 0$.

Conversely, if $A - a_n \to 0$, then for each $\varepsilon > 0$, there is ℓ with $|(A - a_n) - 0| < \varepsilon$ for all $n \geq \ell$. Hence, $|A - a_n| < \varepsilon$ for all $n \geq \ell$, and so $a_n \to A$.

This is so easy that it looks hard.

3.30. (i) Prove that if $|r| < 1$, then $r^n \to 0$.

* In light of Theorem 3.19, we may assume that $r = -s$, where $0 < s < 1$. Given $\varepsilon > 0$, there is ℓ with $|s^n - 0| = |s^n| < \varepsilon$ for all $n \geq \ell$. But $|r^n - 0| = |r^n| = |(-1)^n s^n| = |s^n|$ for all n, and so $|r^n - 0| < \varepsilon$ for all $n \geq \ell$.

(ii) Prove that the geometric series $\Sigma_{n=0}^{\infty} r^n$ converges, with sum $1/(1 - r)$, whenever $|r| < 1$.

* If $r = 0$, then the partial sum $s_n = \Sigma_{i=0}^{n} r^i = 0^0 = 1$ (we made this agreement in the first chapter). Thus, $\{s_n\}$ is the constant sequence of all 1's, and hence $s_n \to 1$. If $r = 0$, then $1/(1 - r) = 1$, as desired.

We may now assume that $-1 < r < 0$. As in the text,

$$s_n = 1/(1 - r) - r^n/(1 - r).$$

By part (i), $r^n \to 0$, and the proof that $s_n \to 1/(1 - r)$ is now completed as in the text.

3.31. Prove, using the definition of convergence, that $⅓ = .3333\cdots$.

* By definition, $B = .3333\cdots$ is the limit of the sequence $\{a_n\}$ with $a_n = 3/10(1 + (1/10) + (1/10)^2 + \cdots + (1/10)^{n-1})$. By the example of the geometric series, we have

$$a_n = (3/10)[1/(1 - r) - r^n/(1 - r)],$$

where $r = 1/10$. Since $r^n/(1 - r) \to 0$, we have $B = 3/10 \cdot 10/9 = \frac{1}{3}$.

3.32. Assume that there are two sequences converging to the same limit: $a_n \to A$ and $b_n \to A$. Prove that

\ldots

$$a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \cdots \to A.$$

* Let $\varepsilon > 0$. Since $a_n \to A$, there is ℓ' with $|a_n - A| < \varepsilon$ for all $n \geq \ell'$; since $b_n \to A$, there is ℓ'' with $|b_n - A| < \varepsilon$ for all $n \geq \ell''$. If ℓ is the larger of ℓ' and ℓ'', then both of these inequalities hold simultaneously.

Define a sequence $\{c_m\}$ by $c_{2n-1} = a_n$ and $c_{2n} = b_n$ (this is the sequence described in the statement). If $L = 2\ell$, then we claim that $|c_m - A| < \varepsilon$ for all $m \geq L$ (and this will complete the proof). Let $m \geq L = 2\ell$. If $m = 2n$, then

$$|c_m - A| = |c_{2n} - A| = |b_n - A| < \varepsilon,$$

because $m = 2n \geq 2\ell$ implies $n \geq \ell$. Similarly, if $m = 2n - 1$, then

$$|c_m - A| = |c_{2n-1} - A| = |a_n - A| < \varepsilon,$$

because $m = 2n - 1 \geq 2\ell$ implies $2n \geq 2\ell + 1$, and hence $n \geq \ell + \frac{1}{2} > \ell$.

3.33. Define a sequence $\{a_n\}$ by $a_{2n-1} = 5$ and $a_{2n} = 1/n$; thus, the sequence begins $5, 1, 5, \frac{1}{2}, 5, \frac{1}{3}, 5, \frac{1}{4}, 5, \cdots$. Prove that $\{a_n\}$ does not converge.

* The interval $(4, 6)$, of radius $\varepsilon = 1$ and center 5, contains no term a_{2n}, but it does contain all odd terms a_{2n-1}. In words, there is no ℓ with $|5 - a_n| < 1$ for all $n \geq \ell$. Therefore, $a_n \not\to 5$.

Now let L be any number other than 5. There is $\varepsilon > 0$ so that $(L - \varepsilon, L + \varepsilon)$ does not contain 5 (for example, take $\varepsilon = \frac{1}{2}|L - 5|$); that is, this interval contains no term a_{2n}. In words, there is no ℓ with $|L - a_n| < 1$ for all $n \geq \ell$. Therefore, $a_n \not\to L$.

Alternatively, one can use Exercise 3.30(ii).

3.34. Define a sequence $\{a_n\}$ by $a_{2n-1} = 1/n$ and $a_{2n} = \frac{1}{2}^n$. The sequence begins: 1, ½, ½, ¼, ⅓, 1/8, ¼, 1/16, ⋯.
(i) Prove that $a_n \to 0$, where $a_n = 1/n$

* Given $\varepsilon > 0$, there is ℓ with $1/n < \varepsilon$ for all $n \geq \ell$, and there is ℓ' with $\frac{1}{2}^n < \varepsilon$ for all $n \geq \ell'$. Thus, if $\ell'' = \ell + \ell'$, then $a_n < \varepsilon$ for all $n \geq \ell''$.

(ii) Prove that the terms a_n do not get "closer and closer" to the limit 0; that is, if $n > \ell$, then it does not follow that $|a_n - 0| < |a_\ell - 0|$.

* Since $2^n > n$ for all $n \geq 1$, we have $\frac{1}{2}^n < 1/n$, for all n. That is, $a_{2n-1} = 1/n > a_{2n} = \frac{1}{2}^n$. But it is easy to prove, by induction, that $2^n > n + 1$ for all $n \geq 2$, so that $\frac{1}{2}^n < 1/(n + 1)$; that is, $a_{2n} < a_{2n+1} = a_{2(n+1)-1}$. Thus, the sequence keeps oscillating up and down.

3.35. (i) Let $\{P_n\}$ be the sequence of inscribed 2^{n+1}-gons in a disk of radius r, and let $\{Q_n\}$ be the sequence of circumscribed 2^{n+1}-gons (notation as in Chapter 2). If p_n is the perimeter of P_n, and if q_n is the perimeter of Q_n, prove that $p_n \to 2\pi r$ and $q_n \to 2\pi r$.

* In Theorem 3.6, we proved that $\text{area}(P_*) \nearrow \text{area}(D)$; moreover, in Theorem 3.13, we saw that $\text{area}(P_n) = \frac{1}{2} h_n p_n$. As $\text{area}(D) = \pi r^2$, Theorem 3.21 gives $\frac{1}{2} h_n p_n \to \pi r^2$, hence, $h_n p_n \to 2\pi r^2$. Now $h_n \to r$, by Exercise 3.11 (and Theorem 3.21), so that Exercise 3.27(ii) gives

$$p_n = h_n p_n / h_n \to 2\pi r^2/r = 2\pi r.$$

In Theorem 3.11, we proved that $\text{area}(Q*) \searrow \text{area}(D)$; moreover, in Theorem 3.13, we saw that $\text{area}(Q_n) = \frac{1}{2} r q_n$. As $\text{area}(D) = \pi r^2$, Theorem 3.21 gives $\frac{1}{2} r q_n \to \pi r^2$, and so Theorem 3.18 gives $q_n \to 2\pi r$.

(ii) Prove that $\sin(\pi/2^n)/(\pi/2^n) \to 1$.

