Solutions Manual For

Basic Probability Theory

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SOLUTIONS TO PROBLEMS

Chapter 1

Section 1.2

2. $D_1 = AB + AC + BC$

$D_2 = ABC^c + AB^cC + A^cBC$

$D_3 = A + B + C$

$D_4 = AB^cC^c + A^cBC^c + A^cB^cC$

$D_5 = (ABC)^c = A^c + B^c + C^c$

where $AB = A \cap B$, $A + B = A \cup B$

4. (a) $x \in A \cap (B - C)$ iff $x \in A$ and $x \in B - C$

iff $x \in A \cap B$ and $x \notin A \cap C$

iff $x \in (A \cap B) - (A \cap C)$

(b) $x \in A - (B \cup C)$ iff $x \in A$ and $x \notin B \cup C$

iff $x \in A$ and $x \notin B$ and $x \notin C$

iff $x \in A - B$ and $x \notin C$

iff $x \in (A - B) - C$

It is true that $(A \cup C) - B \subset (A - B) \cup C$. For if $x \in (A \cup C) - B$ and $x \notin C$ then $x \in A - B$. But the sets need not be equal. For example, if $A = B = C$ then $(A \cup C) - B = A - A = \phi$, and $(A - B) \cup C = \phi \cup A = A$.

6. $A^c \cap B^c = (A \cup B)^c$, which will not be empty unless $A \cup B = \Omega$. Thus A^c and B^c will be disjoint iff $A \cup B = \Omega$. $(A \cap C) \cap (B \cap C) \subset A \cap B = \phi$, hence $A \cap C$ and $B \cap C$ are disjoint. $C \subset (A \cup C) \cap (B \cup C)$, so $A \cup C$ and $B \cup C$ are not disjoint if $C \neq \phi$.

. $\bigcap_{i=1}^{n} A_i \subset A_n$ always. If $x \in A_n$ then $x \in \bigcap_{i=1}^{n} A_i$ since $A_n \subset A_{n-1} \subset \ldots \subset A_1$; hence $\bigcap_{i=1}^{n} A_i = A_n$. $A_i \subset \bigcup_{i=1}^{n} A_i$ always. If $x \in$ some A_i then $x \in A_1$ since $A_n \subset A_{n-1} \subset \ldots \subset A_1$; hence $\bigcup_{i=1}^{n} A_i = A_1$.

. No. For example, let $A_n = (0, \frac{1}{n})$. Note also that $\sum_{i=1}^{n} \frac{1}{2^i} < 1$ for all n, but $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$.

ection 1.3

. $P(A) = P(A-B) + P(A \cap B)$, so any example in which $P(A \cap B) < P(B)$ will do (e.g., let $A = B^c$).

ection 1.4

The probability that the first digit will be > 5, but the second and third will be ≤ 5, is $(4)(6)(6)/10^3 = .144$. Thus the desired probability is $3(.144) = .432$.

The number of outcomes is (24)(18), and the number of favorable cases is 3(5) + 8(7) + 13(6), thus p = 149/432.

(a) The first card may be chosen in 52 ways, the second in 48 since the first face value cannot be repeated, and the third in 44, etc. Thus $p = (52)(48)\ldots(20)(16)/52^{10}$.

(b) The probability that exactly 9 cards will be of the same suit is $4\binom{13}{9}39/\binom{52}{10}$. (First select the suit, and 9 of 13 face values, then the odd card.) Similarly the probability that all 10 cards will be of the same suit is $4\binom{13}{10}/\binom{52}{10}$. Thus $p = (4\binom{13}{9}39 + 4\binom{13}{10})/\binom{52}{10}$.

The total number of positions available to the women is $\binom{m+w}{w}$. The adjacent positions for the women may be selected in $m+w - w + 1 = m+1$ ways. Thus $p = (m+1)/\binom{m+w}{w}$.

7. The probability that at least one is defective is 1 - the probability that none is defective, so $1-p = \binom{75}{15}/\binom{100}{15}$.

8. This is an application of the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, A = {exactly 3 kings}, B = {exactly 3 aces}. Thus $P(A \cup B) = \binom{52}{8}^{-1}(\binom{4}{3}\binom{48}{5} + \binom{4}{3}\binom{48}{5} - \binom{4}{3}\binom{4}{3}\binom{44}{2})$.

10. (a) A sentence of length k must start with a word of length 1 or 2; there is only one possible word of length 1, but there are 2 possible words of length 2. If the first word is of length j, the remainder of the sentence may be completed in N(k-j) ways; the result follows.

(b) Assume $N(k) = \lambda^k$; this will be a solution provided $\lambda^k = \lambda^{k-1} + 2\lambda^{k-2}$, i.e. $\lambda^2-\lambda-2 = 0$, or $\lambda = 2$ or $\lambda = -1$. Thus $A2^k + B(-1)^k$ is a solution. Also N(0) = A+B, N(1) = 2A-B, so A and B are determined by N(0) and N(1). Since N(0) and N(1) determine N(k) for all k, any two solutions that agree when k = 0 and 1 agree everywhere, so that $A2^k + B(-1)^k$ is the general solution. In the present case, A+B = 1, 2A-B = 1, so A = 2/3, B = 1/3.

11. The total number of outcomes is 365^r; the number of favorable cases is $(365)(364)\ldots(365-r+1) = (365)_r$. Thus $p = (365)_r/365^r$.

13. (a) Let A be a subset of $\Omega = \{1,2,\ldots,n\}$. Either $1 \in A$ or $1 \notin A$; this gives us two possibilities. In general, either $k \in A$ or $k \notin A$, $k = 1,2,\ldots,n$. This gives us $2(2)\ldots(2) = 2^n$ ways of choosing A. Alternately, the number of subsets with exactly k members is the number of ways of selecting k distinct integers out of n, namely $\binom{n}{k}$. The total number of subsets is $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$.

(b) The number of ways of selecting subsets A with exactly k members is $\binom{n}{k}$. Having chosen such an A, we have B = A + a subset of A^c. Since there are 2^{n-k} subsets of A^c, B may be chosen in 2^{n-k} ways. The number of pairs of subsets is $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = (1+2)^n = 3^n$.

(a) Let $\Omega = \{1,2,\dots,n\}$. The integer 1 belongs to a set A_1 of the partition, where A_1 contains j other elements ($j = 0,1,\dots,n-1$). Thus A_1 can be chosen in $\binom{n-1}{j}$ ways. Having chosen A_1, we must partition A_1^c; this can be done in $g(n-1-j)$ ways. Thus

$$g(n) = \sum_{j=0}^{n-1} \binom{n-1}{j} g(n-1-j) = \sum_{j=0}^{n-1} \binom{n-1}{n-1-j} g(n-1-j)$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} g(k).$$

(b) Let $h(n) = e^{-1} \sum_{k=0}^{\infty} k^n/k!$. Then

$$\sum_{k=0}^{n-1} \binom{n-1}{k} h(k) = \sum_{k=0}^{n-1} \binom{n-1}{k} e^{-1} \sum_{j=0}^{\infty} \frac{j^k}{j!}$$

$$= e^{-1} \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{k=0}^{n-1} \binom{n-1}{k} j^k \right]$$

$$= e^{-1} \sum_{j=0}^{\infty} \frac{1}{j!} (1+j)^{n-1}$$

$$= e^{-1} \sum_{j=0}^{\infty} \frac{(1+j)^n}{(j+1)!} = h(n)$$

since $\frac{0^n}{0!} = 0$.

Now $h(0) = e^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} = e^{-1} e^{1} = 1$.

Thus g and h satisfy the difference equation of (a), and they agree when $n = 0$. By the form of the difference equation, they agree everywhere.

ion 1.5

"If" is immediate (set $B_{i_r} = A_{i_r}$). For the "only if" part, it suffices to show that if $A_1,\dots,A_n$ are independent, then $P(A_1^c \cap A_2 \cap \dots \cap A_n) = P(A_1^c)P(A_2)\dots P(A_n)$; the result then

1. (continued)

follows by an induction argument. But

$$P(A_1^c \cap A_2 \cap \dots \cap A_n) = P(A_2 \cap \dots \cap A_n) - P(A_1 \cap A_2 \cap \dots \cap A_n)$$

$$= (1-P(A_1))P(A_2)\dots P(A_n) \text{ by independence}$$

$$= P(A_1^c)P(A_2)\dots P(A_n).$$

2. $$\frac{p(k+1)}{p(k)} = \frac{\binom{n}{k+1} p^{k+1} q^{n-k-1}}{\binom{n}{k} p^k q^{n-k}} = \frac{n-k}{k+1}\frac{p}{q}$$

Thus $p(k+1)/p(k)$ is > 1 iff $(n-k)p > (k+1)q$, i.e. iff $k < (n+1)p - 1$

< 1 iff $k > (n+1)p - 1$

$= 1$ iff $k = (n+1)p - 1$. The result follows.

3. (a) Let $A = \{\text{spade is obtained}\}$, $B = \{\text{heart is obtained}\}$; $P(A \cap B) = 0 \neq P(A)P(B)$.

(b) Let $A = \{\text{spade is obtained}\}$, $B = \{\text{ace is obtained}\}$. $P(A \cap B) = 1/52$, $P(A) = 1/4$, $P(B) = 1/13$.

(c) If A and B are independent and mutually exclusive, then either A or B must have probability zero. For $P(A \cap B) = 0$ by disjointness, and $= P(A)P(B)$ by independence. Similarly, if the events A_i, $i \in I$, are independent and disjoint, either all or all but one of the events must have probability zero. (If $P(A_i) \neq 0$, apply the above argument to each A_j, $j \neq i$, to conclude that $P(A_j) = 0$ for all $j \neq i$.)

(d) Let $A = \{\text{spade}\}$, $B = \{\text{spade or heart}\}$; $P(A \cap B) = P(A) = 1/4$, $P(A)P(B) = 1/8$.

6. There are as many terms in (1.5.2) as there are unordered samples of size n out of k, with replacement, i.e. $\binom{k+n-1}{n}$ (see 1.4.4).

7. (a) For a favorable outcome, we must select n_i of the available t_i balls for color C_i, $i = 1,2,\ldots,k$. The total number of outcomes is the number of ways of selecting n distinct objects from a set of t; the result follows.

(b) This is a standard multinomial problem. The probability is

$$\frac{n!}{n_1!n_2!\cdots n_k!} p_1^{n_1}\cdots p_k^{n_k} \text{ where } p_i = t_i/t.$$

8. (a) $P(A \cap A) = P(A)P(A)$, hence $P(A) = (P(A))^2$, so that $P(A) = 0$ or 1.

(b) If $P(A) = 0$, then since $A \cap B$ is a subset of A, $P(A \cap B) = 0$ also. Thus $P(A \cap B) = P(A)P(B)$. If $P(A) = 1$, then $P(A^c) = 0$, hence by the above argument, A^c and B are independent. But then A and B are independent (see Remark 1 or Problem 1 of Section 1.5).

Section 1.6

1. Let X be the number of successes. Then

$$P\{\text{all successes occur consecutively} \mid 4 \le X \le 6\}$$

$$= \frac{\sum_{k=4}^{6} P\{X=k \text{ and all successes occur consecutively}\}}{\sum_{k=4}^{6} P\{X=k\}}$$

$$= (7p^4q^6 + 6p^5q^5 + 5p^6q^4)/\sum_{k=4}^{6}\binom{10}{k}p^kq^{10-k}.$$

2. $P\{X \ge 3 \mid X \ge 1\} = P\{X \ge 3, X \ge 1\}/P\{X \ge 1\} = P\{X \ge 3\}/P\{X \ge 1\}$

$$= \frac{1 - P\{X = 0\} - P\{X = 1\} - P\{X = 2\}}{1 - P\{X = 0\}}$$

$$= \frac{1 - q^n - npq^{n-1} - \binom{n}{2}p^2q^{n-2}}{1 - q^n}.$$

5. We may regard this problem as one of dealing two 13 card hands to players 1 and 2 from a deck with 26 cards, of which 6 are spades. In each case, we are looking for the probability that (say) player 1 received a particular number of spades. Once the number of spades for player 1 is determined, that of player 2 is determined also. Thus,

(a) $\binom{6}{3}\binom{20}{10}/\binom{26}{13} = .36$

(b) $(\binom{6}{2}\binom{20}{11} + \binom{6}{4}\binom{20}{9})/\binom{26}{13} = 2\binom{6}{2}\binom{20}{11}/\binom{26}{13} = .48$

(c) $2\binom{6}{1}\binom{20}{12}/\binom{26}{13} = .15$

(d) $2\binom{20}{13}/\binom{26}{13} = .01$.

6. Let A = {first two balls white}, B = {six white balls in the sample}. If the sampling is done with replacement, then

$$P(A|B) = P(A \cap B)/P(B) = \frac{(2/3)^2\binom{8}{4}(2/3)^4(1/3)^4}{\binom{10}{6}(2/3)^6(1/3)^4}.$$

If the sampling is done without replacement, $P(A|B)$ is the number of ways of selecting 4 positions out of 8 for the white balls (the first 2 positions must be occupied by white balls), divided by the number of ways of selecting 6 positions out of 10; i.e. $\binom{8}{4}/\binom{10}{6} = 1/3$. Note that the answer is the same with replacement as without replacement. Once it is specified that 6 white and 4 black balls are obtained, the problem is simply one of counting arrangements.

8. (a) The probability is $P(AB + CD + AED + CEB)$ where A is the event that the switch labeled 'A' is closed, etc, and + stands for union, product for intersection. Using the expansion formula (1.4.5) for the union of n events, we obtain (writing ab for P(AB), etc.)

$$ab + cd + aed + ceb - abcd - abed - abce - cdea - cdeb$$
$$- abcde + 4abcde - abcde = 2p^2 + 2p^3 - 5p^4 + 2p^5.$$

8. (continued)

(b) $P\{E \text{ open and signal received}\} = P\{E^c(AB + CD)\}$

$$= P\{ABE^c\} + P\{CDE^c\} - P\{ABCDE^c\} = 2p^2q - p^4q,\ q = 1-p.$$

Thus

$$P\{E \text{ open} \mid \text{signal received}\} = \frac{(2p^2-p^4)q}{2p^2 + 2p^3 - 5p^4 + 2p^5}\ .$$

Chapter 2

<u>Section 2.2</u>

2. $\{\omega: a \le R(\omega) < b\} = \{\omega: R(\omega) < b\} - \{\omega: R(\omega) < a\} \in \mathcal{F}$, hence

$\{\omega: a \le R(\omega) \le b\} = \bigcap_{n=1}^{\infty} \{\omega: a \le R(\omega) < b + \frac{1}{n}\} \in \mathcal{F}$ for all real a,b

3. $\{\omega: R_1(\omega) + R_2(\omega) < b\} = \bigcup_{\substack{r,s \text{ rational} \\ r+s < b}} \{\omega: R_1(\omega) < r,\ R_2(\omega) < s\} \in \mathcal{F}$

hence $R_1 + R_2$ is a random variable.

$\{\omega: aR(\omega) < b\} = \{\omega: R(\omega) < \frac{b}{a}\}$ if $a > 0$

$= \{\omega: R(\omega) > \frac{b}{a}\}$ if $a < 0$

$= \phi$ or Ω if $a = 0$.

