SOLUTIONS MANUAL
for
Stochastic Modeling:
Analysis and Simulation

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Contents

Preface

This manual contains solutions to the problems in *Stochastic Modeling: Analysis and Simulation* that do not require computer simulation. For obvious reasons, simulation results depend on the programming language, the pseudorandom-number generators and the random-variate-generation routines in use. The manual does include pseudocode for many of the simulations, however. A companion disk contains SIMAN, SLAM, Fortran and C code for the simulation "cases" in the text.

I had help preparing this manual. Jaehwan Yang checked many of the solutions. Peg Steigerwald provided expert LaTeX typing of the first draft. And Hasan Jap coded the SLAM, Fortran and C cases. Please report any errors, typographical or substantial, to me at

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Chapter 2

Sample Paths

1. The simulation of the self-service system ends at time 129 minutes. Self-service customers 7 and 13 experience delays of 1 and 7 minutes, respectively, while full-service customers 19 and 20 experience delays of 3 and 7 minutes, respectively. All other customers have delays of 0 minutes.

   The simulation of the full-service systems ends at time 124 minutes. Self-service customers 7 and 13 experience delays of 1 and 5 minutes, respectively, while full-service customer 20 experiences a delay of 1 minute. All other customers have delays of 0 minutes.

2. The sample-average gap is 6 minutes. The histogram may resemble an exponential distribution.

3. If the modification does not change any service times, then the new simulation will be the same as the simulation of the self-service system with the following exception:

   At time 84, self-service customer 13 will switch to the full-service queue, shortening the customer's delay to 5 minutes.

   Other troublesome assumptions:

   Some full-service customers might use self-service if the full-service queue is excessively long.

   We have not accounted for how self-service customers pay for their jobs.

4. See Exercise 3.

5.

Inputs:	arrival times	service times
Logic:	first-come-first-served	
system events:	arrival	finish

part	old service times	new service times
1	$8-5=3$	2.7
2	$11-8=3$	2.7
3	$17-11=6$	5.4
4	$20-17=3$	2.7

time	system event	next arrival	next finish	number of parts
0	–	5	–	0
5	arrival	6	5+2.7 = 7.7	1
6	arrival	10	7.7	2
7.7	finish	10	7.7+2.7 = 10.4	1
10	arrival	15	10.4	2
10.4	finish	15	10.4+5.4 = 15.8	1
15	arrival	–	15.8	2
15.8	finish	–	15.8+2.7 = 18.5	1
18.5	finish	–	–	0

6.

cust no.	arrival	finish	collate?	old copy time	new copy time	new finish	
0	0.00	0.00	0.0	0.00	0.00	0.00	
1	0.09	2.33	0.8	2.24	1.79	1.88	
2	0.35	3.09	0.8	0.76	0.61	2.49	
3	0.54	4.39	1.1	1.30	1.43	3.92	
4	0.92	4.52	0.8	0.13	0.10	4.02	
5	2.27	4.62	1.1	0.10	0.11	4.13	
6	3.53	4.64	1.1	0.02	0.02	4.16	
7	3.66	4.80	0.8	0.16	0.13	4.28	
8	4.63	4.86	0.8	0.06	0.05	4.68	
9	7.33	8.24	0.8	0.91	0.73	8.06	
10	7.61	8.50	1.1	0.26	0.29	8.34	
					new makespan =	8.34	(or 8.35)
					old makespan =	8.50	

7. Inputs: Number of hamburgers demanded each day.

   Logic: Order only on odd-numbered days. Only order if ≤ 600 patties. Order 1000 minus number in stock.

   System event: morning count

day	demand	
1	$(415+585)-704$	$=296$
2	$704-214$	$=490$
3	$(214+786)-856$	$=144$
4	$856-620$	$=236$
5	$620-353$	$=267$
6	$353-172$	$=181$
7	$(172+828)-976$	$=24$
8	$976-735$	$=241$
9	$735-433$	$=302$
10	$433-217$	$=216$
11	$(217+783)-860$	$=140$
12	$860-598$	$=262$
13	$(598+402)-833$	$=167$

day	beginning inventory	demand	next order	size of order
1	415	296	2	585
2	704	490	–	–
3	214	144	4	786
4	856	236	–	–
5	620	267	–	–
6	353	181	–	–
7	172	24	8	828
8	976	241	–	–
9	735	302	–	–
10	433	216	–	–
11	217	140	12	783
12	860	262	–	–
13	598	167	–	–
14	431	–	–	–

No sales were lost. The only change is that no order is placed on day 13.

8. The available sample path does not permit us to extract the number and timing of light failures if they remained beyond one year. This is the critical input needed for the simulation.

9. Inputs: arrival times, processing times

   Logic: first-come-first-served

   System events: arrival of a job, finishing a job

   Here is a portion of a sample path that would result from the sequence of rolls 5, 1, 2, . . .

time	system event	number of jobs	next arrival	next finish
0	–	0	30	∞
30	arrival	1	60	30+38 = 68
60	arrival	2	90	68
68	finish	1	90	68+20 = 88
88	finish	0	90	∞
90	arrival	1	120	90+50 = 140

10. Inputs: arrival times, processing times, CPU assignment

    Logic: first-come-first-served at each CPU, random distribution to CPUs

    System Events: job arrival, job completion at A, job completion at B

    Here is a portion of a sample path that would result from the assignment A, A, B, A, . . .

time	system event	number at A	number at B	next arrival	next A	next B
0	–	0	0	40	∞	∞
40	arrival	1	0	80	40+70 = 110	∞
80	arrival	2	0	120	110	∞
110	complete A	1	0	120	110+70 = 180	∞
120	arrival	1	1	160	180	120+70 = 190
160	arrival	2	1	200	180	190

11. No answer provided.

12. No answer provided.

13. No answer provided.

Chapter 3

Basics

1. (a)

$$\begin{aligned}\Pr\{X = 4\} &= F_X(4) - F_X(3) \\ &= 1 - 0.9 = 0.1\end{aligned}$$

   (b)

$$\begin{aligned}\Pr\{X \neq 2\} &= 1 - \Pr\{X = 2\} \\ &= 1 - 0.3 = 0.7\end{aligned}$$

   (c)

$$\begin{aligned}\Pr\{X < 3\} &= \Pr\{X \leq 2\} \\ &= F_X(2) = 0.4\end{aligned}$$

   (d)

$$\begin{aligned}\Pr\{X > 1\} &= 1 - \Pr\{X \leq 1\} = 1 - F_X(1) \\ &= 1 - 0.1 = 0.9\end{aligned}$$

2. (a) 40 minutes = 40/60 hour = 2/3 hour

$$\begin{aligned}\Pr\{Y > 2/3\} &= 1 - \Pr\{Y \leq 2/3\} \\ &= 1 - F_Y(2/3) = 1 - (1 - e^{-2(2/3)}) \\ &= e^{-4/3} \approx 0.264\end{aligned}$$

   (b)

$$\begin{aligned}\Pr\{Y > 1\} &= 1 - \Pr\{Y \leq 1\} \\ &= 1 - F_Y(1) = 1 - (1 - e^{-2(1)}) \\ &= e^{-2} \approx 0.135\end{aligned}$$

(c) 10 minutes = 1/6 hour

$$\begin{aligned}
\Pr\{1/6 < Y \leq 2/3\} &= F_Y(2/3) - F_Y(1/6) \\
&= (1 - e^{-4/3}) - (1 - e^{-1/3}) \\
&= e^{-1/3} - e^{-4/3} \approx 0.453
\end{aligned}$$

(d) 5 minutes = 1/12 hour

$$\begin{aligned}
\Pr\{Y \leq 1/12\} = F_Y(1/12) &= 1 - e^{-2(1/12)} \\
&= 1 - e^{-1/6} \approx 0.154
\end{aligned}$$

3. (a)

$$\begin{aligned}
f_X(a) &= \frac{d}{da} F_X(a) \\
&= \begin{cases} 2a/\delta^2, & 0 \leq a \leq \delta \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(b) δ

(c) $\mathrm{E}[X] = \int_0^\delta a(2a/\delta^2)\, da = (2/3)\delta$

4. (a)

$$\begin{aligned}
F_X(a) &= \int_{-\infty}^{a} f_X(b) db \\
&= \begin{cases} 0, & a < 0 \\ a^3/\beta^3, & 0 \leq a \leq \beta \\ 1, & \beta < a \end{cases}
\end{aligned}$$

(b) 0

(c) $\mathrm{E}[X] = \int_0^\beta a(3a^2/\beta^3)\, da = (3/4)\ \beta$

5. (a)

$$\begin{aligned}
\mathrm{E}[Y] &= \int_0^2 a(3/16\ a^2 + 1/4) da \\
&= 5/4
\end{aligned}$$

(b)

$$\begin{aligned} F_Y(a) &= \int_{-\infty}^{a} f_Y(b)db \\ &= \begin{cases} 0, & a < 0 \\ a^3/16 + a/4, & 0 \le a \le 2 \\ 1, & 2 < a \end{cases} \end{aligned}$$

(c) $1\frac{1}{2}$, because the density function is larger in a neighborhood of $1\frac{1}{2}$ than in a neighborhood of $\frac{1}{2}$.

6. (a)

$$\begin{aligned} \Pr\{X = 4 \mid X \neq 1\} &= \frac{\Pr\{X = 4, X \neq 1\}}{\Pr\{X \neq 1\}} \\ &= \frac{\Pr\{X = 4\}}{\Pr\{X \neq 1\}} = \frac{1 - F_X(3)}{1 - F_X(1)} \\ &= \frac{1 - 0.9}{1 - 0.1} = 1/9 \end{aligned}$$

(b)

$$\begin{aligned} \Pr\{X = 4 \mid X \neq 1, X \neq 2\} &= \frac{\Pr\{X = 4\}}{\Pr\{X \neq 1, X \neq 2\}} \\ &= \frac{1 - F_X(3)}{1 - F_X(2)} = \frac{1 - 0.9}{1 - 0.4} = 1/6 \end{aligned}$$

(c)

$$\begin{aligned} \Pr\{X = 2 \mid X \le 2\} &= \frac{\Pr\{X = 2, X \le 2\}}{\Pr\{X \le 2\}} \\ &= \frac{\Pr\{X = 2\}}{\Pr\{X \le 2\}} = \frac{F_X(2) - F_X(1)}{F_X(2)} \\ &= \frac{0.4 - 0.1}{0.4} = 3/4 \end{aligned}$$

7. (a)

$$\begin{aligned} \Pr\{Y > 2/3 \mid Y > 1/2\} &= \frac{\Pr\{Y > 2/3, Y > 1/2\}}{\Pr\{Y > 1/2\}} \\ &= \frac{\Pr\{Y > 2/3\}}{\Pr\{Y > 1/2\}} = \frac{1 - F_Y(2/3)}{1 - F_Y(1/2)} = \frac{e^{-2(2/3)}}{e^{-2(1/2)}} \\ &= e^{-2(1/6)} \approx 0.72 \end{aligned}$$

(b) $\Pr\{Y > 1 \mid Y > 2/3\} = \frac{e^{-2(1)}}{e^{-2(2/3)}} = e^{-2(1/3)} \approx 0.51$

(c)

$$\begin{aligned}
\Pr\{Y < 1/6 \mid Y < 2/3\} &= \frac{\Pr\{Y < 1/6\}}{\Pr\{Y < 2/3\}} \\
&= \frac{F_Y(1/6)}{F_Y(2/3)} = \frac{1 - e^{-2(1/6)}}{1 - e^{-2(2/3)}} \approx 0.38
\end{aligned}$$

8. (a)

$$\begin{aligned}
\Pr\{W = 3 \mid V = 2\} &= \frac{\Pr\{W = 3, V = 2\}}{\Pr\{V = 2\}} \\
&= \frac{p_{VW}(2,3)}{p_V(2)} = \frac{3/20}{12/20} = 3/12 = 1/4
\end{aligned}$$

(b)

$$\begin{aligned}
\Pr\{W \le 2 \mid V = 1\} &= \frac{\Pr\{W \le 2, V = 1\}}{\Pr\{V = 1\}} \\
&= \frac{\Pr\{W = 1, V = 1\} + \Pr\{W = 2, V = 1\}}{p_V(1)} \\
&= \frac{2/10 + 1/10}{4/10} = \frac{3/10}{4/10} = 3/4
\end{aligned}$$

(c)

$$\begin{aligned}
\Pr\{W \ne 2\} &= 1 - \Pr\{W = 2\} \\
&= 1 - p_W(2) = 1 - 10/20 = 10/20 = 1/2
\end{aligned}$$

9. Joint distribution

		X_2		
		0	1	
X_1	0	0.75	0.05	0.80
	1	0.10	0.10	0.20
		0.85	0.15	

(a)

$$\begin{aligned}
\Pr\{X_2 = 1 \mid X_1 = 0\} &= \frac{\Pr\{X_2 = 1, X_1 = 0\}}{\Pr\{X_1 = 0\}} \\
&= \frac{0.05}{0.80} = 0.0625 \\
\Pr\{X_2 = 1 \mid X_1 = 1\} &= \frac{0.10}{0.20} = 0.50
\end{aligned}$$

Clearly X_1 and X_2 are dependent since $\Pr\{X_2 = 1 \mid X_1 = 0\} \neq \Pr\{X_2 = 1 \mid X_1 = 1\}$.

(b) Let $g(a) = \begin{cases} 100, & a = 0 \\ -20, & a = 1 \end{cases}$

$$\begin{aligned}
\mathrm{E}[g(X_2) \mid X_1 = 1] &= g(0)\Pr\{X_2 = 0 \mid X_1 = 1\} \\
&\quad + g(1)\Pr\{X_2 = 1 \mid X_1 = 1\} \\
&= 100(1 - 0.50) - 20(0.50) \\
&= 40
\end{aligned}$$

10. $\widehat{\mu} = 2.75$ $\widehat{\sigma} = 0.064$ $\widehat{\mathrm{se}} = \frac{\widehat{\sigma}}{\sqrt{6}} \approx 0.026$

$\widehat{\mu} + \widehat{\mathrm{se}} = 2.776$

$\widehat{\mu} - \widehat{\mathrm{se}} = 2.724$

Assuming a mean of 2.776

$$\begin{aligned}
\Pr\{X > 2.90\} &= 1 - \Pr\{X \leq 2.90\} \\
&= 1 - \Pr\left\{\frac{X - 2.776}{0.064} \leq \frac{2.90 - 2.776}{0.064}\right\} \\
&= 1 - \Pr\{Z \leq 1.938\} \\
&\approx 1 - 0.974 = 0.026
\end{aligned}$$

where Z is a standard-normal random variable.

Assuming a mean of 2.724

$$\begin{aligned}
\Pr\{X > 2.90\} &= 1 - \Pr\left\{Z \leq \frac{2.90 - 2.724}{0.064}\right\} \\
&= 1 - \Pr\{Z \leq 2.75\} \approx 1 - 0.997 = 0.003
\end{aligned}$$

11. (a) $\widehat{\mu} = 8.08 \quad \widehat{\sigma} \approx 3.60$
    (b) $\widehat{\text{se}} = \frac{\widehat{\sigma}}{\sqrt{5}} \approx 1.61$
    (c)
$$\widehat{F}(a) = \begin{cases} 0, & a < 3.5 \\ 1/5, & 3.5 \le a < 5.9 \\ 2/5, & 5.9 \le a < 7.7 \\ 3/5, & 7.7 \le a < 11.1 \\ 4/5, & 11.1 \le a < 12.2 \\ 1, & 12.2 \le a \end{cases}$$

12. (a) $\widehat{\mu} \approx 10.98 \quad \widehat{\sigma} \approx 4.81$
    (b) $\widehat{\text{se}} \approx 1.08$
    (c) Use the definition $\widehat{F}(a) = \frac{1}{20}\sum_{i=1}^{20} \mathcal{I}(d_i \le a)$
    (d) try normal

13. (a) $\widehat{\mu} = 0.80 \quad \widehat{\sigma} \approx 0.89$
    (b) $\widehat{\text{se}} \approx 0.16$
    (c)
$$\widehat{F}(a) = \begin{cases} 0, & a < 0 \\ 14/30, & 0 \le a < 1 \\ 23/30, & 1 \le a < 2 \\ 29/30, & 2 \le a < 3 \\ 1, & 3 \le a \end{cases}$$
    (d)
$$\widehat{p}(a) = \begin{cases} 14/30, & a = 0 \\ 9/30, & a = 1 \\ 6/30, & a = 2 \\ 1/30, & a = 3 \\ 0, & \text{otherwise} \end{cases}$$

14. (a) $\widehat{\mu} \approx 6.05 \quad \widehat{\sigma} \approx 0.23$
    (b) $\widehat{\text{se}} \approx 0.05$
    (c) Use the definition $\widehat{F}(a) = \frac{1}{25}\sum_{i=1}^{25} \mathcal{I}(d_i \le a)$
    (d) try normal

15. Let X be a random variable taking values 0 (to represent a “tail”) and 1 (to represent a “head”). A plausible model is that X has a Bernoulli distribution with parameter $\gamma = \Pr\{X = 1\}$. Your date will likely show $\gamma > 1/2$.

16. Let U be a random variable having the uniform distribution of [0, 1].

Let $V = 1 - U$. Then

$$\begin{aligned}\Pr\{V \le a\} &= \Pr\{1 - U \le a\} \\ &= \Pr\{U \ge 1 - a\} = a,\ 0 \le a \le 1.\end{aligned}$$

Therefore, V and U are both uniformly distributed on [0, 1].

Therefore, $Y = -\ln(1-U)/\lambda = -\ln(V)/\lambda$ and $Y = -\ln(U)/\lambda$ must have the same distribution.

17. (a) $U = F(Y) = \frac{a-\alpha}{\beta-\alpha}$

Therefore, $Y = (\beta - \alpha)U + \alpha$

algorithm uniform

1. $U \leftarrow$ `random()`
2. $Y \leftarrow (\beta - \alpha)U + \alpha$
3. return Y

For $\alpha = 0,\ \beta = 4$

U	Y
0.1	0.4
0.5	2.0
0.9	3.6

(b) $U = F(Y) = 1 - e^{-(Y/\beta)^\alpha}$

Therefore,

$$\begin{aligned}1 - U &= e^{-(Y/\beta)^\alpha} \\ \ln(1-U) &= -(Y/\beta)^\alpha \\ Y &= \beta(-\ln(1-U))^{1/\alpha}\end{aligned}$$

algorithm Weibull

1. $U \leftarrow$ `random()`
2. $Y \leftarrow \beta(-\ln(1-U))^{1/\alpha}$
3. return Y

For $\alpha = 1/2,\ \beta = 1$

U	Y
0.1	0.011
0.5	0.480
0.9	5.302

For $\alpha = 2,\ \beta = 1$

U	Y
0.1	0.325
0.5	0.833
0.9	1.517

(c)

$$Y = \begin{cases} 1, & 0 \leq U \leq 0.3 \\ 2, & 0.3 < U \leq 0.4 \\ 3, & 0.4 < U \leq 0.7 \\ 4, & 0.7 < U \leq 0.95 \\ 6, & 0.95 < U \leq 1 \end{cases}$$

Using algorithm discrete_inverse_cdf with $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 6$.

U	Y
0.1	1
0.5	3
0.9	4

(d)

$$Y = \begin{cases} 0, & 0 \leq U \leq 1 - \gamma \\ 1, & 1 - \gamma < U \leq 1 \end{cases}$$

algorithm Bernoulli

1. $U \leftarrow$ `random()`
2. if $\{U \leq 1 - \alpha\}$ then
   $Y \leftarrow 0$
   else
   $Y \leftarrow 1$
   endif
3. return Y

For $\gamma = 1/4$

U	Y
0.1	0
0.5	0
0.9	1

(e)

$$\begin{aligned} p(a) &= F(a) - F(a-1) \\ &= 1 - (1-\gamma)^a - (1 - (1-\gamma)^{a-1}) \\ &= (1-\gamma)^{a-1}\gamma\ ,\ \ a = 1, 2, 3 \ldots \end{aligned}$$

$$Y = \begin{cases} 1, & 0 \le U \le \gamma \\ 2, & \gamma < U \le 1 - (1-\gamma)^2 \\ \vdots & \vdots \\ a, & 1 - (1-\gamma)^{a-1} < U \le 1 - (1-\gamma)^a \\ \vdots & \vdots \end{cases}$$

algorithm geometric
1. $U \leftarrow$ `random()`
   $a \leftarrow 1$
2. until $U \le 1 - (1-\gamma)^a$
   do
      $a \leftarrow a + 1$
   endo
3. $Y \leftarrow a$
4. return Y

For $\gamma = 1/4$

U	Y
0.1	1
0.5	2
0.9	9

18. (a)

$$f(a) = \frac{d}{da}F(a) = \begin{cases} 0, & a < \alpha \\ \frac{1}{\beta-\alpha}, & \alpha \le a \le \beta \\ 0, & \beta < a \end{cases}$$

(b) $f(a) = \frac{d}{da}F(a) = 0 - e^{-(a/\beta)^\alpha}(-\alpha(a/\beta)^{\alpha-1}(1/\beta))$

$$= \begin{cases} \alpha\, \beta^{-\alpha} a^{\alpha-1} e^{-(a/\beta)^\alpha}, & 0 < a \\ 0, & a \le 0 \end{cases}$$

(c) Using the relationship $p(a) = F(a) - F(a-1)$

$$p(a) = \begin{cases} 0.3, & a = 1 \\ 0.1, & a = 2 \\ 0.3, & a = 3 \\ 0.25, & a = 4 \\ 0.05, & a = 6 \end{cases}$$

(d)

$$p(a) = \begin{cases} 1-\gamma, & a=0 \\ \gamma, & a=1 \end{cases}$$

(e) From the solution of Exercise 17(e)

$$p(a) = \begin{cases} \gamma(1-\gamma)^{a-1}, & \text{for } a = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

19. (a)

$$\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} a f(a) da = \int_{\alpha}^{\beta} \left(\frac{a}{\beta-\alpha}\right) da \\
&= \left.\frac{a^2}{2(\beta-\alpha)}\right|_{\alpha}^{\beta} = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{(\beta+\alpha)(\beta-\alpha)}{2(\beta-\alpha)} \\
&= \frac{\beta+\alpha}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} a f(a) da = \int_{0}^{\infty} a\ \alpha\ \beta^{-\alpha} a^{\alpha-1} e^{-(a/\beta)^{\alpha}} da \\
&= \int_{0}^{\infty} a(\alpha/\beta)(a/\beta)^{\alpha-1} e^{-(a/\beta)^{\alpha}} da = (1)
\end{aligned}$$

Let $u = (a/\beta)^{\alpha}$ so that $du/da = (\alpha/\beta)(a/\beta)^{\alpha-1}$ and $a = \beta u^{1/\alpha}$. Then

$$\begin{aligned}
(1) &= \int_{0}^{\infty} \beta u^{1/\alpha}\ \frac{du}{da}\ e^{-u} da \\
&= \beta \int_{0}^{\infty} u^{1/\alpha} e^{-u} du \\
&= \beta \int_{0}^{\infty} u^{(1/\alpha+1)-1} e^{-u} du \\
&= \beta\Gamma(1/\alpha+1) = (\beta/\alpha)\Gamma(1/\alpha)
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{\text{all } a} a\ p_X(a) \\
&= 1(0.3) + 2(0.1) + 3(0.3) + 4(0.25) + 6(0.05) \\
&= 2.7
\end{aligned}$$

(d)

$$
\begin{aligned}
\mathrm{E}[X] &= \sum_{\text{all } a} a\ p_X(a) = 0(1-\gamma) + 1(\gamma) \\
&= \gamma
\end{aligned}
$$

(e)

$$
\begin{aligned}
\mathrm{E}[X] &= \sum_{\text{all } a} a\ p_X(a) = \sum_{a=1}^{\infty} a\gamma(1-\gamma)^{a-1} \\
&= \gamma \sum_{a=1}^{\infty} a(1-\gamma)^{a-1} \qquad (\text{let } q = 1-\gamma) \\
&= \gamma \sum_{a=1}^{\infty} \frac{d}{dq} q^a \\
&= \gamma \frac{d}{dq} \sum_{a=1}^{\infty} q^a \qquad = \gamma \frac{d}{dq} \sum_{a=0}^{\infty} \ q^a \\
& \qquad\qquad\qquad (\text{since } \frac{d}{dq} q^0 = 0) \\
&= \gamma \frac{d}{dq} \left(\frac{1}{1-q} \right) = \frac{\gamma}{(1-q)^2} = \frac{\gamma}{(\gamma)^2} = \frac{1}{\gamma}
\end{aligned}
$$

20. From Exercise 25, $\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$. So if we calculate $\mathrm{E}[X^2]$, we can combine it with the answers in Exercise 19.