* Position P_n, as in Figure 3.23, so that the coordinates of X are $(\cos\vartheta_n/2, \sin\vartheta_n/2)$, where $\sin\vartheta_n/2 = \frac{1}{2}b_n$ and b_n is the length of a side of P_n. Note that $\vartheta_n = 2\pi/2^{n+1} = \pi/2^n$.

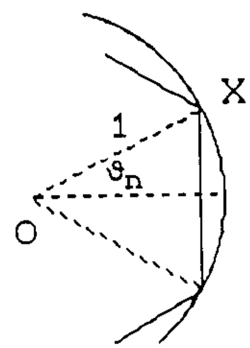


Figure 3.23

Now

$$p_n = 2^{n+1}b_n = 2^{n+2}(\tfrac{1}{2}b_n)$$

$$= 2^{n+2}\sin(\vartheta_n/2) = 2^{n+2}\sin(\pi/2^{n+1}),$$

so that $2^{n+2}\sin(\pi/2^{n+1}) \to 2\pi$, by part (i), and hence

$$2^{n+1}\sin(\pi/2^{n+1}) \to \pi.$$

Finally, Theorem 3.18 gives

$$\sin(\pi/2^{n+1})/(\pi/2^{n+1}) \to 1.$$

3.36. (i) Prove that if a_n lies in $(b - 1/n, b + 1/n)$ for all $n \geq 1$, then $a_n \to b$.

* Given $\varepsilon > 0$, there is ℓ with $1/\ell < \varepsilon$ (because $1/n \to 0$). If $n \geq \ell$, then $1/n < 1/\ell$, so that the interval $(b - 1/n, b + 1/n)$ is inside of the interval $(b - 1/\ell, b + 1/\ell)$. Therefore, if $n \geq \ell$, then a_n in $(b - 1/n, b + 1/n)$ implies a_n in $(b - 1/\ell, b + 1/\ell)$; hence, $|a_n - b| < 1/\ell < \varepsilon$. Therefore, $a_n \to b$.

Alternatively, one can use the Sandwich theorem (Exercise 3.31). For all $n \geq 1$, we have

$$L - 1/n < a_n < L + i/n.$$

Since $L - 1/n \to L$ and $L + 1/n \to L$, we have $a_n \to L$.

(ii). Assume that $f(x)$ is continuous at a point b. If $b \neq 0$, prove that there is some interval $(b - \varepsilon, b + \varepsilon)$ so that $f(a) \neq 0$ for all a in $(b - \varepsilon, b + \varepsilon)$. Hint. Use part (i).

* If the result is false, then for each $\varepsilon > 0$, there is some a_ε in $(b - \varepsilon, b + \varepsilon)$ with $f(a_\varepsilon) = 0$. In particular, for each $n \geq 1$, there is a_n in $(b - 1/n, b + 1/n)$ with $f(a_n) = 0$. By part (i), we have $a_n \to b$. Hence, $\{f(a_n)\}$ is the constant sequence of all 0's, and so $f(a_n) \to 0$. But since $f(x)$ is continuous at b, we have $f(a_n) \to f(b) \neq 0$, a contradiction.

3.37. If $\{a_n\}$ is a decreasing sequence if $a_n \to t$, prove that $a_n \geq t$ for all $n \geq 1$.

* If $\{a_n\}$ is eventually constant, i.e., there is some integer N so that $a_N = a_{N+1} = a_{N+2} = \cdots$, then it is easy to see that $t = a_N$, and so $a_n \geq t$ for all n.

We may assume, therefore, that for every m, there exists $k > m$ with $a_m > a_k$ (strict inequality). Suppose that the result is false; that is, suppose that $t > a_\ell$ for some ℓ. Choose $\varepsilon = t - a_\ell$. Since $a_n \to t$, there is N with $|a_n - t| < \varepsilon$ for $n \geq N$. There is $k \geq N$ with $a_\ell > a_k$, so that

$$t - a_k = t - a_\ell + a_\ell - a_k = \varepsilon + (a_\ell - a_k).$$

Since $a_\ell > a_k$, the term $a_\ell - a_k$ is positive, and so $t - a_k > \varepsilon$. As $k \geq N$, this contradicts $|a_k - t| < \varepsilon$.

Chapter 4

4.1. If r_1 and r_2 are the roots of $f(x) = x^2 + bx + c$, compute b and c in terms of r_1 and r_2.

* By the hint,

$$f(x) = x^2 + bx + c$$

$$= (x - r_1)(x - r_2).$$

$$= x^2 - (r_1 + r_2) + r_1r_2.$$

Therefore, $b = -(r_1 + r_2)$ and $c = r_1r_2$.

4.2. If a, b, and c are odd integers, prove that $ax^2 + bx + c$ has no rational roots.

* If $x = p/q$ is a rational root of $ax^2 + bx + c = 0$, then

$$(\otimes) \qquad ap^2 + bpq + cq^2 = 0.$$

As in the preamble before the proof of Theorem 2.5, we can assume that p and q are not both even. There are now three cases (we shall use Exercise 2.4 throughout).

Case 1. p is even and q is odd.

In this case, $ap^2 + bpq = p(ap + bq)$ is even, so that cq^2 must also be even. But since both c and q are odd, Exercise 2.4(ii) gives oddness of cq^2, a contradiction.

Case 2. p is odd and q is even.

This is similar to the argument in Case I.

Case 3. Both p and q are odd.

In this case, each of the terms in Eq. $(\otimes)$ is odd, so that we have a sum of three odd integers being zero. Transposing, we have the sum of two odd integers being odd, contradicting Exercise 2.4(ii).

An alternative, though more complicated proof, can be based on Exercise 2.8(ii): $\sqrt{b^2 - 4ac}$ is rational if and only if $b^2 - 4ac$ is a perfect square.

If $b^2 - 4ac = m^2$, then m^2 must be odd (being of the form odd - even), and hence m must be odd; say, $m = 2k - 1$. In this case, we have $b^2 - m^2 = 4ac$, so that $(b + m)(b - m) = 4ac$. If we write $b = 2\ell - 1$, then

$$\begin{aligned} 4ac = (b + m)(b - m) &= [2k + 2\ell - 2][2k - 2\ell] \\ &= 4(k + \ell - 1)(k - \ell) \\ &= 4[k^2 - k - \ell^2 + \ell]. \end{aligned}$$

It follows that $k^2 - k - (\ell^2 - \ell) = ac$ is odd. But it is easy to see that $k^2 - k$ is always even, for it factors as $k(k - 1)$; that is, it is a product of two consecutive integers, one of which must be even. Hence, $[k^2 - k] - [\ell^2 - \ell]$ is even, a contradiction.

4.3. Find the points where the line with equation $y = 2x + 2$ intersects the circle of radius 1 and center (0, 0).

* If a point (x, y) lies on the graphs of $x^2 + y^2 = 1$ and of $y = 2x + 2$, then $x^2 + (2x + 2)^2 = 1$; hence, $5x^2 + 8x + 4 = 1$. The quadratic formula gives

$$x = (-8 \pm \sqrt{64 - 60})/10 = (-8 \pm 2)/10 = -1 \text{ or } -.6.$$

If follows that there are two points of intersection: (-1, 0) and (-0.6, 0.8).

4.4. Suppose that a rectangle having sides of lengths x and y has area A and perimeter p. If $p^2 - 16A \geq 0$, show that the roots of the quadratic in Eq. (1) are x and y.

* The quadratic is $2z^2 - pz + 2A$ can be rewritten in terms of x and y:

$$2z^2 - pz + 2A = 2z^2 - (2x + 2y)z + 2xy.$$

Setting either $z = x$ or $z = y$ gives 0.