In any case, $\{\omega: aR(\omega) < b\} \in \mathcal{F}$, so aR is a random variable.

$\{\omega: \sqrt[3]{R(\omega)} < b\} = \{\omega: R(\omega) < b^3\}$, hence $\sqrt[3]{R}$ is a random variable.

<u>Section 2.4</u>

2. $f_2(y) = f_1(-\ln y)\left|\frac{d}{dy}(-\ln y)\right|$

$= \frac{1}{2y}\ ,\ e^{-1} < y < e$

$= 0$ elsewhere.

3. $f_2(y) = f_1(\frac{1}{2}y)\left|\frac{d}{dy}\frac{1}{2}y\right| = 2y^{-2},\ 2 < y < 4$

$= f_1(\sqrt{y})\left|\frac{d}{dy}\sqrt{y}\right| = \frac{1}{2}y^{-3/2},\ y > 4$

$= 0,\ y < 2.$

5. (a) is a special case of (b). To prove (b), let $0 < y < 1$ and pick an x such that $F_1(x) = y$. Then $P\{R_2 \le y\} = P\{R_1 \le x\} = F_1(x) = y$, and the result follows.

We show that $F(x) = \int_{-\infty}^{x} f(t)$ for all x. Pick any x, and let $x_1,\dots,x_n$ be the points of discontinuity of f (or points where F' does not exist) which lie in the interval $(-\infty,x]$. Then

$$\int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x_1} f(t)dt + \int_{x_1}^{x_2} f(t)dt + \dots + \int_{x_{n-1}}^{x_n} f(t)dt + \int_{x_n}^{x} f(t)dt.$$

Now if $x_{i-1} < a < b < x_i$, f is continuous on $[a,b]$ and $f = F'$ on $[a,b]$, so by the fundamental theorem of calculus, $\int_a^b f(t)dt = F(b) - F(a)$. Let $b \to x_i$, $a \to x_{i-1}$. Since F is continuous everywhere, $F(b) - F(a) \to F(x_i) - F(x_{i-1})$. Thus

$$\int_{x_{i-1}}^{x_i} f(t)dt = F(x_i) - F(x_{i-1}) \text{ for all } i. \text{ Similarly,}$$

$$\int_{-\infty}^{x_1} f(t)dt = F(x_1) - \lim_{x \to -\infty} F(x) = F(x_1), \int_{x_n}^{x} f(t)dt = F(x)-F(x_n).$$

Thus

$$\int_{-\infty}^{x} f(t)dt = F(x_1) + F(x_2) - F(x_1) + \dots + F(x_n) - F(x_{n-1}) + F(x) - F(x_n) = F(x).$$

(a) $R_2 = k$ iff $R_1 = .ik\dots$ for some $i = 0,1,\dots,9$

iff $10R_1 = i + k10^{-1} + \dots$ for some $i = 0,1,\dots,9$

iff $i + k10^{-1} \le 10R_1 < i + (k+1)10^{-1}$ for some $i = 0,1,\dots,9$.

(b) In this case $f_1(y) = f(y^2)\left|\frac{d}{dy} y^2\right|$ where f is the uniform density on $[0,1]$; thus $f_1(y) = 2y$. Therefore

$$P\{R_2=k\} = \sum_{i=0}^{9} [(10^{-1}i + 10^{-2}k + 10^{-2})^2 - (10^{-1}i + 10^{-2}k)^2]$$

$$= \sum_{i=0}^{9} [2(10^{-1}i + 10^{-2}k)10^{-2} + 10^{-4}]$$

$$= 10^{-4} \sum_{i=0}^{9} (20i + 2k + 1)$$

$$= 10^{-4} [\frac{20(10)(9)}{2} + 10(2k+1)] = .091 + .002k.$$

9. The equations of motion are $x = (v_o\cos\theta)t$, $y = (v_o\sin\theta)t - \frac{1}{2}gt^2$, g = acceleration of gravity. The projectile returns to earth when $y = 0$, i.e. at time $t_o = (2v_o\sin\theta)/g$. Thus $R = (v_o\cos\theta)t_o = (v_o^2\sin 2\theta)/g$. Since 2θ is uniformly distributed between 0 and π, we obtain, as in Example 2 of Section 2.4,

$$f_R(y) = \frac{2g}{\pi v_o^2}\left[1 - \left(\frac{gy}{v_o^2}\right)^2\right]^{-1/2}, \quad 0 < y < v_o^2/g.$$

Section 2.5

2. $P\{a \le R \le b\} = P\{R \le b\} - P\{R < a\} = F(b) - F(a^-)$

$P\{a \le R < b\} = P\{R < b\} - P\{R < a\} = F(b^-) - F(a^-)$

$P\{a < R < b\} = P\{R < b\} - P\{R \le a\} = F(b^-) - F(a)$.

Section 2.6

1. $P\{a_1 < R_1 \le b_1,\ a_2 < R_2 \le b_2\} = P\{a_1 < R_1 \le b_1,\ R_2 \le b_2\}$

$- P\{a_1 < R_1 \le b_1,\ R_2 \le a_2\} = P\{R_1 \le b_1,\ R_2 \le b_2\}$

$- P\{R_1 \le a_1,\ R_2 \le b_2\} - P\{R_1 \le b_1,\ R_2 \le a_2\}$

$+ P\{R_1 \le a_1,\ R_2 \le a_2\} = F_{12}(b_1,b_2) - F_{12}(a_1,b_2) - F_{12}(b_1,a_2)$

$+ F_{12}(a_1,a_2)$.

Since $F_{12}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{12}(u,v)dudv$, the result follows.

2. By an analysis similar to Problem 1, the desired probability is $F(b_1,b_2,b_3) - F(a_1,b_2,b_3) - F(b_1,a_2,b_3) - F(b_1,b_2,a_3) + F(a_1,a_2,b_3) + F(a_1,b_2,a_3) + F(b_1,a_2,a_3) - F(a_1,a_2,a_3)$. In n dimensions,

$$P\{a_1 < R_1 \le b_1,\ldots,a_n < R_n \le b_n\} = \Delta_{b_1;a_1}\cdots\Delta_{b_n;a_n} F(x_1,\ldots,x_n)$$

where Δ is the difference operator:

$$\Delta_{b_n;a_n} F(x_1,\ldots,x_n) = F(x_1,\ldots,x_{n-1},b_n) - F(x_1,\ldots,x_{n-1},a_n).$$

This may be expressed as $F_0 - F_1 + F_2 - F_3 + \ldots + (-1)^n F_n$, where F_i is the sum of all $\binom{n}{i}$ terms of the form $F(c_1,\ldots,c_n)$, such that $c_k = a_k$ for exactly i integers $\in \{1,2,\ldots,n\}$, and $c_k = b_k$ for the remaining n-i integers.

3. By Problem 1, $P\{-1 < R_1 \le 0,\ 0 < R_2 \le 1\} = F(0,1) - F(-1,1) - F(0,0) + F(-1,0) = 1-1-1+0 = -1 < 0$, a contradiction.

Section 2.7

4. $F_{12\ldots n}(x_1,\ldots,x_n) = P\{R_1 \le x_1,\ldots,R_n \le x_n\} =$

$$\prod_{i=1}^{n} P\{R_i \le x_i\} = \prod_{i=1}^{n} F_i(x_i).$$

5. If R is degenerate at c, and R_1 is an arbitrary random variable, then R and R_1 are independent, since $P\{R \in B, R_1 \in B_1\} = P\{R_1 \in B_1\}$ if $c \in B$, and $= 0$ if $c \notin B$. In particular R and R are independent. Conversely, let R and R be independent. Then $P\{R \le x\} = P\{R \le x, R \le x\} = P\{R \le x\}P\{R \le x\}$, i.e. $F_R(x) = [F_R(x)]^2$ for all x, hence $F_R(x) = 0$ or 1 for all x. If c is the smallest x such that $F_R(x) = 1$ then $F_R(x) = 1$, $x \ge c$; $F_R(x) = 0$, $x < c$. Thus $P\{R=c\} = 1$.

6. Let $g_1(x) = \sin x$, $g_2(x) = x$. If R and sin R are independent, so are $g_1(R)$ and $g_2(\sin R)$, i.e. sin R and sin R are independent, hence by Problem 5, sin R is degenerate. Conversely if sin R is degenerate, R and sin R are independent by the remarks in Problem 5.

7. $P\{R_1 \in B_1,\ldots,R_n \in B_n\} =$

$$\int_{x_1 \in B_1,\ldots,x_n \in B_n} \cdots \int f_{12\ldots n}(x_1,\ldots,x_n)\,dx_1\ldots dx_n$$

$$= \int_{B_1} f_1(x_1)dx_1 \cdots \int_{B_n} f_n(x_n)dx_n = P\{R_1 \in B_1\}\ldots P\{R_n \in B_n\}.$$

Section 2.8

5. The core is described by $x^2 + y^2 \le a^2$, $x^2 + y^2 + z^2 \le 4a^2$. The volume of the core is, in cylindrical coordinates,

$$2\int_0^{2\pi} d\theta \int_0^a r\,dr \int_0^{(4a^2-r^2)^{1/2}} dz = 2\int_0^{2\pi} d\theta \int_0^a r(4a^2-r^2)^{1/2}dr$$

$$= 4\pi\left[-\frac{1}{3}(4a^2-r^2)^{3/2}\right]_0^a = 4\pi\left(\frac{8}{3} - \sqrt{3}\right)a^3.$$

The probability that the worm will not be eaten is

$$\frac{4\pi\left(\frac{8}{3} - \sqrt{3}\right)a^3}{\text{volume of sphere of radius } 2a} = \frac{4\pi\left(\frac{8}{3} - \sqrt{3}\right)a^3}{\frac{4}{3}\pi(2a)^3} = 1 - \frac{3}{8}\sqrt{3}.$$

Thus the probability that it will be eaten is $\frac{3}{8}\sqrt{3}$.

6. The volume of the region is $\displaystyle\iint_{\substack{x^2+y^2 \le 4,\\ x \ge 0}} \int_{z=0}^{3x} dz\,dy\,dx$

$$= \iint_{\substack{x^2+y^2 \le 4,\\ x \ge 0}} 3x\,dx\,dy = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{2} (3r\cos\theta)\,r\,dr\,d\theta = 16.$$

6. (continued)

The desired probability is $\frac{1}{16} \iint\limits_{\substack{x^2+y^2 \le 4,\\ x \ge 0}} \int_{z=0}^{2x} dz dy dx =$

$$\frac{1}{16} \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2} (2r \cos\theta) r \, dr d\theta = \frac{2}{3},$$

as would be expected intuitively since each vertical line from $z = 0$ to $z = 3x$ has 2/3 of its length below the line $z = 2x$.

7. $$1 = \iint\limits_{\substack{x^2+y^2 \le 4,\\ x \ge 0}} \int_{z=0}^{3x} kz^2 dz dy dx = k \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2} 9r^3\cos^3\theta (r \, dr d\theta)$$

$$P\{R_3 \le 2R_1\} = \iint\limits_{\substack{x^2+y^2 \le 4,\\ x \ge 0}} \int_{z=0}^{2x} kz^2 dz dy dx =$$

$$\frac{\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2} \frac{8}{3} r^3 \cos^3\theta \, (r \, dr d\theta)}{\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2} 9r^3\cos^3\theta \, (r \, dr d\theta)} = \frac{8}{27}.$$

8. $P\{T_1 \le b_1, \ldots, T_n \le b_n\} = n! \, P\{T_1 \le b_1, \ldots, T_n \le b_n,$

$R_1 < R_2 < \ldots < R_n\} = n! \, P\{R_1 \le b_1, R_1 < R_2 \le b_2,$

$R_2 < R_3 \le b_3, \ldots, R_{n-1} < R_n \le b_n\}$

$$= n! \int_{-\infty}^{b_1} f(x_1) dx_1 \int_{x_1}^{b_2} f(x_2) dx_2 \cdots \int_{x_{n-1}}^{b_n} f(x_n) dx_n =$$

$$\int_{-\infty}^{b_1} \cdots \int_{-\infty}^{b_n} g(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

8. (continued)

where $g(x_1, \ldots, x_n) = n! \, f(x_1) f(x_2) \ldots f(x_n), \; x_1 < x_2 < \ldots < x_n$

$= 0$ elsewhere.

9. $$P\{R_1 \ge 2R_2 \ge 3R_3\} = \int_{z=0}^{\infty} \int_{y=3z/2}^{\infty} \int_{x=2y}^{\infty} e^{-(x+y+z)} dx dy dz$$

$$= \int_{z=0}^{\infty} e^{-z} \int_{y=3z/2}^{\infty} e^{-3y} dy dz = \int_0^{\infty} \frac{1}{3} e^{-\frac{11}{2}z} dz = \frac{2}{33}.$$

11. $\min\limits_{i \ne j} |x_i - x_j| \ge d, \; x_1 < x_2 < \ldots < x_n$ is equivalent to

$x_{n-1} + d \le x_n \le L, \; x_{n-2} + d \le x_{n-1} \le x_n - d, \ldots,$

$$x_1 + d \le x_2 \le x_3 - d, \; 0 \le x_1 \le x_2 - d.$$

But this is in turn equivalent to $x_{n-1} + d \le x_n \le L$,

$x_{n-2} + d \le x_{n-1} \le L-d, \; x_{n-3} + d \le x_{n-2} \le L-2d, \ldots,$

$x_1 + d \le x_2 \le L-(n-2)d, \; 0 \le x_1 \le L-(n-1)d.$

Hence $P\{\min\limits_{i \ne j} |R_i - R_j| \ge d, \; R_1 < R_2 < \ldots < R_n\} =$

$$\frac{1}{L^n} \int_0^{L-(n-1)d} dx_1 \int_{x_1+d}^{L-(n-2)d} dx_2 \cdots \int_{x_{n-2}+d}^{L-d} dx_{n-1} \int_{x_{n-1}+d}^{L} dx_n$$

$$= \frac{1}{n! L^n} [L-(n-1)d]^n.$$

Thus $P\{\min\limits_{i \ne j} |R_i - R_j| \ge d\} = [\frac{L-(n-1)d}{L}]^n$ if $(n-1)d \le L$

$= 0$ if $(n-1)d > L$.

12. $P\{\underline{W} \in B\} = P\{\underline{R} \in g^{-1}(B)\} = \int \cdots \int_{g^{-1}(B)} f(\underline{x})d\underline{x}$

Let $\underline{y} = g(\underline{x}), \underline{x} = h(\underline{y})$ to obtain

$$\int \cdots \int_B f(h(\underline{y}))|J_h(\underline{y})|d\underline{y},$$ and the result follows.