(a)

$$
\begin{aligned}
\mathrm{E}[X^2] &= \int_{\alpha}^{\beta} \left(\frac{a^2}{\beta-\alpha} \right) da = \left. \frac{a^3}{3(\beta-\alpha)} \right|_{\alpha}^{\beta} \\
&= \frac{\beta^3 - \alpha^3}{3(\beta-\alpha)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Var}[X] &= \frac{\beta^3 - \alpha^3}{3(\beta-\alpha)} - \left(\frac{\beta+\alpha}{2} \right)^2 \\
&= \frac{\beta^3 - \alpha^3}{3(\beta-\alpha)} - \frac{(\beta+\alpha)^2}{4}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4(\beta^3-\alpha^3)}{12(\beta-\alpha)} - \frac{3(\beta+\alpha)^2(\beta-\alpha)}{12(\beta-\alpha)} \\
&= \frac{4(\beta-\alpha)(\beta^2+\beta\alpha+\alpha^2) - 3(\beta^2+2\beta\alpha+\alpha^2)(\beta-\alpha)}{12(\beta-\alpha)} \\
&= \frac{\beta^2-2\beta\alpha+\alpha^2}{12} = \frac{(\beta-\alpha)^2}{12}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathrm{E}[X^2] &= \int_0^\infty a^2 \alpha\beta^{-\alpha} a^{\alpha-1} e^{-(a/\beta)^\alpha} da \\
&= \int_0^\infty a^2 (\alpha/\beta)(a/\beta)^{\alpha-1} e^{-(a/\beta)^\alpha} da \\
&\qquad \text{Using the same substitution as 19(b) gives} \\
&= \int_0^\infty \beta^2 u^{2/\alpha} e^{-u} du = \beta^2 \int_0^\infty u^{2/\alpha} e^{-u} du \\
&= \beta^2 \Gamma(2/\alpha+1) = \frac{2\beta^2}{\alpha}\Gamma(2/\alpha)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Var}[X] &= \frac{2\beta^2}{\alpha}\ \Gamma(2/\alpha) - (\beta/\alpha\ \Gamma(1/\alpha))^2 \\
&= \frac{\beta^2}{\alpha}\ (2\Gamma(2/\alpha) - 1/\alpha\ (\Gamma(1/\alpha))^2)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathrm{E}[X^2] &= 1^2(0.3) + 2^2(0.1) + 3^2(0.3) + 4^2(0.25) + 6^2(0.05) \\
&= 9.2
\end{aligned}
$$

Therefore, $\mathrm{Var}[X] = 9.2 - (2.7)^2 = 1.91$

(d) $\mathrm{E}[X^2] = 0^2(1-\gamma) + 1^2(\gamma) = \gamma$
Therefore, $\mathrm{Var}[X] = \gamma - \gamma^2 = \gamma(1-\gamma)$

(e)

$$
\begin{aligned}
\mathrm{E}[X^2] &= \sum_{a=1}^\infty a^2\ \gamma(1-\gamma)^{a-1} \\
&= \gamma \sum_{a=1}^\infty a^2\ q^{a-1} \qquad (\text{let } q = 1-\gamma)
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma \sum_{a=1}^{\infty} \left(\frac{d^2}{dq^2} q^{a+1} - a q^{a-1} \right) \\
&= \gamma \frac{d^2}{dq^2} \sum_{a=1}^{\infty} q^{a+1} - \frac{1}{\gamma} \qquad \text{(from Exercise 19(e))} \\
&= \gamma \frac{d^2}{dq^2} \sum_{a=-1}^{\infty} q^{a+1} - \frac{1}{\gamma} \qquad \left(\text{since } \frac{d^2}{dq^2} q^0 = \frac{d^2}{dq^2} q = 0 \right) \\
&= \gamma \frac{d^2}{dq^2} \sum_{b=0}^{\infty} q^{b} - \frac{1}{\gamma} = \gamma \frac{d^2}{dq^2} \left(\frac{1}{1-q} \right) - \frac{1}{\gamma} \\
&= \gamma \frac{2}{(1-q)^3} - \frac{1}{\gamma} = \frac{2}{\gamma^2} - \frac{1}{\gamma}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Var}[X] &= \frac{2}{\gamma^2} - \frac{1}{\gamma} - \frac{1}{\gamma^2} = \frac{1}{\gamma^2} - \frac{1}{\gamma} \\
&= \frac{1-\gamma}{\gamma^2}
\end{aligned}
$$

21. No answer provided.

22. You need the solutions from Exercise 17. Using them provides the following inverse cdf functions

    (a) `B1 = -LN (1-A1)* 2`

    (b) `B1 = 4* A1`

    (c) `B1 = 2.257* (-LN (1-A1))** (1/2)`

    All of these distributions have expected value 2, but are otherwise **very** different.

23. Use exponential distributions for the time data, and parameterize them by using the fact that $1/\lambda$ is the expected value.

    <u>interarrival-time gaps</u>

    $\widehat{1/\lambda} = 6$ minutes (sample average)

    Therefore, $\widehat{\lambda} = 1/6$ customer/minute

    <u>self-service customers</u>

    $\widehat{1/\lambda} = 3$ minutes (sample average)

    Therefore, $\widehat{\lambda} = 1/3$ customer/minute

<u>full-service customers</u>

$\widehat{1/\lambda} = 7$ minutes (sample average)

Therefore, $\widehat{\lambda} = 1/7$ customer/minute

<u>all customers</u>

$\widehat{1/\lambda} = 4.6$ minutes (sample average)

Therefore, $\widehat{\lambda} = 1/4.6$ customer/minute

<u>customer type</u>

Use a Bernoulli distribution with $\gamma = \Pr\{\text{full-service customer}\}$

$\widehat{\gamma} = 8/20 = 0.4$

All of the models except one are very good fits, since in fact the interarrival time gaps, customer types and conditonal service times were generated from these distributions! The model for all customers' service times will not be as good since it is actually a mixture of two exponential distributions.

24. Suppose X has mass function p_X

$$\begin{aligned}
\mathrm{E}[aX+b] &= \sum_{\text{all } c}(ac+b)p_X(c) \\
&= a\sum_{\text{all } c} p_X(c) + b\sum_{\text{all } c} p_X(c) \\
&= a\mathrm{E}[X] + b
\end{aligned}$$

Suppose X has density function f_X

$$\begin{aligned}
\mathrm{E}[aX+b] &= \int_{-\infty}^{\infty}(ac+b)f_X(c)dc \\
&= a\int_{-\infty}^{\infty} cf_X(c)dc + b\int_{-\infty}^{\infty} f_X(c)dc \\
&= a\mathrm{E}[X] + b
\end{aligned}$$

25. Suppose X has a density f_X

$$\mathrm{Var}[X] = \int_{-\infty}^{\infty}(a-\mathrm{E}[X])^2 f_X(a)da$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} a^2 f_X(a)da - 2\mathrm{E}[X]\int_{-\infty}^{\infty} a f_X(a)da \\
&\qquad\qquad + (\mathrm{E}[X])^2 \int_{-\infty}^{\infty} f_X(a)da \\
&= \mathrm{E}[X^2] - 2\mathrm{E}[X]\mathrm{E}[X] + (\mathrm{E}[X])^2 \\
&= \mathrm{E}[X^2] - (\mathrm{E}[X])^2
\end{aligned}$$

The proof for discrete-valued X is analogous.

26.

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{a=0}^{\infty} a \Pr\{X=a\} \\
&= \sum_{a=1}^{\infty} \left\{ \sum_{i=1}^{a} \Pr\{X=a\} \right\} \\
&= \sum_{i=1}^{\infty}\sum_{a=i}^{\infty} \Pr\{X=a\} = \sum_{i=0}^{\infty} \left\{ \sum_{a=i+1}^{\infty} \Pr\{X=a\} \right\} \\
&= \sum_{i=0}^{\infty} \Pr\{X>i\}
\end{aligned}$$

27. $\mathrm{E}[X^m] = 0^m(1-\gamma) + 1^m\ \gamma = \gamma$

28. Notice that

$$\begin{aligned}
\mathrm{E}[\mathcal{I}(X_i \le a)] &= 0\ \Pr\{X_i > a\} + 1 \Pr\{X_i \le a\} \\
&= F_X(a)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}[\widehat{F}_X(a)] &= \frac{1}{m}\sum_{i=1}^{m} \mathrm{E}[\mathcal{I}(X_i \le a)] \\
&= \frac{1}{m}\ m\ F_X(a) = F_X(a)
\end{aligned}$$

29.

$$\begin{aligned}
\mathrm{E}[g(Y)] &= \int_{-\infty}^{\infty} g(a) f_Y(a)da \\
&= \int_{\mathcal{A}} 1\ f_Y(a)da + 0 \\
&= \Pr\{Y \in \mathcal{A}\}
\end{aligned}$$

30. (a) $F_X^{-1}(q) = -\ln(1-q)/\lambda$ for the exponential. To obtain the median we set $q = 0.5$. Then $\eta = -\ln(1-0.5)/\lambda \approx 0.69/\lambda$. This compares to $\mathrm{E}[X] = 1/\lambda$.
Notice that

$$\begin{aligned} F_X(1/\lambda) &= 1 - e^{-\lambda(1/\lambda)} \\ &= 1 - e^{-1} \approx 0.63 \end{aligned}$$

showing that the expected value is approximately the 0.63 quantile for all values of λ.

(b) $F_X^{-1}(q) = \alpha + (\beta - \alpha)q$ for the uniform distribution. For $q = 0.5$, $\eta = \alpha + (\beta - \alpha)0.5 = (\beta + \alpha)/2$ which is identical to the $\mathrm{E}[X]$.

31. We want $\Pr\{Y = b\} = 1/5$ for $b = 1, 2, \ldots, 5$.

$$\begin{aligned} \Pr\{Y = b\} &= \Pr\{X = b \mid X \neq 6\} \\ &= \frac{\Pr\{X = b, X \neq 6\}}{\Pr\{X \neq 6\}} \qquad \text{(definition)} \\ &= \frac{\Pr\{X = b\}}{\Pr\{X \neq 6\}} = \frac{1/6}{5/6} = 1/5 \end{aligned}$$

32. (a)

$$\begin{aligned} \Pr\{Y = a\} &= \Pr\left\{Z = a \,\middle|\, V \leq \frac{p_Y(Z)}{cp_Z(Z)}\right\} \\ &= \frac{\Pr\left\{Z = a,\ \ V \leq \frac{p_Y(Z)}{cp_Z(Z)}\right\}}{\Pr\left\{V \leq \frac{p_Y(Z)}{cp_Z(Z)}\right\}} = \frac{(1)}{(2)} \\ (2) &= \sum_{a \in \mathcal{B}} \Pr\left\{V \leq \frac{p_Y(a)}{cp_Z(a)} \mid Z = a\right\} p_Z(a) \\ &= \sum_{a \in \mathcal{B}} \left(\frac{p_Y(a)}{cp_Z(a)}\right) p_Z(a) = \frac{1}{c} \sum_{a \in \mathcal{B}} p_Y(a) = \frac{1}{c} \\ (1) &= \Pr\left\{V \leq \frac{p_Y(a)}{cp_Z(a)} \mid Z = a\right\} p_Z(a) \\ &= \frac{p_Y(a)}{cp_Z(a)} p_Z(a) = \frac{p_Y(a)}{c} \end{aligned}$$

Therefore, $\frac{(1)}{(2)} = p_Y(a)$.

(b) Let $T \equiv$ number of trials until acceptance. On each trial

$$\Pr\{\text{accept}\} = \Pr\left\{V \leq \frac{p_Y(Z)}{cp_Z(Z)}\right\} = \frac{1}{c}$$

Since each trial is independent, T has a geometric distribution with $\gamma = 1/c$. Therefore, $\mathrm{E}[T] = \frac{1}{1/c} = c$.

(c) algorithm 17(c)

1. roll a die to obtain Z
2. $V \leftarrow$ `random()`
3. if $V \leq \frac{p(Z)}{c\ 1/6}$ then
      return $Y \leftarrow Z$
   else
      goto step 1
endif

Here $c = \frac{0.3}{1/6} = 9/5$ because the largest value of p_Y is 0.3. Therefore, the expected number of trials is $9/5 = 1\frac{4}{5}$ (almost 2).

33. (a)

$$\begin{aligned}
\Pr\{Y \leq a\} &= \Pr\left\{Z \leq a \,\middle|\, V \leq \frac{f_Y(Z)}{cf_Z(Z)}\right\} \\
&= \frac{\Pr\left\{Z \leq a, \quad V \leq \frac{f_Y(Z)}{cf_Z(Z)}\right\}}{\Pr\left\{V \leq \frac{f_Y(Z)}{cf_Z(Z)}\right\}} = \frac{(1)}{(2)} \\
(2) &= \int_{-\infty}^{\infty} \Pr\left\{V \leq \frac{f_Y(Z)}{cf_Z(Z)} \mid Z = a\right\} f_Z(a)da \\
&= \int_{-\infty}^{\infty} \Pr\left\{V \leq \frac{f_Y(a)}{cf_Z(a)}\right\} f_Z(a)da \\
&= \int_{-\infty}^{\infty} \frac{f_Y(a)}{cf_Z(a)} f_Z(a)da = \frac{1}{c}\int_{-\infty}^{\infty} f_Y(a)da = \frac{1}{c} \\
(1) &= \int_{-\infty}^{\infty} \Pr\left\{Z \leq a, V \leq \frac{f_Y(Z)}{cf_Z(Z)} \mid Z = b\right\} f_Z(b)db \\
&= \int_{-\infty}^{a} \Pr\left\{V \leq \frac{f_Y(b)}{cf_Z(b)}\right\} f_Z(b)db + \int_{a}^{\infty} 0\ f_Z(b)db \\
&= \int_{-\infty}^{a} \frac{f_Y(b)}{cf_Z(b)} f_Z(b)db = \frac{1}{c}\int_{-\infty}^{a} f_Y(b)db = \frac{1}{c}F_Y(a)
\end{aligned}$$

Therefore, $\frac{(1)}{(2)} = F_Y(a)$.

(b) Let $T \equiv$ number of trials until acceptance. On each trial

$$\Pr\{\text{accept}\} = \Pr\left\{V \leq \frac{f_Y(Z)}{cf_Z(Z)}\right\} = \frac{1}{c}$$

Since each trial is independent, T has a geometric distribution with $\gamma = 1/c$. Therefore, $\mathrm{E}[T] = \frac{1}{1/c} = c$.

(c) Note that $f_Y(a)$ is maximized at $f_Y(1/2) = 1\frac{1}{2}$.
Let $f_Z(a) = 1, \quad 0 \leq a \leq 1$
$c = 1\frac{1}{2}$
Therefore, $cf_Z(a) = 1\frac{1}{2} \geq f_Y(a)$

1. $U \leftarrow$ `random()`
2. $Z \leftarrow U$
3. $V \leftarrow$ `random()`
4. if $V \leq 6Z(1-Z)/(1\frac{1}{2})$ then
   return $Y \leftarrow Z$
   else
   goto step 1
   endif

$\mathrm{E}[T] = 1\frac{1}{2}$

34. (a)

$$\begin{aligned}
F(a) &= \int_\alpha^a \frac{2(b-\alpha)}{(\beta-\alpha)^2} db \\
&= \frac{(a-\alpha)^2}{(\beta-\alpha)^2}, \quad \alpha \leq a \leq \beta \\
U &= \frac{(X-\alpha)^2}{(\beta-\alpha)^2} \\
(X-\alpha)^2 &= U(\beta-\alpha)^2 \\
X &= \alpha + \sqrt{U(\beta-\alpha)^2} = \alpha + (\beta-\alpha)\sqrt{U}
\end{aligned}$$

algorithm
1. $U \leftarrow$ `random()`
2. $X \leftarrow \alpha + (\beta-\alpha)\sqrt{U}$
3. return X

(b) f is maximized at $a = \beta$ giving $f(\beta) = \frac{2}{\beta-\alpha}$. Let $f_Z(a) = \frac{1}{\beta-\alpha}$ for $\alpha \leq a \leq \beta$. Set $c = 2$ so that $cf_Z(a) \geq f(a)$.

Notice that

$$\begin{aligned} \frac{f(a)}{cf_Z(a)} &= \frac{\frac{2(a-\alpha)}{(\beta-\alpha)^2}}{\frac{2}{\beta-\alpha}} \\ &= \frac{a-\alpha}{\beta-\alpha} \end{aligned}$$

algorithm

1. $U \leftarrow$ `random()`
   $Z \leftarrow \alpha + (\beta - \alpha)U$
2. $V \leftarrow$ `random()`
3. if $V \leq \frac{Z-\alpha}{\beta-\alpha}$ then
     return $Y \leftarrow Z$
   else
     goto step 1
   endif

Chapter 4

Simulation

1. An estimate of the probability of system failure within 5 days should be about 0.082.

2. No answer provided.

3. Let $G_1, G_2, \ldots$ represent the interarrival-time gaps between the arrival of jobs. Let B_n represent the type of the nth job; 0 for a job without collating, 1 for a job with collating.

   Let X_n represent the time to complete the nth job that does not require collating.

   Let Z_n represent the time to complete the nth job that does require collating.

   We model all of these random variables as mutually independent.

   Let S_n represent the number of jobs in progress or waiting to start.

   Define the following system events with associated clocks:

   $e_0()$ (initialization)

   $S_0 \leftarrow 0$

   $C_1 \leftarrow F_G^{-1}$ (`random()`)

   $C_2 \leftarrow \infty$

   $e_1()$ (arrival of a job)

   $S_{n+1} \leftarrow S_n + 1$

   if $\{S_{n+1} = 1\}$ then

   $B_n \leftarrow F_B^{-1}$ (`random()`)

   if $\{B_n = 0\}$ then

   $C_2 \leftarrow T_{n+1} + F_X^{-1}$ (`random()`)

   else

$C_2 \leftarrow T_{n+1} + F_Z^{-1}$ (`random()`)

endif

endif

$C_1 \leftarrow T_{n+1} + F_G^{-1}$ (`random()`)

$e_2()$ (finish job)

$S_{n+1} \leftarrow S_n - 1$

if $\{S_{n+1} > 0\}$ then

$B_n \leftarrow F_B^{-1}$ (`random()`)

if $\{B_n = 0\}$ then

$C_2 \leftarrow T_{n+1} + F_X^{-1}$ (`random()`)

else

$C_2 \leftarrow T_{n+1} + F_Z^{-1}$ (`random()`)

endif

endif

4. Let D_n represent the number of hamburgers demanded on the nth day. We model $D_1, D_2, \ldots$ as independent.

   Let S_n represent the number of patties in stock at the beginning of the nth day.

   Define the following system events with associated clocks:

   $e_0()$ (initialization)

   $S_0 \leftarrow 1000$

   $C_1 \leftarrow 1$

   $e_1()$ (morning count)

   if n is odd then

   if $\{S_n \leq 500\}$ then

   $S_{n+1} \leftarrow S_n - F_D^{-1}$ (`random()`) $+(1000 - S_n)$

   else

   $S_{n+1} \leftarrow S_n - F_D^{-1}$ (`random()`)

   endif

   else

$S_{n+1} \leftarrow S_n - F_D^{-1}$ (`random()`)

endif

5. Let X_n represent the time to complete the nth job.

   Let S_n represent the number of jobs in process or waiting in the buffer.

   Define the following system events and associated clocks:

   $e_0()$ (initialization)

   $C_1 \leftarrow 30$

   $C_2 \leftarrow \infty$

   $S_0 \leftarrow 0$

   $e_1()$ (arrival of a job)

   $S_{n+1} \leftarrow S_n + 1$

   if $\{S_{n+1} = 1\}$ then

   $C_2 \leftarrow T_{n+1} + F_X^{-1}(\texttt{random}())$

   endif

   $C_1 \leftarrow T_{n+1} + 30$

   $e_2()$ (finish a job)

   $S_{n+1} \leftarrow S_n - 1$

   if $\{S_{n+1} > 0\}$ then

   $C_2 \leftarrow T_{n+1} + F_X^{-1}(\texttt{random}())$

   endif

6. Let B_n be a Bernoulli random variable representing the CPU assignment of the nth job.

   Let $S_{0,n}$ represent the number of jobs at CPU A, and $S_{1,n}$ similar for CPU B.

   Define the following system events and associated clocks:

   $e_0()$ (initialization)

   $S_{0,n} \leftarrow 0$

   $S_{1,n} \leftarrow 0$

   $C_1 \leftarrow 40$

   $C_2 \leftarrow \infty$

   $C_3 \leftarrow \infty$

$e_1()$ (arrival of a job)

$B \leftarrow F_B^{-1}(\texttt{random}())$

$S_{B,n+1} \leftarrow S_{B,n} + 1$

if $\{S_{B,n+1} = 1\}$ then

$C_{2+B} \leftarrow T_{n+1} + 70$

endif

$C_1 \leftarrow T_{n+1} + 40$

$e_2()$ (finish at CPU A)

$S_{0,n+1} \leftarrow S_{0,n} - 1$

if $\{S_{0,n+1} > 0\}$ then

$C_2 \leftarrow T_{n+1} + 70$

endif

$e_3()$ (finish at CPU B)

$S_{1,n+1} \leftarrow S_{1,n} - 1$

if $\{S_{1,n+1} > 0\}$ then

$C_3 \leftarrow T_{n+1} + 70$

endif

7. No answer provided.

8. (a)

(b)

$$\begin{aligned}
\bar{Y}_1 &= \{0(5-0)+1(6-5)+2(9-6)+1(11-9) \\
&\quad +2(13-11)+3(15-13)+2(16-15) \\
&\quad +1(18-16)+0(19-18)\} \div 19 \\
&= \frac{23}{19} \approx 1.2 \text{ customers waiting for service} \\
\bar{Y}_2 &= \frac{0(3-0)+1(19-3)}{19} = \frac{16}{19} \approx 0.84 \\
&\qquad \text{utilization of the copier}
\end{aligned}$$

(c) $N_{19} = 5$ arrivals

An estimate of average delay is

$$\frac{\int_0^{19} Y_{1,t}\, dt}{N_{19}} = \frac{23}{5} = 4.6 \text{ minutes}$$

9. Based on the available data, the sample-average interarrival-time gap was about 17.6 seconds, and might be modeled as exponentially distributed.

   Let F_G be the cdf of the interarrival-time gaps, which we assume to be independent because students act independently.

   Let Δt be the pick-up interval, in seconds.

   Let S_n represent the number of students waiting after the nth state change.

   Define the following system events and associated clocks:

   $e_0()$ (start of day)

   $S_0 \leftarrow 0$

   $C_1 \leftarrow F_G^{-1}(\texttt{random()})$

   $C_2 \leftarrow \Delta t$

   $e_1()$ (student arrives)

   $S_{n+1} \leftarrow S_n + 1$

   $C_1 \leftarrow T_{n+1} + F_G^{-1}(\texttt{random()})$

   $e_2()$ (pick up)

$S_{n+1} \leftarrow S_n - \min\{S_n, 60\}$

$C_2 \leftarrow C_2 + \Delta t$

record S_{n+1}

The recorded values of S_{n+1} are the number left waiting after each bus pick up. They can be analyzed to examine the effect of $\Delta t = 600$, 720, 900.

Chapter 5

Arrival-Counting Processes

1. (a)

$$\Pr\{Y_2 = 5\} = \frac{e^{-2(2)}(2(2))^5}{5!} \approx 0.16$$

(b)

$$\Pr\{Y_4 - Y_3 = 1\} = \Pr\{Y_1 = 1\} = \frac{e^{-2}2}{1!} \approx 0.271$$

(c)

$$\Pr\{Y_6 - Y_3 = 4 \mid Y_3 = 2\} = \Pr\{Y_3 = 4\}$$

$$= \frac{e^{-2(3)}(2(3))^4}{4!} \approx 0.134$$

(d)

$$\Pr\{Y_5 = 4 \mid Y_4 = 2\} = \Pr\{Y_5 - Y_4 = 2 \mid Y_4 = 2\}$$

$$= \Pr\{Y_1 = 2\} = \frac{e^{-2}2^2}{2!} \approx 0.271$$

2. No answer provided.

3. $\lambda = 2/\text{hour}$. Let t be measured in hours from 6 a.m.