4.5. Find and estimate the roots of $10^{-50}x^2 + 2x - 9$.

* Now $\sqrt{4 + 36/10^{50}}$ is very close to $\sqrt{4} = 2$. Therefore, the root $x = [-2 - \sqrt{4 + 36/10^{50}}]/(2/10^{50}) \approx -4/(2/10^{50}) = -2\times10^{50}$. To calculate the second root, use the variant expression:

$$x = [-2 + \sqrt{4 + 36/10^{50}}]/(2/10^{50})$$

$$= 18/(2 + \sqrt{4 + 36/10^{50}})$$

$$\approx 18/4 = 4.5.$$

4.6. Suppose a cannon on the ground makes an angle of 45°.
(i) If the initial velocity of a shell is v_0 = 80 feet per second, what is the horizontal distance R the shell travels when it hits the ground?

* The quadratic equation is $-16t^2 + 80(1/\sqrt{2})t = 0$, so that the time t it takes to hit the ground is $t = 5/\sqrt{2}$ seconds. Substituting this value of t into Eq. (3) of Chapter 4, one finds $R = 80(1/\sqrt{2})5(1/\sqrt{2}) = 200$ feet.

(ii) What should the initial velocity v_0 of a shell be in order that it hit a target 400 feet away?

* In this problem, Eqs. (2) and (3) are:

$$400 = v_0 t\cos 45^\circ = v_0 t/\sqrt{2}$$

$$0 = -16t^2 + v_0 t\sin 45^\circ = -16t^2 + v_0 t/\sqrt{2},$$

where t is the time needed to hit the ground. Hence, $16t^2 = 400$ and $t = 5$. Therefore, $400 = v_0 5/\sqrt{2}$, and $v_0 = 80\sqrt{2}$.

\ldots

4.7. Prove, for any numbers $a_1, a_2, \cdots, a_n$, that

$$[(a_1 + a_2 + \cdots + a_n)/n]^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)/n.$$

* Set $b_1 = b_2 = \cdots = b_n = 1$ in Cauchy's inequality (Theorem 4.4):

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le$$
$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2),$$

to obtain

$$(a_1 + a_2 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Now multiply both sides by $1/n^2$.

4.8. Rewrite each of the following complex numbers in the form $a + bi$.

(i) $(3 + 4i)(2 - i)$.

* $10 + 5i$

(ii) $(1 + i)^2$.

* $2i$

(iii) $[1/\sqrt{2} + (1/\sqrt{2})i]^2$.

* i

(iv) $(3 + 4i)(3 - 4i)$.

* 25

(v) $(3 + 4i)/(1 + i)$.

* $\frac{1}{2}(7 + i)$

4.9. Prove the *cancellation law* for complex numbers: if u, v, and z are complex numbers with $zu = zv$, and if $z \neq 0$, then $u = v$.

* Since $zu = zv$, we have $[(1/z)z]u = [(1/z)z]v$, and so $u = v$.

4.10. Show that if u and v are nonzero complex numbers, then their product uv is also nonzero.

* If $uv = 0$, then $0 = (1/u)uv = [(1/u)u]v = v$. This contradicts the assumption that $v \neq 0$.

4.11. Prove that if z is a complex number on the unit circle, then $1/z = \bar{z}$.

* For z is on the unit circle, $z = a + bi$, where $a^2 + b^2 = 1$. But $z\bar{z} = a^2 + b^2 = 1$, and so $\bar{z} = 1/z$.

4.12. Let $z = a + bi$. Prove that if z is a real number, then $z = \bar{z}$; conversely, if $z = \bar{z}$, then z is a real number.

* Let $z = s + ti$. If z is a real number, then $t = 0$, that is, $z = s + 0i$; hence, $\bar{z} = s - 0i = s + 0i = z$. Conversely, if $z = \bar{z}$, then $s + ti = s - ti$. One equation of complex numbers gives two equations of real numbers: $s = s$, of course, and $t = -t$. Hence $t = 0$ and $z = s + ti = s + 0i = s$; therefore, z is real.

4.13. If $z = a + bi$, show that $(x - z)(x - \bar{z})$ is a quadratic polynomial having real coefficients.

* $$(x - z)(x - \bar{z}) = (x - a - ib)(x - a + ib)$$

$$= (x - a)^2 - (ib)^2 = (x - a)^2 + b^2 = x - 2ax + a^2 + b^2.$$

4.14. If z and w are complex numbers, prove that

$$\overline{z + w} = \bar{z} + \bar{w} \quad \text{and} \quad \overline{zw} = \bar{z} \cdot \bar{w}.$$

* Let $z = s + ti$ and $w = p + qi$. Now $z + w = (s + p) + (t + q)i$, so that

$$\overline{z + w} = (s + p) - (t + q)i;$$

$$\bar{z} + \bar{w} = (s - ti) + (p - qi) = (s + p) - (t + q)i.$$

Also, $zw = (sp - tq) + (sq + tp)i$, so that

$$\overline{zw} = (sp - tq) - (sq + tp)i,$$

and this is equal to $\bar{z}\,\bar{w} = (s - ti)(p - qi)$.

4.15. Let $f(x) = ax^2 + bx + c$, where the coefficients a, b, and c are complex. Prove that if the roots of $f(x) = 0$ are not real numbers, then they are complex conjugates.

* Quadratic formula: If $D = b^2 - 4ac$, then $f(x)$ has a complex root if and only if $D < 0$; write $D = -E$ in this case. The roots are

$$-b/2a + i\,\sqrt{E}/2a) \text{ and } -b/2a - i\,\sqrt{E}/2a,$$

and these are conjugates.

Alternatively, suppose that $f(z) = 0$. Thus,

$$0 = \bar{0} = \overline{az^2 + bz + c} = \overline{az^2} + \overline{bz} + \bar{c} = \bar{a}\,\bar{z}^2 + \bar{b}\,\bar{z} + \bar{c}$$

$$= a\bar{z}^2 + b\bar{z} + c, \text{ because } a, b, c \text{ are real}$$

$$= f(\bar{z}).$$

Therefore, $\bar{z}$ is a root of $f(x)$.

4.16. If u is a root of a cubic polynomial $f(x)$ having real coefficients, then its conjugate $\bar{u}$ is also a root of $f(x)$.

* If $f(x) = x^3 + bx^2 + cx + d$, then

$$0 = u^3 + bu^2 + cu + d.$$

Repeated application of Exercise 4.14 gives

$$0 = \overline{0} = \overline{(u^3 + bu^2 + cu + d)}$$

$$= \bar{u}^3 + \overline{bu^2} + \overline{cu} + \bar{d}$$

$$= \bar{u}^3 + b\bar{u}^2 + c\bar{u} + d \quad (b, c, d \text{ are real})$$

$$= f(\bar{u}).$$

Therefore, $\bar{u}$ is a root of $f(x)$. (This result and proof can be generalized to polynomials of any degree having real coefficients.)

4.17. If ζ is a cube root of unity, show that $\bar{\zeta} = \zeta^2$.

* $\zeta = -\frac{1}{2} + i\sqrt{3}/2$, and computing ζ^2 gives $-\frac{1}{2} - i\sqrt{3}/2$.

4.18. Find the square roots of $3 - 4i$.

* $\pm(2 - i)$.

4.19. Find the roots of $x^2 + (2 + i)x + 2i = 0$.

* Using the quadratic formula and the previous exercise,

$$x = \tfrac{1}{2}(-2 - i \pm \sqrt{3 - 4i}) = \tfrac{1}{2}(-2 - i \pm (2 - i)) = -i \text{ or } -2.$$

4.20. Prove the binomial theorem for complex numbers: If z and w are complex numbers, then for all $n \geq 0$,

$$(z + w)^n = \Sigma_{r=0}^{n} \binom{n}{r} z^r w^{n-r}.$$

* The proof is identical to that of Lemma 1.18 (if one replaces x by z throughout) and Corollary 1.21.