13. $f_{12}(x,y) = \frac{1}{2\pi b^2} e^{-(x^2+y^2)/2b^2}$

$x = r\cos\theta$, $y = r\sin\theta$, so $J_h(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Thus $f^*_{12}(r,\theta) = \frac{1}{2\pi b^2} re^{-r^2/2b^2}$, $0 < \theta < 2\pi$, $r > 0$. Evaluate the individual densities of R_o and θ_o by

$$\int_{-\infty}^{\infty} f^*_{12}(r,\theta)d\theta, \int_{-\infty}^{\infty} f^*_{12}(r,\theta)dr$$ to obtain

$$f_{R_o}(r) = \frac{1}{b^2} re^{-r^2/2b^2}, \ r > 0; \ f_{\theta_o}(\theta) = \frac{1}{2\pi}, \ 0 < \theta < 2\pi.$$

Therefore $f^*_{12}(r,\theta) = f_{R_o}(r)f_{\theta_o}(\theta)$, proving independence.

14. $f_{34}(z,w) = f_{12}(x,y)\left|\frac{\partial(x,y)}{\partial(z,w)}\right|$ where $z = xy$, $w = y$, i.e.

$x = \frac{z}{w}$, $y = w$.

The Jacobian is $\begin{vmatrix} 1/w & -z/w^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{w}$.

Thus $f_{34}(z,w) = \frac{1}{w} f_1(\frac{z}{w})f_2(w)$, $z,w > 0$.

Hence

$$f_3(z) = \int_{-\infty}^{\infty} f_{34}(z,w)dw = \int_0^{\infty} \frac{1}{w} f_1(\frac{z}{w})f_2(w)dw.$$

Note: The equations $z = xy$, $w = y$ define a one to one mapping of $\{(x,y): x > 0, y > 0\}$ onto $\{(z,w): z > 0, w > 0\}$.

15. $R = R_1 + R_2/(1+R_2)$. The density of $R_2/(1+R_2)$ is

$g(y) = f_2(y/(1-y))\left|\frac{d}{dy}(y/(1-y))\right| = 1/(1-y)^2$, $0 \le y \le \frac{1}{2}$,

hence R_1 and $R_2/(1+R_2)$ have joint density $f(x,y) = 1/(1-y)^2$, $0 \le x \le 1$, $0 \le y \le \frac{1}{2}$. Thus

$$P\{R \le \tfrac{1}{2}\} = \int_0^{1/2} dx \int_0^{1/2-x} (1-y)^{-2}dy = -\frac{1}{2} + \ln 2.$$

16. The speed of the particle is $(R_1^2 + R_2^2)^{1/2}$, hence $T = (R_1^2 + R_2^2)^{-1/2}$. Thus

$$P\{T \le t\} = P\{R_1^2 + R_2^2 \ge 1/t^2\} = \iint_{x^2+y^2 \ge 1/t^2} (2\pi)^{-1} e^{-(x^2+y^2)/2} dxdy$$

$$= (2\pi)^{-1}\int_0^{2\pi} d\theta \int_{1/t^2}^{\infty} re^{-r^2/2}dr = e^{-1/t^2}, \ t > 0.$$

Thus $f_T(t) = 2t^{-3}e^{-1/t^2}$, $t > 0$; $f_T(t) = 0$, $t \le 0$.

Section 2.9

1. (a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$= x + x^2[-\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \cdots]$$

If $|x| \le \frac{1}{2}$, $\left| -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \cdots \right| \le \frac{1}{2} + \frac{1}{2}(\frac{1}{3}) + (\frac{1}{2})^2 \frac{1}{4} + \cdots$

$$\le \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1,$$

and the result follows.

1. (continued)

(b) $\ln(1 - \frac{x_n}{n})^n = n \ln(1 - \frac{x_n}{n}) = n(-\frac{x_n}{n} + \theta \frac{x_n^2}{n^2}) \to -\lambda$.

Thus $(1 - \frac{x_n}{n})^n \to e^{-\lambda}$.

2. (a) $P\{R \geq 1\} = 1 - P\{R = 0\} = 1 - e^{-\lambda} \geq .99$, so $e^{-\lambda} \leq .01$, or $\lambda = .001n \geq -\ln .01 = \ln 100 = 4.6$. Thus $n \geq 4600$.

(b) $P\{R < 3\} = e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2}) = 5e^{-2}$, hence $P\{R \geq 3\} = 1-5e^{-2}$.

3. (a) $P\{R_1 = 1\} = P\{R_1 = 1, R_2 = 1\} + P\{R_1 = 1, R_2 = 2\}$

$= .4 + .3 = .7$

$P\{R_2 = 1\} = P\{R_1 = 1, R_2 = 1\} + P\{R_1 = 2, R_2 = 1\}$

$= .4 + .2 = .6$

$P\{R_1 = 1, R_2 = 1\} = .4 \neq P\{R_1 = 1\}\, P\{R_2 = 1\}$, hence R_1 and R_2 are not independent.

(b) $P\{R_1R_2 \leq 2\} = 1 - p_{12}(2,2) = .9$.

Chapter 3

Section 3.2

1. $E(R^n) = \int_{-\infty}^{\infty} x^n \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 0$ if n is odd, by symmetry.

If n is even, $E(R^n) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^n e^{-x^2/2}\, dx = (y = \frac{1}{2}x^2)$

$$\frac{2}{\sqrt{2\pi}} \int_0^{\infty} (2y)^{n/2} e^{-y} (2y)^{-1/2} dy =$$

$$\frac{2}{\sqrt{2\pi}} 2^{(n-1)/2} \int_0^{\infty} y^{(n-1)/2} e^{-y} dy = \frac{2^{n/2}}{\sqrt{\pi}} \Gamma(\frac{n+1}{2}) =$$

$$\frac{2^{n/2}}{\sqrt{\pi}} (\frac{n-1}{2}) \Gamma(\frac{n-1}{2}) = \frac{2^{n/2}}{\sqrt{\pi}} (\frac{n-1}{2})(\frac{n-3}{2}) \cdots \frac{5}{2}\frac{3}{2}\frac{1}{2} \Gamma(\frac{1}{2})$$

$$= (n-1)(n-3)\ldots(5)(3)(1).$$

3. (a) $E(R_1R_2) = \int_0^{\infty}\int_0^{\infty} xye^{-x}e^{-y} dxdy = (\int_0^{\infty} e^{-x}dx)^2 = 1$.

(b) $E(R_1 - R_2) = \int_0^{\infty}\int_0^{\infty} (x-y)e^{-x}e^{-y} dxdy = 1 - 1 = 0$.

(c) $E|R_1 - R_2| = \int_0^{\infty}\int_0^{\infty} |x-y| e^{-x}e^{-y} dxdy$

$$= \iint_A (x-y)e^{-x}e^{-y} dxdy + \iint_B (y-x)e^{-y}e^{-x} dydx$$

where A: $x,y \geq 0$, $x \geq y$, and B: $x,y \geq 0$, $x < y$

$$= \text{(by symmetry) } 2\int_0^{\infty} e^{-x}[\int_0^x (x-y)e^{-y}dy]dx$$

$$= 2\int_0^{\infty} e^{-x}[x(1-e^{-x}) + xe^{-x} + e^{-x} - 1]dx$$

$$= 2\int_0^{\infty} [xe^{-x} + e^{-2x} - e^{-x}]dx = 2(1 + \tfrac{1}{2} - 1) = 1.$$

4. $E[\max(R_1,R_2)] = 2\int_{-1}^{1} x\,dx \int_{-1}^{x} \frac{1}{4}\,dy = \frac{1}{2}\int_{-1}^{1} x(x+1)\,dx = \frac{1}{3}$.

Alternately, $F_1(x) = F_2(x) = \frac{1}{2}(x+1)$, $-1 \le x \le 1$. Hence if $R_3 = \max(R_1,R_2)$, $F_3(x) = F_1(x)F_2(x) = \frac{1}{4}(x+1)^2$, $-1 \le x \le 1$. Thus $f_3(x) = \frac{1}{2}(x+1)$, $-1 \le x \le 1$. Consequently

$$E(R_3) = \int_{-\infty}^{\infty} x\, f_3(x)\,dx = \frac{1}{2}\int_{-1}^{1} x(x+1)\,dx = \frac{1}{3}.$$

5. $E[C(R)] = \int_0^3 2xe^{-x}dx + \int_3^{\infty} [2 + 6(x-3)]xe^{-x}dx$

$$= \int_0^{\infty} 2xe^{-x}dx + 6\int_3^{\infty} (x-3)(x-3+3)e^{-(x-3)}e^{-3}dx$$

$$= 2 + 6e^{-3}(2+3) = 2 + 30e^{-3} \approx 3.5.$$

6. (a) $P\{\text{at least one fails}\} = 1 - P\{\text{neither fails}\} =$

$$1 - P\{R_1 > T,\ R_2 > T\} = 1 - \left(\int_T^{\infty} \lambda e^{-\lambda x}dx\right)^2 = 1 - e^{-2\lambda T}.$$

(b) If R is the "down time" then

$R = T - \max(R_1,R_2)$ if $R_1 \le T$ and $R_2 \le T$

$= 0$ if either $R_1 > T$ or $R_2 > T$

$$E(R) = \int_0^T\int_0^T [T-\max(x,y)]\,\lambda e^{-\lambda x}\,\lambda e^{-\lambda y}\,dx\,dy$$

$$= \text{(by symmetry) } 2\int_0^T \lambda e^{-\lambda x}\,dx \int_0^x (T-x)\,\lambda e^{-\lambda y}\,dy =$$

$$2\lambda \int_0^T (T-x)\,e^{-\lambda x}(1 - e^{-\lambda x})\,dx.$$

7. $E(R) = np\sum_{k=1}^{n} k\,\frac{(n-1)!}{k!(n-k)!}\,p^{k-1}(1-p)^{n-k}$

$$= np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} = np\sum_{r=0}^{n-1}\binom{n-1}{r}p^r(1-p)^{n-r-1}$$

$$= np(p + 1-p)^{n-1} = np.$$

<u>Section 3.3</u>

2. $E[(R-m)^n] = \int_{-\infty}^{\infty} (x-m)^n \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}\,dx.$

Let $y = \frac{x-m}{\sigma}$ to obtain $\sigma^n \int_{-\infty}^{\infty} \frac{y^n}{\sqrt{2\pi}} e^{-y^2/2}\,dy$, which is σ^n times the n^{th} moment of a random variable that is normal with mean 0 and variance 1. Thus, by Problem 1, Section 3.2,

$E[(R-m)^n] = 0$, n odd

$= \sigma^n (n-1)(n-3)\ldots(5)(3)(1)$, n even.

3. $E(R_1R_2) = E(\cos\theta\sin\theta) = \int_0^{2\pi} \frac{1}{2\pi}\cos x \sin x\,dx$

$$= \frac{1}{4\pi}\int_0^{2\pi} \sin 2x\,dx = 0$$

$$E(R_1) = \int_0^{2\pi}\frac{1}{2\pi}\cos x\,dx = 0,\quad E(R_2) = \int_0^{2\pi}\frac{1}{2\pi}\sin x\,dx = 0$$

$E[(R_1+R_2)^2] = E(R_1^2) + E(R_2^2) + 2E(R_1R_2) = E(R_1^2) + E(R_2^2)$

and since $E(R_1) = E(R_2) = E(R_1+R_2) = 0$,

$$\mathrm{Var}(R_1+R_2) = \mathrm{Var}\,R_1 + \mathrm{Var}\,R_2.$$

Since $P\{R_1^2 \le 1/4,\ R_2^2 \le 1/4\} = 0 \ne P\{R_1^2 \le 1/4\}P\{R_2^2 \le 1/4\}$, R_1 and R_2 are not independent.

4. $-|R| \le R \le |R|$, so by properties 2 and 3,

$$-E(|R|) \le E(R) \le E(|R|),\ \text{i.e. } |E(R)| \le E(|R|).$$

5. $R^n = (R-m+m)^n = \sum_{k=0}^{n}\binom{n}{k}(R-m)^k m^{n-k}$.

Thus $\alpha_n = E(R^n) = \sum_{k=0}^{n}\binom{n}{k}m^{n-k}\beta_k$, assuming $\beta_1,\ldots,\beta_{n-1}$ are finite and β_n exists. From this result and properties 8 and 9 we conclude that α_n is finite iff β_n is finite.

Section 3.4

2. $aE(R_1) + bE(R_2) = c$, hence $a(R_1 - E(R_1)) + b(R_2 - E(R_2)) = 0$, and the result follows.

3. Let $g(x) = E[(xR_1+R_2)^2] = E(R_1^2)x^2 + 2E(R_1R_2)x + E(R_2^2)$. Assume R_1 not essentially 0; otherwise the result is immediate. Now equality holds in the Schwarz inequality iff the discriminant of g is 0, i.e. iff the equation $g(x) = 0$ has a real repeated root. But this happens iff $g(x) = 0$ for some x, i.e. iff for some x we have $xR_1 + R_2 = 0$ (with probability 1). Therefore, equality holds iff R_1 and R_2 are linearly dependent.

Section 3.5

1. $I_{A_1}, \ldots, I_{A_n}$ are independent iff

$$P\{I_{A_1} = i_1, \ldots, I_{A_n} = i_n\} = P\{I_{A_1} = i_1\} \ldots P\{I_{A_n} = i_n\}$$

for all $i_1, \ldots, i_n = 0$ or 1, i.e. iff

$$P(B_1 \cap B_2 \cap \cdots \cap B_n) = P(B_1)P(B_2)\ldots P(B_n)$$

where for each k, B_k = either A_k or A_k^c. This is equivalent to the independence of $A_1, \ldots, A_n$ (see Problem 1, Section 1.5).

(a) $I_\Omega(\omega) = 1$ since all points ω belong to Ω,

$I_\phi(\omega) = 0$ since no points ω belong to ϕ.

(b) $I_{A \cap B}(\omega) = 1$ iff $\omega \in A \cap B$

iff $I_A(\omega) = I_B(\omega) = 1$

iff $I_A(\omega) I_B(\omega) = 1$

$I_{A \cup B}(\omega) = 1$ iff $\omega \in A \cup B$

iff $I_A(\omega) = 1$ or $I_B(\omega) = 1$

iff $I_A(\omega) + I_B(\omega) - I_{A \cap B}(\omega) = 1$

2. (continued)

(c) $I_{\bigcup_{i=1}^{\infty} A_i}(\omega) = 1$ iff $\omega \in$ exactly one A_i (by disjointness)

iff $\sum_{i=1}^{\infty} I_{A_i}(\omega) = 1$

(d) Let A_n expand to A. If $\omega \in A$ then eventually $\omega \in A_n$, hence $I_{A_n}(\omega)$ is eventually 1, so $I_{A_n}(\omega) \to I_A(\omega)$. If $\omega \notin A$ then $I_{A_n}(\omega) \equiv 0$, hence $I_{A_n}(\omega) \to I_A(\omega)$. The contracting case is handled similarly.

4. Let A_i = {trial i results in success and trial $i+1$ in failure}, $i = 1, 2, \ldots, n-1$. Then $R_0 = \sum_{i=1}^{n-1} I_{A_i}$, hence

$$E(R_0) = \sum_{i=1}^{n-1} P(A_i) = (n-1)p(1-p).$$

6. Let A_i = {box i is empty}. Then $R = \sum_{i=1}^{50} I_{A_i}$, hence $E(R) = \sum_{i=1}^{50} P(A_i)$. But $P(A_i) = P\{$all balls go into a box other than $i\} = (\frac{49}{50})^{100}$. Hence $E(R) = 50\,(\frac{49}{50})^{100}$.