(a)

$$\begin{aligned}\Pr\{Y_4 = 9 \mid Y_2 = 6\} &= \Pr\{Y_4 - Y_2 = 3 \mid Y_2 = 6\} \\ &= \Pr\{Y_2 = 3\} \\ &= \frac{e^{-2(2)}(2(2))^3}{3!} \approx 0.195\end{aligned}$$

(b) The expected time between arrivals is

$$1/\lambda = 1/2 \text{ hour}$$

Let G be the gap between two successive arrivals. Then G is exponentially distributed with $\lambda = 2$.

$$\begin{aligned}
\Pr\{G > 1\} &= 1 - \Pr\{G \le 1\} \\
&= 1 - (1 - e^{-2(1)}) \\
&= e^{-2} \\
&\approx 0.135
\end{aligned}$$

(c) Because of the memoryless property

$$\mathrm{E}[R_0] = \mathrm{E}[G] = 1/2 \text{ hour}$$

$$\Pr\{R_0 \le 1/4\} = \Pr\{G \le 1/4\} = 1 - e^{-2(1/4)}$$

$$\approx 0.393$$

(d)

$$\Pr\{T_{13} \le 7\} = 1 - \sum_{j=0}^{12} \frac{e^{-2(7)}(2(7))^j}{j!}$$

$$\approx 0.641$$

(e) Let λ_0 be the arrival rate for urgent patients.

$$\begin{aligned}
\lambda_0 = 0.14\lambda &= 0.28 \text{ patients/hour} \\
\Pr\{Y_{0,12} > 6\} &= 1 - \Pr\{Y_{0,12} \le 6\} \\
&= 1 - \sum_{j=0}^{6} \frac{e^{-0.28(12)}(0.28(12))^j}{j!} \\
&\approx 0.055
\end{aligned}$$

(f) Let λ_2 be the overall arrival rate.

$$\lambda_2 = \lambda + 4 = 6/\text{hour}$$

$$\begin{aligned}\Pr\{Y_{2,6} > 30\} &= 1 - \Pr\{Y_{2,6} \le 30\} \\ &= 1 - \sum_{j=0}^{30} \frac{e^{-6(6)}(6(6))^j}{j!} \approx 0.819\end{aligned}$$

4. We want the smallest m such that

$$\Pr\{Y_{30} - Y_6 > m\} \le 0.05$$

$$\Pr\{Y_{30} - Y_6 > m\} = 1 - \sum_{j=0}^{m} \frac{e^{-24}\ 24^j}{j!}$$

$m = 32$ does it.

5. If we assume that the accident rate is still $\lambda = 1/\text{week}$, then

$$\begin{aligned}\Pr\{Y_{t+24} - Y_t \le 20\} &= \Pr\{Y_{24} \le 20\} \\ &= \sum_{j=0}^{20} \frac{e^{-24}\ 24^j}{j!} \approx 0.24\end{aligned}$$

Therefore, there is nearly a 1 in 4 chance of seeing 20 or fewer accidents even if the rate is still 1/week. This is not overwhelming evidence in favor of a lower rate.

6. t corresponds to square meters of metal and $\lambda = 1/50$ defect/meter^2

(a)

$$\begin{aligned}\Pr\{Y_{200} \ge 7\} &= 1 - \Pr\{Y_{200} \le 6\} \\ &= 1 - \sum_{j=0}^{6} \frac{e^{-\frac{1}{55}(200)}(\frac{200}{50})^j}{j!} \\ &\approx 0.111\end{aligned}$$

(b) For an out-of-control process $\lambda \ge 4/50$. For $\lambda = 4/50$ we want

$$\Pr\{Y_{200} > c\} \ge 0.95$$

$$\begin{aligned}
\Pr\{Y_{200} > c\} &= 1 - \Pr\{Y_{200} \le c\} \\
&= 1 - \sum_{j=0}^{c} \frac{e^{-\frac{4}{50}(200)}(\frac{4(200)}{50})^j}{j!} \\
&= 1 - \sum_{j=0}^{c} \frac{e^{-16}(16)^j}{j!} \ge 0.95
\end{aligned}$$

Using trial and error, or the approximation from Exercise 2, $c = 9$.

$$\Pr\{Y_{200} > 9\} \approx 0.957$$
$$\Pr\{Y_{200} > 10\} \approx 0.923$$

We will declare an "in-control" process "out-of-control" if $\lambda = 1/50$ and $\{Y_{200} > 9\}$.

$$\Pr\{Y_{200} > 9\} = 1 - \sum_{j=0}^{9} \frac{e^{-\frac{1}{50}(200)}(\frac{200}{50})^j}{j!} \approx 0.008$$

7. Suppose we model the arrival of students to the bus stop as a Poisson arrival process with expected time between arrivals of

$$\widehat{1/\lambda} = 17.6 \text{ seconds}$$

giving $\widehat{\lambda} = 1/17.6$ students/second ≈ 3.4 students/minute.

Now we can compute the probability of more than 60 students arriving during each proposed time interval

$$\begin{aligned}
\Pr\{Y_{15} > 60\} &= 1 - \Pr\{Y_{15} \le 60\} \\
&= 1 - \sum_{j=0}^{60} \frac{e^{-3.4(15)}(3.4(15))^j}{j!} \\
&\approx 0.094 \\
\Pr\{Y_{12} > 60\} &\approx 0.002 \\
\Pr\{Y_{10} > 60\} &\approx 0.000
\end{aligned}$$

Ignoring students left behind on one pick up that add to the next pick up, we see that there is nearly a 1 in 10 chance of filling the bus when pick up is every 15 minutes. The carryover will only make the problem worse. Pick up every 12 minutes effectively eliminates the problem. Every 10 minutes is more often than needed.

8. (a) Restricted to periods of the day when the arrival rate is roughly constant, a Poisson process is appropriate to represent a large number of customers acting independently.

   (b) Not a good approximation, since most arrivals occur during a brief period just prior to the start, and only a few before or after this period. Therefore, arrivals do not act independently.

   (c) Not a good approximation if patients are scheduled. We do not have independent increments because patients are anticipated. (May be a good approximation for a walk-in clinic, however.)

   (d) Not a good approximation because the rate of finding bugs will decrease over time.

   (e) Probably a good approximation since fires happen (largely) independently, and there are a large number of potential customers (buildings).

9. (a)

$$\begin{aligned} c &= 60+72+68 = 200 \text{ total arrivals} \\ d &= 3+3+3 = 9 \text{ total hours observed} \\ \widehat{\lambda} &= c/d \approx 22 \text{ customers/hour} \\ \widehat{\text{se}} &= \sqrt{\frac{\widehat{\lambda}}{d}} \approx 1.6 \text{ customers/hour} \end{aligned}$$

   (b) Let $Y_t \equiv$ number of arrivals by time t, and suppose $\lambda = 22$/hour.

$$\Pr\{Y_3 \le 56\} = \sum_{j=0}^{56} \frac{e^{-22(3)}(22(3))^j}{j!} \approx 0.12$$

   Since this probability is rather small, we might conclude that Fridays are different (have a lower arrival rate).

   At $\lambda = 22 + \widehat{\text{se}} = 23.6$

$$\Pr\{Y_3 \le 56\} \approx 0.04$$

   and at $\lambda = 22 - \widehat{\text{se}} = 20.4$

$$\Pr\{Y_3 \le 56\} \approx 0.29$$

   So less than or equal to 56 could be quite rare if $\lambda = 23.6$. This is further evidence that Fridays could be different.

10. Let

$$\begin{aligned} \lambda_0 &= 8/\text{hour} \\ \lambda_1 &= 1/18/\text{hour} \qquad p_1 = 0.005 \\ \lambda_2 &= 1/46/\text{hour} \qquad p_2 = 0.08 \end{aligned}$$

(a) $\lambda = \lambda_0 + \lambda_1 + \lambda_2 = 8\frac{32}{414}$ surges/hour
By the superposition property, the arrival of **all** surges is also a Poisson process. Therefore, $\mathrm{E}[Y_8] = \lambda 8 \approx 64.6$ surges.

(b) By the decomposition property, the "small" and "moderate" surges can be decomposed into Poisson processes of computer-damaging surges.

$$\begin{aligned} \lambda_{10} &= p_1\lambda_1 = 1/3600/\text{hour} \\ \lambda_{20} &= p_2\lambda_2 = 1/575/\text{hour} \end{aligned}$$

These processes can be superposed to give $\lambda_3 = \lambda_{10} + \lambda_{20} \approx 0.0020$. Therefore, $\mathrm{E}[Y_8] = 8\lambda_3 \approx 0.016$ computer-damaging surge.

(c) Using the arrival rate from part (b)

$$\Pr\{Y_8 = 0\} = e^{-\lambda_3 8} \approx 0.98$$

11. No answer provided.

12. Let $\{Y_t^{(0)}; t \geq 0\}$ be a Poisson arrival process representing the arrival of requests that require printing; $\lambda^{(0)} = 400$/hour.

$\{Y_t^{(1)}; t \geq 0\}$ similarly represents the arrival of requests that do not require printing; $\lambda^{(1)} = 1000$/hour.

(a) $Y_t = Y_t^{(0)} + Y_t^{(1)}$ is Poisson with rate $\lambda = \lambda^{(0)} + \lambda^{(1)} = 1400$/hour.

$$\Pr\{Y_{1.5} > 2000\} = \sum_{m=2001}^{\infty} \frac{e^{-1400(1.5)}(1400(1.5))^m}{m!} \approx 0.985$$

(b) Let $Y_t^{(A)}$ and $Y_t^{(B)}$ represent the arrivals to computers A and B, respectively.

$$Y_t^{(A)} \text{ is Poisson with } \lambda^{(A)} = (1/2)\lambda = 700$$

$$Y_t^{(B)} \text{ is Poisson with } \lambda^{(B)} = (1/2)\lambda = 700$$

and they are independent, therefore,

$$
\begin{aligned}
&\Pr\{Y_{1.5}^{(A)} > 1000, Y_{1.5}^{(B)} > 1000\} \\
&\qquad = \Pr\{Y_{1.5}^{(A)} > 1000\} \Pr\{Y_{1.5}^{(B)} > 1000\} \\
&\qquad = \left\{ \sum_{m=1001}^{\infty} \frac{e^{-700(1.5)}(700(1.5))^m}{m!} \right\}^2 \approx 0.877
\end{aligned}
$$

13. Let $\{Y_t; t \geq 0\}$ model the arrival of autos with $\lambda = 1$/minute. Then $\{Y_{0,t}; t \geq 0\}$ which models the arrival of trucks is Poisson with $\lambda_0 = (0.05)\lambda = 0.05$/min, and $\{Y_{1,t}; t \geq 0\}$ which models all others is Poisson with $\lambda_1 = (0.95)\lambda = 0.95$/min.

(a)

$$
\begin{aligned}
\Pr\{Y_{0,60} \geq 1\} &= \sum_{m=1}^{\infty} \frac{e^{-(0.05)(60)}[(0.05)(60)]^m}{m!} \\
&= 1 - \frac{e^{-3}(3)^0}{0!} = 1 - e^{-3} \approx 0.95
\end{aligned}
$$

(b) Since $Y_{0,t}$ is independent of $Y_{1,t}$ what happened to $Y_{0,t}$ is irrelevant. Therefore, $10 + \mathrm{E}[Y_{1,60}] = 10 + 60(.95) = 67$.

(c)

$$
\begin{aligned}
&\Pr\{Y_{0,60} = 5 \mid Y_{60} = 50\} \\
&\qquad = \frac{\Pr\{Y_{0,60} = 5, Y_{60} = 50\}}{\Pr\{Y_{60} = 50\}} \\
&\qquad = \Pr\{Y_{0,60} = 5, Y_{1,60} = 45\} \frac{1}{\frac{e^{-60}(60)^{50}}{50!}} \\
&\qquad = \Pr\{Y_{0,60} = 5\} \Pr\{Y_{1,60} = 45\} \frac{50!}{e^{-60}(60)^{50}} \\
&\qquad = \frac{e^{-3}(3)^5}{5!} \; \frac{e^{-57}(57)^{45}}{45!} \; \frac{50!}{e^{-60}(60)^{50}} \\
&\qquad = \frac{50!}{5!45!} \left(\frac{3}{60}\right)^5 \left(\frac{57}{60}\right)^{45} \\
&\qquad \approx 0.07
\end{aligned}
$$

14. The rate of errors after the nth proofreading is

$$
\lambda_n = \frac{\lambda}{2^n} = \frac{1}{2^n} \text{ error/1000 words}
$$

We want $\Pr\{Y_{200} = 0\} \geq 0.98$

$$\begin{aligned}
\Pr\{Y_{200} = 0\} &= e^{-\lambda_n(200)} \\
&= e^{-\frac{200}{2^n}} \geq 0.98 \\
\Rightarrow -\frac{200}{2^n} &\geq \ln(0.98) \\
2^n &\geq -\frac{200}{\ln(0.98)} \approx 9900 \\
&\Rightarrow n = 14 \text{ times} \\
&\quad \text{since } 2^{13} = 8192 \\
&\quad\quad 2^{14} = 16,384
\end{aligned}$$

15. (a) The rate at which sales are made does not depend on the time of year; there is no seasonal demand. Also, the market for A, B and C is neither increasing nor decreasing.

(b) Let $Y_t \equiv$ total sales after t weeks.

By the superposition property Y_t is a Poisson process with rate $\lambda = 10 + 10 = 20$/week.

$$\begin{aligned}
\Pr\{Y_1 > 30\} &= \sum_{n=31}^{\infty} \frac{e^{-20(1)}\ (20(1))^n}{n!} \\
&= 1 - \sum_{n=0}^{30} \frac{e^{-20}20^n}{n!} \approx 0.013
\end{aligned}$$

(c) Let $P \equiv$ person hours/order.

Then

$$\begin{aligned}
\mathrm{E}[P] &= 25(0.2) + 15(0.7) + 40(0.1) \\
&= 19.5 \text{ person-hours}
\end{aligned}$$

Since $\mathrm{E}[Y_4] = 20(4) = 80$ orders we expect $80(19.5) = 1560$ person-hours per month.

(d) Let $Y_t^{(B)} \equiv$ Louise's sales of B after t weeks.

The decomposition property implies that $Y_t^{(B)}$ is Poisson with rate $\lambda^{(B)} = (0.7)10 = 7$ per week.

$$
\begin{aligned}
\Pr\{Y_2^{(B)} - Y_1^{(B)} > 5, Y_1^{(B)} > 5\} & \\
&= \Pr\{Y_2^{(B)} - Y_1^{(B)} > 5\} \Pr\{Y_1^{(B)} > 5\} \\
&= \Pr\{Y_1^{(B)} > 5\} \Pr\{Y_1^{(B)} > 5\} \\
&= \left(1 - \sum_{n=0}^{5} \frac{e^{-7} 7^n}{n!}\right)^2 \approx 0.489
\end{aligned}
$$

16. (a) 10 seasons = 30 months

$$
\begin{aligned}
\widehat{\lambda} &= \frac{8}{30} \approx 0.27 \text{ hurricanes/month} \\
\widehat{\text{se}} &= \sqrt{\frac{\widehat{\lambda}}{30}} \approx 0.09 \text{ hurricanes/month}
\end{aligned}
$$

(b) $\Pr\{Y_3 = 0\} = e^{-\widehat{\lambda}(3)} \approx 0.45$

17. (a) The rate at which calls are placed does not vary from 7 a.m.–6 p.m.

(b) Let $Y_t^{(D)} \equiv$ number of long-distance calls placed by hour t, where $t = 0$ corresponds to 7 a.m. Then $Y_t^{(D)}$ is Poisson with rate $\lambda^{(D)} = (1000)(0.13) = 130$/hour.

$$
\begin{aligned}
\Pr\{Y_8^{(D)} - Y_7^{(D)} > 412\} &= \Pr\{Y_1^{(D)} > 412\} \\
&= \sum_{n=413}^{\infty} \frac{e^{-130}(130)^n}{n!} \\
&= 1 - \sum_{n=0}^{412} \frac{e^{-130} 130^n}{n!} \\
&\approx 0
\end{aligned}
$$

(Notice that $t = 8$ corresponds to 3 p.m.)

(c) Expected number of local calls is

$$
\mathrm{E}[Y_{11}^{(L)}] = \lambda^{(L)}(11) = (870)(11) = 9570 \text{ calls}
$$

Expected number of long-distance calls is

$$
\mathrm{E}[Y_{11}^{(D)}] = \lambda^{(D)}(11) = 130(11) = 1430 \text{ calls}
$$

expected revenue = (0.08) (9570) + (0.13) (1430) = \$951.50

(d)

$$\Pr\{Y_1^{(D)} > 300 \mid Y_1 = 1200\}$$

$$= \sum_{n=301}^{1200} \binom{1200}{n} (0.13)^n (0.87)^{1200-n} \approx 0$$

18. Let $\{Y_t^{(T)}; t \geq 0\}$ represent arrival of trucks to the restaurant.

$$\text{Poisson with rate } \lambda^{(T)} = 10(0.1) = 1/\text{hour}$$

Let $\{Y_t^{(C)}; t \geq 0\}$ represent arrival of cars to the restaurant.

$$\text{Poisson with rate } \lambda^{(C)} = 20(0.1) = 2/\text{hour}$$

(a) $Y_t = Y_t^{(T)} + Y_t^{(C)}$ is Poisson with rate $\lambda = \lambda^{(T)} + \lambda^{(C)} = 3/\text{hour}$

$$\mathrm{E}[Y_1] = \lambda(1) = 3 \text{ cars and trucks}$$

(b) $\Pr\{Y_1 = 0\} = e^{-3(1)} \approx 0.05$

(c) Let $C \equiv$ number of passengers in a car.

$$\mathrm{E}[C] = (0.3)(1) + (0.5)(2) + (0.2)(3) = 1.9$$

Let $P \equiv$ number of passengers arriving in 1 hour.

$$\begin{aligned} \mathrm{E}[P] &= \mathrm{E}\left[Y_1^{(T)} + \mathrm{E}[C]Y_1^{(C)}\right] \\ &= 1 + (1.9)2 = 4.8 \text{ passengers} \end{aligned}$$

19. Traffic engineer's model: $\{Y_t; t \geq 0\}$ models the number of accidents and is Poisson with rate $\lambda = 2/\text{week}$.

(a)

$$\begin{aligned} \Pr\{Y_2 \geq 20\} &= \sum_{m=20}^{\infty} \frac{e^{-2(2)}(2(2))^m}{m!} \\ &= 1 - \sum_{m=0}^{19} \frac{e^{-4}4^m}{m!} \approx 1.01 \times 10^{-8} \end{aligned}$$

(b)

$$
\begin{aligned}
& \Pr\{Y_{52} - Y_{48} = 20, Y_{48} = 80 \mid Y_{52} = 100\} \\
&= \frac{\Pr\{Y_{52} - Y_{48} = 20, Y_{48} = 80, Y_{52} = 100\}}{\Pr\{Y_{52} = 100\}} \\
&= \frac{\Pr\{Y_{52} - Y_{48} = 20\} \Pr\{Y_{48} = 80\}}{\Pr\{Y_{52} = 100\}} \\
&= \frac{\Pr\{Y_{4} = 20\} \Pr\{Y_{48} = 80\}}{\Pr\{Y_{52} = 100\}} \\
&= \frac{e^{-2(4)}(2(4))^{20}}{20!} \; \frac{e^{-2(48)}(2(48))^{80}}{80!} \; \frac{100!}{e^{-2(52)} \; (2(52))^{100}} \\
&= \frac{100!}{20!80!} \left(\frac{4}{52}\right)^{20} \left(\frac{48}{52}\right)^{80} = \binom{100}{20} \left(\frac{1}{13}\right)^{20} \left(\frac{12}{13}\right)^{80} \\
&\approx 5 \times 10^{-5}
\end{aligned}
$$

(c) $\Pr\{Y_2 = 0\} = e^{-2(2)} \approx 0.02$

20. (a)

$$
\begin{aligned}
\Lambda(t) &= \int_0^t \lambda(a)da \\
\Lambda(t) &= \int_0^t 1da = t \ , \quad \text{for } 0 \le t < 6 \\
\Lambda(t) &= \Lambda(6) + \int_0^t 2da = 6 + 2a \mid_6^t \\
&= 6 + 2t - 12 = 2t - 6 \ , \quad \text{for } 6 \le t < 13 \\
\Lambda(t) &= \Lambda(13) + \int_{13}^t (1/2)da \\
&= 20 + (1/2)a \mid_{13}^t = 20 + (1/2)t - 13/2 \\
&= (1/2)(t + 27), \quad \text{for } 13 < t \le 24
\end{aligned}
$$

(b) 8 a.m. $\rightarrow$ hour 2
2 p.m. $\rightarrow$ hour 8

$$
\begin{aligned}
\Lambda(8) - \Lambda(2) &= (2(8) - 6) - 2 \\
&= 8
\end{aligned}
$$

$$\begin{aligned}
\Pr\{\mathcal{Y}_8 - \mathcal{Y}_2 > 12\} &= \sum_{m=13}^{\infty} \frac{e^{-8}8^m}{m!} \\
&= 1 - \sum_{m=0}^{12} \frac{e^{-8}8^m}{m!} \approx 0.06
\end{aligned}$$

$$\mathrm{E}[\mathcal{Y}_8 - \mathcal{Y}_2] = 8 \text{ patients}$$

(c) 10 a.m. $\rightarrow$ hour 4
$\Lambda(4) - \Lambda(2) = 2$

$$\begin{aligned}
\Pr\{\mathcal{Y}_4 = 9 \mid \mathcal{Y}_2 = 6\} &= \Pr\{\mathcal{Y}_4 - \mathcal{Y}_2 = 3 \mid \mathcal{Y}_2 = 6\} \\
&= \Pr\{\mathcal{Y}_4 - \mathcal{Y}_2 = 3\} \\
&= \frac{e^{-2}2^3}{3!} \approx 0.18
\end{aligned}$$

(d)

$$\begin{aligned}
\Pr\{\mathcal{Y}_{1/4} > 0\} &= 1 - \Pr\{\mathcal{Y}_{1/4} = 0\} \\
&= 1 - \left(\frac{e^{-1/4}(1/4)^0}{0!}\right) \\
&= 1 - e^{-1/4} \approx 0.22
\end{aligned}$$

(since $\Lambda(1/4) = 1/4$)

(e) 1 p.m. $\rightarrow$ hour 7

$$\Lambda(7) = 2(7) - 6 = 8$$

$$\Pr\{\mathcal{Y}_7 \geq 13\} = \sum_{m=13}^{\infty} \frac{e^{-8}8^m}{m!} \approx 0.06$$

(from part (b))

21. (a) Compute the average number of arrivals (and its se) for each hour

hour	average	(se)	estimates
1	144.2	(12.3)	λ_1
2	229.4	(7.2)	λ_2
3	382.6	(7.9)	λ_3
4	96.0	(2.7)	λ_4

$$\lambda(t) = \begin{cases} 144, & 0 \le t < 1 \qquad \text{(after rounding)} \\ 229, & 1 \le t < 2 \\ 383, & 2 \le t < 3 \\ 96, & 3 \le t \le 4 \end{cases}$$

(b)

$$\begin{aligned}
\Lambda(t) &= \int_0^t \lambda(a)da \\
\Lambda(t) &= \int_0^t 144da = 144t \qquad 0 \le t < 1 \\
\Lambda(t) &= \Lambda(1) + \int_1^t 229da \\
&= 144 + 229(t-1) \\
&= 229t - 85 \qquad 1 \le t < 2 \\
\Lambda(t) &= \Lambda(2) + \int_2^t 383da \\
&= 373 + 383(t-2) \\
&= 383t - 393, \quad 2 \le t < 3 \\
\Lambda(t) &= \Lambda(3) + \int_3^t 96da \\
&= 756 + 96(t-3) \\
&= 96t + 468 \qquad 3 \le t \le 4
\end{aligned}$$

(c)

$$\begin{aligned}
&\mathrm{E}[\mathcal{Y}_{3.4} - \mathcal{Y}_{1.75}] \\
&\quad = \Lambda(3.4) - \Lambda(1.75) \\
&\quad = (96(3.4) + 468) - (229(1.75) - 85) \\
&\quad = 478.65 \approx 479 \text{ cars}
\end{aligned}$$

(d)

$$\begin{aligned}
&\Pr\{\mathcal{Y}_{3.4} - \mathcal{Y}_{1.75} > 700\} \\
&\quad = \sum_{m=701}^{\infty} \frac{e^{-479}(479)^m}{m!} \\
&\quad = 1 - \sum_{m=0}^{700} \frac{e^{-479}(479)^m}{m!} \approx 0
\end{aligned}$$

22. The algorithms given here are direct consequences of the definitions, and not necessarily the most efficient possible.

    (a) Recall that the inverse cdf for the exponential distribution with parameter λ is

    $$X = -\ln(1-U)/\lambda$$

    algorithm Erlang
    $a \leftarrow 0$
    for $i \leftarrow 1$ to n
    do
        $a \leftarrow a - \ln$ (1-`random()`)$/\lambda$
    enddo
    return $X \leftarrow a$

    (b) algorithm binomial
    $a \leftarrow 0$
    for $i \leftarrow 1$ to n
    do
        $U \leftarrow$ `random()`
        if $\{U \geq 1-\gamma\}$ then
            $a \leftarrow a+1$
        endif
    enddo
    return $X \leftarrow a$

    (c) algorithm Poisson
    $a \leftarrow 0$
    $b \leftarrow -\ln$(1-`random()`)$/\lambda$
    while $\{b < t\}$
    do
        $b \leftarrow b - \ln$(1-`random()`)$/\lambda$
        $a \leftarrow a+1$
    enddo
    return $X \leftarrow a$

    (d) algorithm nspp
    $t \leftarrow \Lambda(t)$
    $a \leftarrow 0$
    $b \leftarrow -\ln$(1-`random()`)

while $\{b < t\}$
do
  $b \leftarrow b - \ln(1\text{-}\texttt{random()})$
  $a \leftarrow a + 1$
enddo
return $X \leftarrow a$

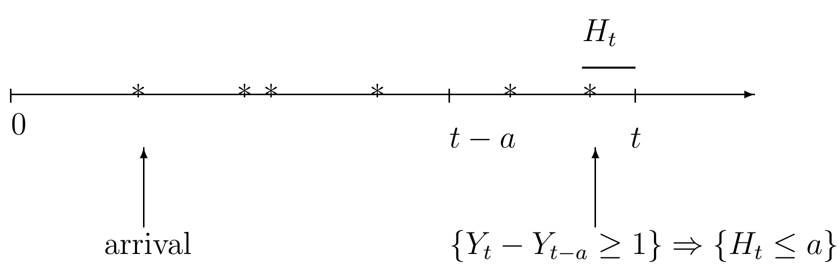
The first step in algorithm nspp converts the t to the time scale for the rate-1 stationary process.