4.21. Prove that if u, v, and w are roots of a cubic polynomial $x^3 + bx^2 + cx + d$, then $b = -(u + v + w)$ and $d = -uvw$.

* $$x^3 + bx^2 + cx + d = (x - u)(x - v)(x - w)$$

$$= x^3 - (u + v + w)x^2 + (uv + uw + vw)x - uvw.$$

4.22. (i) Find $\sqrt{8 + 15i}$.

* In polar form, $z = 8 + 15i = 17(\cos 62^\circ + i\sin 62^\circ)$; 62° is the angle θ in the first quadrant with $\cos\theta = 8/17$ and $\sin\theta = 15/17$.

\approx

$$\sqrt{8 + 15i} \approx \sqrt{17}(\cos 31^\circ + i\sin 31^\circ) \approx 3.533 + 2.123i.$$

(Of course, the other square root is the negative of this one.) One can also use the method of Theorem 4.10.

(ii) Find $\sqrt[4]{8 + 15i}$.

\approx

* $$\sqrt[4]{8 + 15i} \approx \sqrt[4]{17}(\cos 15.5^\circ + i\sin 15.5^\circ) \approx 1.964 + .542i.$$

The other fourth roots are obtained by multiplying this one by i, -1, and $-i$. (The problem has been ambiguously worded, however, so that a student with only one fourth root has solved it.)

4.23. Find an 8th root of $9 - 7i$.

\approx

* $$9 - 7i \approx \sqrt{130}(\cos 322^\circ + i\sin 322^\circ);$$

322° is the angle θ in the fourth quadrant with $\cos\theta = 9/\sqrt{130}$ and $\sin\theta = -7/\sqrt{130}$.

$$\sqrt[8]{9 - 7i} \approx 1.356(\cos 40^\circ + i\sin 40^\circ) \approx 1.403 + .875i.$$

\approx

(The other eighth roots are obtained from this one by multiplying by the powers of $e^{2\pi i/8}$.)

4.24. Show that cos 1.25664 + isin 1.25664 is a fifth root of unity ($1.25664 \underset{\text{approx}}{\approx} 2\pi/5 = 72°$).

* By De Moivre's theorem, $(e^{i72})^5 = e^{i360} = 1$.

4.25. Prove that if ζ is a complex cube root of unity, then

$$\zeta(1 - \zeta^2)(1 - \zeta)^2 = 3i\sqrt{3}.$$

* $$\zeta(1 - \zeta^2)(1 - \zeta)^2 = \zeta(1 + \zeta)(1 - \zeta)^3$$

[because $1 - \zeta^2 = (1 + \zeta)(1 - \zeta)$]

$$= (\zeta + \zeta^2)(1 - \zeta)^3$$

$$= -(1 - \zeta)^3$$

[because $\zeta^2 + \zeta + 1 = 0$]

$$= -1 + 3\zeta - 3\zeta^2 + \zeta^3$$

$$= 3\zeta - 3\zeta^2$$

[because $\zeta^3 = 1$]

$$= 3(\zeta - \bar{\zeta})$$

$$= 3i\sqrt{3}$$

[because $\zeta = -½ \pm i\sqrt{3}/2$].

4.26. For every positive integer n, show that there is a polynomial $f_n(x)$ of degree n and with integer coefficients so that $\cos n\theta = f_n(\cos\theta)$.

* By De Moivre's theorem,

$$\cos nx + i\sin nx = (\cos x + i\sin x)^n.$$

The binomial theorem gives

$$(\cos x + i\sin x)^n = \Sigma_{j=0}^{n} \binom{n}{j} \cos^{n-j}x\, i^j \sin^j x$$

and so cos nx is the real part of this. The real part is the sum of all the terms with j even:

$$\cos nx = \Sigma_{j \text{ even}} \binom{n}{j} \cos^{n-j}x\, (-1)^{j/2} (\sin^2 x)^{j/2}.$$

But $\sin^2 x = 1 - \cos^2 x$, and so this substitution gives a formula for cos nx as a polynomial in cos x.

4.27. (i) Prove that $2\cos \theta = e^{i\theta} + e^{-i\theta}$.

* $\quad e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta) = 2\cos\theta.$

(ii) Use De Moivre's theorem to give a new proof of Exercise 2.51:

$$2\cos(n+1)\theta = (2\cos\theta)(2\cos n\theta) - 2\cos(n-1)\theta.$$

* By (i),

$$\begin{aligned}(2\cos\theta)(2\cos n\theta) &= (e^{i\theta} + e^{-i\theta})(e^{in\theta} + e^{-in\theta}) \\ &= e^{i(n+1)\theta} + e^{i(1-n)\theta} + e^{i(n-1)\theta} + e^{-i(n+1)\theta} \\ &= e^{i(n+1)\theta} + e^{-i(n+1)\theta} + e^{i(n-1)\theta} + e^{i(1-n)\theta} \\ &= 2\cos(n+1)\theta + 2\cos(n-1)\theta.\end{aligned}$$

4.28. (i) It can be proved that if $z = a + ib$, then

$$e^z = e^a e^{ib} = e^a(\cos b + i\sin b).$$

In contrast to real exponentiation, show that e^z can be a negative real number.

* If $z = i\pi$, then $e^{i\pi} = -1$.

(ii) If z and w are complex numbers, prove that

$$e^z e^w = e^{z+w}.$$

* Let $z = a + ib$ and $w = c + id$. Then

$$e^z e^w = e^a e^{ib} e^c e^{id} = e^a e^c e^{ib} e^{id}$$

$$= e^{a+c} e^{ib+id} = e^{a+c} e^{i(b+d)} = e^{z+w}.$$

(iii) If $w = e^z$, where z is a complex number, then define $\log(w) = z$. In contrast to real logarithms, show that -1 has a logarithm; indeed, show that it has infinitely many logarithms.

* We know that $e^{i\pi} = -1$, so that $\log(-1) = i\pi$. On the other hand, log is not single valued, for $(e^{i\pi})^{2n-1} = -1$ for all integers n, and so $\log(-1) = (2n - 1)i\pi$ for every integer n.

4.29. For any complex number z, define $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$. In contrast to real cosines, show that $\cos z > 1$ is possible.

* If b is real, then $\cos(ib) = \frac{1}{2}(e^{i(ib)} + e^{-i(ib)})$

$$= \tfrac{1}{2}(e^{-b} + e^{b}) = \cosh(b).$$

But $\cosh(b) \geq 1$ (indeed, we saw in Chapter 1 that there is equality if and only if $b = 0$).

4.30. (i) Write $1/(e^{i\theta} - 1)$ in the form $a + ib$.

* $$1/(e^{i\theta} - 1) = 1/(\cos\theta - 1 + i\sin\theta)$$

$$= (\cos\theta - 1 - i\sin\theta)/(2 - 2\cos\theta)$$

Hence, $a = (\cos\theta - 1)/(2 - 2\cos\theta)$ and $b = \sin\theta/(2 - 2\cos\theta)$.

(ii) Prove, for all $n \geq 1$ and for all θ, that

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = [e^{i(n+1)\theta} - 1]/[e^{i\theta} - 1].$$

* The proof is identical to the proof of Exercise 1.2.

(iii) Prove the identity: $(2 - 2\cos\theta)(\sin\theta + \sin 2\theta + \sin 3\theta) = -(\cos 4\theta - 1)\sin\theta + \sin 4\theta(\cos\theta - 1)$.