Section 3.6

1. (a) $P\{-.5 \le R \le 4\} = P\{\frac{-.5-1}{3} \le R^* \le \frac{4-1}{3}\}$

$= F^*(1) - F^*(-.5) = F^*(1) - 1 + F^*(.5)$

$=$ (from the table) $.841 - 1 + .691 = .532$.

(b) $P\{R \ge c\} = P\{R^* \ge \frac{c-1}{3}\} = 1 - F^*(\frac{c-1}{3}) = F^*(\frac{1-c}{3}) = .9$.

From the table, $\frac{1-c}{3} = 1.28$, or $c = -2.84$.

2. $P\{|R-m| \geq k\sigma\} = P\{|R^*| \geq k\} = P\{R^* \leq -k\} + P\{R^* \geq k\} =$
$F^*(-k) + 1 - F^*(k) = 2(1 - F^*(k))$, which does not depend on m or σ. From the table, $F^*(1.96) = .975$, hence

$$P\{|R-m| \geq 1.96\,\sigma\} = 2(.025) = .05.$$

Section 3.7

2. (a) $P\{R_n \neq 0\} \to 0$ as $n \to 0$

For $P\{R_n \neq 0\} = P\{R_n = e^n\} = \frac{1}{n} \to 0.$

(b) $E(R_n{}^k) = \sum_x x^k\, P\{R_n = x\} = 0\, P\{R_n = 0\} + e^{nk}\, P\{R_n = e^n\}$

$= \frac{1}{n} e^{nk} \to \infty$ as $n \to \infty$ for any $k > 0$.

3. Apply the weak law of large numbers with $\varepsilon = -\frac{m}{2} > 0$. Then

$$P\{\frac{R_1+\ldots+R_n}{n} \geq \frac{m}{2}\} = P\{\frac{R_1+\ldots+R_n}{n} - m \geq -\frac{m}{2}\}$$

$$\leq P\{|\frac{R_1+\ldots+R_n}{n} - m| \geq \varepsilon\} \text{ by (1.3.9)}$$

$$\to 0 \text{ as } n \to \infty.$$

If K is any negative number, $\frac{nm}{2} < K$ for large n, hence

$$P\{R_1+\ldots+R_n < K\} \geq P\{R_1+\ldots+R_n < \frac{nm}{2}\} \to 1.$$

Thus for large n, the probability that your total losses after n trials will exceed $|K|$ is overwhelming.
Moral: Do not gamble (at least not if your average gain on a given trial is negative). The weak law of large numbers, known colloquially as the Law of Averages, predicts that you are very likely to be wiped out.

Section 4.2

2. Restrict x and y to be ≥ 0 throughout. Then

$$C = \{(x,y):\ x+y \leq 2\},\ C_x = \{y: y \leq 2-x\},\ 0 \leq x \leq 2;$$
$$C_x = \phi,\ x > 2$$

$$P_x(C_x) = 1,\ 0 \leq x \leq 1$$
$$= \frac{2-x}{x},\ 1 \leq x \leq 2$$
$$= 0,\ x > 2.$$

3. By (4.2.3), $P\{4 \leq R_1+R_2 \leq 6\} =$

$$\sum_{n=1}^{\infty} p_n \int_{\{y:4 \leq n+y \leq 6,\ y \geq 0\}} ne^{-ny}dy = p_1(e^{-3}-e^{-5})$$
$$+ p_2(e^{-4}-e^{-8}) + p_3(e^{-3}-e^{-9}) + p_4(1-e^{-8}) + p_5(1-e^{-5}$$

4. $P\{R_2 \in B | R_1 = x_i\} = \dfrac{P\{R_1 = x_i,\ R_2 \in B\}}{P\{R_1 = x_i\}}$

$$= \frac{p(x_i) \int_B f_i(y)dy}{p(x_i)} \text{ by (4.2.2).}$$

Thus $P\{R_2 \in B | R_1 = x_i\} = \int_B f_i(y)dy = P_{x_i}(B)$.

5. If $0 \leq y \leq 1$, $P\{R_2 \leq y\} = \int_{-\infty}^{\infty} f_1(x)\ P\{R_2 \leq y | R_1 = x\}dx$

$$= \int_1^{\infty} \frac{1}{x^2} (\tfrac{y}{x})dx = \frac{1}{2}\, y.$$

5. (continued)

Let $y > 1$. $P\{R_2 \le y \mid R_1 = x\} = 1$ if $1 \le x \le y$

$= \frac{y}{x}$ if $x > y$.

If $y > 1$, $P\{R_2 \le y\} = \int_1^y \frac{1}{x^2}(1)dx + \int_y^\infty \frac{1}{x^2}(\frac{y}{x})dx$

$$= 1 - \frac{1}{y} + \frac{1}{2y} = 1 - \frac{1}{2y}.$$

Thus $f_2(y) = \frac{1}{2}$, $0 \le y \le 1$

$= \frac{1}{2y^2}$, $y > 1$.

Section 4.3

1. $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{y=x}^{\infty} e^{-y}dy = e^{-x}$, $x \ge 0$.

Thus $h(y|x) = \frac{f(x,y)}{f_1(x)} = e^{x-y}$, $0 \le x \le y$

$= 0$ elsewhere.

Therefore $P\{R_2 \le y \mid R_1 = x\} = \int_{-\infty}^{y} h(y|x)dy = e^x \int_x^y e^{-y}dy = 1 - e^{x-y}$, $y \ge x$

$= 0$, $y < x$.

3. (a) is a special case of (b). In (b),

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \frac{1}{\text{area } C}\int_{C_x} dy = \frac{\text{length } C_x}{\text{area } C}.$$

Thus $h(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{1}{\text{length } C_x}$ if $y \in C_x$, i.e. given $R_1 = x$, R_2 is uniformly distributed on C_x.

4. If $R_1 = x$ then $R_2 - R_1 \le z$ iff $R_2 \le x+z$. Thus

$$P\{R_3 \le z \mid R_1 = x\} = P\{R_2 \le x+z \mid R_1 = x\}$$

$$= \int_{-\infty}^{x+z} h(y|x)dy = \int_x^{x+z} e^{x-y}dy \text{ (see Problem 1)}$$

$$= e^x(e^{-x} - e^{-(x+z)}) = 1 - e^{-z}.$$

The conditional density of R_3 given $R_1 = x$ is $\frac{d}{dz}(1-e^{-z}) = e^{-z}$, $z \ge 0$, $x \ge 0$.

$$P\{1 \le R_3 \le 2 \mid R_1 = x\} = \int_1^2 e^{-z}dz = e^{-1} - e^{-2}, \ x \ge 0.$$

Note that R_1 and R_3 are independent but R_1 and R_2 are not.

Section 4.4

2. $E(R_0^{-n} \mid R_1 = x_1, \ldots, R_n = x_n) = \int_{-\infty}^{\infty} \lambda^{-n} h(\lambda \mid x_1, \ldots, x_n)d\lambda$

$$= \frac{(1+x)^{n+1}}{n!}\int_0^\infty e^{-\lambda(1+x)}d\lambda = \frac{(1+x)^n}{n!}, \quad x = \sum_{i=1}^n x_i.$$

3. $h(y|x) = 1/x$, $0 \le x \le 1$, $0 \le y \le x$

$= 1$, $1 \le x \le 2$, $0 \le y \le 1$

$= \frac{1}{3-x}$, $2 \le x \le 3$, $x-2 \le y \le 1$.

Thus $E(R_2 \mid R_1 = x) = \int_{-\infty}^{\infty} yh(y|x)dy$

$= \frac{1}{x}\int_0^x ydy = \frac{1}{2}x$, $0 \le x \le 1$

$= \int_0^1 ydy = \frac{1}{2}$, $1 \le x \le 2$

$= \frac{1}{3-x}\int_{x-2}^{1} ydy = \frac{1-(x-2)^2}{2(3-x)} = \frac{x-1}{2}$, $2 \le x \le 3$

Note that all computations may be avoided by making use of Problem 3b in Section 4.3.

4. Let R_1 be the number of ones, R_2 the number of twos. Given that $R_1 = k$, R_2 has the binomial distribution with parameters n-k and 1/5 (see Example 1, Section 2.9). Thus $E(R_2|R_1=k) = \frac{1}{5}(n-k)$.

5. (a) $P\{R_1 \geq 1 | R_1+R_2 \leq 3\} = \dfrac{P\{R_1 \geq 1,\ R_1+R_2 \leq 3\}}{P\{R_1+R_2 \leq 3\}} = \dfrac{3/8}{7/8} = \dfrac{3}{7}$.

(b) $E(R_1|R_1+R_2 \leq 3) = \dfrac{E(R_1 I_{\{R_1+R_2 \leq 3\}})}{P\{R_1+R_2 \leq 3\}}$

$$= \frac{8}{7}\int_{x=0}^{2}\int_{y=0}^{2} x I_{\{x+y \leq 3\}} f(x,y)\,dx\,dy$$

$$= \frac{8}{7}\int_{\substack{0 \leq x \leq 2,\ 0 \leq y \leq 2 \\ x+y \leq 3}} \frac{1}{4}\, x\,dx\,dy = \frac{2}{7}\left[\int_0^1 x\,dx \int_0^2 dy + \int_1^2 x\,dx \int_0^{3-x} dy\right]$$

$$= \frac{2}{7}\left(1 + \frac{13}{6}\right) = \frac{19}{21}.$$

6. $\sum_{n=1}^{\infty} P(B_n)\, E(R|B_n) = \sum_{n=1}^{\infty} \dfrac{P(B_n)E(RI_{B_n})}{P(B_n)}$

$$= \sum_{n=1}^{\infty} E(RI_{B_n}) = E\left(\sum_{n=1}^{\infty} RI_{B_n}\right) = E(R) \text{ since } \sum_{n=1}^{\infty} I_{B_n} = 1.$$

(It can be shown that $E(\sum_{n=1}^{\infty} RI_{B_n}) = \sum_{n=1}^{\infty} E(RI_{B_n})$ if $E(R)$ exists.)

9. $P\{T-t_0 \leq x | T > t_0\} = \dfrac{P\{t_0 < T \leq t_0+x\}}{P\{T > t_0\}} = \dfrac{e^{-t_0} - e^{-(t_0+x)}}{e^{-t_0}} = 1-e^{-x}$.

Thus the (conditional) waiting time starting from t_0 has the same density (e^{-x}, $x \geq 0$) as the original waiting time T, i.e. the bulb "does not remember" that it has already burned for t_0 units of time.

11. $E(R_1^2+R_2^2|R_1=x) = x^2 + E(R_2^2|R_1=x)$ (cf. Problem 5, Section 4.3

Now $E(R_2^2|R_1=x) = \int_{-\infty}^{\infty} y^2 h(y|x)\,dy = \int_{-\infty}^{\infty} y^2 f_2(y)\,dy = E(R_2^2)$ by independence. Thus

$$E(R_1^2+R_2^2|R_1=x) = x^2 + \int_{-1}^{0} \frac{1}{2} y^2\,dy + \int_{0}^{\infty} \frac{1}{2} y^2 e^{-y}\,dy$$

$$= x^2 + \frac{1}{6} + 1 = x^2 + \frac{7}{6}.$$

12. (a) $P\{R_1=x | y < R_2 < y+dy\} = \dfrac{P\{R_1=x,\ y < R_2 < y+dy\}}{P\{y < R_2 < y+dy\}} =$

$$\frac{P\{R_1=x\}P\{y < R_2 < y+dy | R_1=x\}}{\sum_{x'} P\{R_1=x'\}P\{y < R_2 < y+dy | R_1=x'\}} \sim \frac{P\{R_1=x\}h(y|x)\,dy}{\sum_{x'} P\{R_1=x'\}h(y|x')\,dy}$$

(b) $P\{R_1 \in A,\ R_2 \in B\} = \sum_{x \in A} P\{R_1=x\} \int_B h(y|x)\,dy$ by (4.2.2).

But $\int_B f_2(y)\, P\{R_1 \in A | R_2=y\}\,dy =$

$$\int_B \sum_{x'} P\{R_1=x'\}h(y|x') \sum_{x \in A} \frac{P\{R_1=x\}h(y|x)}{\sum_{x'} P\{R_1=x'\}h(y|x')}\,dy =$$

$$\sum_{x \in A} P\{R_1=x\} \int_B h(y|x)\,dy.$$

13. $P\{R_1 \in A,\ R_2 \in B\} = \int_A f_1(x)\, P\{R_2 \in B | R_1=x\}\,dx$

$$= \int_A f_1(x) \sum_{y \in B} p(y|x)\,dx = \sum_{y \in B} \int_A \frac{f_1(x)p(y|x)\,dx}{p_2(y)}\ p_2(y)$$

$= \sum_{y \in B} P\{R_2=y\}\, P\{R_1 \in A | R_2=y\}$, which is the appropriate version of the theorem of total probability.

14. $P\{R_1=x_1,\ldots,R_n=x_n \mid R=\lambda\} = \lambda^x(1-\lambda)^{n-x}$, $x = \sum_{i=1}^{n} x_i$, $x_i = 0$ or 1.

Thus $P\{R_1=x_1,\ldots,R_n=x_n\} = \int_0^1 \lambda^x(1-\lambda)^{n-x}d\lambda = \beta(1+x,n-x+1)$.

By Problem 13, the conditional density of R given $R_1=x_1,\ldots,R_n=x_n$ is

$$\frac{\lambda^x(1-\lambda)^{n-x}}{\beta(1+x,n-x+1)}, \quad 0 \le \lambda \le 1.$$

Hence $E(R \mid R_1=x_1,\ldots,R_n=x_n) = \int_0^1 \frac{\lambda(\lambda^x(1-\lambda)^{n-x})}{\beta(1+x,n-x+1)}\,d\lambda$

$$= \frac{\beta(2+x,n-x+1)}{\beta(1+x,n-x+1)} = \frac{\Gamma(2+x)}{\Gamma(1+x)}\frac{\Gamma(n+2)}{\Gamma(n+3)} = \frac{x+1}{n+2}.$$

15. (a) The probability of error is

$P(\text{heads})P\{R \in S \mid \text{heads}\} + P(\text{tails})\,P\{R \notin S \mid \text{tails}\}$

$$= p\int_S f_0(x)dx + (1-p)\int_{S^c} f_1(x)dx = p\int_S f_0(x)dx +$$

$$(1-p)[1 - \int_S f_1(x)dx] = \int_S [pf_0(x)-(1-p)f_1(x)]dx + 1-p.$$

(b) Let $L(x) = \frac{f_1(x)}{f_0(x)}$. If $L(x) > \frac{p}{1-p}$, the integrand is < 0, so to minimize the probability of error, we should put $x \in S$. If $L(x) < \frac{p}{1-p}$, the integrand is > 0, so take $x \notin S$. If $L(x) = \frac{p}{1-p}$, do anything.