23.

$$
\begin{aligned}
&\mathrm{E}[Y_{t+\Delta t} - Y_t] \\
&\quad = \mathrm{E}[Y_{\Delta t}] \quad \text{(time stationarity)} \\
&\quad = \sum_{k=0}^{\infty} k \frac{e^{-\lambda \Delta t}(\lambda \Delta t)^k}{k!} \quad \text{(by definition)} \\
&\quad = \sum_{k=1}^{\infty} \frac{e^{-\lambda \Delta t}(\lambda \Delta t)^k}{(k-1)!} \\
&\quad = e^{-\lambda \Delta t}(\lambda \Delta t) \sum_{k=1}^{\infty} \frac{(\lambda \Delta t)^{k-1}}{(k-1)!} \\
&\quad = e^{-\lambda \Delta t}(\lambda \Delta t) \sum_{j=0}^{\infty} \frac{(\lambda \Delta t)^{j}}{j!} \\
&\quad = e^{-\lambda \Delta t}(\lambda \Delta t) e^{\lambda \Delta t} \\
&\quad = \lambda \Delta t
\end{aligned}
$$

24.

$$
\begin{aligned}
\Pr\{H_t = t\} &= \Pr\{Y_t = 0\} = e^{-\lambda t} \\
\Pr\{H_t \le a\} &= \Pr\{Y_t - Y_{t-a} \ge 1\} \quad (\text{ at least 1 arrival between } t-a \text{ and } t \text{ }) \\
&= 1 - \Pr\{Y_t - Y_{t-a} < 1\} \\
&= 1 - e^{-\lambda a} \\
\Pr\{H_t = t\} &= e^{-\lambda t} \\
\Pr\{H_t \le a\} &= 1 - e^{-\lambda a}\ ,\ 0 \le a \le t
\end{aligned}
$$



Notice that $\lim_{t\to\infty} \Pr\{H_t = t\} = \lim_{t\to\infty} e^{-\lambda t} = 0$. Therefore, $\lim_{t\to\infty} \Pr\{H_t \le a\} = 1 - e^{-\lambda a}\ ,\ a \ge 0$.

25.

$$\begin{aligned}
\mathrm{E}[L_t] &= \mathrm{E}[H_t + R_t] \\
&= \mathrm{E}[H_t] + \mathrm{E}[R_t] \\
&= \mathrm{E}[H_t] + \mathrm{E}[G] \quad \text{(memoryless)} \\
&> \mathrm{E}[G]
\end{aligned}$$

since certainly $\mathrm{E}[H_t] > 0$.

26. Now $p_G(a) = 1/60$ for $a = 1, 2, \ldots, 60$. Therefore, $\delta = \mathrm{E}[G] = 61/2 = 30\frac{1}{2}$ months.

$$\eta = \mathrm{E}[R] = \mathrm{E}[d(e^{iG} - 1)]$$

$$= \sum_{a=1}^{60} d(e^{ia} - 1) p_G(a)$$

$$= d\left(\frac{1}{60}\sum_{a=1}^{60} e^{ia} - 1\right)$$

For the specific case $i = 0.06/12 = 0.005$

$$\approx d\left(\frac{70.14}{60} - 1\right) = d(0.169)$$

$$= 50,700 \qquad \text{for } d = 300,000$$

Therefore the long-run rate of return is

$$\frac{\eta}{\delta} \approx \$1,662.29 \text{ per month}$$

$$\approx \$19,947.54 \text{ per year}$$

This is about \$52 per year less than the other model.

27. We suppose that the time to burn-out of lights are independent (almost certainly true) and identically distributed (true if same brand of bulb and same usage pattern). Therefore the replacements form a renewal arrival process.

    Estimate δ by $\widehat{\delta} = 248.7$ hours (sample average).

    Therefore, the long-run replacement rate is

$$\frac{1}{\delta} \approx 4.021 \times 10^{-4} \text{ bulbs/hour}$$

$$\approx 35 \text{ bulbs/year}$$

28. We model it as a renewal-reward process. Therefore, the modeling approximations are: the times between orders are independent and time-stationary random variables, and the dollar amount of the orders are also independent and time-stationary random variables. These approximations are reasonable except on special holidays (e.g., Mother's Day) when the rate and size of orders changes.

    $\delta = 15$ minutes

    $\eta = \$27$

    The long-run revenue/minute is $\eta/\delta = \$1.8$/minute.

29. As given in the hint,

$$\begin{aligned}
\delta &= \int_0^c a\ f_X(a)da + c\int_c^{30} f_X(a)da \\
&= \frac{c^3}{1350} + c\left(1 - \frac{c^2}{900}\right) \\
&= c - \frac{c^3}{2700} \text{ days}
\end{aligned}$$

    The reward (cost) depends on whether failure occurs before c.

$$\begin{aligned}
\eta &= 1000\int_0^c f_X(a)da + 300\int_c^{30} f_X(a)da \\
&= \frac{7c^2}{9} + 300
\end{aligned}$$

    Therefore, the long-run cost is

$$\frac{\eta}{\delta} = -300\left(\frac{7c^2+2700}{c(c^2-2700)}\right) \text{ per day}$$

To minimize the cost we take the derivative w.r.t. c and set it equal to 0, then solve for c. Of the 3 solutions, the only one in the range [0, 30] is $c = 15.9$ for which

$$\frac{\eta}{\delta} \approx \$34.46 \text{ per day}$$

Compare this to $c = 30$ for which

$$\frac{\eta}{\delta} = \$50 \text{ per day}$$

Therefore, it is worthwhile to replace early.

30. Suppose we wait for n patrons before beginning a tour. Then $\delta = n$ minutes is the expected time between tours. The expected cost for a tour of size n is

$$\begin{aligned}\eta &= 10 + (0 + (0.50) + (2)(0.50) + (3)(0.50) \\ &\qquad + \cdots + (n-1)(0.50)) \\ &= 10 + (0.50)\frac{(n(n-1))}{2}\end{aligned}$$

(a) The long-run cost/minute is

$$\frac{\eta}{\delta} = \frac{10}{n} + \frac{(0.50)(n-1)}{2}$$

A plot as a function of n reveals that η/δ decreases until $n = 6$, then increases. At $n = 6, \eta/\delta = \$2.92$ per minute.

(b) $\delta = 1(6) = 6$ minutes

31. To prove the result for decomposition of m processes, do the decomposition in pairs, as follows:

(i) Decompose the original process into 2 subprocesses with probabilities γ_1 and $1-\gamma_1$. Clearly both subprocesses are Poisson, the first with rate $\lambda_1 = \gamma_1\lambda$, and the latter with rate $\lambda' = (1-\gamma_1)\lambda$.

(ii) Decompose the λ' process into two subprocesses with probabilities $\gamma_2/(1-\gamma_1)$ and $1-\gamma_2/(1-\gamma_1)$. Clearly both subprocesses are Poisson, the first with rate

$$\lambda_2 = \lambda'\left(\frac{\gamma_2}{1-\gamma_1}\right) = \gamma_2\lambda$$

and the latter with rate

$$\lambda'' = \left(1 - \left(\frac{\gamma_2}{1-\gamma_1}\right)\right)\lambda'$$

(iii) Continue by decomposing the λ'' process, etc.

To prove the result for superposition of m processes, superpose them two at a time:

(i) Superpose the λ_1 and λ_2 processes to obtain a Poisson process with rate $\lambda' = \lambda_1 + \lambda_2$.

(ii) Superpose the λ' and λ_3 processes to obtain a Poisson process with rate $\lambda'' = \lambda' + \lambda_3 = \lambda_1 + \lambda_2 + \lambda_3$.

(iii) Continue

To prove the result for the superposition of m nonstationary processes, we prove that for the superposition process

$$\Pr\{\mathcal{Y}_{\tau+\Delta\tau} - \mathcal{Y}_\tau = h \mid \mathcal{Y}_\tau = k\}$$

$$= \frac{e^{-[\Lambda(\tau+\Delta\tau)-\Lambda(\tau)]}[\Lambda(\tau+\Delta\tau) - \Lambda(\tau)]^h}{h!}$$

For clarity we prove only the case $m = 2$.

Let $\mathcal{Y}_{i,\tau}$ be the NSPP with rate $\Lambda_i(\tau)$, for $i = 1, 2$. Therefore,

$$\begin{aligned}
&\Pr\{\mathcal{Y}_{\tau+\Delta\tau} - \mathcal{Y}_\tau = h \mid \mathcal{Y}_\tau = k\} \\
&= \sum_{\ell=0}^{h} \Pr\{\mathcal{Y}_{1,\tau+\Delta\tau} - \mathcal{Y}_{1,\tau} = \ell, \mathcal{Y}_{2,\tau+\Delta\tau} - \mathcal{Y}_{2,\tau} = h-\ell \mid \mathcal{Y}_{1,\tau} + \mathcal{Y}_{2,\tau} = k\} \\
&= \sum_{\ell=0}^{h} \Pr\{\mathcal{Y}_{1,\tau+\Delta\tau} - \mathcal{Y}_{1,\tau} = \ell\} \Pr\{\mathcal{Y}_{2,\tau+\Delta\tau} - \mathcal{Y}_{2,\tau} = h-\ell\} \\
&= \sum_{\ell=0}^{h} \frac{e^{-\Delta t_1}(\Delta t_1)^\ell}{\ell!} \; \frac{e^{-\Delta t_2}(\Delta t_2)^{h-\ell}}{(h-\ell)!} \qquad (1)
\end{aligned}$$

where we use the fact that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are independent NSPPs, and where $\Delta t_i = \Lambda_i(\tau + \Delta\tau) - \Lambda_i(\tau)$. Let $\Delta t = \Delta t_1 + \Delta t_2 = \Lambda(\tau + \Delta\tau) - \Lambda(\tau)$. Then (1) simplifies to

$$\frac{e^{-\Delta t}(\Delta t)^h}{h!} \sum_{\ell=0}^{h} \frac{h!}{\ell!(h-\ell)!} \left(\frac{\Delta t_1}{\Delta t}\right)^\ell \left(\frac{\Delta t_2}{\Delta t}\right)^{h-\ell} = \frac{e^{-\Delta t}(\Delta t)^h}{h!} \cdot 1$$

32. No answer provided.

33. No answer provided.

34. No answer provided.

Chapter 6

Discrete-Time Processes

1. $\Box$ session `1 2 3 2 3 3 3 3 3 2 3 2 2 3 2 4`

$$\begin{aligned}
\Pr\{\Box\} &= p_1 p_{12} p_{23} p_{32} p_{23} p_{33} p_{33} p_{33} p_{33} p_{32} p_{23} p_{32} p_{22} p_{23} p_{32} p_{24} \\
&= p_1 p_{12} (p_{23})^4 (p_{32})^4 (p_{33})^4 p_{22} p_{24} \\
&= 1\ (0.95)(0.63)^4 (0.36)^4 (0.4)^4 (0.27)(0.1) \\
&\approx 1.7 \times 10^{-6}
\end{aligned}$$

$*$ session `1 2 2 3 3 2 3 3 3 4`

$$\begin{aligned}
\Pr\{*\} &= p_1 p_{12} p_{22} p_{23} p_{33} p_{32} p_{23} p_{33} p_{33} p_{34} \\
&= p_1 p_{12} p_{22} (p_{23})^2 (p_{33})^3 p_{32} p_{34} \\
&= 1\ (0.95)(0.27)(0.63)^2 (0.4)^3 (0.36)(0.24) \\
&\approx 5.6 \times 10^{-4}
\end{aligned}$$

2. (a) 2, 2, 3, 2, 4,...
   (b) 3, 3, 2, 3, 2, 3, 4,...

3. (a) 1, 1, 1, 1, 1,...
   (b) 2, 2, 1, 1, 1, 1, 2,...

4. (a) $p_{12} = 0.4$
   (b) $p_{11}^{(2)} = 0.68$
   (c)$p_{21} p_{11} = 0.48$

5. recurrent: $\{2, 3, 4\}$
   transient: $\{1\}$

6. (a) irreducible, all states recurrent

   (b) irreducible subset {2, 5} are recurrent

   {1, 3, 4} are transient

   (c) irreducible, all states recurrent

7. (a)

   6(a)

   $\underline{n = 2}$

$$\begin{pmatrix} 0.54 & 0.08 & 0.09 & 0.08 & 0.21 \\ 0.27 & 0.19 & 0.24 & 0.02 & 0.28 \\ 0.08 & 0.26 & 0.32 & 0.18 & 0.16 \\ 0.05 & 0.10 & 0.01 & 0.81 & 0.03 \\ 0.31 & 0.15 & 0.18 & 0.02 & 0.34 \end{pmatrix}$$

   $\underline{n = 5}$

$$\begin{pmatrix} 0.27496 & 0.16438 & 0.19033 & 0.14404 & 0.22629 \\ 0.34353 & 0.14109 & 0.16246 & 0.12082 & 0.23210 \\ 0.33252 & 0.11926 & 0.11720 & 0.23690 & 0.19412 \\ 0.12455 & 0.11640 & 0.06041 & 0.60709 & 0.09155 \\ 0.33137 & 0.14745 & 0.17200 & 0.11146 & 0.23772 \end{pmatrix}$$

   $\underline{n = 20}$

$$\begin{pmatrix} 0.2664964441 & 0.1393997993 & 0.1400484926 & 0.2628192095 & 0.1912360548 \\ 0.2675253416 & 0.1395887257 & 0.1406686163 & 0.2602309763 & 0.1919863403 \\ 0.2632018511 & 0.1388746569 & 0.1382171161 & 0.2707695205 & 0.1889368557 \\ 0.2490755395 & 0.1364940963 & 0.1301154884 & 0.3054031203 & 0.1789117568 \\ 0.2678558364 & 0.1396398961 & 0.1408494124 & 0.2594398254 & 0.1922150308 \end{pmatrix}$$

   6(b)

   $\underline{n = 2}$

$$\begin{pmatrix} 0.01 & 0.55 & 0.14 & 0.01 & 0.29 \\ 0 & 0.74 & 0 & 0 & 0.26 \\ 0 & 0.35 & 0.09 & 0 & 0.56 \\ 0 & 0 & 0.3 & 0 & 0.7 \\ 0 & 0.65 & 0 & 0 & 0.35 \end{pmatrix}$$

$\underline{n=5}$

$$\begin{pmatrix} 0.00001 & 0.68645 & 0.00521 & 0.00001 & 0.30832 \\ 0 & 0.71498 & 0 & 0 & 0.28502 \\ 0 & 0.69230 & 0.00243 & 0 & 0.30527 \\ 0 & 0.6545 & 0.0081 & 0 & 0.3374 \\ 0 & 0.71255 & 0 & 0 & 0.28745 \end{pmatrix}$$

$\underline{n=20}$

$$\begin{pmatrix} 1.0\times 10^{-20} & 0.7142857122 & 7.554699530\times 10^{-11} & 1.0\times 10^{-20} & 0.2857142881 \\ 0 & 0.7142857146 & 0 & 0 & 0.2857142858 \\ 0 & 0.7142857134 & 3.486784401\times 10^{-11} & 0 & 0.2857142871 \\ 0 & 0.7142857108 & 0.0000000001162261467 & 0 & 0.2857142895 \\ 0 & 0.7142857147 & 0 & 0 & 0.2857142859 \end{pmatrix}$$

6(c)

$\underline{n=2}$

$$\begin{pmatrix} 0.03 & 0.67 & 0.04 & 0.15 & 0.11 \\ 0.02 & 0.74 & 0 & 0.08 & 0.16 \\ 0.03 & 0.55 & 0.09 & 0.24 & 0.09 \\ 0 & 0.8 & 0 & 0 & 0.2 \\ 0.01 & 0.83 & 0.01 & 0.03 & 0.12 \end{pmatrix}$$

$\underline{n=5}$

$$\begin{pmatrix} 0.01739 & 0.75399 & 0.00343 & 0.07026 & 0.15493 \\ 0.01728 & 0.75624 & 0.00242 & 0.06838 & 0.15568 \\ 0.01737 & 0.75119 & 0.00461 & 0.07201 & 0.15482 \\ 0.0170 & 0.7578 & 0.0024 & 0.0670 & 0.1558 \\ 0.01736 & 0.75702 & 0.00259 & 0.06860 & 0.15443 \end{pmatrix}$$

$\underline{n=20}$

$$\begin{pmatrix} 0.01727541956 & 0.7564165848 & 0.002467917087 & 0.06836130311 & 0.1554787760 \\ 0.01727541955 & 0.7564165848 & 0.002467917078 & 0.06836130306 & 0.1554787760 \\ 0.01727541956 & 0.7564165848 & 0.002467917098 & 0.06836130312 & 0.1554787759 \\ 0.01727541955 & 0.7564165848 & 0.002467917078 & 0.06836130306 & 0.1554787759 \\ 0.01727541955 & 0.7564165848 & 0.002467917080 & 0.06836130306 & 0.1554787759 \end{pmatrix}$$

7(b)

6(a)

$$\boldsymbol{\pi} = \begin{pmatrix} 0.2616580311 \\ 0.1386010363 \\ 0.1373056995 \\ 0.2746113990 \\ 0.1878238342 \end{pmatrix}$$

6(b)

$$\boldsymbol{\pi} = \begin{pmatrix} 0.7142857143 \\ 0.2857142857 \end{pmatrix}$$

6(c)

$$\boldsymbol{\pi} = \begin{pmatrix} 0.01727541955 \\ 0.7564165844 \\ 0.002467917078 \\ 0.06836130306 \\ 0.1554787759 \end{pmatrix}$$

8. $\mathcal{T} = \{1, 3, 4\}$

   $\mathcal{R}_1 = \{2\}, \mathcal{R}_2 = \{5\}, \mathcal{R}_3 = \{6\}$

$$\boldsymbol{\alpha}' = \begin{pmatrix} 0.7366666667 & 0.09666666666 & 0.1666666667 \end{pmatrix}$$

9. No answer provided.

10. In the answer we use all the data. However, it might also be reasonable to have different models for free throws and shots from the floor.

    (a) Let $\{S_n; n = 0, 1, \ldots\}$ represent the sequence of shots, where $\mathcal{M} = \{0, 1\}$ and 0 corresponds to a miss, 1 corresponds to a made shot.

    If each shot is independent with probability p of being made, then S_n is a Markov chain with one-step transition matrix

$$\mathbf{P} = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}$$

    An estimate of p is

$$\widehat{p} = \frac{30}{45} \approx 0.67$$

with

$$\widehat{\text{se}} = \sqrt{\frac{\widehat{p}(1-\widehat{p})}{45}} \approx 0.07$$

(b) Now let S_n be a Markov chain with

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$$

In the sample, let n_{ij} be the number of times the transition (i, j) occurred, and $n_i = \sum_{j=0}^{1} n_{ij}$. Then $\widehat{p}_{ij} = n_{ij}/n_i$. The observed values were

n_{ij}	0	1	n_i
0	6	9	15
1	8	21	29

Therefore,

$$\widehat{\mathbf{P}} = \begin{pmatrix} 0.40 & 0.60 \\ 0.28 & 0.72 \end{pmatrix}$$

An estimate of the se of $\widehat{p}_{ij}$ is $\sqrt{\frac{\widehat{p}_{ij}(1-\widehat{p}_{ij})}{n_i}}$. Therefore, the matrix of $\widehat{\text{se}}$'s is

$$\widehat{\text{se}} = \begin{pmatrix} 0.13 & 0.13 \\ 0.08 & 0.08 \end{pmatrix}$$

(c) Let S_n be a Markov chain with state space $\mathcal{M} = \{0, 1, 2, 3\}$ corresponding to the two most recent shots:

0 = miss, miss

1 = miss, made

2 = made, miss

3 = made, made

The observed values were

n_{ij}	0	1	2	3	n_i
0	2	4	0	0	6
1	0	0	1	8	9
2	5	4	0	0	9
3	0	0	6	13	19

$$\widehat{\mathbf{P}} = \begin{pmatrix} 0.33 & 0.67 & 0 & 0 \\ 0 & 0 & 0.11 & 0.89 \\ 0.56 & 0.44 & 0 & 0 \\ 0 & 0 & 0.32 & 0.68 \end{pmatrix}$$

$$\widehat{\text{se}} = \begin{pmatrix} 0.19 & 0.19 & 0 & 0 \\ 0 & 0 & 0.10 & 0.10 \\ 0.17 & 0.17 & 0 & 0 \\ 0 & 0 & 0.11 & 0.11 \end{pmatrix}$$

(d) Assume a made shot, followed by 4 missed shots, followed by a made shot.

Under model (a) : $(1-\widehat{p})^4\widehat{p} \approx 0.008$

Under model (b) : $\widehat{p}_{10}(\widehat{p}_{00})^3\widehat{p}_{01} \approx 0.011$

Under model (c) : In this case we need to know the result of the previous two shots.

If (made, made): $\widehat{p}_{32}\widehat{p}_{20}(\widehat{p}_{00})^2\widehat{p}_{01} \approx 0.013$

If (miss, made): $\widehat{p}_{12}\widehat{p}_{20}(\widehat{p}_{00})^2\widehat{p}_{01} \approx 0.005$

Thus, a run of this length is slightly more likely under the first case of model (c).

This is only one small bit of evidence that favors model (c). Essentially, we look to see if the additional information we account for in model (b) substantially changes the transition probabilities from model (a). If it does, then we see if the transition probabilities change when moving from (b) to (c). We stop when the additional information no longer changes the probabilities. For example,

event	model (a)	model (b)	model (c)
make a shot	$\widehat{p} = 0.67$	$\widehat{p}_{01} = 0.60$ $\widehat{p}_{11} = 0.72$	$\widehat{p}_{01} = 0.67$ $\widehat{p}_{21} = 0.44$ $\widehat{p}_{13} = 0.89$ $\widehat{p}_{33} = 0.68$

Notice that model (b) seems to indicate that there is some difference depending on whether or not Robinson made the most recent shot, compared to model (a). Model (c) indicates some additional differences based on the last two shots. So maybe there is a "hot hand," but because of the large standard errors we cannot be sure.

11. See Chapter 9, Section 9.1.9.

12. See Chapter 9, Section 9.1.3.

13. No answer provided.

14. <u>Case 6.3</u>

    S_n represents the destination of the nth rider

    $\mathcal{M} = \{1, 2, \ldots, m\}$ one state for each region of the city

    $p_{ij} \equiv$ probability a rider picked up in region i has destination in region j

    $p_i \equiv$ probability driver starts the day in region i

    <u>Case 6.4</u>

    S_n represents the nth key typed

$$\mathcal{M} = \left\{ \underbrace{1, 2, \ldots, 26}_{\text{letters}} \underbrace{27, 28, \ldots, 36}_{\text{numbers}} \underbrace{37, 38, \ldots, 42}_{\text{punctuation}} \underbrace{43, \ldots}_{\text{space and others}} \right\}$$

$$\equiv \{A, B, \ldots, Z, 0, 1, \ldots, 9, ;, \cdot, ``, `, ?, \prime, \text{ space}, \ldots\}$$

    $p_{ij} \equiv$ probability next key typed is j given last one was i

    $p_i \equiv$ probability first letter typed is i

    <u>Case 6.5</u>

    S_n represents the brand of the nth toothpaste purchase

    $\mathcal{M} = \{1, 2, 3, 4, \ldots, m\}$ where each state represents a brand of toothpaste

    $p_{ij} \equiv$ probability a consumer next purchases brand j given they currently use i

    $p_i \equiv$ probability a consumer uses brand i at the beginning of the study

    <u>Case 6.6</u>

    S_n represents the high temperature on the nth day

    $\mathcal{M} = \{-20, -19, -18, \ldots, 109, 110\}$ is temperature in Fahrenheit

    $p_{ij} \equiv$ probability high temperature tomorrow is j given today's high is i

    $p_i \equiv$ probability temperature is i on the first day of the study

    <u>Case 6.7</u>

    S_n represents the task type of the nth task

    $\mathcal{M} = \{1, 2, \ldots, m\}$ where each state represents a task such as "move," "place," "change tool," etc.

    $p_{ij} \equiv$ probability next task is j given current task is i

    $p_i \equiv$ probability first task is type i

Case 6.8

S_n represents accident history for years $n-1$ and n

$\mathcal{M} = \{1, 2, 3, 4\}$ as defined in Case 6.8

15. No answer provided.

16. (a)

$$\pi_2 = 1/4 = 0.25$$

(b) No answer provided.