* By part (ii), $1 + e^{i\theta} + e^{i2\theta} + e^{i3\theta} = (e^{i4\theta} - 1)/(e^{i\theta} - 1)$. Rewrite the right hand side, using part (i) and De Moivre's theorem, and multiply both sides by $2 - 2\cos\theta$. The imaginary parts of the two sides are equal, and this is the desired identity.

4.31. (i) If $\cos 3\alpha$ is positive, show that there is an acute angle β with $3\alpha = 3\beta$ or $3\alpha = 3(\beta + 90°)$; moreover, show that the collection of numbers $\cos\beta$, $\cos(\beta + 120°)$, $\cos(\beta + 240°)$ coincides with the collection of numbers $\cos(\beta + 90°)$, $\cos(\beta + 210°)$, $\cos(\beta + 330°)$.

* Each collection of cosine values consists of

$$\cos\beta,\ -\tfrac{1}{2}\cos\beta + \sqrt{3}/2\sin\beta, \text{ and } -\tfrac{1}{2}\cos\beta - \sqrt{3}/2\sin\beta.$$

(ii) If $\cos 3\alpha$ is negative, show that there is an acute angle β with $3\alpha = 3(\beta + 30°)$ or $3\alpha = 3(\beta + 60°)$; moreover, show that the collection of numbers $\cos(\beta + 30°)$, $\cos(\beta + 150°)$, $\cos(\beta + 270°)$ coincides with the collection of numbers $\cos(\beta + 60°)$, $\cos(\beta + 180°)$, $\cos(\beta + 270°)$.

* Each collection of cosine values consists of

$$-\sin\beta,\ \sqrt{3}/2\cos\beta - \tfrac{1}{2}\sin\beta,\ -\sqrt{3}/2\cos\beta - \tfrac{1}{2}\sin\beta.$$

4.32. Consider the polynomial $f(X) = X^3 + X^2 - 36$ that arose in the castle problem in Chapter 2.

(i) Show that 3 is a root of $f(X)$ and find the other two roots as roots of the quadratic $f(X)/(X - 3)$.

$$* \qquad X^3 + X^2 - 36 = (X - 3)(X^2 + 4X + 12).$$

The other roots are thus $-2 \pm i\sqrt{8}$.

(ii) Use the cubic formula to find the root 3 of $f(X)$.

* The substitution $X = x - ⅓$ gives the reduced cubic

$$\tilde{f}(x) = x^3 - \tfrac{1}{3}x - 970/27.$$

Thus, $q = -⅓$, $r = -970/27$, $R = 940896/27^2$, and

$$g^3 = \tfrac{1}{2}(970/27 + \sqrt{940896}/27)$$

$$= (1/54)(970 + \sqrt{940896}) \approx 35.926;$$

hence,

$$g \approx 3.2996.$$ approx

It follows that $h = -3q/g = 1/9g \approx ..0336$, and so

$$g + h \approx 3.3332 \approx 3\tfrac{1}{3}.$$

A root of $f(x)$ is thus $3⅓ - ⅓ = 3$.

(iii) Show that the discriminant of $f(X)$ is negative, and find its real root (which is 10/3) using cosh.

* The discriminant formula gives a negative value, so that $f(x)$ has complex roots. In the reduced cubic $\tilde{f}(x) = x^3 - ⅓x - 970/27$, we have $q = -⅓$ and $r = -970/27$, so that

$$-4q/3 = 4/9 > 0.$$

It follows that we are in the hyperbolic cosine case, and $t = ⅔$.

approx

Now $\cosh \beta = -4r/t^3 = 485$, so that $\beta \approx 6.8773$ and $\beta/3 \approx 2.29243$. But $\cosh(2.29243) \approx 4.999 \approx 5$, so that $t\cosh(\beta/3) \approx \frac{2}{3} \times 5 = 10/3$.

4.33. (i) Show, for all a and b and for all $j \geq 1$, that

$$a^j - b^j = (a - b)(a^{j-1} + a^{j-2}b + a^{j-3}b^2 + \cdots + ab^{j-2} + b^{j-1}).$$

* After multiplying out, all terms cancel except $a^j - b^j$.

(ii) If $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial of degree n. For any number u, show that there is some polynomial $q(x)$ of degree $n - 1$ with

$$f(x) = (x - u)q(x) + f(u).$$

* $f(x) - f(u) = \Sigma_{j=0}^{n} c_j x^j - \Sigma_{j=0}^{n} c_j u^j = \Sigma_{j=0}^{n} c_j(x^j - u^j)$; now use part (i) to factor $x^j - u^j = (x - u)h_j(x)$ for all $j \geq 1$, where $h_j(x)$ is some suitable polynomial. Hence,

$$f(x) - f(u) = (x - u)q(x) + c_0(x^0 - u^0) = (x - u)q(x),$$

where $q(x) = \Sigma_{j=1}^{n} h_j(x)$.

(iii). Show that u is a root of $f(x)$ if and only if $x - u$ is a factor of $f(x)$.

* If u is a root of $f(x)$, then $f(u) = 0$ and $f(x) = (x - u)q(x)$, by (ii).

Conversely, if $f(x) = (x - u)(q(x)$, then $f(u) = (u - u)(q(u) = 0$, and so u is a root of $f(x)$.

4.34. Show that every cubic polynomial $f(x) = x^3 + bx^2 + cx + d$ having real coefficients b, c, and d has at least one real root.

* Suppose that $f(x)$ has a nonreal root u; by Exercise 4.16, $\bar{u}$ is also a root of $f(x)$. By Exercise 4.33(iii), we have

$$f(x) = (x - u)(x - \bar{u})(x - v),$$

where v is another root of $f(x)$. Now the constant term is $d = u\bar{u}v$, so that $v = d/u\bar{u}$ is real (for $u\bar{u} \neq 0$ and is real).

4.35. Show that if $\cos 3\alpha = r$, then the roots of $f(x) = 4x^3 - 3x - r$ are $\cos \alpha$, $\cos(\alpha + 120°)$, and $\cos(\alpha + 240°)$.

* Since $r = \cos 3\alpha = \cos 3(\alpha + 120°) = \cos 3(\alpha + 240°)$, the triple angle formula (Corollary 4.16(ii)) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ shows that all three cosines are roots of $f(x)$.

4.36 Find the roots of $f(x) = x^3 - 3x + 1$.

* Since $f(x)$ is already reduced, we can use the cubic formula at once. Hence, $q = -3$, $r = 1$, and $R = -3$. It follows that $g^3 = -\frac{1}{2} + i\sqrt{3}/2 = \zeta$; since ζ is a cube root of unity, g must be a 9th root of unity. By De Moivre's theorem,

$$g = \cos 40° + i\sin 40°.$$

Now $h = -q/3g = 1/g = \bar{g} = \cos 40° - i\sin 40°$, and so a root is $g + h = 2\cos 40° \cong 1.532$.
approx

4.37. Find the roots of $f(x) = x^3 - 9x + 28$.

* $q = -9$, $r = 28$, $R = 676 = 26^2$, $g = -1$, $h = -3$, and the roots are:

$$-4,\ 2 + i\sqrt{3},\ 2 - i\sqrt{3}.$$

The complex roots emerge most quickly if one writes

$$f(x) = (x + 4)(x^2 - 4x + 7)$$

and then uses the quadratic formula on the quadratic factor.

4.38. Find the roots of $f(x) = x^3 - 24x^2 - 24x - 25$.

* The reduced polynomial is $\tilde{f}(x) = x^3 - 216x - 1241$. Here, $q = -216$, $r = -1241$, $R = 47089$, $\sqrt{R} = 217$, $g^3 = 729$, $g = 9$, and $h = 8$. Therefore, a root of $\tilde{f}(x)$ is $g + h = 17$, and so a root of $f(x)$ is $17 + 8 = 25$. One can factor $f(x) = (x - 25)(x^2 + x + 1)$, and so the other two roots of $f(x)$ are $-\frac{1}{2} \pm i\sqrt{3}/2$.