For the example, $L(x) = \dfrac{e^{-(x-m_1)^2/2\sigma^2}}{e^{-(x-m_0)^2/2\sigma^2}}$

$$L(x) > \frac{p}{1-p} \text{ iff } \frac{(x-m_0)^2-(x-m_1)^2}{2\sigma^2} > \ln\frac{p}{1-p}, \text{ i.e.}$$

$$x \in S \text{ iff } x > \frac{\sigma^2}{m_1-m_0}\ln\frac{p}{1-p} + \frac{m_1+m_0}{2}, \text{ assuming } m_0 < m_1.$$

16. $E(R \mid R \ge 2) = E(RI_{\{R \ge 2\}})/P\{R \ge 2\} = \sum_{k=2}^{n} kp_R(k)/\sum_{k=2}^{n} p_R(k) =$

$$(np - 1p_R(1))/(1-p_R(0) - p_R(1)) = \frac{np - npq^{n-1}}{1-q^n-npq^{n-1}}.$$

17. $E(R_2 \mid 2 \le R_2 \le 4) = E(R_2 I_{\{2 \le R_2 \le 4\}})/P\{2 \le R_2 \le 4\} =$

$$\frac{E(R_1^2 I_{\{\sqrt{2} \le R_1 \le 2\}}) + E(3I_{\{6 < R_1 \le 10\}})}{P\{\sqrt{2} \le R_1 \le 2\} + P\{6 < R_1 \le 10\}}$$

$$= \frac{\int_{\sqrt{2}}^{2} \frac{x^2}{10}dx + \int_6^{10}\frac{3}{10}dx}{\frac{1}{10}(2-\sqrt{2}+4)} = \frac{\frac{1}{3}(8-2^{3/2})+12}{6-\sqrt{2}}.$$

Alternately, $P\{R_2=3\} = 4/10$, and after removing this discontinuity from the distribution function of R_2, we obtain $f_2(y) = \frac{dF_2(y)}{dy} = 1/20y^{1/2}$, $0 < y \le 36$. Thus

$$E(R_2 I_{\{2 \le R_2 \le 4\}}) = 3P\{R_2=3\} + \int_2^4 y\frac{1}{20y^{1/2}}dy =$$

$$\frac{12}{10} + \frac{1}{30}(8-2^{3/2}) \text{ as above.}$$

19. (a) If R is absolutely continuous,

$$E[(\theta^*-\theta)^2] = \int_{-\infty}^{\infty} E[(\theta^*-\theta)^2 \mid R=x]f_R(x)dx.$$

To minimize this, it is sufficient to minimize $E[(\theta^*-\theta)^2 \mid R=x]$ for each x. But since $\theta^* = d(R)$, we hav

$$E[(\theta^*-\theta)^2 \mid R=x] = E[(d(x)-\theta)^2 \mid R=x] = d^2(x)$$

$$- 2E(\theta \mid R=x)d(x) + E(\theta^2 \mid R=x).$$

19. (a) (continued)

Since $y^2-2Ay+B$ is a minimum when $y = A$, we have $d(x) = E(\theta|R=x)$. If R is discrete,

$$E[(\theta^*-\theta)^2] = \sum_x E[(\theta^*-\theta)^2|R=x]\, p_R(x),$$

and the same argument applies.

(b) Clearly $d(x) = 1$ if $1 < x \le 3$, $d(x) = -1$ if $-3 \le x < -1$. If $-1 \le x \le 1$, $P\{\theta=1|R=x\} = P\{\theta=1\}f_R(x|\theta=1)/f_R(x)$ (see Problem 12). Given $\theta = 1$, R is uniformly distributed between -1 and 3, so $P\{\theta=1|R=x\} = (1/2)(1/4)/(\frac{1}{2} f_R(x|\theta=1) + \frac{1}{2} f_R(x|\theta=-1)) = (1/8)/(1/8 + 1/8) = 1/2$. Thus $P\{\theta=-1|R=x\} = 1/2$ also, so that $d(x) = E(\theta|R=x) = 0$. With probability 1/2, $(\theta^*-\theta)^2 = 0$, and with probability 1/2, $(\theta^*-\theta)^2 = 1$, hence the minimum value of $E[(\theta^*-\theta)^2]$ is 1/2.

20. The conditional density of θ given $R = x$ is (see Problem 13)

$$h_\theta(\lambda|x) = f_\theta(\lambda)P\{R=x|\theta=\lambda\}/P\{R=x\} = e^{-\lambda}e^{-\lambda}\frac{\lambda^x}{x!} \Big/ \int_0^\infty e^{-\lambda}e^{-\lambda}\frac{\lambda^x}{x!}\,d\lambda$$

$$= 2^{x+1}\lambda^x e^{-2\lambda}/x!$$

Thus $E(\theta|R=x) = \int_0^\infty \lambda h_\theta(\lambda|x)d\lambda = (x!)^{-1}2^{x+1}\int_0^\infty \lambda^{x+1}e^{-2\lambda}d\lambda$

$$= (x+1)!\,2^{x+1}/x!\,2^{x+2} = \frac{1}{2}(x+1).$$

Section 5.2

2. $N_1(s) = N_2(s) = \frac{1}{3s}(e^s-1) + \frac{2}{3s}(1-e^{-s})$, all s

$N_0(s) = N_1(s)N_2(s) = \frac{1}{9s^2}(e^{2s} + 2e^{s}-3-4e^{-s}+4e^{-2s})$

$f_0(x) = \frac{1}{9}[(x+2)u(x+2) + 2(x+1)u(x+1) - 3xu(x) - 4(x-1)u(x-$

$+ 4(x-2)u(x-2)]$.

5. $N_{R_i}(s) = \int_0^\infty \lambda e^{-\lambda x}e^{-sx}\,dx = \frac{\lambda}{s+\lambda}$, Re $s > -\lambda$

$N_0(s) = (\frac{\lambda}{s+\lambda})^n$, so $f_0(x) = \frac{\lambda^n x^{n-1}e^{-\lambda x}}{(n-1)!}u(x)$.

6. If $R = \tan\theta$, $f_R(y) = f_\theta(\arctan y)\left|\frac{d}{dy}\arctan y\right| = \frac{1}{\pi(1+y^2)}$

(The same result is obtained if θ is uniformly distributed between 0 and π, or 0 and 2π.)

7. For $x \ge 0$, $f(x) = u(x) - xu(x) + (x-1)u(x-1)$. Thus the Lapl transform of $f(x)u(x)$ is $N_1(s) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-s}$ (all s).

The Laplace transform $N_2(s)$ of $f(x)u(-x)$ is that of $f(-x)u(x)$ (= $f(x)u(x)$) with s replaced by -s (see Property 3, Section i.e. $N_2(s) = -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{s}$ (all s). Thus

$$N_R(s) = N_1(s) + N_2(s) = \frac{1}{s^2}(e^s+e^{-s}) - \frac{2}{s^2}.$$

Let $s = iu$ to obtain $M_R(u) = -\frac{1}{u^2}(2\cos u) + \frac{2}{u^2}$

$$= \frac{2(1-\cos u)}{u^2}.$$

8. (a) By (5.2.1), $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(u)e^{iux}\,du$, or

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(x)\,e^{-iux}\,dx\ .$$

(We may replace e^{iux} by e^{-iux} since M and f are real valued.) Multiply both sides by k to obtain the desired result.

(b) $\int_{-\infty}^{\infty} e^{-|x|}e^{-iux}\,dx = \frac{2}{1+u^2}$ (see the discussion after (5.2.1)).

This is nonnegative and integrable. Thus $ke^{-|u|}$ is a characteristic function; since $e^0 = 1$, the appropriate k is 1. $M(u) = 1 - |u|$, $M(u) = 0$, $|u| > 1$, is a characteristic function by Problem 7.

10. $M'(u) = \int_{-\infty}^{\infty} -x \sin ux \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin ux\, d(e^{-x^2/2}) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,u\cos ux\,dx = -uM(u).$$

The differential equation $dy/dx = -xy$ may be written as $\frac{1}{y}\frac{dy}{dx} = -x$. Integrate to obtain $\ln y = -\frac{x^2}{2} + C$, or $y = Ce^{-x^2/2}$. Thus $M(u) = e^{-u^2/2}$ (note M(0) is always 1). If R is normal (m,σ^2) then $R^* = (R-m)/\sigma$ is normal (0,1), so

$$E(e^{-iuR}) = E(e^{-iu(m+\sigma R^*)})$$

$$= e^{-ium} M_{R^*}(u\sigma) = e^{-ium}\, e^{-u^2\sigma^2/2}\ .$$

<u>Section 5.3</u>

1. No. If so, then $f(x) = 2e^{-x}u(x) - [u(x) - u(x-1)]$

$$= 2e^{-x} - 1,\ 0 \le x \le 1$$
$$= 2e^{-x},\ x > 1$$
$$= 0,\ x < 0.$$

This is negative for x near 1 and < 1, an impossibility.

2. $N_R(s) = \int_a^b f(x)e^{-sx}dx$. For any particular s, $|e^{-sx}|$ has some largest value for $x \in [a,b]$, say K. Then

$$|N_R(s)| \le \int_a^b K|f(x)|\,dx < \infty.$$

3. No. Let R be uniformly distributed between 0 and 1. Then $N_R(s) = \frac{1-e^{-s}}{s}$, so $M_R(u) = \frac{1-e^{-iu}}{iu}$. Now

$$|M_R(u)| = \frac{1}{|u|}\ |1 - \cos u + i \sin u|$$

$$= \sqrt{2}\,\frac{(1-\cos u)^{1/2}}{|u|} = \left|\frac{\sin \frac{1}{2}u}{\frac{1}{2}u}\right|,\ \text{and thus}$$

$$\int_{-\infty}^{\infty} |M_R(u)|\,du = \infty.$$

<u>Section 5.4</u>

2. Let $\Omega = \{\omega_1, \omega_2\}$, $p(\omega_1) = p(\omega_2) = \frac{1}{2}$.

Let $R(\omega_1) = 1$, $R(\omega_2) = 0$.

If n is even set $R_n(\omega_1) = 1$, $R_n(\omega_2) = 0$.

If n is odd set $R_n(\omega_1) = 0$, $R_n(\omega_2) = 1$.

Then $P\{R_n = 0\} = P\{R_n = 1\} = 1/2$ for all n, and

$$P\{R=0\} = P\{R=1\} = 1/2.$$

2. (continued)

Thus $F_n(x) = F(x)$ for all n and all x, so $R_n \xrightarrow{d} R$.

But if $0 < \epsilon < 1$, $P\{|R_n - R| \geq \epsilon\} = P\{R_n \neq R\} = 0$ if n is even

$= 1$ if n is odd.

Thus $R_n \overset{P}{\nrightarrow} R$.

3. If $\epsilon > 0$, then $P\{|R_n - c| \geq \epsilon\} = P\{R_n \geq c + \epsilon\} + P\{R_n \leq c - \epsilon\}$

$$= 1 - P\{R_n < c + \epsilon\} + P\{R_n \leq c - \epsilon\}$$

$$\leq 1 - P\{R_n \leq c + \tfrac{\epsilon}{2}\} + P\{R_n \leq c - \epsilon\} = 1 - F_n(c + \tfrac{\epsilon}{2})$$

$$+ F_n(c - \epsilon) \to 1 - 1 + 0 = 0.$$

4. If $\epsilon > 0$, $P\{|R_n| \geq \epsilon\} = P\{R_n = e^n\}$ for large enough n

$= \frac{1}{n} \to 0$. Thus $R_n \xrightarrow{P} 0$.

But $E(R_n^k) = 0\, P\{R_n = 0\} + e^{nk}\, P\{R_n = e^n\} = \frac{1}{n} e^{nk} \to \infty$.

5. (a) $P\{|Q_n - p| \leq .001\} = P\{|\frac{R_n}{n} - p| \leq .001\}$ where R_n is the number of "A" voters

$$= P\left\{\left|\frac{R_n - np}{(np(1-p))^{1/2}}\right| \leq \frac{.001 n^{1/2}}{(p(1-p))^{1/2}}\right\} \sim$$

$$P\left\{|R^*| \leq \frac{.001 n^{1/2}}{(p(1-p))^{1/2}}\right\}$$

where R^* is normal with mean 0 and variance 1. Now $P\{|R^*| \leq a\} = F^*(a) - F^*(-a) = 2F^*(a) - 1$. Thus, with

$$a = \frac{.001 n^{1/2}}{(p(1-p))^{1/2}}, \quad 2F^*(a) - 1 \geq .99 \text{ or } F^*(a) \geq .995.$$

5. (continued)

From the table, $a \geq 2.6$, or $n \geq (2600)^2 p(1-p)$. The largest possible value of $p(1-p)$ occurs at $p = 1/2$, so $n \geq (2600)^2 \frac{1}{4} = (1300)^2 = 1{,}690{,}000$.

(b) $2F^*(b) - 1 \geq .95$, $b = \frac{.01 n^{1/2}}{(p(1-p))^{1/2}}$, thus $F^*(b) \geq .975$, or $b \geq 1.96$. Therefore $n \geq (196)^2 \frac{1}{4} = (98)^2 = 9604$.

6. (a) $\frac{d}{dx} \frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2} = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} (1 + \frac{1}{x^2})$. Thus

$\frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2}$ and $\int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} (1 + \frac{1}{t^2})\, dt$ have the same derivative, hence differ by a constant, necessarily 0 (let $x \to \infty$). Since

$$\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} (1 + \frac{1}{t^2})\, dt \geq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt,$$

the result follows.

(b) By (a),
$$\frac{\frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2}}{\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt} = 1 + \frac{\frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{t^2} e^{-t^2/2}\, dt}{\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt}$$

The ratio of integrals on the right is $\leq \frac{1}{x^2}$, and therefore $\to 0$ as $x \to \infty$, proving the result.

8. $P\{7940 \le R \le 8080\} \sim P\{\frac{7940-8000}{40} \le R^* \le \frac{8080-8000}{40}\}$

$$= P\{-1.5 \le R^* \le 2\} = F^*(2) - F^*(-1.5) =$$

$$F^*(2) + F^*(1.5) - 1 = .977 + .933 - 1 = .91.$$

Chapter 6

Section 6.1

1. We specify

$$P\{(R_1,\ldots,R_n) \in B_n\} = \int_{B_n}\!\!\cdots\!\int f_1(x_1)\ldots f_n(x_n)dx_1\ldots dx_n.$$

It follows from this that

$$P\{(R_1,\ldots,R_k) \in B_k\} =$$

$$P\{(R_1,\ldots,R_k) \in B_k, -\infty < R_{k+1} < \infty,\ldots,-\infty < R_n < \infty\} =$$

$$\int_{B_k}\!\!\cdots\!\int f_1(x_1)\ldots f_k(x_k)dx_1\ldots dx_k.$$

Thus the probability measures P_n are consistent, hence we can construct a probability space on which we can define independ random variables $R_1, R_2, \ldots$, with R_n having density f_n, $n = 1,2,\ldots$

2. $P\{R_n = 1$ for infinitely many $n\} =$

$$\lim_{n\to\infty} \lim_{m\to\infty} P[\bigcup_{k=n}^{m} \{R_k=1\}]$$ as in Section 6.1.

But

$$P(\bigcup_{k=n}^{m} \{R_k=1\}) = 1 - P(\bigcap_{k=n}^{m} \{R_k=0\}) = 1 - q^{m-n+1} \to 1$$

as $m \to \infty$, so $P\{R_n=1$ for infinitely many $n\} = 1$.