(c) Under the binomial model with probability of $\gamma = \pi_2$

$$\begin{aligned}
\Pr\{Z_{15} \le 1\} &= \sum_{j=0}^{1} \frac{15!}{j!(15-j)!} \pi_2^j (1-\pi_2)^{15-j} \\
&= 0.9904 \text{ when } \pi_2 = 0.01 \\
&= 0.0802 \text{ when } \pi_2 = 0.25
\end{aligned}$$

For the Markov-chain model we have to look at all possible sample paths:

• Pr{no defectives}

$$\begin{aligned}
\Pr\{\text{no defectives}\} &= p_1(p_{11}p_{11}\cdots p_{11}) \\
&= \pi_1\, p_{11}^{14} = a
\end{aligned}$$

• Pr {1 defective} has several cases:

first item defective

$$\Pr = p_2 p_{21} p_{11} \cdots p_{11} = \pi_2 p_{21} p_{11}^{13} = b$$

last item defective

$$\Pr = p_1 p_{11} \cdots p_{11} p_{12} = \pi_1 p_{11}^{13} p_{12} = c$$

some other item defective

$$\begin{aligned}
\Pr &= p_1 p_{11} p_{11} \cdots p_{12} p_{21} p_{11} p_{11} \cdots p_{11} \\
&= \pi_1 p_{11}^{12} p_{12} p_{21} = d
\end{aligned}$$

and there are 13 possible "other ones"

$$\begin{aligned}
\Pr\{\leq 1 \text{ defective}\} &= a + b + c + 13d \\
&= p_{11}^{12}(\pi_1 p_{11}^2 + \pi_2 p_{21} p_{11} + \pi_1 p_{11} p_{12} \\
&\qquad\qquad + 13\pi_1 p_{12} p_{21}) \\
&= p_{11}^{12}(\pi_1(p_{11}^2 + p_{11}p_{12} + 13p_{12}p_{21}) + \\
&\qquad\qquad \pi_2 p_{21} p_{11})
\end{aligned}$$

For (6.11) this ≈ 0.9169

For 16(a) this ≈ 0.1299

Notice that the Markov-chain model gives a lower probability of accepting a good process, and a higher probability of accepting a bad process.

17. (a) $\mathcal{M} = \{1, 2, 3, 4\} = \{A, B, C, D\}$ is the location of the AGV

    n represents the number of trips

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}$$

    (b) $p_{24}^{(5)} \approx 0.1934$

    (c) $\pi_4 = 3/16 = 0.1875$

18. (a)

$$(2,000,000)p_{12} + (2,000,000)p_{22} + (2,000,000)p_{32} = 2,107,800$$

    (b) If "today" is 1994, then 16 years have passed

$$(2,000,000)(p_{11}^{(16)} + p_{21}^{(16)} + p_{31}^{(16)}) \approx 2,019,566$$

19. (a) $\mathcal{M} = \{0, 1, 2, \ldots, k\}$ is the number of prisoners that remain in the prison

    n is the number of months

    Let

$$b(i, \ell) = \binom{i}{\ell} p^{\ell}(1-p)^{i-\ell}$$

    which is the probability ℓ prisoners are paroled if there are i in prison.

    Then $\mathbf{P}$ has the following form

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ b(1,1) & b(1,0) & 0 & 0 & \cdots & 0 \\ b(2,2) & b(2,1) & b(2,0) & 0 & \cdots & 0 \\ b(3,3) & b(3,2) & b(3,1) & b(3,0) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ b(k,k) & b(k,k-1) & b(k,k-2) & b(k,k-3) & \cdots & b(k,0) \end{pmatrix}$$

(b) $p_{k0}^{(m-1)}$

With $k = 6, p = 0.1$ and $m = 12$ we obtain $p_{60}^{(11)} \approx 0.104$.

(c) $\sum_{\ell=0}^{k} \ell p_{k\ell}^{(m)}$

With $k = 6, p = 0.1$ and $m = 12$ we obtain $\sum_{\ell=0}^{6} \ell p_{6\ell}^{(12)} \approx 1.7$.

(d) For each prisoner, the probability that they have not been paroled by m months is

$$q = (1-p)^m$$

Therefore, $1 - q$ is the probability they have been paroled before m months.

The prison will close if all have been paroled. Since the prisoners are independent

$$\Pr\{\text{all paroled before } m\} = (1-q)^k = (1-(1-p)^m)^k$$

20. (a) $\mathcal{M} = \{1, 2, 3\} \equiv$ {good and declared good, defective but declared good, defective and declared defective}

$n \equiv$ number of items produced

$$\mathbf{P} = \begin{pmatrix} 0.995 & 0.005(0.06) & 0.005(0.94) \\ 0.495 & 0.505(0.06) & 0.505(0.94) \\ 0.495 & 0.505(0.06) & 0.505(0.94) \end{pmatrix}$$

(b) $\pi_2 = 0.0006$

21. (a)

$$\begin{aligned} \mathcal{M} &= \{1, 2, 3, 4\} \\ &= \{(\text{no, no}), (\text{no, accident}), (\text{accident, no}), (\text{accident, accident})\} \end{aligned}$$

n counts number of years

$$\mathbf{P} = \begin{pmatrix} 0.92 & 0.08 & 0 & 0 \\ 0 & 0 & 0.97 & 0.03 \\ 0.92 & 0.08 & 0 & 0 \\ 0 & 0 & 0.97 & 0.03 \end{pmatrix}$$

(b) $\pi_1 \approx 0.85$

(c)

$$\begin{aligned} & 2000(500\pi_1 + 625(\pi_2 + \pi_3) + 800\pi_4) \\ \approx \quad & 2000(500(0.8499) + 625(0.0739 + 0.0739) + 800(0.0023)) \\ \approx \quad & \$1,038,298 \end{aligned}$$

22. The proof is by induction.

For $i \in \mathcal{A}$ and $j \in \mathcal{B}$, $f_{ij}^{(1)} = p_{ij}$. Thus,

$$\mathbf{F}_{\mathcal{AB}}^{(1)} = \mathbf{P}_{\mathcal{AA}}^{(0)} \mathbf{P}_{\mathcal{AB}}$$

Now suppose that the result is true for all $n \leq k$, for some $k > 1$.

$$\begin{aligned} f_{ij}^{(k+1)} &= \sum_{\ell \in \mathcal{A}} \Pr\{S_{k+1} = j, S_k \in \mathcal{A}, \ldots, S_2 \in \mathcal{A}, S_1 = \ell \mid S_0 = i\} \\ &= \sum_{\ell \in \mathcal{A}} \Pr\{S_{k+1} = j, S_k \in \mathcal{A}, \ldots, S_2 \in \mathcal{A} \mid S_1 = \ell, S_0 = i\} \\ &\qquad\qquad\qquad \times \Pr\{S_1 = \ell \mid S_0 = i\} \\ &= \sum_{\ell \in \mathcal{A}} \Pr\{S_k = j, S_{k-1} \in \mathcal{A}, \ldots, S_1 \in \mathcal{A} \mid S_0 = \ell\} p_{i\ell} \\ &= \sum_{\ell \in \mathcal{A}} f_{\ell j}^{(k)} p_{i\ell} = \sum_{\ell \in \mathcal{A}} p_{i\ell} f_{\ell j}^{(k)} \end{aligned}$$

Therefore, in matrix form and using the induction hypothesis

$$\begin{aligned} \mathbf{F}_{\mathcal{AB}}^{(k+1)} &= \mathbf{P}_{\mathcal{AA}} \mathbf{F}_{\mathcal{AB}}^{(k)} \\ &= \mathbf{P}_{\mathcal{AA}} \mathbf{P}_{\mathcal{AA}}^{(k-1)} \mathbf{P}_{\mathcal{AB}} \\ &= \mathbf{P}_{\mathcal{AA}}^{(k)} \mathbf{P}_{\mathcal{AB}} \end{aligned}$$

Therefore, the result is true for $n = k + 1$, and is thus true for any n.

23. Let $X \equiv$ the number of times the process is in state j.

Then

$$X = \sum_{n=0}^{\infty} \mathcal{I}(S_n = j)$$

and clearly

$$\mathcal{I}(S_0 = j) = \begin{cases} 0, & i \neq j \\ 1 & i = j \end{cases}$$

Thus,

$$\begin{aligned}
\mu_{ij} = \mathrm{E}[X] &= \mathrm{E}\left[\sum_{n=0}^{\infty} \mathcal{I}(S_n = j)\right] \\
&= \mathcal{I}(i = j) + \sum_{n=1}^{\infty} \mathrm{E}[\mathcal{I}(S_n = j)] \\
&= \mathcal{I}(i = j) + \sum_{n=1}^{\infty} \left\{0(1 - p_{ij}^{(n)}) + 1p_{ij}^{(n)}\right\} \\
&= \mathcal{I}(i = j) + \sum_{n=1}^{\infty} p_{ij}^{(n)}
\end{aligned}$$

In matrix form

$$\begin{aligned}
\mathbf{M} &= \mathbf{I} + \sum_{n=1}^{\infty} \mathbf{P}_{\mathcal{TT}}^{(n)} \\
&= \mathbf{I} + \sum_{n=1}^{\infty} \mathbf{P}_{\mathcal{TT}}^{n} \\
&= (\mathbf{I} - \mathbf{P}_{\mathcal{TT}})^{-1}
\end{aligned}$$

24. $\mathcal{T} = \{1, 2, 3\}$ and

$$\mathbf{P}_{\mathcal{TT}} = \begin{pmatrix} 0 & 0.95 & 0.01 \\ 0 & 0.27 & 0.63 \\ 0 & 0.36 & 0.40 \end{pmatrix}$$

Therefore,

$$\mathbf{M} = (\mathbf{I} - \mathbf{P}_{\mathcal{TT}})^{-1} \approx \begin{pmatrix} 1 & 2.72 & 2.87 \\ 0 & 2.84 & 2.98 \\ 0 & 1.70 & 3.46 \end{pmatrix}$$

The expected session length is

$$\mu_{11} + \mu_{12} + \mu_{13} \approx 1 + 2.72 + 2.87 \approx 6.6$$

25. (a) We first argue that

$$V_{ij} = \begin{cases} 1, & \text{if } S_1 = j \\ 1 + V_{hj}, & \text{if } S_1 = h \neq j \end{cases}$$

Clearly if the process moves to state j on the first step then $\{V_{ij} = 1\}$. If the process moves to some other state $h \neq j$, then the Markov property implies that the future evolution of the process is as if the process started in state h.

Therefore,

$$\begin{aligned}
\mathrm{E}[V_{ij}] &= \sum_{h=1}^{m} \mathrm{E}[V_{ij} \mid S_1 = h] \Pr\{S_1 = h \mid S_0 = i\} \\
&= 1\ p_{ij} + \sum_{h \neq j} \mathrm{E}[1 + V_{hj}] p_{ih} \\
&= p_{ij} + \sum_{h \neq j} (1 + \nu_{hj}) p_{ih} \\
&= p_{ij} + \sum_{h=1}^{m} p_{ih}(1 + \nu_{hj}) - p_{ij}(1 + \nu_{jj}) \\
&= \sum_{h=1}^{m} p_{ih}(1 + \nu_{hj}) - p_{ij}\nu_{jj}
\end{aligned}$$

In matrix form

$$\begin{aligned}
\mathbf{V} &= \mathbf{P}(\mathbf{1}\ \mathbf{1}' + \mathbf{V}) - \mathbf{PD} \\
&= \mathbf{P}(\mathbf{V} - \mathbf{D}) + \mathbf{1}\ \mathbf{1}'
\end{aligned}$$

(b) Since

$$\mathbf{V} = \mathbf{PV} - \mathbf{PD} + \mathbf{1}\ \mathbf{1}'$$

$$\begin{aligned}
\boldsymbol{\pi}'\mathbf{V} &= \boldsymbol{\pi}'\mathbf{PV} - \boldsymbol{\pi}'\mathbf{PD} + \boldsymbol{\pi}'\mathbf{1}\ \mathbf{1}' \\
&= \boldsymbol{\pi}'\mathbf{V} - \boldsymbol{\pi}'\mathbf{D} + \mathbf{1}'
\end{aligned}$$

Therefore,

$$\boldsymbol{\pi}'\mathbf{D} = \mathbf{1}'$$

or term by term

$$\pi_i \nu_{ii} = 1$$

or

$$\nu_{ii} = \frac{1}{\pi_i}$$

26. Let $\mathcal{M} = \{0, 1, 2, \ldots, 10\}$ represent the number of cherries a player has in the basket. The one-step transition matrix is

$$\mathbf{P} = \begin{pmatrix} 3/7 & 1/7 & 1/7 & 1/7 & 1/7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/7 & 0 & 1/7 & 1/7 & 1/7 & 1/7 & 0 & 0 & 0 & 0 & 0 \\ 3/7 & 0 & 0 & 1/7 & 1/7 & 1/7 & 1/7 & 0 & 0 & 0 & 0 \\ 1/7 & 2/7 & 0 & 0 & 1/7 & 1/7 & 1/7 & 1/7 & 0 & 0 & 0 \\ 1/7 & 0 & 2/7 & 0 & 0 & 1/7 & 1/7 & 1/7 & 1/7 & 0 & 0 \\ 1/7 & 0 & 0 & 2/7 & 0 & 0 & 1/7 & 1/7 & 1/7 & 1/7 & 0 \\ 1/7 & 0 & 0 & 0 & 2/7 & 0 & 0 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 0 & 0 & 0 & 0 & 2/7 & 0 & 0 & 1/7 & 1/7 & 2/7 \\ 1/7 & 0 & 0 & 0 & 0 & 0 & 2/7 & 0 & 0 & 1/7 & 3/7 \\ 1/7 & 0 & 0 & 0 & 0 & 0 & 0 & 2/7 & 0 & 0 & 4/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\mathcal{T} = \{0, 1, \ldots, 9\}$ and $\mathcal{R} = \{10\}$.

The expected number of spins is

$$\mu_{00} + \mu_{01} + \cdots + \mu_{09} \approx 15.8$$

which is the sum of the first row of

$$
\begin{aligned}
\mathbf{M} &= (\mathbf{I} - \mathbf{P}_{\mathcal{T}\mathcal{T}})^{-1} \\
&= \left(\begin{matrix}
5.642555330 & 1.237076756 & 1.468321920 & 1.508490981 & 1.699310675 \\
4.430331574 & 2.051427403 & 1.388800859 & 1.464830124 & 1.619923520 \\
4.189308265 & 0.9910305820 & 2.185061681 & 1.373952904 & 1.557546460 \\
3.624339776 & 1.132212235 & 1.088206223 & 2.150572935 & 1.430445774 \\
3.326241706 & 0.7736368034 & 1.211218915 & 1.044607959 & 2.189326947 \\
2.944765281 & 0.7518819338 & 0.8321534392 & 1.159204127 & 1.064213430 \\
2.502145104 & 0.5682528341 & 0.7588874323 & 0.7378123674 & 1.120907044 \\
2.094007860 & 0.4917825124 & 0.5692601344 & 0.6742348636 & 0.6995152842 \\
1.721601833 & 0.3844024235 & 0.4797861159 & 0.4846075656 & 0.6262492773 \\
1.404367293 & 0.3172345401 & 0.3724060269 & 0.4081372440 & 0.4426201777
\end{matrix}\right. \\
&\qquad \left.\begin{matrix}
1.105741430 & 1.019364871 & 0.9134948380 & 0.6768445448 & 0.5307779547 \\
1.191625516 & 1.002037338 & 0.9081983520 & 0.6745406751 & 0.5394859830 \\
1.126134901 & 1.081735894 & 0.8876763410 & 0.6647276567 & 0.5371821134 \\
1.104626156 & 1.008894626 & 0.9654729638 & 0.6442056457 & 0.5318856274 \\
1.000579145 & 0.9847916242 & 0.8926316951 & 0.7239042017 & 0.5145580951 \\
1.792814119 & 0.8807446128 & 0.8711229501 & 0.6584135873 & 0.6004421813 \\
0.6677006009 & 1.639625786 & 0.7440222642 & 0.5960365279 & 0.5210550255 \\
0.7822967692 & 0.5336608097 & 1.520642295 & 0.5051593083 & 0.4773941689 \\
0.4032312936 & 0.6566735018 & 0.4237868366 & 1.301420130 & 0.3978731088 \\
0.3814764240 & 0.2980980700 & 0.5649684897 & 0.2410233088 & 1.212223756
\end{matrix}\right)
\end{aligned}
$$

The distribution of the number of spins is $f_{0,10}^{(n)}$, $n = 1, 2, \ldots$ with $\mathcal{A} = \{0, 1, \ldots, 9\}$ and $\mathcal{B} = \{10\}$ which is the $(1, 1)$ element of $\mathbf{F}_{\mathcal{AB}}^{(n)} = \mathbf{P}_{\mathcal{AA}}^{(n-1)}\mathbf{P}_{\mathcal{AB}}$.

A table for n from 1 to 25 is

n	$f_{0,10}^{(n)}$
1	0
2	0
3	.02915451895
4	.05997501041
5	.06538942107
6	.06399544408
7	.06018750691
8	.05585344577
9	.05162197768
10	.04764238654
11	.04395328102
12	.04054295131
13	.03739618725
14	.03449304106
15	.03181517402
16	.02934515930
17	.02706689761
18	.02496550906
19	.02302726490
20	.02123949969
21	.01959053092
22	.01806958294
23	.01666671664
24	.01537276451
25	.01417927081

There are a variety of ways to revise the spinner.

Advanced Exercise

Let $X_i \equiv$ number of spins by player i

Then the total number of spins is

$$2\min\{X_1, X_2\} = 2Z$$

$$\begin{aligned}
\Pr\{Z \leq n\} &= 1 - \Pr\{Z > n\} \\
&= 1 - \Pr\{\min\{X_1 X_2\} > n\} \\
&= 1 - \Pr\{X_1 > n, X_2 > n\}
\end{aligned}$$

$$\begin{aligned} &= 1 - \Pr\{X_1 > n\} \Pr\{X_2 > n\} \\ &= 1 - (\Pr\{X_1 > n\})^2 \end{aligned}$$

since spins are independent and identically distributed. Notice that $\Pr\{X_1 > n\} = 1 - p_{0,10}^{(n)}$. Then letting T be the total number of spins

$$\begin{aligned} \Pr\{T \le 2n\} &= \Pr\{Z \le n\} \\ &= 1 - (1 - p_{0,10}^{(n)})^2 \end{aligned}$$

which implies that

$$\begin{aligned} \Pr\{T = 2n\} &= \Pr\{T \le 2n\} - \Pr\{T \le 2(n-1)\} \\ &= (1 - p_{0,10}^{(n)})^2 - (1 - p_{0,10}^{(n-1)})^2 \end{aligned}$$

This result can be used to calculate the distribution or expectation. You will find that $\mathrm{E}[T] < \mathrm{E}[2X_1]$.

27. (a)

$$\begin{aligned} \mathcal{M} &= \{1,2,3,4,5\} \\ &\equiv \{\text{insert, withdraw, deposit, information, done}\} \end{aligned}$$

n represents the number of transactions

$$\mathbf{P} = \begin{pmatrix} 0 & 0.5 & 0.4 & 0.1 & 0 \\ 0 & 0 & 0.05 & 0.05 & 0.9 \\ 0 & 0.05 & 0 & 0.05 & 0.9 \\ 0 & 0.05 & 0.05 & 0 & 0.9 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Both properties are rough approximations at best. Customers are likely to do only 1 of each type of transaction, so all of their previous transactions influence what they will do next. Also, the more transactions they have made, the more likely they are to be done, violating stationarity.

(c) Need $f_{15}^{(n)}$ for $n = 1, 2, \ldots, 20$ with $\mathcal{A} = \{1,2,3,4\}$ and $\mathcal{B} = \{5\}$.

n	$f_{15}^{(n)}$
1	0
2	0.9
3	0.09
4	0.009
5	0.0009
6	0.00009
7–20	nearly 0

(d) We need $100\mu_{12}$ because μ_{12} is the expected number of times in the withdraw state.

$$
\begin{aligned}
\mathcal{T} &= \{1,2,3,4\} \quad \mathcal{R} = \{5\} \\
\mathbf{M} &= (\mathbf{I} - \mathbf{P}_{\mathcal{T}\mathcal{T}})^{-1} \\
&= \begin{pmatrix} 1.0 & 0.5291005291 & 0.4338624339 & 0.1481481481 \\ 0 & 1.005291005 & 0.05291005291 & 0.05291005291 \\ 0 & 0.05291005291 & 1.005291005 & 0.05291005291 \\ 0 & 0.05291005291 & 0.05291005291 & 1.005291005 \end{pmatrix}
\end{aligned}
$$

Therefore, $100\mu_{12} \approx 100(0.5291) = \$52.91$.

28. Clearly $\Pr\{H = 1\} = 1 - p_{ii} = \gamma$. Suppose the result is correct for all $a \leq n$ for some $n > 1$. Then

$$
\begin{aligned}
\Pr\{H = n+1\} &= \Pr\{S_{n+1} \neq i, S_n = i, S_{n-1} = i, \ldots, S_1 = i \mid S_0 = i\} \\
&= \Pr\{S_{n+1} \neq i, S_n = i, \ldots, S_2 = i \mid S_1 = i, S_0 = i\} \Pr\{S_1 = i \mid S_0 = i\} \\
&= \Pr\{S_n \neq i, S_{n-1} = i, \ldots, S_1 = i \mid S_0 = i\}(1-\gamma) \\
&= (1-\gamma)^{n-1}\, \gamma(1-\gamma) = (1-\gamma)^n \gamma
\end{aligned}
$$

from the induction hypothesis. Therefore, it is correct for any a.