4.39. (i) Find the roots of $x^3 - 15x - 4$ using the cubic formula.

* We have $q = -15$, $r = -4$, $R = 484$. The roots are:

$$g + h = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

$$\zeta g + \zeta^2 h = \zeta \cdot \sqrt[3]{2 + \sqrt{-121}} + \zeta^2 \cdot \sqrt[3]{2 - \sqrt{-121}}$$

$$\zeta^2 g + \zeta h = \zeta^2 \cdot \sqrt[3]{2 + \sqrt{-121}} + \zeta \cdot \sqrt[3]{2 - \sqrt{-121}}.$$

It is not at all clear how to estimate the roots in the form above. From Theorem 4.24, however, we do know that all the roots are real numbers because the discriminant $-27r^2 - 4q^3 = -27 \times (-4)^2 - 4(-15)^3 = 13068$ is positive.

(ii) Find the roots using the trigonometric formula.

approx approx approx approx

* $t = \sqrt{20} \approx 4.47$ and $\cos 3\alpha \approx .179$. Hence $3\alpha \approx 80°$ and $\alpha \approx 27°$. Hence, $\cos 27° \approx .891$, $\cos 147° \approx -.839$, $\cos 267° \approx -.052$. Hence, the roots are:

$$\sqrt{20}\cos 27° \approx 4.47 \times .891 \approx 3.98 \approx 4;$$

$$\sqrt{20}\cos 147° \approx 4.47 \times (-.839) \approx -3.750;$$

$$\sqrt{20}\cos 267° \approx 4.47 \times (-.052) \approx -.232.$$

Once one sees that 4 is a root, it is simplest to use long division to find the other two roots. Since

$$x^3 - 15x^2 - 4 = (x - 4)(x^2 + 4x + 1),$$

the other two roots are $-2 \pm \sqrt{3}$.

4.40. Find the roots of $x^3 - 6x + 4$.

* Roots are 2, $-1 + \sqrt{3}$, $-1 - \sqrt{3}$.

4.41. Find the roots of $x^4 - 15x^2 - 20x - 6$.

* This is a realistic problem; the implementation of the formula is rather long (but see the remark at the end of this solution).

Set $L = -15$, $M = -20$, and $N = -6$. Then Eq. (16) is:

$$j^6 + 2Lj^4 + (L^2 - 4N)j^2 - M^2 = j^6 - 30j^4 + 249j^2 - 400.$$

We now make this cubic (in j^2) reduced with the substitution $j^2 = y + 10$, to obtain

approx $$F(y) = y^3 - 51y + 90.$$

Apply the cubic formula: $q = -51$, $r = 90$, $R = -11552$. Thus, $\sqrt{R} \approx 107.5i$, $g^3 \approx \frac{1}{2}(-90 + 107.5i) \approx -45 + 53.8i$, and De Moivre's theorem gives a cube root: $g \approx 3 + 2.8i$. Now $h = -q/3g = 51/3[3 + 2.8i.] = 17/(3 + 2.8i.) \approx 3 - 2.8i = \bar{g}$. Therefore, $g + h = g + \bar{g} = 6$ is a root of $F(y)$ (alternatively, one could have checked for integer roots of $F(y)$ using Theorem B). Therefore,

$$j^2 = 16$$

(for $j^2 = y + 10$) and $j = 4$. Eqs. (15) are thus

$$2m = 16 - 15 - 20/4 = -4$$

$$2\ell = 16 - 15 + 20/4 = 6.$$

Thus, $m = -2$, $\ell = 3$, and we have the factorization

$$x^4 - 15x^2 - 20x - 6 = (x^2 + 4x + 3)(x^2 - 4x - 2).$$

The quadratic formula applied to each of the two factors gives the desired roots:

$$-3, -1, 2 + \sqrt{6}, 2 - \sqrt{6}.$$

In particular, all the roots are real.

Remark. It is natural to use Theorem B to check first for rational, hence integral roots. One finds that -1 and -3 are roots (the candidates are ± 1, ± 2, ± 3, and ± 6). Therefore, long division gives

$$x^4 - 15x^2 - 20x - 6 = (x + 1)(x + 3)(x^2 - 4x - 2).$$

The other two roots are thus $2 + \sqrt{6}$ and $2 - \sqrt{6}$.

One now understands the unpopularity of the quartic formula.

4.42. (i) Show that $\cos 2\theta = (1 - \tan^2\theta)/(1 + \tan^2\theta)$.

$*$ $(1 - \tan^2\theta)/(1 + \tan^2\theta) = (1 - \sin^2\theta/\cos^2\theta)/(1 + \sin^2\theta/\cos^2\theta)$

$$= (\cos^2\theta - \sin^2\theta)/(\cos^2\theta + \sin^2\theta)$$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos 2\theta.$$

(ii). If r is a rational number, show that the only rational values of $\tan r\pi$ are 0 and ± 1.

$*$ If $\tan r\pi$ is rational, then part (i) shows that $\cos 2r\pi$ is rational. By Theorem 4.25, $\cos 2r\pi = 0, \pm\frac{1}{2}$, or ± 1. Hence, $\sin 2r\pi = 1, \pm\frac{1}{2}$, or 0. If $\cos 2r\pi = 0$, then $\tan 2r\pi = \sin 2r\pi/\cos 2r\pi$ is not defined. In the other two cases, we have $\sin 2r\pi/\cos 2r\pi = \pm 1$ or 0.

4.43. (i) Give an example of two positive irrationals whose sum is rational.

* $\sqrt{2} + 1$ and $-\sqrt{2} + 2$.

(ii) Show that $\sqrt{2} + \sqrt{3}$ is a root of $x^4 - 10x^2 + 1$.

* $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$, $(\sqrt{2} + \sqrt{3})^4 = 49 + 20\sqrt{6}$, and so

$(\sqrt{2} + \sqrt{3})^4 - 10(\sqrt{2} + \sqrt{3})^2 + 1 = 49 + 20\sqrt{6} + 5 + 2\sqrt{6} + 1 = 0.$

(iii) Use Theorem B to show that $\sqrt{2} + \sqrt{3}$ is irrational.

* Were $\sqrt{2} + \sqrt{3}$ rational, then Theorem B would say it is either 1 or -1.

4.44. (i) Prove that $\log_5 6$ is irrational.

* If $\log_5 6 = r/s$, where r and s are positive integers, then $5^{r/s} = 6$ and $5^r = 6^s$. This cannot be, for 5^r is odd and 6^s is even.

(ii) Prove that $\log_6 15$ is irrational.

* If $\log_6 15 = r/s$, where r and s are positive integers, then $6^{r/s} = 15$ and $6^r = 15^s$. This cannot be, for 6^r is even and 15^s is odd.

Glossary of Boring Terms

G.1. (i) For any formula φ, prove that $\varphi \vee (\sim\varphi)$ is a tautology.

* $\varphi \vee \psi$ is true if at least one of φ or ψ is true.

(ii) For any formula φ, prove that $\varphi \wedge (\sim\varphi)$ is always false.

* $\varphi \wedge \psi$ is false if either φ or ψ is false.

G.2 Prove that $\sim(\sim\varphi)$ is logically equivalent to φ.

* If φ is true, then $\sim\varphi$ is false, and so $\sim(\sim\varphi)$ is true; if φ is false, then $\sim\varphi$ is true, and so $\sim(\sim\varphi)$ is false.