$P\{\lim_{n\to\infty} R_n=1\} = P\{R_n=1$ for sufficiently large $n\}$

$= 1 - P\{R_n=0$ for infinitely many $n\} = 1 - 1 = 0$

by the above argument.

Section 6.2

4. Let A be the average time required. If $p \neq q$, there is a positive probability $|p-q|$ of never returning to 0 (see 6.2.7)). Thus $A \geq \infty\,(|p-q|) = \infty$. If $p = q$, then regardless of the result of the first trial, the average number of trials required (after the first) to return to 0 is infinite by the remark after the statement of Problem 3. The result follows. Note: A more precise analysis may be found in Problem 5 of Section 6.3.

5. In the gambler's ruin problem starting at $x > 0$, the probability of eventually reaching 0 is 1 if $q \geq p$, and $(q/p)^x$ if $q < p$ (see (6.2.6)). By symmetry, the probability of reaching b starting from 0 is 1 if $p \geq q$, and $(p/q)^b$ if $p < q$.

Section 6.3

2. (a) $h_{2n} = \frac{2}{n}\binom{2n-2}{n-1}(\frac{1}{2})^{2n-2}(\frac{1}{2})^2 = \frac{u_{2n-2}}{2n}$

(b) $$\frac{u_{2n}}{u_{2n-2}} = \frac{(2n)!}{n!n!}(\tfrac{1}{2})^{2n} \Big/ \frac{(2n-2)!}{(n-1)!(n-1)!}(\tfrac{1}{2})^{2n-2} = \frac{(2n)(2n-1)}{n^2}\frac{1}{4}$$
$$= 1 - \frac{1}{2n}$$

Thus

$$h_{2n} = \frac{u_{2n-2}}{2n} = u_{2n-2}\left(1 - \frac{u_{2n}}{u_{2n-2}}\right) = u_{2n-2} - u_{2n}.$$

3. $P\{S_1 \neq 0, \ldots, S_{2n} \neq 0\} = 1 - P\{$at least one return in the first

$2n$ steps$\} = 1 - h_2 - h_4 - \cdots - h_{2n}$

$= 1 - (u_0 - u_2) - (u_2 - u_4) - \cdots - (u_{2n-2} - u_{2n})$ by Problem 2

$= u_{2n}$ (note $u_0 = 1$).

$P\{S_1 \neq 0, \ldots, S_{2n-1} \neq 0\} = P\{S_1 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} = 0\} +$

$P\{S_1 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} \neq 0\} = h_{2n} + u_{2n}$.

4. $P\{S_1 \geq 0, \ldots, S_{2n} \geq 0\} = 1 - P\{S_i < 0$ for some $i = 1, 2, \ldots, 2n\}$

$$= 1 - \sum_{i=1,3,5,\ldots}^{2n-1} P\{\text{first passage through } -1 \text{ occurs at time}$$

But $P\{$first passage through -1 at time $i\} = P\{$first passage through $+1$ at time $i\} =$ (by (6.3.6), with $i = 2k+1$)

$$\frac{1}{1+2k}\binom{1+2k}{k}(\tfrac{1}{2})^{1+2k} = h_{2k+2} = h_{i+1} \text{ (see Problem 2a).}$$

Thus $P\{S_1 \geq 0, \ldots, S_{2n} \geq 0\} = 1 - h_2 - h_4 - \cdots - h_{2n} = u_{2n}$ as in Problem 3.

8. (a) Say the insects meet after j steps. If the spider walks steps east and b steps north, the fly must walk n-a steps west and n-b steps south. But $a+b = j$ and $(n-a)+(n-b) =$ so $2n-j = j$, or $n = j$. Thus $a+b = n$, which means that they must meet on the diagonal D.

(b) The probability that they will meet with the spider takin a steps east and n-a steps north (and the fly taking n-a steps west and a steps south) is $[\binom{n}{a}(\frac{1}{2})^n]^2$. Thus the probability that they will meet is

$$(\tfrac{1}{2})^{2n}\sum_{a=0}^{n}\binom{n}{a}^2 = \binom{2n}{n}(\tfrac{1}{2})^{2n} \text{ by Problem 7.}$$

(Note that this is really a random walk problem; tilt the picture so that the line from the spider to the fly is the axis.)

Section 6.4

3. $$(1-z)A(z) = (1-z)\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n z^{n+1}$$

$$= a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots = \sum_{n=0}^{\infty}(a_n - a_{n-1})z^n.$$

3. (continued)

Thus

$$\lim_{z \to 1} (1-z)A(z) = \sum_{n=0}^{\infty} (a_n - a_{n+1}) = a_0 + (a_1 - a_0) + (a_2 - a_1) + \dots + (a_n - a_{n-1}) + \dots$$

$$= \lim_{n \to \infty} a_n.$$

4. The generating function of $R + k$ is $E(z^{R+k}) = z^k A(z)$; the generating function of kR is $E(z^{kR}) = E[(z^k)^R] = A(z^k)$. Now

$$F(n) = P\{R \leq n\} = \sum_{k=0}^{n} p_k \cdot 1, \quad p_k = P\{R=k\}.$$

Thus $\{F(n)\}$ is the convolution of $\{p_0, p_1, \dots\}$ and $\{1, 1, \dots, \}$, so by Theorem 1, the generating function of $\{F(n)\}$ is

$$A(z) \sum_{n=0}^{\infty} z^n = \frac{A(z)}{1-z}.$$

5. (a) $P\{R=k\} = P\{k-1$ failures followed by a success$\}$

$$= q^{k-1}p, \quad k = 1, 2, \dots$$

(b) $$N_R(s) = \sum_{k=1}^{\infty} e^{-sk} P\{R=k\} = \frac{p}{q} \sum_{k=1}^{\infty} (qe^{-s})^k =$$

$$\frac{pe^{-s}}{1-qe^{-s}}, \quad |qe^{-s}| < 1.$$

N_R is analytic at $s = 0$, hence (see Section 5.3)

$$E(R) = -N_R'(0) = -[-pe^{-s}/(1-qe^{-s})^2]_{s=0} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$E(R^2) = N_R''(0) = [(1-qe^{-s})^2(pe^{-s}) + 2pe^{-s}(1-qe^{-s})(qe^{-s})]/(1-qe^{-s})^4 \text{ at } s = 0, \text{ i.e.}$$

$$(p^3 + 2p^2q)/p^4 = (-p^3 + 2p^2)/p^4 = 2p^{-2} - p^{-1}.$$

5. (b) (continued)

Thus

$$\text{Var } R = E(R^2) - (ER)^2 = \frac{1-p}{p^2}.$$

The generating function of R is $N_R(s)$ with $z = e^{-s}$, i.e. $A(z) = pz/1-qz$. We may compute that

$$A'(z) = \frac{p}{(1-qz)^2}, \quad A''(z) = \frac{2pq}{(1-qz)^3}.$$

Thus by (6.4.1) and (6.4.2),

$$E(R) = \frac{p}{(1-q)^2} = \frac{1}{p}, \quad \text{Var } R = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

7. $P\{R=k\} = P\{R_1 = \dots = R_k = 1, R_{k+1} = 0\} +$

$$P\{R_1 = \dots = R_k = 0, R_{k+1} = 1\} = p^k q + q^k p.$$

Thus

$$E(R) = \sum_{k=1}^{\infty} k(p^k q + q^k p).$$

But $\sum_{k=1}^{\infty} kq^{k-1}p$ is the mean of a random variable with the geometric distribution, i.e. $1/p$. Thus

$$E(R) = p \sum_{k=1}^{\infty} kp^{k-1}q + q \sum_{k=1}^{\infty} q^{k-1}p = \frac{p}{q} + \frac{q}{p}.$$

8. $P\{N_1=j, N_2=k\} = P\{T_1=j, T_2=k-j\} =$

$p^2 q^{k-2}$, $j = 1, 2, \dots$, $k = 2, 3, \dots$, $j < k$ (Problem 6b).

$$E(N_1N_2) = E(T_1(T_1+T_2)) = E(T_1^2) + E(T_1)E(T_2)$$

$$= E(T_1^2) + [E(T_1)]^2$$

$$E(N_1)E(N_2) = E(N_1)(2E(N_1)) = 2[E(T_1)]^2.$$

8. (continued)

Thus

$$\mathrm{Cov}(N_1,N_2) = E(N_1N_2) - E(N_1)E(N_2) = E(T_1^2) - (E(T_1))^2$$
$$= \mathrm{Var}\ T_1$$

$$\rho(R_1,R_2) = \frac{\mathrm{Cov}(N_1,N_2)}{\sigma_1\sigma_2} = \frac{\sigma_1^{\ 2}}{\sigma_1\sigma_2} = \frac{\sigma_1}{\sigma_2} = \frac{\sigma_1}{\sqrt{2}\,\sigma_1} = \frac{1}{\sqrt{2}}$$

since $\sigma_2^{\ 2} = \mathrm{Var}\ N_2 = 2\ \mathrm{Var}\ N_1$.

Section 6.5

1. If $M > 0$, $P\{\sum_{n=1}^{\infty} T_n \le M\} \le P\{T_1 + \dots + T_n \le \frac{n}{2\lambda}\}$ if $\frac{n}{2\lambda} \ge M$

$\to 0$ as $n \to \infty$ by the Weak Law of Large Numbers (note $E(T_i) = 1/\lambda$).

Thus

$$P\{\sum_{n=1}^{\infty} T_n < \infty\} = P[\bigcup_{M=1}^{\infty} \{\sum_{n=1}^{\infty} T_n \le M\}] \le \sum_{M=1}^{\infty} P\{\sum_{n=1}^{\infty} T_n \le M\} = 0.$$

3. (a) $P\{R_t = 1, R_{t+\tau} = 1\} = P\{R_t = 1\}\,P\{R_{t+\tau} = 1 | R_t = 1\}$.

But $P\{R_t = 1\} = P\{R_0 = 1\}P\{R_t = 1|R_0 = 1\}$

$+ P\{R_0 = -1\}P\{R_t = 1|R_0 = -1\} = \frac{1}{2}$ P{even number of customers in $(0,t]$} + $\frac{1}{2}$ P{odd number of customers in $(0,t]$} $= \frac{1}{2}$

$P\{R_{t+\tau} = 1 | R_t = 1\}$ = P{even number of customers in $(t,t+\tau]$} $= \frac{1}{2}(1+e^{-2\lambda\tau})$ by Problem 2.

Thus $P\{R_t = 1, R_{t+\tau} = 1\} = \frac{1}{4}(1 + e^{-2\lambda\tau})$.

3. (a) (continued)

Similarly, $P\{R_t = -1, R_{t+\tau} = -1\} = \frac{1}{4}(1 + e^{-2\lambda\tau})$

$P\{R_t = 1, R_{t+\tau} = -1\} = \frac{1}{4}(1 - e^{-2\lambda\tau})$

$P\{R_t = -1, R_{t+\tau} = 1\} = \frac{1}{4}(1 - e^{-2\lambda\tau})$.

(b) $K(t,\tau) = E(R_tR_{t+\tau}) - E(R_t)E(R_{t+\tau}) = E(R_tR_{t+\tau})$

$= \sum_{x,y} xy\,P\{R_t = x, R_{t+\tau} = y\}$

$= P\{R_t = 1, R_{t+\tau} = 1\} + P\{R_t = -1, R_{t+\tau} = -1\}$

$- P\{R_t = 1, R_{t+\tau} = -1\} - P\{R_t = -1, R_{t+\tau} = 1\}$

$= \frac{1}{2}(1 + e^{-2\lambda\tau}) - \frac{1}{2}(1 - e^{-2\lambda\tau})$

$= e^{-2\lambda\tau}$.

Section 6.6

1. If $P(A_n) = 1$, $n = 1,2,\dots$ then $P(\bigcap_{n=1}^{\infty} A_n) = 1 - P(\bigcup_{n=1}^{\infty} A_n^{\ c})$

and

$$P(\bigcup_{n=1}^{\infty} A_n^{\ c}) \le \sum_{n=1}^{\infty} P(A_n^{\ c}) = 0.$$

This fails for an uncountable intersection. For example, let be uniformly distributed between 0 and 1, and take $A_t = \{R \ne$ $0 \le t \le 1$. Each A_t has probability 1, but $\bigcap_{0 \le t \le 1} A_t = \phi$, hence has probability 0.

2. Given $\epsilon > 0$, choose m so that $\frac{1}{m} < \epsilon$. Then

$P\{|R_k - R| \ge \epsilon$ for at least one $k \ge n\} \le$

$P\{|R_k - R| \ge \frac{1}{m}$ for at least one $k \ge n\} \to 0$.

3. If $0 < \epsilon \le 1$, $P\{|R_{nm}| \ge \epsilon\} = P\{R_{nm}=1\} = \frac{1}{n} \to 0$, so $R_{nm} \xrightarrow{P} 0$. But for any ω and any n, $R_{nm}(\omega)$ is 1 for exactly one $m = 1,2,\ldots,n$ and 0 for the other m. Thus $\lim R_{nm}(\omega)$ never exists.

5. By Theorem 1, $\lim_n \sup A_n = [-1,1)$, $\lim_n \inf A_n = \{0\}$.

6. $\lim\inf A_n = \{(x,y): x^2+y^2 < 1\}$, $\lim\sup A_n =$

$$\{(x,y): x^2+y^2 \le 1\} - \{(0,1),(0,-1)\}.$$

Proof: (a) If $x^2+y^2 < 1$ then eventually the distance from (x,y) to $(\frac{(-1)^n}{n}, 0)$ is < 1, hence $(x,y) \in A_n$; thus $x^2+y^2 < 1$ implies $(x,y) \in \lim_n \inf A_n$.

(b) If $x^2+y^2 = 1$ but $(x,y) \ne (0,1)$ or $(0,-1)$, say $x > 0$. Then $(x,y) \in A_n$ for all even n since the distance from (x,y) to $(\frac{1}{n},0)$ is < 1; but $(x,y) \notin A_n$ for odd n since the distance from (x,y) to $(\frac{-1}{n}, 0)$ is > 1. Thus $(x,y) \in \lim_n \sup A_n$, $(x,y) \notin \lim\inf A_n$ (similar reasoning for $x < 0$).

(c) If $x^2+y^2 > 1$ then eventually $(x,y) \notin A_n$. Also, (0,1) and (0,-1) are in none of the A_n since the distance from (0,1) and (0,-1) to $(\frac{(-1)^n}{n}, 0)$ is > 1. Thus such points are not in $\lim_n \sup A_n$. The result follows from (a), (b) and (c).

7. Let $x = \lim_n \sup x_n$. Then $\lim_n \sup A_n = (-\infty,x)$ or $(-\infty,x]$. For if $y \in A_n$ for infinitely many n then $x_n > y$ for infinitely many n, hence $\lim_n \sup x_n \ge y$. Thus $\lim_n \sup A_n \subset (-\infty,x]$. If $y < x$ then $x_n > y$ for infinitely many n, so $y \in \lim\sup A_n$. Thus $(-\infty,x) \subset \lim_n \sup A_n$, and the result follows. (The same analysis is valid for lim inf, with "eventually" replacing "for infinitely many n".)