29. Since $f_{jj}^{(1)} = p_{jj}$, $\mathbf{R}_{\mathcal{BB}}^{(1)} = \mathbf{P}_{\mathcal{BB}}$.

We can write

$$
\begin{aligned}
f_{jj}^{(2)} &= \sum_{h \in \mathcal{A}} \Pr\{S_2 = j, S_1 = h \mid S_0 = j\} \\
&= \sum_{h \in \mathcal{A}} \Pr\{S_2 = j \mid S_1 = h, S_0 = j\} \Pr\{S_1 = h \mid S_0 = j\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{h\in\mathcal{A}} \Pr\{S_1 = j \mid S_0 = h\} p_{jh} \\
&= \sum_{h\in\mathcal{A}} p_{hj} p_{jh} = \sum_{h\in\mathcal{A}} p_{jh} p_{hj}
\end{aligned}
$$

In matrix form

$$
\mathbf{R}^{(2)}_{\mathcal{BB}} = \mathbf{P}_{\mathcal{BA}} \mathbf{P}_{\mathcal{AB}}
$$

For $n \geq 3$

$$
\begin{aligned}
f^{(n)}_{jj} &= \Pr\{S_n = j, S_{n-1} \in \mathcal{A}, \ldots, S_1 \in \mathcal{A} \mid S_0 = j\} \\
&= \sum_{h\in\mathcal{A}} \Pr\{S_n = j, S_{n-1} \in \mathcal{A}, \ldots, S_1 = h \mid S_0 = j\} \\
&= \sum_{h\in\mathcal{A}} \Pr\{S_n = j, S_{n-1} \in \mathcal{A}, \ldots, S_2 \in \mathcal{A} \mid S_1 = h, S_0 = j\} \\
&\qquad\qquad\qquad\qquad \times \Pr\{S_1 = h \mid S_0 = j\} \\
&= \sum_{h\in\mathcal{A}} \Pr\{S_{n-1} = j, S_{n-2} \in \mathcal{A}, \ldots, S_1 \in \mathcal{A} \mid S_0 = h\} p_{jh} \\
&= \sum_{h\in\mathcal{A}} f^{(n-1)}_{hj} p_{jh} = \sum_{h\in\mathcal{A}} p_{jh} f^{(n-1)}_{hj}
\end{aligned}
$$

In matrix form

$$
\mathbf{R}^{(n)}_{\mathcal{BB}} = \mathbf{P}_{\mathcal{BA}} (\mathbf{P}^{n-2}_{\mathcal{AA}} \mathbf{P}_{\mathcal{AB}})
$$

In the customer-behavior model

$$
\mathbf{P}_{\mathcal{AA}} = \begin{pmatrix} 0 & 0.01 & 0.04 \\ 0 & 0.40 & 0.24 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\mathbf{P}_{\mathcal{BA}} = (0 \qquad 0.63 \qquad 0.10)
$$

$$
\mathbf{P}_{\mathcal{AB}} = \begin{pmatrix} 0.95 \\ 0.36 \\ 0 \end{pmatrix}
$$

Therefore

$$\mathbf{R}^{(5)}_{\mathcal{BB}} = f^{(5)}_{22} \approx 0.015$$

30. Let $p(a) = \Pr\{X_n = a\} = \frac{e^{-2}2^a}{a!}, a = 0, 1, \ldots$.
For given (r, s), the state space is

$$\mathcal{M} = \{r, r+1, \ldots, s\}$$

Careful thinking shows that

$$p_{ij} = \begin{cases} p(i-j), & j = r, r+1, \ldots, i \\ 0, & j = i+1, i+2, \ldots, s-1 \\ 1 - \sum_{h=r}^{i} p_{ih}, & j = s \end{cases}$$

with the exception that $p_{ss} = 1 - \sum_{h=r}^{s-1} p_{sh}$.
The one-step transition matrix for $(r, s) = (4, 10)$ is

$$\mathbf{P} = \left(\begin{array}{cccc} 0.1353352832 & 0 & 0 & 0 \\ 0.2706705664 & 0.1353352832 & 0 & 0 \\ 0.2706705664 & 0.2706705664 & 0.1353352832 & 0 \\ 0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353352832 \\ 0.09022352214 & 0.1804470442 & 0.2706705664 & 0.2706705664 \\ 0.03608940886 & 0.09022352214 & 0.1804470442 & 0.2706705664 \\ 0.01202980295 & 0.03608940886 & 0.09022352214 & 0.1804470442 \end{array} \right.$$

$$\left. \begin{array}{ccc} 0 & 0 & 0.8646647168 \\ 0 & 0 & 0.5939941504 \\ 0 & 0 & 0.3233235840 \\ 0 & 0 & 0.1428765398 \\ 0.1353352832 & 0 & 0.0526530176 \\ 0.2706705664 & 0.1353352832 & 0.0165636087 \\ 0.2706705664 & 0.2706705664 & 0.1398690889 \end{array} \right)$$

The associated steady-state distribution is

$$\boldsymbol{\pi} = \begin{pmatrix} 0.1249489668 \\ 0.1250496100 \\ 0.1256593229 \\ 0.1258703295 \\ 0.1188385832 \\ 0.09050677047 \\ 0.2891264170 \end{pmatrix}$$

Therefore, the long-run expected inventory position is

$$\sum_{h=r}^{s} h\pi_h = \sum_{h=4}^{10} h\pi_h \approx 7.4$$

Similarly, for the $(r, s) = (3, 12)$ policy, $\mathbf{P} =$

$$\begin{pmatrix}
0.1353352832 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8646647168 \\
0.2706705664 & 0.1353352832 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5939941504 \\
0.2706705664 & 0.2706705664 & 0.1353352832 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3233235840 \\
0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353352832 & 0 & 0 & 0 & 0 & 0 & 0.1428765398 \\
0.09022352214 & 0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353352832 & 0 & 0 & 0 & 0 & 0.0526530176 \\
0.03608940886 & 0.09022352214 & 0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353352832 & 0 & 0 & 0 & 0.0165636087 \\
0.01202980295 & 0.03608940886 & 0.09022352214 & 0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353352832 & 0 & 0 & 0.0045338057 \\
0.003437086558 & 0.01202980295 & 0.03608940886 & 0.09022352214 & 0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353352832 & 0 & 0.0010967192 \\
0.0008592716393 & 0.003437086558 & 0.01202980295 & 0.03608940886 & 0.09022352214 & 0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353352832 & 0.0002374476 \\
0.0001909492532 & 0.0008592716393 & 0.003437086558 & 0.01202980295 & 0.03608940886 & 0.09022352214 & 0.1804470442 & 0.2706705664 & 0.2706705664 & 0.1353817816
\end{pmatrix}$$

$$\boldsymbol{\pi} = \begin{pmatrix} 0.09091028492 \\ 0.09091052111 \\ 0.09089895276 \\ 0.09087291478 \\ 0.09094611060 \\ 0.09138954271 \\ 0.09154300364 \\ 0.08642895347 \\ 0.06582378587 \\ 0.2102759303 \end{pmatrix}$$

$$\sum_{h=3}^{12} h\pi_h \approx 7.9$$

Therefore, policy B has the largest long-run inventory position.

31. We show (6.5) for $k = 2$. The general proof is analogous.

$$\begin{aligned}
&\Pr\{S_{n+2} = j \mid S_n = i, S_{n-1} = a, \ldots, S_0 = z\} \\
&= \sum_{h=1}^{m} \Pr\{S_{n+2} = j, S_{n+1} = h \mid S_n = i, S_{n-1} = a, \ldots, S_0 = z\} \\
&= \sum_{h=1}^{m} \Pr\{S_{n+2} = j \mid S_{n+1} = h, S_n = i, \ldots, S_0 = z\} \\
&\qquad\qquad \times \Pr\{S_{n+1} = h \mid S_n = i, \ldots, S_0 = z\}
\end{aligned}$$

We now apply the Markov property

$$\begin{aligned}
&= \sum_{h=1}^{m} \Pr\{S_{n+2} = j \mid S_{n+1} = h\} \Pr\{S_{n+1} = h \mid S_n = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_{n+2} = j, S_{n+1} = h \mid S_n = i\} \\
&= \Pr\{S_{n+2} = j \mid S_n = i\}
\end{aligned}$$

We show (6.6) for $k = 2$. The general proof is analogous.

$$\begin{aligned}
&\Pr\{S_{n+2} = j \mid S_n = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_{n+2} = j, S_{n+1} = h \mid S_n = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_{n+2} = j \mid S_{n+1} = h, S_n = i\} \Pr\{S_{n+1} = h \mid S_n = i\}
\end{aligned}$$

We now apply the Markov and stationarity properties

$$\begin{aligned}
&= \sum_{h=1}^{m} \Pr\{S_2 = j \mid S_1 = h\} \Pr\{S_1 = h \mid S_0 = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_2 = j, S_1 = h \mid S_0 = i\} \\
&= \Pr\{S_2 = j \mid S_0 = i\}
\end{aligned}$$

independent of n.

32.

$$
\begin{aligned}
p_{ij}^{(n)} &= \Pr\{S_n = j \mid S_0 = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_n = j, S_k = h \mid S_0 = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_n = j \mid S_k = h, S_0 = i\} \Pr\{S_k = h \mid S_0 = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_n = j \mid S_k = h\} p_{ih}^{(k)} \\
&= \sum_{h=1}^{m} \Pr\{S_{n-k} = j \mid S_0 = h\} p_{ih}^{(k)} \\
&= \sum_{h=1}^{m} p_{hj}^{(n-k)}\ p_{ih}^{(k)} \\
&= \sum_{h=1}^{m} p_{ih}^{(k)} p_{hj}^{(n-k)}
\end{aligned}
$$

33. No answer provided.

Chapter 7

Continuous-Time Processes

1. $\pi_1 = 5/7, \quad \pi_2 = 2/7$

2. (a) (i)

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 2/5 & 0 & 2/5 & 1/5 & 0 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 3/6 & 1/6 & 0 & 0 & 2/6 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(ii) all recurrent

(iii)

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{175}{282} \\ \frac{20}{141} \\ \frac{10}{141} \\ \frac{25}{282} \\ \frac{11}{141} \end{pmatrix}$$

(b) (i)

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(ii) $\mathcal{T} = \{1, 2, 3\} \quad \mathcal{R} = \{4, 5\}$

(iii) $\boldsymbol{\pi} = \binom{5/7}{2/7}$

(c) (i)

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) $\mathcal{T} = \{2,3,4\} \quad \mathcal{R}_1 = \{1\}, \quad \mathcal{R}_2 = \{5\}$

(iii) $\pi_{\mathcal{R}_1} = 1 \quad \pi_{\mathcal{R}_2} = 1$

3. (a) (i) For $i = 1, 2, \ldots, 5$

$$\begin{aligned}
\frac{dp_{i1}(t)}{dt} &= -p_{i1}(t) + 2p_{i2}(t) + p_{i3}(t) \\
\frac{dp_{i2}(t)}{dt} &= p_{i1}(t) - 5p_{i2}(t) + p_{i4}(t) \\
\frac{dp_{i3}(t)}{dt} &= 2p_{i2}(t) - 4p_{i3}(t) \\
\frac{dp_{i4}(t)}{dt} &= p_{i2}(t) - 6p_{i4}(t) + 5p_{i5}(t) \\
\frac{dp_{i5}(t)}{dt} &= 3p_{i3}(t) + 2p_{i4}(t) - 5p_{i5}(t)
\end{aligned}$$

(ii)

$$\mathbf{P}(t) = e^{-6t} \sum_{n=0}^{\infty} \mathbf{Q}^n \frac{(6t)^n}{n!}$$

where

$$\mathbf{Q} = \begin{pmatrix} 5/6 & 1/6 & 0 & 0 & 0 \\ 1/3 & 1/6 & 1/3 & 1/6 & 0 \\ 1/6 & 0 & 1/3 & 0 & 1/2 \\ 1/2 & 1/6 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 5/6 & 1/6 \end{pmatrix}$$

(b) (i) For $i = 1, 2, \ldots, 5$

$$\begin{aligned}
\frac{dp_{i1}(t)}{dt} &= -p_{i1}(t) + 2p_{i2}(t) + p_{i3}(t) \\
\frac{dp_{i2}(t)}{dt} &= p_{i1}(t) - 4p_{i2}(t)
\end{aligned}$$

$$\begin{aligned}
\frac{dp_{i3}(t)}{dt} &= 2p_{i2}(t) - 4p_{i3}(t) \\
\frac{dp_{i4}(t)}{dt} &= -2p_{i4}(t) + 5p_{i5}(t) \\
\frac{dp_{i5}(t)}{dt} &= 3p_{i3}(t) + 2p_{i4}(t) - 5p_{i5}(t)
\end{aligned}$$

(ii)

$$\mathbf{P}(t) = e^{-5t} \sum_{n=0}^{\infty} \mathbf{Q}^n \frac{(5t)^n}{n!}$$

where

$$\mathbf{Q} = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 & 0 \\ 2/5 & 1/5 & 2/5 & 0 & 0 \\ 1/5 & 0 & 1/5 & 0 & 3/5 \\ 0 & 0 & 0 & 3/5 & 2/5 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) (i) For $i = 1, 2, \ldots, 5$

$$\begin{aligned}
\frac{dp_{i1}(t)}{dt} &= 2p_{i2}(t) + p_{i3}(t) \\
\frac{dp_{i2}(t)}{dt} &= -4p_{i2}(t) \\
\frac{dp_{i3}(t)}{dt} &= 2p_{i2}(t) - 4p_{i3}(t) \\
\frac{dp_{i4}(t)}{dt} &= -2p_{i4}(t) \\
\frac{dp_{i5}(t)}{dt} &= 3p_{i3}(t) + 2p_{i4}(t)
\end{aligned}$$

(ii)

$$\mathbf{P}(t) = e^{-4t} \sum_{n=0}^{\infty} \mathbf{Q}^n \frac{(4t)^n}{n!}$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4. We give a simulation for the proposed system.

   Let $\mathcal{M} = \{1, 2, 3, 4\}$ be as defined in Case 7.2.

   Define the following random variables:

   $G \equiv$ time gap between regular calls

   $H \equiv$ time gap between calls to the chair

   $X \equiv$ time to answer regular calls

   $Z \equiv$ time to answer chair's calls

   Define the following system events.

   $e_0()$ (start of the day)

   $S_0 \leftarrow 1$
   $C_1 \leftarrow F_G^{-1}(\texttt{random()})$
   $C_2 \leftarrow F_H^{-1}(\texttt{random()})$
   $C_3 \leftarrow \infty$
   $C_4 \leftarrow \infty$

   $e_1()$ (regular call)

   if $\{S_n = 1\}$ then
   $S_{n+1} \leftarrow 2$
   $C_3 \leftarrow T_{n+1} + F_X^{-1}\ (\texttt{random()})$
   endif
   $C_1 \leftarrow T_{n+1} + F_G^{-1}\ (\texttt{random()})$

   $e_2()$ (chair call)

   if $\{S_n = 1\}$ then
   $S_{n+1} \leftarrow 3$
   $C_4 \leftarrow T_{n+1} + F_Z^{-1}\ (\texttt{random()})$
   else
   if $\{S_n = 2\}$ then
   $S_{n+1} \leftarrow 4$
   $Z \leftarrow F_Z^{-1}\ (\texttt{random()})$
   $C_4 \leftarrow T_{n+1} + Z$
   $C_3 \leftarrow C_3 + Z$
   endif

endif

$C_2 \leftarrow T_{n+1} + F_H^{-1}$ (`random()`)

$e_3()$ (end regular call)

$S_{n+1} \leftarrow 1$

$e_4()$ (end chair call)

if $\{S_n = 3\}$ then

$S_{n+1} \leftarrow 1$

else

$S_{n+1} \leftarrow 2$

endif

5. (a) $\mathcal{M} = \{0, 1, 2, 3\}$ represents the number of vans in use at time t days. Approximate the time between requests and the time in use as exponentially distributed. Therefore, we have a Markov-process model with

$$\mathbf{G} = \begin{pmatrix} -\frac{8}{7} & \frac{8}{7} & 0 & 0 \\ \frac{1}{2} & -\frac{23}{14} & \frac{8}{7} & 0 \\ 0 & 1 & -\frac{15}{7} & \frac{8}{7} \\ 0 & 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix}$$

(b)

$$\boldsymbol{\pi} = \begin{pmatrix} 0.1267709745 \\ 0.2897622274 \\ 0.3311568313 \\ 0.2523099667 \end{pmatrix}$$

The rate at which requests are denied is $8/7\ \pi_3 \approx 0.29$ requests/day.

(c) $0\pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3 \approx 1.71$ vans

6. (7.1) for a and n integer

$$\begin{aligned} \Pr\{X > n + a \mid X > n\} &= \frac{\Pr\{X > n + a, X > n\}}{\Pr\{X > n\}} \\ &= \frac{\Pr\{X > n + a\}}{\Pr\{X > n\}} \\ &= \frac{(1-\gamma)^{n+a}}{(1-\gamma)^n} \\ &= (1-\gamma)^a = \Pr\{X > a\} \end{aligned}$$

(7.2) is proved similarly

$$\begin{aligned}
\Pr\{Z > t + b \mid Z > t\} &= \frac{\Pr\{Z > t + b\}}{\Pr\{Z > t\}} \\
&= \frac{e^{-\lambda(t+b)}}{e^{-\lambda t}} \\
&= e^{-\lambda b} \\
&= \Pr\{Z > b\}
\end{aligned}$$

7. For this model

$$\mathbf{G} = \begin{pmatrix}
-1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
1 & -3/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 1 & -3/2 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 1 & -3/2 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

giving

$$\boldsymbol{\pi} = \begin{pmatrix}
0.1290322581 \\
0.06451612903 \\
0.09677419355 \\
0.1451612903 \\
0.2177419355 \\
0.1532258065 \\
0.1209677419 \\
0.07258064516
\end{pmatrix}$$

The long-run lost sales rate is

$$\lambda \pi_0 \approx 0.129 \text{ sales/day}$$

The average inventory level is

$$\sum_{j=0}^{7} j\pi_j \approx 3.56 \text{ units}$$

8. See Exercise 7 for an analysis of the (3,7) policy.

   For the (2,7) policy

$$\mathbf{G} = \begin{pmatrix} -1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & -3/2 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & -3/2 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{\pi} = \begin{pmatrix} 0.2394366197 \\ 0.05633802817 \\ 0.08450704225 \\ 0.1267605634 \\ 0.1901408451 \\ 0.1901408451 \\ 0.07042253521 \\ 0.04225352113 \end{pmatrix}$$

   $\lambda\pi_0 \approx 0.239$

   $\sum_{j=0}^{7} j\pi_j \approx 3.03$

   Therefore, (2,7) carries a slightly lower inventory level but with more than double the lost sales rate.

9. To solve this problem we must evaluate $p_{02}(t)$ for $t = 17520$ hours (2 years). Using the uniformization approach with $n^* = 1000$, and carrying 10 digits of accuracy in all calculations,

$$\tilde{p}_{02}(t) = 0.56$$

   with truncation error $\leq 0.72 \times 10^{-8}$.

   The reason that this is dramatically larger than the simulation estimate of 0.02 obtained in Chapter 4 is the difference in repair-time distribution. In Chapter 4 the distribution is uniform on [4, 48] hours. Here the distribution is exponential on $[0, \infty)$ hours. The exponential model is pessimistic because it allows repair times longer than 48 hours.

10. Let $\mathcal{M} = \{0, 1, 2\}$ represent the number of **failed** machines, so that $\{Y_t; t \geq 0\}$ is the number of failed machines at time t hours. When $\{Y_t = 0\}$, the failure rate is 2(0.01), since each machine fails at rate 0.01/hour and there are two machines. When $\{Y_t = 1\}$

the failure rate is 0.02/hour, as stated in the problem. When $\{Y_t = 1 \text{ or } 2\}$ the repair rate is 1/24/hour $\approx$ 0.04/hour.

Therefore,

$$\mathbf{G} = \begin{pmatrix} -0.02 & 0.02 & 0 \\ 0.04 & -0.06 & 0.02 \\ 0 & 0.04 & -0.04 \end{pmatrix}$$

and

$$\pi_0 \approx 0.57, \quad \pi_1 \approx 0.29, \quad \pi_2 \approx 0.14$$

(a) $\pi_2 \approx 0.14$ or 14% of the time.

(b) $\pi_1 + \pi_2 \approx 0.43$ or 43% of the time.

11. Let X_{ij} be the total time in j given $\{Y_0 = i\}$.

Let Z_j be the time spent in j on a single visit.

Then

$$\mu_{ij} = \mathrm{E}[X_{ij}] = \mathrm{E}[Z_j]\mathcal{I}(i = j) + \sum_{k \in \mathcal{T}, k \neq i} \mathrm{E}[X_{kj}]p_{ik}$$

because if we start in j we must spend Z_j there, and we can condition on the first state visited after state i; the Markov property implies that once the process leaves i for k, the fact that it started in i no longer matters. But notice that

$$\mathrm{E}[X_{kj}] = \mu_{kj}$$

Therefore,

$$\mu_{ij} = 1/g_{jj}\ \mathcal{I}(i = j) + \sum_{k \in \mathcal{T}, k \neq i} \mu_{kj}\ p_{ik}$$

Then noting that $p_{ik} = g_{ik}/g_{ii}$ and $g_{jj} = g_{ii}$ if $i = j$, we have

$$\mu_{ij} = \frac{\mathcal{I}(i = j)}{g_{ii}} + \sum_{k \in \mathcal{T}, k \neq i} \mu_{kj}\ \frac{g_{ik}}{g_{ii}}$$

or

$$g_{ii}\ \mu_{ij} = \mathcal{I}(i = j) + \sum_{k \in \mathcal{T}, k \neq i} g_{ik}\ \mu_{kj}$$

12. Let Z_i be the time initially spent in state i.

Then

$$\nu_{ij} = \mathrm{E}[V_{ij}] = \mathrm{E}[Z_i] + \sum_{k \neq i, j} \mathrm{E}[V_{kj}]p_{ik}$$

by conditioning on the first state visited after i (if it is not j), and recalling that the Markov property implies that once the process leaves i for k the fact that it started in i no longer matters. But notice that

$$\mathrm{E}[V_{kj}] = \nu_{kj}$$

Therefore,

$$\nu_{ij} = 1/g_{ii} + \sum_{k \neq i,j} \nu_{kj}\ p_{ik}$$

Then noting that $p_{ik} = g_{ik}/g_{ii}$

$$\nu_{ij} = 1/g_{ii} + \sum_{k \neq i,j} \nu_{kj}\ g_{ik}/g_{ii}$$

or

$$g_{ii}\nu_{ij} = 1 + \sum_{k \neq i,j} g_{ik}\ \nu_{kj}$$

13. (a) We use the embedded Markov chain and holding times to parameterize the Markov process. The state space and embedded Markov chain are given in the answer to Exercise 27, Chapter 6.

We take t in minutes, so that $1/\psi_1 = 1/2,\ 1/\psi_2 = 1,\ 1/\psi_3 = 2, 1/\psi_4 = 1$ and ψ_5 is ∞. Therefore,

$$\mathbf{G} = \begin{pmatrix} -\psi_1 & 0.5\psi_1 & 0.4\psi_1 & 0.1\psi_1 & 0 \\ 0 & -\psi_2 & 0.05\psi_2 & 0.05\psi_2 & 0.9\psi_2 \\ 0 & 0.05\psi_3 & -\psi_3 & 0.05\psi_3 & 0.9\psi_3 \\ 0 & 0.05\psi_3 & 0.05\psi_3 & -\psi_4 & 0.9\psi_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 & 0.8 & 0.2 & 0 \\ 0 & -1 & 0.05 & 0.05 & 0.9 \\ 0 & 0.025 & -0.5 & 0.025 & 0.45 \\ 0 & 0.05 & 0.05 & -1 & 0.9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) We might expect the time to perform each type of transaction to be nearly constant, implying that the exponential distribution is not appropriate. The only way to be certain, however, is to collect data.

(c) We need $1 - p_{15}(t)$ for $t = 4$ minutes. Using the uniformization approach with $n^* = 16$ and carrying 10 digits of accuracy in all calculations,

$$1 - \tilde{p}_{15}(4) = 1 - 0.88 = 0.12$$

with truncation error ≤ 0.004.

(d) From Exercise 11

$$g_{ii}\ \mu_{ij} = \mathcal{I}(i=j) + \sum_{k\in\mathcal{T},k\neq i} g_{ik}\ \mu_{kj}$$

In this problem $\mathcal{T} = \{1,2,3,4\}$ and we need $\mu_{11}+\mu_{12}+\mu_{13}+\mu_{14}$. There will be 16 equations in 16 unknowns. To set up these equations it helps to rewrite the result as

$$-\mathcal{I}(i=j) = \sum_{k=1}^{i-1} g_{ik}\ \mu_{kj} - g_{ii}\ \mu_{ij} + \sum_{k=i+1}^{4} g_{ik}\ \mu_{kj}$$

for $i = 1,2,3,4$ and $j = 1,2,3,4$. Letting $\mathbf{x} = (\mu_{11}\ \mu_{21}\ \mu_{31}\ \mu_{41}\ \mu_{12}\ \mu_{22}\ldots\mu_{44}\ )'$ we can write the system of equations as

$$\mathbf{Ax} = \mathbf{b}$$

with

$$\mathbf{A} = \left(\begin{array}{cccccccc}
-2 & 1 & 0.8 & 0.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 1 & 0.8 & 0.2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0.05 & 0.05 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0.05 & 0.05 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.025 & -0.5 & 0.025 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.025 & -0.5 & 0.025 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.05 & 0.05 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.05 & 0.05 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0.8 & 0.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 1 & 0.8 & 0.2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0.05 & 0.05 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0.05 & 0.05 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.025 & -0.5 & 0.025 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.025 & -0.5 & 0.025 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.05 & 0.05 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.05 & 0.05 & -1
\end{array}\right)
$$

$$
\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}
$$

for which the solution is

$$\mathbf{x} = \begin{pmatrix} 0.5000000000 \\ 0 \\ 0 \\ 0 \\ 0.5291005291 \\ 1.005291005 \\ 0.05291005291 \\ 0.05291005291 \\ 0.8677248677 \\ 0.1058201058 \\ 2.010582011 \\ 0.1058201058 \\ 0.1481481481 \\ 0.05291005291 \\ 0.05291005291 \\ 1.005291005 \end{pmatrix}$$

Thus, $\mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} \approx 2.04$ minutes.

14. (a) Let $\mathcal{M} = \{0, 1, 2, 3\}$ represent the state of the system, where 0 corresponds to an empty system; 1 to the voice-mail being busy but the operator idle; 2 to the operator being busy and the voice mail idle; and 3 to both the voice mail and operator being busy. Then $\{Y_t; t \geq 0\}$ is the state of the system at time t hours.

$$\mathbf{G} = \begin{pmatrix} -20 & 20 & 0 & 0 \\ 30 & -50 & 0 & 20 \\ 10 & 0 & -30 & 20 \\ 0 & 10 & 50 & -60 \end{pmatrix}$$

(b) Need $\pi_2 + \pi_3$

$\boldsymbol{\pi}' = (17/41, 8/41, 10/41, 6/41)$

Therefore $\pi_2 + \pi_3 = 16/41 \approx 0.39$ or 39% of the time.

15. Let $\lambda \equiv$ failure rate, $\tau \equiv$ repair rate, and $\mathcal{M} = \{0, 1, 2\}$ represent the number of failed computers. Then

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 \\ \tau & -(\lambda + \tau) & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

and $\mathcal{T} = \{0, 1\}$ are the transient states.