G.3. Prove that $\varphi \Leftrightarrow \psi$ and $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ are logically equivalent.

* The truth table for $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ is the same as that of $\varphi \Leftrightarrow \psi$.

G.4. Show that $\varphi \Rightarrow \psi$ is logically equivalent to $(\sim\varphi) \vee \psi$.

* The truth table for $(\sim\varphi) \vee \psi$ is the same as that of $\varphi \Rightarrow \psi$.

G.5. Show that $\sim(\varphi \Rightarrow \psi)$ is logically equivalent to $\varphi \wedge (\sim\psi)$.

* Both have the same truth table:

φ	T	T	F	F
ψ	T	F	T	F
	F	T	F	F

G.6. Show that $(\varphi \vee \psi) \wedge [\sim(\varphi \wedge \psi)]$ is logically equivalent to $[\varphi \wedge (\sim\psi)] \vee [\psi \wedge (\sim\varphi)]$.

* Both have the same truth table:

φ	T	T	F	F
ψ	T	F	T	F
	F	T	T	F

G.7. Show that $\varphi \Leftrightarrow \psi$ and $\sim[(\varphi \wedge (\sim\psi)) \vee (\psi \wedge (\sim\varphi))]$ are logically equivalent.

* Both have the same truth table:

φ	T	T	F	F
ψ	T	F	T	F
	T	F	F	T

G.8. Given statements φ and ψ, define $\varphi \downarrow \psi$ by the truth table

φ	T	T	F	F
ψ	T	F	T	F
$\varphi \downarrow \psi$	F	F	F	T

(i) Show that $(\varphi \downarrow \psi) \Leftrightarrow (\sim\varphi) \wedge (\sim\psi)$.

* Both have the same truth table.

(ii) Show that $\sim\varphi \Leftrightarrow \varphi \downarrow \varphi$.

* Both have the same truth table.

(iii) Show that $\varphi \wedge \psi \Leftrightarrow (\sim\varphi) \downarrow (\sim\psi)$.

* Both have the same truth table.

(iv) Show that $\varphi \vee \psi \Leftrightarrow \sim(\varphi \downarrow \psi)$.

* Both have the same truth table.

G.9. (*Associativity*). For any formulas φ, ψ, and ϑ, prove that

$\varphi \vee (\psi \vee \vartheta) \Leftrightarrow (\varphi \vee \psi) \vee \vartheta$ and $\varphi \wedge (\psi \wedge \vartheta) \Leftrightarrow (\varphi \wedge \psi) \wedge \vartheta$

are tautologies.

* Both have the same truth table.

φ	T	T	T	T	F	F	F	F
ψ	T	T	F	F	T	T	F	F
ϑ	T	F	T	F	T	F	T	F
	T	T	T	T	T	T	T	F

G.10. (*Distributivity*) Let φ, ψ, and ϑ be formulas.
(i) Prove that $\varphi \vee (\psi \wedge \vartheta) \Leftrightarrow (\varphi \vee \psi) \vee (\varphi \vee \vartheta)$ is a tautology.

* Both have the same truth table

φ	T	T	T	T	F	F	F	F
ψ	T	T	F	F	T	T	F	F
ϑ	T	F	T	F	T	F	T	F
	T	T	T	T	T	F	F	F

(ii) Prove that $\varphi \wedge (\psi \vee \vartheta) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)$ is a tautology.

* Both have the same truth table.

φ	T	T	T	T	F	F	F	F
ψ	T	T	F	F	T	T	F	F
ϑ	T	F	T	F	T	F	T	F
	T	T	T	F	F	F	F	F

G.11. (i) Define a sequence $r_1, r_2, \cdots, r_n, \cdots$, where r_n is the world record for the fastest mile ever run in or before year n. Show that this sequence converges.

* This sequence is decreasing, i.e., $r_n \geq r_{n+1}$ for all $n \geq 1$, and it is bounded below by 0.

(ii) Conclude that there is a time t so that runners can run a mile in a time arbitrarily close to t, yet no runner will ever run a mile even a millisecond faster than t.

* Exercise 3.30 (in the section on sequences) states the theorem that every decreasing sequence that is bounded below does converge to some some number t, and it proves that $r_n \geq t$ for all $n \geq 1$.

G.12. (i) Prove that $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

* If $a = a'$ and $b = b'$, the result is obvious. For the converse, assume that $\{a, \{a, b\}\} = \{a', \{a', b'\}\}$. There are two cases:

I. $a = a'$ and $\{a, b\} = \{a', b'\}$;

II. $a = \{a', b'\}$ and $\{a, b\} = a'$.

In case I, $a = a'$, so that $\{a, b\} = \{a', b'\} = \{a, b'\}$. Therefore,

$$\{a, b\} - \{a\} = \{a, b'\} - \{a\}.$$

If $a = b$, the left side is empty, hence the right side is also empty, and so $a = b'$; therefore, $b = b'$. If $a \neq b$, then the left side is $\{b\}$; the right side is also nonempty, so that it equals $\{b'\}$. Therefore, $b = b'$, as desired.

In case II, $a = \{a', b'\} = \{\{a, b\}, b'\}$. Hence,

$$a \in \{a, b\} \in \{\{a, b\}, b'\} = a,$$

contradicting the axiom that $a \in x \in a$ is always false. Therefore, case II cannot occur.

(ii) Give a formal proof that $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

* As it says in the text, "Once in your life, go into a secluded room and actually write a complete formal proof." I have already done this; it is now the reader's turn.

(iii) Prove that $[a, b] = [a', b']$ if and only if $a = a'$ and $b = b'$.

* If $a = a'$ and $b = b'$, the result is obvious. For the converse, assume that $\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$. There are two cases: $a = b$ and $a \neq b$.

If $a = b$, then $\{a, b\} = \{a\}$ and so $\{\{a\}, \{a, b\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$, the set whose only element is $\{a\}$. It follows that $\{\{a\}\} = \{\{a'\}, \{a', b'\}\}$ is a 1-element set, so that $\{a'\} = \{a', b'\}$; this, in turn, gives $a' = b'$ (for $\{a', b'\}$ is a 1-element set). Therefore, $b = a = a' = b'$, as desired.

If $a \neq b$, then $\{a\} \neq \{a, b\}$, and so $\{\{a\}, \{a, b\}\}$ is a 2-element set; therefore, $\{\{a'\}, \{a', b'\}\}$ is also a 2-element set, so that $\{a'\} \neq \{a', b'\}$, and so $a' \neq b'$. The given equation

$$\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$$

gives $\{a\} = \{a'\}$ and $\{a, b\} = \{a', b'\}$ (the other possibility $\{a\} = \{a', b'\}$ and $\{a, b\} = \{a'\}$ cannot occur because $\{a\}$ has exactly one element and $\{a', b'\}$ has exactly 2 elements). It follows that $a = a'$, and so $\{b\} = \{a, b\} - \{a\} = \{a, b'\} - \{a\} = \{b'\}$. Therefore, $b = b'$, as desired.

G.13. For any subset X of a set U, prove that $X \cup (\sim X) = U$ and $X \cap (\sim X) = \emptyset$.

* If $u \in U$, then either $u \in X$ or $u \notin X$; that is, $X \cup (\sim X) = U$. There is no $u \in U$ with $u \in X$ and $u \notin X$.

G.14. Define $X - Y = \{u \in X: u \notin Y\}$. If $\varphi = \varphi(u)$ and $\psi = \psi(u)$ are formulas, show that $V(\varphi \Rightarrow \psi) = \sim[V(\varphi) - V(\psi)]$.