7. (continued)

Examples: If $x_n = \frac{1}{n}$ then $\lim_n \sup A_n = \lim_n \inf A_n = (-\infty,x]$, $x = 0$. If $x_n = -\frac{1}{n}$ then $\lim\sup A_n = \lim\inf A_n = (-\infty,x)$, $x = 0$.

10. This is answered by the argument of Problem 9.

11. If $0 < \epsilon \le 1$, $P\{|R_n| \ge \epsilon\} = P\{R_n = 1\} = p_n$, hence $R_n \xrightarrow{P} 0$ iff $\lim_{n\to\infty} p_n = 0$. But $R_n \xrightarrow{a.s.} 0$ iff $\sum_{n=1}^{\infty} p_n < \infty$; this follows from Problem 9.

13. Let $\Omega = \{a,b\}$, $P\{a\} = p$, $P\{b\} = 1-p$; take $R_n(a) = 0$ for all n, $R_n(b) = 1$ for all n, $R(a) = R(b) = 0$. Then $P\{R_n \to R\} = P\{a\} = p$, which may be specified arbitrarily. There are many other possible examples.

Section 7.1

2. This follows from $\Pi^{n+1} = \Pi\,\Pi^n$, and an induction argument.

3. Let $S = \{-1,0,1\}$, $p_{0,-1} = p_{-1,1} = p_{1,0} = 1$, and let $g(x) = x^2$. Let the initial distribution be $p_0 = p_1 = p_{-1} = \frac{1}{3}$.

$$P\{R_3^{\,2} = 0 \mid R_1^{\,2} = 1,\ R_2^{\,2} = 1\} = P\{R_3=0 \mid R_1 = -1,\ R_2=1\} = 1.$$

But

$$P\{R_3^{\,2}=0 \mid R_2^{\,2}=1\} = \frac{P\{R_2^{\,2}=1, R_3^{\,2}=0\}}{P\{R_2^{\,2}=1\}} = \frac{P\{R_2=1, R_3=0\}}{P\{R_2=1\} + P\{R_2=-1\}}$$

$$= \frac{P\{R_2=1\}}{P\{R_2=1\} + P\{R_2=-1\}} < 1,$$ so $\{g(R_n)\}$ does not have the Markov property.

Section 7.2

2. $P\{R_n=i_n \mid R_{n+1}=i_{n+1},\ldots,R_{n+k}=i_{n+k}\} = P(A \mid B \cap C)$,

$A = \{R_n=i_n\}$, $B = \{R_{n+2}=i_{n+2},\ldots,R_{n+k}=i_{n+k}\}$, $C = \{R_{n+1}=i_{n+1}\}$.

But

$$P(A \mid B \cap C) = \frac{P(A \cap B \mid C)}{P(B \mid C)} = \frac{P(A \mid C)P(B \mid A \cap C)}{P(B \mid C)} .$$

Now $P(B \mid A \cap C) = P(B \mid C)$ by Problem 1, so $P(A \mid B \cap C) = P(A \mid C)$, the desired result.

Section 7.3

2. If i is essential and i leads to j, then since the equivalence class C of i is closed, we must have $j \in C$. But then i and j are equivalent, hence j leads to i. Conversely, if the condition is satisfied and i leads to j, then j leads to i, so that i and j are equivalent. Therefore $j \in C$, so C is closed.

3. Let $i \in C$, and assume i leads to $j \notin C$. There is a positive probability of reaching j from i, and once having reached j we cannot return to i. Thus there is a positive probability of never returning to i, hence $f_{ii} < 1$ and i is transient.

5. (a) Set $p_{ij} = p_j = P\{R_n=j\}$ for all $i,j \in S$.

(b) S forms a single aperiodic recurrent class. (Given that $R_0 = j$, the probability of never returning to j is

$$P\{R_n \neq j,\ n = 1,2,\ldots\} = \lim_{k\to\infty} [P\{R_1 \neq j\}]^k = 0.)$$

6. $p_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)}$, $n = 1,2,\ldots$ (with $f_{ii}^{(0)} = 0$)

$p_{ii}^{(0)} - 1 = 0 = f_{ii}^{(0)} p_{ii}^{(0)}$

Thus the sequence $\{p_{ii}^{(0)} - 1, p_{ii}^{(1)}, p_{ii}^{(2)}, \ldots\}$ is the convolution of $\{f_{ii}^{(n)}\}$ and $\{p_{ii}^{(n)}\}$, so $U(z) - 1 = H(z)U(z)$.

Section 7.4

3. $$\frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} = \sum_{r=1}^{d} \frac{1}{n} \sum_{\substack{k=1 \\ k\equiv r \bmod d}}^{n} p_{ij}^{(k)}$$

$$= \frac{1}{d}\sum_{r=1}^{d} \frac{d}{n} \sum_{t=0}^{[\frac{n-r}{d}]} p_{ij}^{(td+r)} .$$

By Theorem 2d and the fact that $a_n \to a$ implies $\frac{1}{n}\sum_{k=1}^{n} a_k \to a$, this $\to \frac{1}{d}\sum_{r=1}^{d} f_{ij}^{*}(r)\, d/\mu_j$ where $f_{ij}^{*}(r)$ is the probability of reaching j from i in a number of steps that is $\equiv r \bmod d$. Th

$$\frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} \to \frac{1}{\mu_j}\sum_{r=1}^{d} f_{ij}^{*}(r) = \frac{f_{ij}}{\mu_j} .$$

Section 7.5

1. $p_{ii}^{(nd)} = \sum_{k \in C} p_{ik}^{(nd-1)} p_{ki} = $ (if $i \in C_r$)

$$\sum_{k \in C_{r-1}} p_{ik}^{(nd-1)} p_{ki} .$$

By Theorem 2c of Section 7.4, $p_{ik}^{(nd-1)} \to d/\mu_k$, $k \in C_{r-1}$. By Fatou's lemma,

$$\frac{d}{\mu_i} = \liminf_n p_{ii}^{(nd)} \geq \sum_{k \in C_{r-1}} \frac{d}{\mu_k} p_{ki} = \sum_{k \in C} \frac{d}{\mu_k} p_{ki} .$$

But $\sum_{i \in C} \frac{1}{\mu_i} = 1$ by the discussion in the text. It follows as in Theorem 1a that

$$\frac{1}{\mu_i} = \sum_{k \in C} \frac{1}{\mu_k} p_{ki} .$$

If we assign probability 0 to states outside of C, we have a stationary distribution for the chain. Now any stationary distribution for the chain must assign probability 0 to states not in C. (If $\sum_i v_i p_{ij}^{(n)} = v_j$, $j \notin C$, let $n \to \infty$; since j is transient or recurrent null, $p_{ij}^{(n)} \to 0$ so $v_j = 0$.) Now a stationary distribution $\{v_j\}$ for the chain also induces a stationary distribution on each cyclically moving subclass D of C, relative to Π^d, namely $\{dv_j, j \in D\}$. (Note that $\sum_{j \in D} v_j = \frac{1}{d}$ for each subclass D, because of the cyclic movement.) By the argument in the text, $dv_j = d/\mu_j$, and the result follows.

Chapter 8

Section 8.2

2. (a) If $\theta_1 < \theta_2$,

$$p_{\theta_2}(x)/p_{\theta_1}(x) = \frac{e^{-n\theta_2}\theta_2^{x_1+\dots+x_n}/x_1!\dots x_n!}{e^{-n\theta_1}\theta_1^{x_1+\dots+x_n}/x_1!\dots x_n!} \quad x_1,\dots,x_n = 0,1,\dots$$

$$= e^{-n(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_1}\right)^{t(x)} \quad \text{where } t(x) = \sum_{k=1}^{n} x_k .$$

(b) If $\theta_1 < \theta_2$,

$$p_{\theta_2}(x)/p_{\theta_1}(x) = \frac{\theta_2^{t(x)}(1-\theta_2)^{n-t(x)}}{\theta_1^{t(x)}(1-\theta_1)^{n-t(x)}}$$

$$= \left(\frac{1-\theta_2}{1-\theta_1}\right)^n \left(\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}\right)^{t(x)} \quad \text{where } t(x) = x_1 + \dots + x_n,\ x_i = 0 \text{ or } 1.$$

Remark: The MLR with $t(x) = x$ holds when $p_\theta(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, the probability function of a binomial random variable with parameters n and θ, $0 \leq \theta \leq 1$; the argument is exactly as above.

For the sake of definiteness, we give the form of the UMP test at level α in case (b). We have

$$\varphi(x) = 1 \text{ if } \sum_{k=1}^{n} x_k > c$$

$$= 0 \text{ if } \sum_{k=1}^{n} x_k < c$$

$$= a \text{ if } \sum_{k=1}^{n} x_k = c .$$

where c is chosen $\in \{0,1,\dots,n\}$ so that $P_{\theta_0}\{x:t(x) > c\} + aP_{\theta_0}\{x:t(x)=c\} = \alpha$, i.e.

$$\sum \binom{n}{k}\theta_0^k(1-\theta_0)^{n-k} + a\binom{n}{c}\theta_0^c(1-\theta_0)^{n-c} = \alpha$$

2. (continued)

(c)

$$\frac{P_{\theta+1}(x)}{P_\theta(x)} = \frac{\binom{\theta+1}{x}\binom{N-\theta-1}{n-x}}{\binom{\theta}{x}\binom{N-\theta}{n-x}} = \frac{\theta+1}{\theta+1-x}\,\frac{N-\theta-n+x}{N-\theta},$$

which is an increasing function of $t(x) = x$.

(d) If $\theta_1 < \theta_2$,

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{(2\pi\theta_2)^{-n/2}\, e^{-\sum_{k=1}^{n} x_k^2/2\theta_2}}{(2\pi\theta_1)^{-n/2}\, e^{-\sum_{k=1}^{n} x_k^2/2\theta_1}}$$

$$= \left(\frac{\theta_1}{\theta_2}\right)^{n/2} \exp\left[\left(\frac{1}{2\theta_1} - \frac{1}{2\theta_2}\right)t(x)\right]$$

where $t(x) = \sum_{k=1}^{n} x_k^2$.

4. The test is of the form: reject H_0 if $\sum_{k=1}^{n} x_k > c$, accept H_0 if $\sum_{k=1}^{n} x_k < c$, where $c = n\theta_0 + \sqrt{n}\,\sigma N_\alpha$, $\alpha \leq .05$ (see Example 3 of Section 8.2). From the table of the normal distribution function, $N_\alpha \geq 1.64$. Also,

$$.03 \geq \beta = F^*\left(\frac{c-n\theta_1}{\sqrt{n}\,\sigma}\right).$$

Let M_β be the number such that $F^*(M_\beta) = \beta$.

4. (continued)

Then $c = n\theta_1 + \sqrt{n}\,\sigma M_\beta$, $M_\beta \leq M_{.03} = -1.88$. Thus

$$\frac{c-n\theta_0}{\sqrt{n}\,\sigma} \geq 1.64$$

$$\frac{c-n(\theta_0+\sigma)}{\sqrt{n}\,\sigma} \leq -1.88.$$

Subtract the second equation from the first to obtain $\sqrt{n} \geq 3.52$, or $n \geq 12.4$. Thus the minimum value of n is 13.

7. By Problem 6, $\beta = 1-Q(\theta_1) = (1-\alpha)(\theta_0/\theta_1)^n = (1-\alpha)2^{-n}$ so the set of admissible risk points is $\{(\alpha,(1-\alpha)2^{-n}): 0 \leq \alpha \leq 1\}$. The upper boundary of the risk set is $\{(1-\alpha,1-(1-\alpha)2^{-n}): 0 \leq \alpha \leq 1\}$ $\{(\alpha,1-\alpha 2^{-n}): 0 \leq \alpha \leq 1\}$. Thus the risk set is $\{(\alpha,\beta): 0 \leq \alpha \leq 1,\ (1-\alpha)2^{-n} \leq \beta \leq 1-\alpha 2^{-n}\}$.

8. By Problem 2(b), we reject if $x_1 + x_2 + x_3 > c$, accept if $x_1 + x_2 + x_3 < c$.

k	$P_{1/4}\{x: x_1 + x_2 + x_3 = k\}$	$P_\theta\{x: x_1 + x_2 + x_3 = k\}$
0	$27/64$	$(1-\theta)^3$
1	$3(1/4)(3/4)^2 = 27/64$	$3\theta(1-\theta)^2$
2	$3(1/4)^2(3/4) = 9/64$	$3\theta^2(1-\theta)$
3	$1/64$	θ^3

Thus we take $c = 2$. We reject if $x_1 + x_2 + x_3 = 3$, accept if $x_1 + x_2 + x_3 = 0$ or 1, and if $x_1 + x_2 + x_3 = 2$ we reject with probability a, where $1/64 + 9a/64 = .1$, or $a = .6$. The power function is $Q(\theta) = \theta^3 + .6(3)\theta^2(1-\theta) = (9\theta^2-4\theta^3)/5$.

9. If φ is admissible let φ_λ be a LRT with the same error probabilities (Theorem 4). By the first proof of the Neyman-Pearson Lemma, φ_λ is Bayes with $c_1 = c_2 = 1$, $p = \lambda/1+\lambda$. (When $\lambda = \infty$ we have $p = 1$ and $\alpha_\lambda = 0$, hence $B(\varphi_\lambda) = 0$, so φ_λ is still Bayes in this case.) Since $\alpha = \alpha_\lambda$ and $\beta = \beta_\lambda$, φ is also Bayes by (8.2.3). Conversely, if φ is inadmissible and $c_1, c_2 > 0$, $0 < p < 1$, (8.2.3) shows that φ cannot be Bayes.

Let R be uniformly distributed between 0 and θ, and let $H_0: \theta = 1$, $H_1: \theta = 2$. Let $\varphi_1 \equiv 0$, and let $\varphi_2(x) = 0$, $0 \le x \le 1$; $\varphi_2(x) = 1$, $1 < x \le 2$. Then $\alpha(\varphi_1) = 0$, $\beta(\varphi_1) = 1$, $\alpha(\varphi_2) = 0$, $\beta(\varphi_2) = 1/2$. φ_1 and φ_2 are Bayes when $p = 1$ since $\beta(\varphi_1) = B(\varphi_2) = 0$. But φ_1 is inadmissible since φ_2 is better than φ_1.

11. Assume first that $\beta(\varphi) > 0$, hence φ is of size α by Problem 10. Since α is most powerful, it is admissible, hence by Problem 9, φ is a Bayes solution for some c_1, c_2 and p. But if $\lambda = pc_1/(1-p)c_2$, examination of the way the Bayes solution was constructed shows that $\varphi(x)$ must be 1 for $x > \lambda$, and $\varphi(x) = 0$ for $x < \lambda$, except for x in a set of Lebesgue measure 0. (If for example, $\varphi'(x) \le 1-\delta$ and $L(x) > \lambda$ on a set of positive Lebesgue measure, $B(\varphi')$ would be $> B(\varphi)$). If $\beta(\varphi) = 0$ then φ is a Bayes solution with $p = 0$ since in this case $B(\varphi) = 0$ by (8.2.3). Thus the above argument still applies.