We need $\mu_{00} + \mu_{01}$. Using the result in Exercise 11

$$
\begin{array}{rlllll}
\lambda\mu_{00} = & 1 & & +\lambda\mu_{10} & & \\
\lambda\mu_{01} = & & & & & \lambda\mu_{11} \\
(\lambda+\tau)\mu_{10} = & & \tau\mu_{00} & & & \\
(\lambda+\tau)\mu_{10} = & 1 & & & +\tau\mu_{01} &
\end{array}
$$

or

$$
\begin{array}{rllll}
-1 = & -\lambda\mu_{00} & +\lambda\mu_{10} & & \\
0 = & & & -\lambda\mu_{01} & \lambda\mu_{11} \\
0 = & \tau\mu_{00} & -(\lambda+\tau)\mu_{10} & & \\
-1 = & & & \tau\mu_{01} & -(\lambda+\tau)\mu_{11}
\end{array}
$$

The solution is

$$
\begin{bmatrix} \mu_{00} & \mu_{01} \\ \mu_{10} & \mu_{11} \end{bmatrix} = \begin{bmatrix} \frac{\lambda+\tau}{\lambda^2} & \frac{1}{\lambda} \\ \frac{\tau}{\lambda^2} & \frac{1}{\lambda} \end{bmatrix}
$$

Therefore, $\mathrm{E}[TTF] = \frac{\lambda+\tau}{\lambda^2} + \frac{1}{\lambda} = \frac{2\lambda+\tau}{\lambda^2}$.

System	E[TTF]
A	111,333.33
B	105,473.68
C	100,200.00
D	86,358.28

System A has the largest $\mathrm{E}[TTF]$.

16. (a) Let H_i represent the holding time on a visit to state i.

    Let $F_{N|L=i}$ represent the cdf implied by the ith row of $\mathbf{P}$, the transition matrix of the embedded Markov chain.

    1. $n \leftarrow 0$
       $T_0 \leftarrow 0$
       $S_0 \leftarrow s$
    2. $i \leftarrow S_n$
    3. $T_{n+1} \leftarrow T_n + F_{H_i}^{-1}$ (`random()`)
    4. $S_{n+1} \leftarrow S_n + F_{N|L=i}^{-1}$ (`random()`)
    5. $n \leftarrow n+1$
       goto 2

    (b) Replace step 3 with

    3. $T_{n+1} \leftarrow T_n + \mathrm{E}[H_i]$

    (c) No clocks need to be maintained. When we use 16(b), we only need to generate Markov-chain transitions and we eliminate variability due to the holding times.

17. Let

N be the number of copies a customer wants

L be the total number of lost sales

Q be the size of the pending order

We assume customers take all available copies if there are fewer available than they want. We assume only one pending order at a time.

The system events below implement parts (a)–(c).

$e_0()$ (**initial inventory level**)

$S_0 \leftarrow s$	(initially s copies)
$C_1 \leftarrow F_G^{-1}(\texttt{random}())$	(set clock for first demand)
$C_2 \leftarrow \infty$	(no order arrival pending)

$e_1()$ (**demand from customer**)

$N \leftarrow F_N^{-1}(\texttt{random}())$	(number demanded)
if $\{S_n \geq N\}$ then	(if enough in inventory)
$\quad S_{n+1} \leftarrow S_n - N$	(reduce inventory N copies)
else	(not enough in inventory)
$\quad L \leftarrow L + (N - S_n)$	(record lost sales)
$\quad S_{n+1} \leftarrow 0$	(reduce inventory to zero)
endif	
if $\{S_{n+1} \leq r \text{ and } C_2 = \infty\}$ then	(if reorder point and no order pending)
$\quad C_2 \leftarrow T_{n+1} + F_R^{-1}(\texttt{random}())$	(set clock for lead time)
$\quad Q \leftarrow s - S_{n+1}$	(set order quantity)
endif	
$C_1 \leftarrow T_{n+1} + F_G^{-1}(\texttt{random}())$	(set clock for next demand)

$e_2()$ **(order arrives from manufacturer)**

$S_{n+1} \leftarrow S_n + Q$ (add order to inventory)
if $\{S_{n+1} \le r\}$ then (if still below reorder)
$C_2 \leftarrow T_{n+1} + F_R^{-1}(\texttt{random}())$ (set clock for lead time)
$Q \leftarrow s - S_{n+1}$ (set order quantity)
endif

18.

$$\begin{aligned}
\Pr\{H \le t\} &= \sum_{n=1}^{\infty} \Pr\{H \le t \mid X = n\} \Pr\{X = n\} \\
&= \sum_{n=1}^{\infty} \Pr\left\{\sum_{i=1}^{n} J_i \le t\right\} \gamma(1-\gamma)^{n-1}
\end{aligned}$$

since X and J_i are independent. But $\sum_{i=1}^{n} J_i$ has an Erlang distribution with parameter λ and n phases. Therefore,

$$\begin{aligned}
\Pr\{H \le t\} &= \sum_{n=1}^{\infty} \left\{1 - \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}\right\} \gamma(1-\gamma)^{n-1} \\
&= 1 - \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j} \gamma(1-\gamma)^{n-1} \\
&= 1 - \gamma e^{-\lambda t} \sum_{n=1}^{\infty} (1-\gamma)^{n-1} \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} \\
&= 1 - \gamma e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \sum_{n=j+1}^{\infty} (1-\gamma)^{n-1} \\
&= 1 - \gamma e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \left(\sum_{n=0}^{\infty} (1-\gamma)^n - \sum_{n=0}^{j-1} (1-\gamma)^n \right) \\
&= 1 - \gamma e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \left(\frac{1}{\gamma} - \frac{1-(1-\gamma)^j}{\gamma} \right) \\
&= 1 - e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t(1-\gamma))^j}{j!} \\
&= 1 - e^{-\lambda t}\, e^{\lambda t(1-\gamma)} \\
&= 1 - e^{-\lambda t \gamma}
\end{aligned}$$

an exponential distribution.

This result shows that we can represent the holding time in a state (which is exponentially distributed) as the sum of a geometrically distributed number of exponentially distributed random variables with a common rate. This is precisely what uniformization does.

19.

$$\begin{aligned} p_{ij}(t) &= e^{-g^* t} \sum_{n=0}^{\infty} q_{ij}^{(n)} \frac{(g^* t)^n}{n!} \\ &= \tilde{p}_{ij}(t) + e^{-g^* t} \sum_{n=n^*+1}^{\infty} q_{ij}^{(n)} \frac{(g^* t)^n}{n!} \\ &\geq \tilde{p}_{ij}(t) \end{aligned}$$

because $g^* > 0,\ t > 0$ and $q_{ij}^{(n)} \geq 0$.

Notice that

$$\begin{aligned} & e^{-g^* t} \sum_{n=n^*+1}^{\infty} q_{ij}^{(n)} \frac{(g^* t)^n}{n!} \\ &\qquad = 1 - e^{-g^* t} \sum_{n=0}^{n*} q_{ij}^{(n)} \frac{(g^* t)^n}{n!} \\ &\qquad \leq 1 - e^{-g^* t} \sum_{n=0}^{n*} \frac{(g^* t)^n}{n!} \end{aligned}$$

because $q_{ij}^{(n)} \leq 1$.

20.

$$\begin{aligned} & \Pr\{Z_1 \leq \min_{j \neq 1} Z_j, H > t\} \\ &= \Pr\{Z_1 \leq Z_2, \ldots, Z_1 \leq Z_k, Z_1 > t, \ldots, Z_k > t\} \\ &= \Pr\{t < Z_1 \leq M\} \\ &= \int_t^{\infty} \Pr\{t < Z_1 \leq M \mid M = a\} \lambda_M e^{-\lambda_M a} da \\ &= \int_t^{\infty} (e^{-\lambda_1 t} - e^{-\lambda_1 a}) \lambda_M e^{-\lambda_M a} da \\ &= e^{-\lambda_1 t} \int_t^{\infty} \lambda_M e^{-\lambda_M a} da \end{aligned}$$

$$
\begin{aligned}
& - \frac{\lambda_M}{\lambda_1 + \lambda_M} \int_t^{\infty} (\lambda_1 + \lambda_M) e^{-(\lambda_1+\lambda_M)a} da \\
= & \; e^{-\lambda_1 t} e^{-\lambda_M t} - \frac{\lambda_M}{\lambda_1 + \lambda_M} e^{-(\lambda_1+\lambda_M)t} \\
= & \left(1 - \frac{\lambda_M}{\lambda_1 + \lambda_M}\right) e^{-(\lambda_1+\lambda_M)t} \\
= & \left(\frac{\lambda_1}{\lambda_H}\right) e^{-\lambda_H t}
\end{aligned}
$$

21. (a)

$$
\mathbf{G} = \begin{pmatrix}
-\frac{1}{12} & \frac{0.95}{12} & \frac{0.1}{12} & \frac{0.4}{12} \\
0 & -\frac{73}{1500} & (\frac{73}{1500})(\frac{0.63}{0.73}) & (\frac{73}{1500})(\frac{0.10}{0.73}) \\
0 & (\frac{1}{50})(\frac{.36}{.60}) & -\frac{1}{50} & (\frac{1}{50})(\frac{0.24}{0.60}) \\
\frac{1}{3} & 0 & 0 & -\frac{1}{3}
\end{pmatrix}
$$

Notice that for row 2

$$
1/g_{22} = \frac{15}{1 - p_{22}} = \frac{15}{0.73} = \frac{1500}{73}
$$

so $g_{22} = 73/1500$ to account for transitions from state 2 to itself. Similarly for row 3. The steady state probabilities are the same:

$$
\boldsymbol{\pi} = \begin{pmatrix} 0.08463233821 \\ 0.2873171709 \\ 0.6068924063 \\ 0.02115808455 \end{pmatrix}
$$

(b) No answer provided.

(c) The only change in the generator matrix is that the last row becomes (0, 0, 0, 0).

22. Let $G_1, G_2, \ldots$ be the times between visits to state 1. Because the process is semi-Markov, these are independent, time-stationary random variables.

Let R_n be the time spent in state 1 on the nth visit. Then $R_1, R_2, \ldots$ are also independent with $\mathrm{E}[R_n] = \tau_1$. Therefore we have a renewal-reward process.

From Exercise 25, Chapter 4, the expected number of states visited between visits to 1 is $1/\xi_1$. The fraction of those visits that are to state j has expectation ξ_j/ξ_1. Therefore,

$$
\begin{aligned}
\mathrm{E}[G_n] &= \tau_1 + \sum_{j \neq 1} (\xi_j/\xi_1)\tau_j \\
&= 1/\xi_1 \sum_{j=1}^{m} \xi_j \; \tau_j
\end{aligned}
$$

Thus, the long-run reward rate (fraction of time spent in state 1) is

$$\pi_1 = \frac{\mathrm{E}[R_n]}{\mathrm{E}[G_n]} = \frac{\xi_1 \ \tau_1}{\sum_{j=1}^m \xi_j \ \tau_j}$$

A similar argument applies for all other states.

23. We need to show that

$$\pi_i = \frac{\xi_i / g_{ii}}{\sum_{j=1}^m \ \xi_j / g_{jj}} = \frac{\xi_i / g_{ii}}{d} \qquad (1)$$

satisfies $\boldsymbol{\pi}' \ \mathbf{G} = \mathbf{0}$.

Select the jth equation

$$\sum_{k \neq j} \pi_k \ g_{kj} - \pi_j \ g_{jj} = 0$$

Substituting (1)

$$\begin{aligned} & \sum_{k \neq j} \left(\frac{\xi_k / g_{kk}}{d} \right) g_{kj} - \left(\frac{\xi_j / g_{jj}}{d} \right) g_{jj} \\ &= \ \frac{1}{d} \sum_{k \neq j} \left(\frac{g_{kj}}{g_{kk}} \right) \xi_k - \frac{1}{d} \xi_j \\ &= \ \frac{1}{d} \left(\sum_{k \neq j} p_{kj} \ \xi_k - \xi_j \right) \stackrel{?}{=} 0 \end{aligned}$$

For the equality to hold, we must have

$$\sum_{k \neq j} p_{kj} \ \xi_k = \xi_j \qquad (2)$$

But for a Markov process $p_{jj} = 0$, so (2) is equivalent to

$$\sum_{k=1}^m p_{kj} \ \xi_k = \xi_j$$

which is guaranteed because $\boldsymbol{\xi}$ is the steady-state distribution of the embedded Markov chain.

24. No answer provided.

Chapter 8

Queueing Processes

1. Let μ be the production rate. Sample average production time $= 31.9/10 = 3.19$ minutes. Therefore, $\widehat{\mu} = \frac{1}{3.19} \approx 0.31$ parts/minute. With 2 machines the rate is $2\widehat{\mu} = 0.62$ parts/minute.

$$F_X(a) = \begin{cases} 0, & a < 0 \\ 1 - e^{-0.31a}, & a \geq 0 \end{cases}$$

2.

3. • Self-service system

   We can approximate the self-service and full-service copiers as independent M/M/1 queues.

   <u>self-service queue</u>

   $\widehat{\lambda} = 1/10$ customer/minute

   $\widehat{1/\mu} = 3$ minutes so $\widehat{\mu} = 1/3$ customer/minute

   $\rho = \frac{\widehat{\lambda}}{\widehat{\mu}} = 3/10$

   $w_q = \frac{\rho^2}{\lambda(1-\rho)} = 9/7 \approx 1.3$ minutes

   <u>full-service queue</u>

   $\widehat{\lambda} = 1/15$ customer/minute

   $\widehat{1/\mu} = 7$ minutes so $\widehat{\mu} = 1/7$ customer/minute

   $\rho = 7/15$

   $w_q = \frac{49}{8} \approx 6.1$ minutes

   • Full-service system

   We can approximate it as a single M/M/2 queue. Based on the sample average of all customers

   $\widehat{\lambda} = 1/6$ customer/minute

   $\widehat{1/\mu} = 4.6$ minutes so $\widehat{\mu} = 1/4.6$ customer/minute

   $\rho = \frac{\widehat{\lambda}}{2\widehat{\mu}} \approx 0.38$

   $\ell_q = \frac{\pi_2 \rho}{(1-\rho)^2} \approx 0.13$ customers

   $w_q = \frac{\ell_q}{\lambda} \approx 0.79$ minutes

   Notice that the full-service system is actually superior for both types of customers.

   • Concerns

   For the self-service system we modeled the service times of self-service customers as exponentially distributed, and the service times of full-service customers as exponentially distributed. For the full-service system (M/M/2) we modeled the service times of **all** customers together as exponentially distributed. These are incompatible models, but still o.k. as approximations.

4. (a) We approximate the process of potential arrivals as Poisson, and the service times as exponentially distributed.

$$\lambda_i = \begin{cases} 20(1 - i/16), & i = 0, 1, 2, \ldots, 15 \\ 0, & i = 16, 17, \ldots \end{cases}$$

$$\mu_i = \begin{cases} 10, & i = 1, 2, \ldots, 16 \\ 0, & i = 17, 18, \ldots \end{cases}$$

The lost-customer rate in state i is $20(i/16)$. Therefore, the long-run lost customer rate is

$$\begin{aligned} \eta = \sum_{i=0}^{16} 20(i/16)\pi_i &= \frac{20}{16} \sum_{i=0}^{16} i\pi_i \\ &= \frac{20}{16}\,\ell \\ &\approx 10 \text{ customers/hour} \end{aligned}$$

(b)

$$\ell_q = \sum_{j=2}^{16} (j-1)\pi_j \approx 8.3 \text{ customers}$$

(c) Customers are served at a rate of

$$20 - \eta \approx 10 \text{ customers/hour}$$

Therefore, long-run revenue is

$$(0.50)(10) - 4 = \$1 \text{ per hour}$$

5. (a) We approximate IE and MS as independent M/M/1 queues.

- IE

$\lambda = 20$ letters/day
$\mu = 25$ letters/day
$w = w_q + 1/\mu = 4/25 + 1/25 = 1/5$ day

- MS

$\lambda = 15$ letters/day
$w = w_q + 1/\mu = 3/50 + 1/25 = 1/10$ day

(b) We approximate IE + MS as an M/M/2 queue.
$\lambda = 20 + 15 = 35$ letters/day
$w = w_q + 1/\mu = 49/1275 + 1/25 = 4/51 \approx 0.08$ day
By forming a typing pool, performance improves for both departments.

6. Adopting the modeling approximations stated in the exercise we have a Markovian queue with

$\lambda_i = 45$ customers/hour, $i = 0, 1, 2, \ldots$

$$\mu_i = \begin{cases} \mu, & i = 1, 2, \\ 2\mu, & i = 3, 4, \ldots \end{cases}$$

with $\mu = 30$ customers/hour.

(a) We derive the general result when the second agent goes on duty when there are k customers in the system.

$$d_j = \begin{cases} \left(\frac{\lambda}{\mu}\right)^j, & j = 0, 1, \ldots, k-1 \\ \frac{\lambda^j}{\mu^{k-1}(2\mu)^{j-k+1}}, & j = k, k+1, \ldots \end{cases}$$

$$= \begin{cases} \left(\frac{\lambda}{\mu}\right)^j, & j = 0, 1, \ldots, k-1 \\ \left(\frac{\lambda}{\mu}\right)^j \frac{1}{2^{j-k+1}}, & j = k, k+1, \ldots \end{cases}$$

$$\begin{aligned} \sum_{j=0}^{\infty} d_j &= \sum_{j=0}^{k-1} (\lambda/\mu)^j + \sum_{j=k}^{\infty} \frac{(\lambda/\mu)^j}{2^{j-k+1}} \\ &= \sum_{j=0}^{k-1} (\lambda/\mu)^j + \frac{(\lambda/\mu)^k}{2} \sum_{j=0}^{\infty} \left(\frac{\lambda}{2\mu}\right)^j \\ &= \sum_{j=0}^{k-1} (\lambda/\mu)^j + \frac{(\lambda/\mu)^k}{2(1 - \lambda/(2\mu))} \end{aligned}$$

$$\pi_0 = \frac{1}{\sum_{j=0}^{\infty} dj} = 2/23 \qquad \text{when } k = 3$$

(b)

$$\begin{aligned} \sum_{j=3}^{\infty} \pi_j &= 1 - \pi_0 - \pi_1 - \pi_2 \\ &= 1 - 2/23 - (2/23)(45/30) - (2/23)(45/30)^2 \\ &= 27/46 \approx 0.59 \end{aligned}$$

(c)

$$\begin{aligned}
\ell_q &= \pi_2 + \sum_{j=3}^{\infty}(j-2)\pi_j \\
&= \pi_0 d_2 + \sum_{j=3}^{\infty}(j-2)\pi_0 d_j \\
&= \pi_0 d_2 + \pi_0 \sum_{j=3}^{\infty}(j-3)\frac{(\lambda/\mu)^j}{2^{j-k+1}} \\
&= \pi_0 d_2 + \pi_0 \frac{(\lambda/\mu)^3}{2} \sum_{j=0}^{\infty} j \left(\frac{\lambda}{2\mu}\right)^j \\
&= \pi_0 d_2 + \pi_0 \frac{(\lambda/\mu)^3}{2} \left(\frac{\lambda}{2\mu}\right) \frac{1}{(1-\lambda/(2\mu))^2} \\
&= \pi_0 \left((\lambda/\mu)^2 + \frac{(\lambda/\mu)^3}{2} \quad \frac{(\lambda/(2\mu))}{(1-\lambda/(2\mu))^2}\right) \\
&= 2/23 \left((45/30)^2 + \frac{(45/30)^3}{2} \quad \frac{(45/60)}{(1-45/60)^2}\right) \\
&= 45/23 \approx 1.96 \text{ customers}
\end{aligned}$$

(d) We show that the current trigger is adequate.
$w_q = \frac{\ell_q}{\lambda} = \frac{45}{23}\left(\frac{1}{45}\right) = \frac{1}{23}$
$w = w_q + \frac{1}{\mu} = \frac{1}{23} + \frac{1}{30} = \frac{53}{690}$
$\ell = \lambda w = 45\left(\frac{53}{690}\right) = \frac{159}{46} \approx 3.5 < 5.$

7. Define $D_{i-1} = X_{i-1} = 0$.

   For $i = 1$ we have $D_1 = \max\{0, -G_1\} = 0$ which is clearly correct since the first customer arrives to find the system empty.

   Suppose (8.36) holds for all $i = 1, 2, \ldots, n$ for some $n > 1$.

$$D_{n+1} = (G_1 + \cdots + G_n + D_n + X_n) - (G_1 + \cdots + G_{n+1}) \tag{1}$$

   which is the difference between when customer n departed and customer $n+1$ arrived, provided (1) is ≥ 0.

   Therefore,

$$D_{n+1} = \max\{0, D_n + X_n - G_{n+1}\}$$

   and the result holds for $n+1$ also.

8. (a) We approximate each system as an M/M/1 queue.

   One-person system

   $\lambda = 24$ cars/hour

   $\mu = 30$ cars/hours

   $w_q = 2/15 \approx 0.13$ hour

   $w = w_q + 1/\mu = 1/6 \approx 0.17$ hour

   Two-person system

   $\mu = 48$ cars/hour

   $w_q = 1/48 \approx 0.02$ hour

   $w = w_q + 1/\mu = 1/24 \approx 0.04$ hour

   (b) Need π_0 for each case

   One person $\pi_0 = 1/5$

   Two person $\pi_0 = 1/2$

   (c) The fraction of time that the intercom is blocked is

$$\begin{aligned} \eta = \sum_{j=3}^{\infty} \pi_j &= 1 - \pi_0 - \pi_1 - \pi_2 \\ &= 1 - \sum_{j=0}^{2} (1-\rho)\rho^j \end{aligned}$$

   One person $\eta \approx 0.51$

   Two person $\eta \approx 0.13$

   So in the one-person system the intercom is blocked over half the time.

   (d) Now we model the system as an M/M/1/6 queue.

   • One-person system

$$\pi_6 \approx 0.066$$

   so the lost-customer rate is $\lambda\pi_6 \approx 1.6$ customers/hour.

   • Two-person system

$$\pi_6 \approx 0.008$$

   so $\lambda\pi_6 \approx 0.2$ customer/hour.

9. This is one possible approach to the problem:

   Since the user population is relatively large, treat the arrival process as Poisson with rate

$$\begin{aligned} \lambda &= (10000)(0.7) = 7000 \text{ calls/day} \\ &\approx 292 \text{ calls/hour} \\ &\approx 4.86 \text{ calls/minute} \end{aligned}$$

We have assumed a 24-hour day, but perhaps a 12 or 18-hour day is better.

$1/\mu = 2.5$ minutes so $\mu = 0.4$ calls/minute.

If we treat the connect times as exponentially distributed, then we have an M/M/s/s queue, where s is the number of ports.

A worthwhile table is π_s vs. s, since π_s is the probability a user cannot connect.

s	π_s
5	0.63
6	0.56
7	0.49
8	0.43
9	0.37
10	0.31
11	0.25
12	0.20
13	0.16
14	0.12
15	0.09
16	0.06
17	0.04
18	0.03
19	0.02
20	0.01

10. (a) If we model the arrival of calls as Poisson, service times as exponentially distributed, and no reneging, then we have an M/M/2 queue.

    $\lambda = 20$ calls/hour

    $1/\mu = 3$ minutes so $\mu = 20$ calls/hour

    (b) To keep up $\rho = \frac{\lambda}{2\mu} < 1$ or $\lambda < 2\mu = 40$ calls/hour.

    (c) We want the largest λ such that

    $$w_q = \frac{\pi_2 \rho}{(1-\rho)^2} \leq 4/60 \text{ hour}$$

    By trial-and-error, $\lambda \approx 30$ calls/hour.

    (d) We want the largest λ such that

    $$\sum_{j=8}^{\infty} \pi_j = 1 - \sum_{j=0}^{7} \pi_j \leq 0.15$$

    By trial-and-error $\lambda \approx 31$ calls/hour.

(e) Let the reneging rate be $\beta = 1/5$ call/minute = 12 calls/hour for customers on hold.

$$\mu_i = \begin{cases} i\mu, & i = 1, 2 \\ 2\mu + (i-2)\beta, & i = 3, 4, \ldots \end{cases}$$

with $\mu = 20$.

(f) $\ell_q = \sum_{j=3}^{\infty}(j-2)\pi_j$

Therefore, we need the π_j for $j \geq 3$.

$$d_j = \begin{cases} \frac{(\lambda/\mu)^j}{j!}, & j = 0, 1, 2 \\ \frac{\lambda^j}{2\mu^2 \prod_{i=3}^{j}(2\mu+(i-2)\beta)}, & j = 3, 4, \ldots \end{cases}$$

$$\pi_0 = \frac{1}{\sum_{j=0}^{\infty} dj} \approx \frac{1}{\sum_{j=0}^{20} dj} \approx \frac{1}{2.77} \approx 0.36$$

since $\sum_{j=0}^{n} d_j$ does not change in the second decimal place after $n \geq 20$. Therefore

$$\ell_q \approx \sum_{j=3}^{20}(j-2)\pi_j \approx 0.137 \text{ calls on hold}$$

11. (a) $\mathcal{M} = \{0, 1, 2, \ldots, k+m\}$ is the number of users connected or in the wait queue.

$$\begin{aligned} \lambda_i &= \lambda, \quad i = 0, 1, \ldots \\ \mu_i &= \begin{cases} i\mu, & i = 1, 2, \ldots, k \\ k\mu + (i-k)\gamma, & i = k+1, k+2, \ldots, k+m \end{cases} \end{aligned}$$

(b) $\ell_q = \sum_{j=k+1}^{m}(j-k)\pi_j$

(c) $\lambda\pi_{k+m}$ (60 minutes/hour)

(d) The quantities in (b) and (c) are certainly relevant. Also w_q, the expected time spent in the hold queue.