* We use the observation in the text that if $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are true for every x, then $V(\alpha) = V(\beta)$. By Exercise G.4, $\varphi \Rightarrow \psi$ and $(\sim\varphi) \vee \psi$ are logically equivalent; hence, $V(\varphi \Rightarrow \psi) = V((\sim\varphi) \vee \psi) = V((\sim\varphi) \cup V(\psi) = \sim V(\varphi) \cup V(\psi)$. By a De Morgan law, $\sim V(\varphi) \cup V(\psi) = \sim[V(\varphi) \cap \sim V(\psi)] = \sim[V(\varphi) - V(\psi)]$.

G.15. Show that $V(\varphi \Leftrightarrow \psi) = \sim[V(\varphi) \,\Delta\, V(\psi)]$.

* We use the observation in the text that if $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are true for every x, then $V(\alpha) = V(\beta)$. By Exercise G.7, $\varphi \Leftrightarrow \psi$ and $\sim[(\varphi \wedge (\sim\psi)) \vee (\psi \wedge (\sim\varphi))]$ are logically equivalent. It follows that $V(\varphi \Leftrightarrow \psi) = \sim V([\varphi \wedge (\sim\psi)] \vee [\psi \wedge (\sim\varphi)]) = \sim([V(\varphi) \cap V(\sim\psi)] \cup [V(\psi) \cap V(\sim\varphi)]) = \sim[V(\varphi) \,\Delta\, (\sim V(\psi))]$.

G.16. Show that $V(\varphi \downarrow \psi) = \sim V(\varphi) \cap \sim V(\psi)$, where $\varphi = \varphi(x)$ and $\psi = \psi(x)$.

* Since $\varphi \downarrow \psi$ is defined as $(\sim\varphi) \wedge (\sim\psi)$, it follows that $V(\varphi \downarrow \psi) = V((\sim\varphi) \wedge (\sim\psi)) = V(\sim\varphi) \cap V(\sim\psi) = \sim V(\varphi) \cap \sim V(\psi)$.

G.17. (*Associativity*). For any subsets X, Y, and Z of a set U, prove that $X \cup (Y \cup Z) = (X \cup Y) \cup Z$ and $X \cap (Y \cap Z) = (X \cap Y) \cap Z$.

* The proof involves showing that each side is contained in the other. It is not very enlightening, for it hinges on the definition of the connective "or" given in the text. For example, if $u \in X \cup (Y \cup Z)$, then $u \in X$ or $u \in Y \cup Z$; that is, $u \in X$ or $u \in Y$ or $u \in Z$. Hence, $u \in X \in Y$ or $u \in Z$, and so $u \in (X \cup Y) \cup Z$. The reverse inclusion is proved similarly.

The proof for $\cap$ is proved in the same way, hinging on the definition of the connective "and".

Remark. If one knew that every subset of U is of the form $V(\varphi)$ for some formula $\varphi = \varphi(n)$, then one could use Exercise G.9 together with the observation in the text that if $\alpha = \alpha(n)$ and $\beta = \beta(n)$ are true for every n, then $V(\alpha) = V(\beta)$. This is, in fact, true, but readers may not be able to prove it (it involves describing subsets of U by 2-valued sequences).

G.18. (*Distributivity*) For any subsets X, Y, and Z of a set U, prove that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ and $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

* This proof is similar to that of Exercise G.17, showing that each side is contained in the other; again, it is just a question of looking at the definitions of the connectives.
Remark. If one knew that every subset of U is of the form $V(\varphi)$ for some formula $\varphi = \varphi(n)$, then one could use Exercise G.10 together with the observation in the text that if $\alpha = \alpha(n)$ and $\beta = \beta(n)$ are true for every n, then $V(\alpha) = V(\beta)$. This is, in fact, true, but readers may not be able to prove it (it involves describing subsets of U be 2-valued sequences).

G.19. Prove that the sequence $\{(-1)^n\}$ does not converge to 1.

* As in the text, one must show

$$(\exists \varepsilon > 0)(\forall N)(\exists n \geq N)[|(-1)^n - 1| \geq \varepsilon].$$

Choose $\varepsilon = 2$ (the distance between -1 and 1; of course, any smaller positive choice of ε works as well). For any choice of integer N, choose an odd number $n \geq N$. With these choices, $|(-1)^n - 1| = |-1 - 1| = 2 \geq \varepsilon$, as desired.

G.20. (i) If $f_n(x) = x^n$, prove that $f_n(x)$ converges pointwise to $L(x)$ on $[0, 1]$, where $L(x) = 0$ if $0 \leq x < 1$ and $L(1) = 1$.

* By Theorem 3.20, $x^n \to 0$ whenever $0 \leq x < 1$, while the sequence x^n is just the constant sequence $a_n = 1$ when $x = 1$, and hence it converges to 1. Therefore, $f_n(x)$ converges pointwise to $L(x)$ on $[0, 1]$.

(ii) Write the definition of pointwise convergence with quantifiers, and then write its negation.

* With quantifiers, the definition of pointwise convergence is:

$$(\forall \varepsilon > 0)(\forall c \in [a, b])(\exists N)(\forall n \geq N)[|f_n(c) - L(c)| < \varepsilon].$$

The negation of pointwise convergence is:

$$(\exists \varepsilon > 0)(\exists c \in [a, b])(\forall N)(\exists n \geq N)[|f_n(c) - L(c)| \geq \varepsilon].$$

G.21. (i) Write the definition of uniform convergence with quantifiers.

* $$(\forall \varepsilon > 0)(\exists N)(\forall c \in [a, b])(\forall n \geq N)[|f_n(c) - L(c)| < \varepsilon].$$

Comparing the definitions of pointwise and uniform convergence, one sees that the middle two quantifiers are reversed (the text remarks earlier that the order of the quantifiers is important).

(ii) Show that if $f_n(x)$ converges uniformly to $L(x)$ on $[a, b]$, then $f_n(x)$ converges pointwise to $L(x)$ on $[a, b]$.

* The second basic principle about quantifiers given in the text is $(\exists N)(\forall c \in [a, b])$ implies $(\forall c \in [a, b])(\exists N)$, and this applies here.

(ii) Write the negation of uniform convergence.

* $$(\exists \varepsilon > 0)(\forall N)(\exists c \in [a, b])(\exists n \geq N)[|f_n(c) - L(c)| \geq \varepsilon].$$

(iii) If $f_n(x) = x^n$, prove that $f_n(x)$ does not converge uniformly to $L(x)$ on $[0, 1]$, where $L(x) = 0$ if $0 \leq x < 1$ and $L(1) = 1$.

* Choose $\varepsilon = ⅓$; given N, choose $c = (½)^{1/N}$ and $n = N$. Then

$|f_n(c) - L(c)| = |c^n - 0| = c^n = \frac{1}{2} > \frac{1}{3} = \varepsilon$, as desired. Therefore, this sequence does not converge uniformly to $L(x)$.

G.22. Prove that the empty set $\varnothing$ is a subset of every set X.

*The negation of

$$(\forall x \in \varnothing)[x \in \varnothing \Rightarrow x \in X]$$

is

$$(\exists x \in \varnothing)[(x \in \varnothing) \vee (x \notin X)].$$

But $(\exists x \in \varnothing)\varphi(x)$ is false for any formula $\varphi(x)$, because there is no $x \in \varnothing$. Hence, the negation of $(\forall x \in \varnothing)[x \in \varnothing \Rightarrow x \in X]$ is false, and the original formula is true.

(ii) Prove that there is only one empty set.

* Suppose ϑ is a second empty set; that is, ϑ also has no elements. As in part (i), $\vartheta \subset X$ for every set X; in particular, $\vartheta \subset \varnothing$; similarly, $\varnothing \subset \vartheta$, and so $\vartheta = \varnothing$.