12. Part (a) follows from the discussion after Theorem 3; (b) follows from (a) and Theorem 3. Part (c) holds since every LRT is Bayes (see the first proof of the Neyman-Pearson Lemma and the solution to Problem 9).

13. If $\alpha(\varphi_1) < \alpha(\varphi_2)$ but $\beta(\varphi_1) > \beta(\varphi_2)$, both statements are false. Numerical examples can be produced easily.

<u>Section 8.3</u>

1. (a) $\frac{\partial}{\partial\theta} \ln f_\theta(x_1,\dots,x_n) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i = 0$,

so $\hat\theta = -n/\sum_{i=1}^{n} \ln x_i$.

(b) $\frac{\partial}{\partial\theta} \ln f_\theta(x_1,\dots,x_n) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0$, so $\hat\theta = \bar{x}$.

(c) $f_\theta(x_1,\dots,x_n) = 1/\theta^n$ if $0 \le x_i \le \theta$ for all i

$= 0$ elsewhere.

Thus $\hat\theta = \max(x_1,\dots,x_n)$.

2. $\frac{\partial}{\partial\theta} \ln f_\theta(x) = \frac{1}{\theta} - \frac{2\theta}{x^2+\theta^2} = 0$, so $x^2 = \theta^2$. Thus $\hat\theta = |x|$.

3. $\frac{\partial}{\partial\theta} \ln p_\theta(x) = \frac{r}{\theta} - \frac{x-r}{1-\theta} = 0$, so $\hat\theta = \frac{r}{x}$.

4. $\rho(\theta) = E_\theta[(\frac{R}{n} - \theta)^2] = \frac{1}{n^2} \mathrm{Var}_\theta R = \frac{\theta(1-\theta)}{n}$. Note that $\max_{0 \le \theta \le 1} \rho(\theta) = \frac{1}{4n}$, which is larger than the risk of the minimax estimate.

7. By (8.3.2),

$$\psi(x) = \frac{\int_0^\infty e^{-\theta} \frac{e^{-\theta}\theta^x}{x!} \theta \, d\theta}{\int_0^\infty e^{-\theta} \frac{e^{-\theta}\theta^x}{x!} \, d\theta} = \frac{\Gamma(x+2)2^{-(x+2)}}{\Gamma(x+1)2^{-(x+1)}}$$

$$= \frac{x+1}{2}$$

7. (continued)

$$\rho_\psi(\theta) = E_\theta[(\tfrac{R+1}{2} - \theta)^2]$$

$$= \frac{1}{4} E_\theta[(R-\theta + 1-\theta)^2] = \frac{1}{4}[\mathrm{Var}_\theta R + (1-\theta)^2]$$

$$= \frac{1}{4}[\theta + (1-\theta)^2] = \frac{1}{4}(\theta^2 - \theta + 1)$$

$$B(\psi) = \int_0^\infty e^{-\theta} \frac{1}{4}(\theta^2-\theta+1)d\theta = \frac{1}{4}(2-1+1) = \frac{1}{2}.$$

The maximum likelihood estimate of θ is found by differentiating $\ln[e^{-\theta}\theta^x/x!]$ with respect to θ and setting the result equal to zero; we obtain $\hat{\theta} = x$. The risk function using $\hat{\theta}$ is $E_\theta[(R-\theta)^2] = \theta$, hence $B(\hat{\theta}) = \int_0^\infty \theta e^{-\theta} d\theta = 1 > B(\psi)$.

Section 8.4

4. $f_\theta(x,y) = \frac{1}{2\pi\sigma\tau} \exp[\frac{-(x-\theta)^2}{2\sigma^2} + \frac{(y-\theta)^2}{2\tau^2}]$;

since

$$\frac{(x-\theta)^2}{\sigma^2} + \frac{(y-\theta)^2}{\tau^2} = \frac{x^2}{\sigma^2} + \frac{y^2}{\tau^2} - 2\theta(\frac{x}{\sigma^2} + \frac{y}{\tau^2}) + \theta^2(\frac{1}{\sigma^2} + \frac{1}{\tau^2}),$$

the result follows.

5. (a) The results may be tabulated as follows:

	$a(\theta)$	$b(x)$	$c_1(\theta)$	$t_1(x)$	$c_2(\theta)$	$t_2(x$
(i)	$(1-\theta)^n$	$\binom{n}{x}$	$\ln\theta-\ln(1-\theta)$	x	--	--
(ii)	$e^{-\theta}$	$\frac{1}{x!}$	$\ln\theta$	x	--	--
(iii)	$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\mu^2/2\sigma^2}$	1	$-\frac{1}{2\sigma^2}$	x^2	$\frac{\mu}{\sigma^2}$	x
(iv)	$\frac{1}{\Gamma(\theta_1)\theta_2^{\theta_1}}$	1	θ_1-1	$\ln x$	$-\frac{1}{\theta_2}$	x
(v)	$\frac{1}{\beta(\theta_1,\theta_2)}$	1	θ_1-1	$\ln x$	θ_2-1	$\ln(1$
(vi)	$\frac{\theta^r}{(1-\theta)^r}$	$\binom{x-1}{r-1}$	$\ln(1-\theta)$	x	--	--

(b) It follows quickly from the Factorization Theorem that

$$(\sum_{j=1}^n t_1(R_j), \ldots, \sum_{j=1}^n t_k(R_j)) \text{ is sufficient.}$$

6. If φ is any test let $\varphi'(x) = E[\varphi(R)|T=t(x)]$. Then φ' is a test based on T, and $E_\theta\varphi'(R) = E_\theta\varphi(R)$ for all θ, hence $(\alpha(\varphi'),\beta(\varphi')) = (\alpha(\varphi),\beta(\varphi))$.

Section 8.5

1. $\gamma(\theta) = \sum_{k=0}^\infty (-1)^k\theta^k/k!$, so the UMVUE is

$$\sum_{i=0}^T \frac{T!}{i!} \frac{(-1)^{T-i}}{(T-i)!\,n^{T-i}} = (1 - \frac{1}{n})^T.$$

3. $\bar{R}$ is sufficient by Example 3, Section 8.4. Now $\bar{R}$ is normal $(\theta, \sigma^2/n)$, hence

$$E_\theta g(\bar{R}) = \int_{-\infty}^{\infty} \left(\frac{n}{2\pi\sigma^2}\right)^{1/2} g(y) e^{-(y-\theta)^2 n/2\sigma^2}\, dy$$

$$= \left(\frac{n}{2\pi\sigma^2}\right)^{1/2} e^{-n\theta^2/2\sigma^2} \int_{-\infty}^{\infty} g(y) e^{-ny^2/2\sigma^2} e^{n\theta y/\sigma^2}\, dy.$$

If $E_\theta g(\bar{R}) = 0$ for all $\theta > 0$ then $g(y)e^{-ny^2/2\sigma^2} = 0$ for all y, hence $g(y) = 0$ for all y (except on a set of Lebesgue measure 0). Thus as in Problem 2a, $P_\theta\{g(\bar{R}) = 0\} = 1$, hence $\bar{R}$ is complete.

Since $E(\bar{R}) = \theta$, $\bar{R}$ is a UMVUE of θ; since $\frac{\sigma^2}{n} = \text{Var}\,\bar{R} = E[(\bar{R})^2] - (E\bar{R})^2 = E[(\bar{R})^2] - \theta^2$, $(\bar{R})^2 - \frac{\sigma^2}{n}$ is a UMVUE of θ^2.

7. $E(R_1 \cdots R_j \mid \sum_{i=1}^n R_i = k) = P\{R_1 = \cdots = R_j = 1 \mid \sum_{i=1}^n R_i = k\}$

$$= P\{R_1 = \cdots = R_j = 1, \sum_{i=j+1}^n R_i = k-j\} / P\{\sum_{i=1}^n R_i = k\}$$

$$= \theta^j \binom{n-j}{k-j} \theta^{k-j} (1-\theta)^{n-j-(k-j)} / \binom{n}{k} \theta^k (1-\theta)^{n-k}$$

$$= \frac{\binom{n-j}{k-j}}{\binom{n}{k}} = \frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)}$$ as in Example 1.

8. $E(R_1R_2) = E(R_1)E(R_2) = \theta^2$, hence $E(R_1R_2 \mid \sum_{i=1}^n R_i = k)$ is an unbiased estimate of θ^2 based on a complete sufficient statistic. By Example 2, Section 8.5, $E(R_1R_2 \mid \sum_{i=1}^n R_i = k) = k(k-1)/n^2$.

10. Assume ψ is a best estimate of θ. Let $\psi'(x) \equiv \theta_0$. Then $\rho_\psi(\theta) \le \rho_{\psi'}(\theta) = (\theta-\theta_0)^2$, hence $\rho_\psi(\theta_0) = E_{\theta_0}[(\psi(R)-\theta_0)^2] = 0$. Consequently $\psi(R) \equiv \theta_0$. But θ_0 is arbitrary, so this is a contradiction.

12. (a) $E_\theta[\frac{\partial}{\partial\theta} \ln f_\theta(R)] = \int_{-\infty}^{\infty} [\frac{\partial}{\partial\theta} \ln f_\theta(x)] f_\theta(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{1}{f_\theta(x)} \frac{\partial f_\theta(x)}{\partial x} f_\theta(x)\,dx = \frac{\partial}{\partial\theta} \int_{-\infty}^{\infty} f_\theta(x)\,dx = \frac{\partial}{\partial\theta}(1) = 0$$

(b) $\gamma'(\theta) = \frac{\partial}{\partial\theta} E_\theta \psi(R) = \frac{\partial}{\partial\theta} \int_{-\infty}^{\infty} \psi(x) f_\theta(x)\,dx$

$$= \int_{-\infty}^{\infty} \psi(x) \frac{\partial f_\theta(x)}{\partial\theta} \frac{1}{f_\theta(x)} f_\theta(x)\,dx$$

$$= \int_{-\infty}^{\infty} \psi(x) (\frac{\partial}{\partial\theta} \ln f_\theta(x)) f_\theta(x)\,dx$$

$$= E_\theta[\psi(R) \frac{\partial}{\partial\theta} \ln f_\theta(R)].$$

(c) By the Schwarz inequality,

$$[\text{Cov}_\theta(\psi(R), \frac{\partial}{\partial\theta} \ln f_\theta(R))]^2 \le \text{Var}_\theta \psi(R)\ \text{Var}_\theta \frac{\partial}{\partial\theta} \ln f_\theta$$

The result follows from (a) and (b).

13. The sample variance is not changed by replacing R_i by $R_i - \mu$, so we may assume without loss of generality that $\mu = 0$. Then

$$E[(R_i - \bar{R})^2] = E[(R_i - \frac{1}{n}\sum_{j=1}^n R_j)^2]$$

$$= E(R_i^2) - \frac{2}{n} E(R_i^2) + \frac{1}{n^2} \sum_{j=1}^n E(R_j^2)$$

$$= \sigma^2 (1 - \frac{2}{n} + \frac{1}{n}) = (\frac{n-1}{n})\sigma^2.$$

Section 8.6

3. (a) $T = \sqrt{n}\, R_1/\sqrt{R_2}$ where R_1 is normal (0,1) and R_2 is chi-square (n). Thus $T^2 = nR_1^2/R_2 = R_1^2/(R_2/n)$. R_1^2 is chi-square (1), so that T^2 is F(1,n).

(b) $1/R = (R_2/n)/(R_1/m)$ where R_1 is chi-square (m) and R_2 is chi-square (n), and the result follows.

(c) This is immediate from the fact that a chi-square (n) random variable is representable as $W_1^2 + \dots + W_n^2$ where the W_i are independent and normal (0,1).

4. (a) $\sum_{i=1}^{n} \left(\frac{R_i-\mu}{\sigma}\right)^2 = \frac{W}{\sigma^2}$ is chi-square with n degrees of freedom. If h_n is the chi-square (n) density and a and b are chosen so that $\int_a^b h_n(x)dx = 1-\alpha$ then $P\{a \le \frac{W}{\sigma^2} \le b\} = 1-\alpha$. Therefore $[\frac{W}{b},\frac{W}{a}]$ is a confidence interval for σ^2 with confidence coefficient $1-\alpha$.

(b) If V^2 is the sample variance, nV^2/σ^2 is chi-square with n-1 degrees of freedom. Thus if a and b are chosen so that $\int_a^b h_{n-1}(x)dx = 1-\alpha$ then $P\{a \le \frac{nV^2}{\sigma^2} \le b\} = 1-\alpha$. Therefore $[\frac{nV^2}{b}, \frac{nV^2}{a}]$ is a confidence interval for σ^2 with confidence coefficient $1-\alpha$.

5. $\bar{R}_i-\mu_i$ is normal $(0,\sigma^2/n_i)$, $i = 1,2$, hence

$$\frac{\bar{R}_1-\bar{R}_2 - (\mu_1-\mu_2)}{\sigma\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{1/2}}$$

is normal (0,1). (Note that this result may be used to construct confidence intervals if σ^2 is known.) Since

$$\frac{n_1V_1^2}{\sigma^2} + \frac{n_2V_2^2}{\sigma^2}$$

is chi-square $(n_1 - 1 + n_2 - 1)$ by Problem 3c,

$$\left|\frac{(n_1+n_2-2)n_1n_2}{n_1+n_2}\right|^{1/2} \frac{(\bar{R}_1-\bar{R}_2 - (\mu_1-\mu_2))}{(n_1V_1^2 + n_2V_2^2)^{1/2}}$$

is $t(n_1+n_2-2)$ and the result follows.

6. $n_iV_i^2/\sigma_i^2$ is chi-square (n_i-1), $i = 1,2$, hence

$$\frac{n_2V_2^2/(n_2-1)\sigma_2^2}{n_1V_1^2/(n_1-1)\sigma_1^2} \text{ is } F(n_2-1,n_1-1).$$

If S^2 denotes the corrected sample variance $\frac{n}{n-1} V^2 = \frac{1}{n-1} \sum_{i=1}^{n} (R_i-\bar{R})^2$ then

$$\frac{S_2^2}{S_1^2}\,\frac{\sigma_1^2}{\sigma_2^2} \text{ is } F(n_2-1,n_1-1)$$

and this allows construction of confidence intervals in the usual way.

7. (a) $E_\theta\varphi_k(R) = P_\theta\{k \notin C(R)\} = 1-P_\theta\{k \in C(R)\}$.

If H_0 is true, $k = \gamma(\theta)$ hence $P_\theta\{k \in C(R)\} \geq 1-\alpha$ and the result follows.

(b) $P_\theta\{\gamma(\theta) \in C(R)\} = P_\theta\{x: \varphi_k(x) = 0\}$ where $k = \gamma(\theta)$

(note φ_k exists for each k of the form $\gamma(\theta)$, by hypothesis)

$= 1-P_\theta\{x: \varphi_k(x) = 1\}$ since the tests are nonrandomized

$= 1-E_\theta\varphi_k(R)$.

But when the true parameter is θ then the null hypothesis that $\gamma(\theta) = k$ is true, hence $E_\theta\varphi_k(R) \leq \alpha$ and the result follows.