12. We approximate the system as an M/M/3/20/20 with $\tau = 1$ program/minute, and μ = 4 programs/minute

$$\lambda_i = \begin{cases} (20-i)\tau, & i = 0, 1, \ldots, 19 \\ 0, & i = 20, 21, \ldots \end{cases}$$

$$\mu_i = \begin{cases} i\mu, & i = 1, 2 \\ 3\mu, & i = 3, 4, \ldots, 20 \end{cases}$$

$$d_j = \begin{cases} \frac{\prod_{i=0}^{j-1}(20-i)\tau}{\mu^j \;\; j!}, & j = 1, 2 \\ \frac{\prod_{i=0}^{j-1}(20-i)\tau}{3!\mu^j \;\; 3^{3-j}}, & j = 3, 4, \ldots, 20 \end{cases}$$

$$= \begin{cases} \frac{(\tau/\mu)^j}{j!} \;\; \prod_{i=0}^{j-1}(20-i), & j = 1, 2 \\ \frac{(\tau/\mu)^j}{6 \;\; 3^{3-j}} \;\; \prod_{i=0}^{j-1}(20-i), & j = 3, 4, \ldots, 20 \end{cases}$$

$\sum_{j=0}^{20} d_j \approx 453.388$

$$\boldsymbol{\pi} = \begin{pmatrix} 0.002205615985 \\ 0.01102807993 \\ 0.02619168983 \\ 0.03928753475 \\ 0.05565734086 \\ 0.07420978784 \\ 0.09276223484 \\ 0.1082226072 \\ 0.1172411578 \\ 0.1172411579 \\ 0.1074710613 \\ 0.08955921783 \\ 0.06716941339 \\ 0.04477960892 \\ 0.02612143853 \\ 0.01306071927 \\ 0.005441966361 \\ 0.001813988787 \\ 0.0004534971966 \\ 0.00007558286608 \\ 0.000006298572176 \end{pmatrix}$$

(a) $\sum_{j=3}^{\infty} \pi_j = 1 - \pi_0 - \pi_1 - \pi_2 \approx 0.96$

(b) Need w.

$$\begin{aligned} \ell &= \sum_{j=0}^{20} j \;\pi_j \approx 8.22 \text{ jobs} \\ \lambda_{\text{eff}} &= \sum_{j=0}^{20} \lambda_j \pi_j = \sum_{j=0}^{20} (20-j)\pi_j \approx 9.78 \\ w &= \ell/\lambda_{\text{eff}} \approx 0.84 \text{ minutes} \end{aligned}$$

(c)

$$\ell_q = \sum_{j=4}^{20} (j-3)\pi_j \approx 6.20 \text{ jobs waiting}$$

(d)

$$3\pi_0 + 2\pi_1 + \pi_2 + 0(\pi_3 + \pi_4 + \cdots + \pi_{20}) \approx 0.055 \text{ idle computers}$$

(e)

$$\pi_0 \approx 0.02 \text{ or } 2\% \text{ of the time}$$

(f)

$$(d)/3 \approx 0.018 \text{ or } 1.8\% \text{ of the time}$$

13. No answer provided.

14. (a) For the M/M/1/c, let $\rho = \lambda/\mu$.

$$d_j = \begin{cases} (\lambda/\mu)^j, & j = 0, 1, 2, \ldots, c \\ 0, & j = c+1, c+2, \ldots \end{cases}$$

Therefore,

$$\begin{aligned} \pi_0 &= \left(\sum_{i=0}^{c} d_i\right)^{-1} = \frac{1}{\sum_{i=0}^{c} \rho^i} \\ \pi_j &= \pi_0 d_j = \pi_0 \rho^j \ , \ j = 0, 1, \ldots, c \end{aligned}$$

(b) For the M/M/c/c

$$d_j = \begin{cases} \frac{(\lambda/\mu)^j}{j!}, & j = 0, 1, \ldots, c \\ 0, & j = c+1, c+2, \ldots \end{cases}$$

Therefore,

$$\pi_0 = \left(\sum_{i=0}^{c} d_i\right)^{-1} = \frac{1}{\sum_{i=0}^{c} (\lambda/\mu)^i / i!}$$

$$\pi_j = \pi_0 d_j = \pi_0 \frac{(\lambda/\mu)^j}{j!}, \quad j = 0, 1, \ldots, c$$

(c) For the M/M/s/c with $s \le c$

$$d_j = \begin{cases} \frac{(\lambda/\mu)^j}{j!}, & j = 0, 1, \ldots, s \\ \frac{(\lambda/\mu)^j}{s!\ s^{j-s}}, & j = s+1, \ldots, c \\ 0, & j = c+1, c+2, \ldots \end{cases}$$

Therefore,

$$\pi_j = \pi_0 d_j = \pi_0 \frac{(\lambda/\mu)^j}{j!}, \quad j = 0, 1, \ldots, s$$

and

$$\begin{aligned} \pi_j = \pi_0 d_j &= \pi_0 \frac{(\lambda/\mu)^j}{s!\ s^{j-s}} = \pi_0 \frac{(\lambda/\mu)^s}{s!} \left(\frac{\lambda}{s\mu}\right)^{j-s} \\ &= \pi_s \rho^{j-s}, \quad j = s+1, s+2, \ldots, c \end{aligned}$$

Thus,

$$\begin{aligned} \Pr\{L = j \mid L \le s\} &= \frac{\Pr\{L = j\}}{\Pr\{L \le s\}} \\ &= \frac{\frac{\pi_0 (\lambda/\mu)^j}{j!}}{\sum_{i=0}^{s} \pi_0 \frac{(\lambda/\mu)^i}{i!}} \\ &= \frac{(\lambda/\mu)^j / j!}{\sum_{i=0}^{s} (\lambda/\mu)^i / i!} \end{aligned}$$

Also,

$$\begin{aligned} \Pr\{L = s + j \mid L \ge s\} &= \frac{\Pr\{L = s + j\}}{\Pr\{L \ge s\}} \\ &= \frac{\pi_s \rho^j}{\sum_{i=s}^{c} \pi_s \rho^{i-s}} \\ &= \frac{\rho^j}{\sum_{i=0}^{c-s} \rho^i} \end{aligned}$$

15. No answer provided.

16.

$$\begin{aligned}\Lambda(t) &= \int_0^t \lambda(t)dt \\ &= \begin{cases} \frac{t^3}{3}, & 0 \le t < 10 \\ \frac{2000}{3} + \frac{(t-20)^3}{3}, & 10 \le t < 20 \end{cases}\end{aligned}$$

Therefore, the expected number of arrivals in [9, 10] is

$$\Lambda(10) - \Lambda(9) = \frac{271}{3} \approx 90.3$$

which is also the expected number in the $M(t)/D/\infty$ queue at time $t = 10$.

For the stationary model $M/D/\infty$ with $\lambda = 100$ and $\mu = 1$

$$\ell = \frac{\lambda}{\mu} = 100$$

The stationary model predicts congestion to be over 10% greater than it should be. This happens because $\lambda(t)$ increases sharply toward its peak rate, then declines, while the $M/D/\infty$ model uses a constant rate. Thus, the $M(t)/D/\infty$ is not at its peak rate long enough to achieve so much congestion.

17. Notice that

$$\Pr\{W_q = 0\} = \sum_{j=0}^{s-1} \pi_j$$

And

$$\begin{aligned}\Pr\{W_q > a\} &= \sum_{j=0}^{\infty} \Pr\{W_q > a \mid L = j\}\ \pi_j \\ &= \sum_{j=s}^{\infty} \Pr\{W_q > a \mid L = j\}\ \pi_j\end{aligned}$$

since no waiting occurs if there are fewer than s in the system. But, for $j \ge s$,

$$\Pr\{W_q > a \mid L = j\} = \sum_{n=0}^{j-s} \frac{e^{-s\mu a}(s\mu a)^n}{n!}$$

which is $\Pr\{T > a\}$, where T has an Erlang distribution with parameter $s\mu$ and $j-s+1$ phases. This follows because **each** of the $j-s+1$ customers (including the new arrival) will be at the front of the queue of waiting customers for an exponentially distributed time with parameter $s\mu$. Therefore,

$$\begin{aligned}
\Pr\{W_q > a\} &= \sum_{j=s}^{\infty} \left\{ \sum_{n=0}^{j-s} \frac{e^{-s\mu a}(s\mu a)^n}{n!} \right\} \pi_j \\
&= \sum_{j=s}^{\infty} \left\{ \sum_{n=0}^{j-s} \frac{e^{-s\mu a}(s\mu a)^n}{n!} \right\} \frac{\pi_0(\lambda/\mu)^j}{s!\ s^{j-s}}
\end{aligned}$$

But notice that

$$\begin{aligned}
\frac{\pi_0(\lambda/\mu)^j}{s!\ s^{j-s}} &= \pi_0 \frac{(\lambda/\mu)^s}{s!} (\lambda/(s\mu))^{j-s} \\
&= \pi_s\ \rho^{j-s}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Pr\{W_q > a\} &= \pi_s \sum_{j=s}^{\infty} \left\{ \sum_{n=0}^{j-s} \frac{e^{-s\mu a}(s\mu a)^n}{n!} \right\} \rho^{j-s} \\
&= \pi_s \sum_{j=0}^{\infty} \left\{ \sum_{n=0}^{j} \frac{e^{-s\mu a}(s\mu a)^n}{n!} \right\} \rho^{j} \\
&= \pi_s \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \frac{e^{-s\mu a}(s\mu a)^n}{n!} \rho^j \\
&= \pi_s e^{-s\mu a} \sum_{n=0}^{\infty} \frac{(s\mu a)^n}{n!}\ \sum_{j=n}^{\infty} \rho^j \\
&= \pi_s e^{-s\mu a} \sum_{n=0}^{\infty} \frac{(s\mu a)^n}{n!} \left\{ \frac{\rho^n}{1-\rho} \right\} \\
&= \frac{\pi_s e^{-s\mu a}}{1-\rho} \sum_{n=0}^{\infty} \frac{(s\mu a \rho)^n}{n!} \\
&= \frac{\pi_s e^{-s\mu a}}{1-\rho}\ e^{s\mu a \rho} \\
&= \frac{\pi_s e^{-(s\mu - s\mu\rho)a}}{1-\rho} \\
&= \frac{\pi_s e^{-(s\mu - \lambda)a}}{1-\rho}
\end{aligned}$$

Finally,

$$\begin{aligned}
\Pr\{W_q > a \mid W_q > 0\} &= \frac{\Pr\{W_q > a\}}{\Pr\{W_q > 0\}} \\
&= \frac{(\pi_s e^{-(s\mu-\lambda)a})/(1-\rho)}{\sum_{j=s}^{\infty} \pi_j}
\end{aligned}$$

But

$$\sum_{j=s}^{\infty} \pi_j = \pi_s \sum_{j=s}^{\infty} \rho^{j-s} = \pi_s \sum_{j=0}^{\infty} \rho^j = \frac{\pi_s}{1-\rho}$$

so

$$\Pr\{W_q > a \mid W_q > 0\} = e^{-(s\mu-\lambda)a}$$

18. Model the system as an M/M/s queue with $\lambda = 1/3$ customer/minute, $\mu = 1/2$ customer per minute, and s the number of ATMs.

    They want

$$\begin{aligned}
\Pr\{W_q > 5 \mid W_q > 0\} &= e^{-(s/2-1/3)5} \\
&= \exp(-(5/2)s + 5/3) \\
&\leq 0.15
\end{aligned}$$

    Since $\exp(-5/2 + 5/3) \approx 0.43$ and $\exp(-5 + 5/3) \approx 0.03$, 2 ATMs are adequate.

19. (a) Since $\varepsilon_a = 1$ for the exponential distribution

$$\begin{aligned}
w_q = w_q(\lambda, 1, \mu, \varepsilon_s, 1) &= \left(\frac{1+\varepsilon_s}{2}\right) w_q(\lambda, \mu, 1) \\
&= \left(\frac{1+\varepsilon_s}{2}\right) \frac{\rho^2}{\lambda(1-\rho)}
\end{aligned}$$

    But, $\varepsilon_s = \frac{\sigma^2}{(1/\mu)^2} = \mu^2\sigma^2$. Thus,

$$\begin{aligned}
w_q &= \left(\frac{1+\mu^2\sigma^2}{2}\right)\left(\frac{\rho^2}{\lambda(1-\rho)}\right) \\
&= \frac{\rho^2 + \rho^2\mu^2\sigma^2}{2\lambda(1-\rho)} \\
&= \frac{\rho^2 + \lambda^2\sigma^2}{2\lambda(1-\rho)}
\end{aligned}$$

(b) For the M/M/1

$$w_q(\lambda, \mu, 1) = \frac{\rho^2}{\lambda(1-p)}$$

For the M/D/1

$$w_q(\lambda, 1, \mu, 0, 1) = \frac{\rho^2}{2\lambda(1-\rho)} = \frac{1}{2} w_q(\lambda, \mu, 1)$$

20. No answer provided.

21. Let `osu.edu` be station 1, and `eng.ohio-state.edu` be station 2 in a network of queues. Then

	$j=1$	$j=2$
a_{0j}	2	1
$\mu^{(j)}$	3	2
$s^{(j)}$	1	1

$$\mathbf{R} = \begin{pmatrix} 0 & 0.2 \\ 0 & 0 \end{pmatrix}$$

Therefore,

$\lambda^{(1)} = a_{01} = 2/\text{second}$

$\lambda^{(2)} = a_{02} + 0.2\ \lambda^{(1)} = 1.4/\text{second}$

We approximate it as a Jackson network.

(a) $\ell_q^{(1)} = \frac{\rho_1^2}{1-\rho_1} = \frac{(2/3)^2}{1-2/3} = 4/3$ messages

$w_q^{(1)} = \frac{\ell_q^{(1)}}{\lambda^{(1)}} = 2/3$ second

(b) $\ell_q^{(2)} = \frac{\rho_2^2}{1-\rho_2} = \frac{(1.4/2)^2}{1-(1.4/2)} \approx 1.63$ messages

$w_q^{(2)} = \frac{\ell_q^{(2)}}{\lambda^{(2)}} \approx 1.2$ seconds

(c) (12 K/message) (1.63 messages) = 19.56K

(d) Let h be the inflation factor, so that

$$\begin{aligned} \lambda^{(1)} &= ha_{01} = 2h \\ \lambda^{(2)} &= ha_{02} + 0.2\lambda^{(1)} \\ &= h + 0.2(2h) \\ &= 1.4h \end{aligned}$$

The expected time to final delivery is

$$\begin{aligned}
w^{(1)} + w^{(2)} &= w_q^{(1)} + \frac{1}{\mu^{(1)}} + w_q^{(2)} + \frac{1}{\mu^{(2)}} \\
&= \frac{\rho_1^2}{\lambda^{(1)}(1-\rho_1)} + \frac{1}{3} + \frac{\rho_2^2}{\lambda^{(2)}(1-\rho_2)} + \frac{1}{2} \\
&= \frac{(2h/3)^2}{2h(1-2h/3)} + \frac{1}{3} + \frac{(1.4h/2)^2}{1.4h(1-1.4h/2)} + \frac{1}{2}
\end{aligned}$$

A plot of this function shows that $h = 1.21$ (or 21% increase) is the most that can be allowed before it exceeds 5 seconds.

22. (8.24)

$$\begin{aligned}
\mu^{(2)}(1-\rho_1)\rho_2(1-\rho_2) &= \lambda(1-\rho_1)(1-\rho_2) \\
\mu^{(2)}\rho_2 &= \lambda \\
\lambda &= \lambda
\end{aligned}$$

(8.25)

$$\begin{aligned}
&\lambda\rho_1^{i-1}(1-\rho_1)(1-\rho_2) + \mu^{(2)}\rho_1^{i}(1-\rho_1)\rho_2(1-\rho_2) \\
&\qquad = (\lambda + \mu^{(1)})\rho_1^{i}(1-\rho_1)(1-\rho_2) \\
&\lambda + \mu^{(2)}\rho_1\rho_2 = (\lambda + \mu^{(1)})\rho_1 \\
&\lambda + \lambda\rho_1 = \lambda\rho_1 + \lambda
\end{aligned}$$

(8.26)

$$\begin{aligned}
&\mu^{(1)}\rho_1(1-\rho_1)\rho_2^{j-1}(1-\rho_2) + \mu^{(2)}(1-\rho_1)\rho_2^{j+1}(1-\rho_2) \\
&\qquad = (\lambda + \mu^{(2)})(1-\rho_1)\rho_2^{j}(1-\rho_2) \\
&\mu^{(1)}\rho_1 + \mu^{(2)}\rho_2^2 = (\lambda + \mu^{(2)})\rho_2 \\
&\lambda + \lambda\rho_2 = \lambda\rho_2 + \lambda
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \rho_1^{i}(1-\rho_1)\rho_2^{j}(1-\rho_2) \\
&\qquad = (1-\rho_1)(1-\rho_2)\sum_{i=0}^{\infty}\rho_1^{i}\sum_{j=0}^{\infty}\rho_2^{j}
\end{aligned}$$

$$
\begin{aligned}
&= (1-\rho_1)(1-\rho_2)\sum_{i=0}^{\infty} \rho_1^i \frac{1}{1-\rho_2} \\
&= (1-\rho_1)(1-\rho_2)\frac{1}{1-\rho_1}\ \frac{1}{1-\rho_2} \\
&= 1
\end{aligned}
$$

23. Let B be a Bernoulli random variable that takes the value 1 with probability r. The event $\{B = 1\}$ indicates that a product fails inspection.

    The only change occurs in system event e_3.

$e_3()$	(**complete inspection**)
$S_{2,n+1} \leftarrow S_{2,n} - 1$	(one fewer product at inspection)
if $\{S_{2,n+1} > 0\}$ then	(if another product then start)
$C_3 \leftarrow T_{n+1} + F_Z^{-1}(\texttt{random}())$	(set clock for completion)
endif	
$B \leftarrow F_B^{-1}(\texttt{random}())$	
if $\{B = 1\}$ then	(product fails inspection)
$S_{1,n+1} \leftarrow S_{1,n} + 1$	(one more product at repair)
if $\{S_{1,n+1} = 1\}$ then	(if only one product then start)
$C_2 \leftarrow T_{n+1} + F_X^{-1}(\texttt{random}())$	(set clock for completion)
endif	
endif	

24. The balance equations are

$$
\begin{aligned}
\underline{\text{rate in}} &= \underline{\text{rate out}} \\
(1-r)\mu^{(2)}\pi_{(0,1)} &= \lambda\pi_{(0,0)} \\
\lambda\pi_{(i-1,0)} + r\mu^{(2)}\pi_{(i-1,1)} + (1-r)\mu^{(2)}\pi_{(i,1)} &= (\lambda+\mu^{(1)})\pi_{(i,0)},\ i>0 \\
\mu^{(1)}\pi_{(1,j-1)} + (1-r)\mu^{(2)}\pi_{(0,j+1)} &= (\lambda+\mu^{(2)})\pi_{(0,j)},\ j>0 \\
\lambda\pi_{(i-1,j)} + \mu^{(1)}\pi_{(i+1,j-1)} + r\mu^{(2)}\pi_{(i-1,j+1)} + (1-r)\mu^{(2)}\pi_{(i,j+1)} &= \delta\pi_{(i,j)},\ i,j>0
\end{aligned}
$$

where $\delta = \lambda + \mu^{(1)} + \mu^{(2)}$.

The steady-state probabilities are

$$\pi_{(i,j)} = (1-\rho_1)\rho_1^i(1-\rho_2)\rho_2^j$$

where

$$\rho_i = \frac{\lambda^{(i)}}{\mu^{(i)}} = \frac{\lambda/(1-r)}{\mu^{(i)}}$$

Verification follows exactly as in Exercise 22.

25. To remain stable requires

$$\rho_i = \frac{\lambda^{(i)}}{\mu^{(i)}} = \frac{5/(1-r)}{\mu^{(i)}} = \frac{5}{\mu^{(i)}(1-r)} < 1 \text{ for } i = 1, 2$$

For $i = 1$

$$\rho_1 = \frac{5}{6(1-r)} < 1$$

Therefore, $r < 1/6$.

For $i = 2$

$$\rho_2 = \frac{5}{8(1-r)} < 1$$

Therefore $r < 3/8$. So we require $r < 1/6$.

26.

$$\begin{aligned}
b_{jk} &= r_{jk}\varepsilon_d^{(j)} + (1-r_{jk}) \\
&= r_{jk}\varepsilon_a^{(j)} + (1-r_{jk}) \\
&= r_{jk}\sum_{i=0}^{m}\left(\frac{a_{ij}}{\lambda^{(j)}}\right)b_{ij} + (1-r_{jk}) \\
&= \sum_{i=1}^{m} r_{jk}\left(\frac{a_{ij}}{\lambda^{(j)}}\right)b_{ij} + r_{jk}\left(\frac{a_{0j}}{\lambda^{(j)}}\right)b_{0j} + (1-r_{jk}) \\
&= \sum_{i=1}^{m} r_{jk}\left(\frac{r_{ij}\lambda^{(i)}}{\lambda^{(j)}}\right)b_{ij} + d_{jk} \\
&= \sum_{i=1}^{m} r_{jk}\; c_{ij}b_{ij} + d_{jk}
\end{aligned}$$

27. Approximate the system as a Jackson network of 2 queues with the following parameters:

i	a_{0i}	$\mu^{(i)}$
1	20	30
2	0	30

$$\mathbf{R} = \begin{pmatrix} 0 & 0.9 \\ 0 & 0 \end{pmatrix}$$

Therefore,

$\lambda^{(1)} = a_{01} = 20$

$\lambda^{(2)} = r_{12}\ \lambda^{(1)} = 18$

and each station is an M/M/1 queue.

(a) For each station individually, we want the minimum c such that $\sum_{j=0}^{c} \pi_j \geq 0.95$. For the M/M/1, $\pi_j = (1-\rho)\rho^j$, so

$$\sum_{j=0}^{c}(1-\rho)\rho^j = (1-\rho)\left(\frac{1-\rho^{c+1}}{1-\rho}\right) \leq 0.95$$

Therefore,

$$\begin{aligned} \rho^{c+1} &\geq 0.05 \\ c+1 &\geq \frac{\ln(0.05)}{\ln(\rho)} \\ c &\geq \frac{\ln(0.05)}{\ln(\rho)} - 1 \end{aligned}$$

i	$\rho_i = \lambda^{(i)}/\mu^{(i)}$	c
1	2/3	7
2	3/5	5

(b)

$$\begin{aligned} \ell^{(2)} &= \lambda^{(2)} w^{(2)} = \lambda^{(2)}(w_q^{(2)} + 1/\mu^{(2)}) \\ &= \lambda^{(2)}\left(\frac{\rho_2^2}{\lambda^{(2)}(1-\rho_2)} + \frac{1}{\mu^{(2)}}\right) \\ &= 3/2 \text{ jobs} \end{aligned}$$

(c) $\rho_2 = 3/5 = 0.6$

(d) We now have a network of 3 queues, with queue 3 being the rework station for station 1.

$$\mathbf{R} = \begin{pmatrix} 0 & 0.9 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$$

$\lambda^{(1)} = 20$
$\lambda^{(2)} = 0.9\ \lambda^{(1)} + 0.5\lambda^{(3)}$
$\lambda^{(3)} = 0.1\ \lambda^{(1)}$
Therefore,
$\lambda^{(1)} = 20$
$\lambda^{(2)} = 18 + 1 = 19$
$\lambda^{(3)} = 2$
For station 1 nothing changes. For station 2

$$\begin{aligned} \rho_2 &= 19/30 \approx 0.63 \\ c &= 6 \\ \ell^{(2)} &= \lambda^{(2)} \left(\frac{\rho_2^2}{\lambda^{(2)}(1-\rho_2)} + \frac{1}{\mu^{(2)}} \right) \\ &= 19/11 \approx 1.7 \text{ jobs} \end{aligned}$$

(e) This change has no impact on the results from part (d), but would change the performance of the rework stations.

28. Model as a Jackson network of infinite-server queues, with a queue corresponding to each type of transaction.

transaction	i	a_{0j}	$\mu^{(j)}$ per hour
log on	1	1000 or 1500	20
fetch	2	0	20
read	3	0	5
log off	4	0	100

$$\mathbf{R} = \begin{pmatrix} 0 & 0.79 & 0.01 & 0.2 \\ 0 & 0.17 & 0.63 & 0.2 \\ 0 & 0.16 & 0.4 & 0.44 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_1 = \begin{pmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\mathbf{A}_2 = \begin{pmatrix} 1500 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{\Lambda}[(\mathbf{I} - \mathbf{R})']^{-1}\mathbf{A} = \begin{pmatrix} 1.0 & 0 & 0 & 0 \\ 1.197381672 & 1.510574018 & 0.4028197382 & 0 \\ 1.273917422 & 1.586102719 & 2.089627392 & 0 \\ 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix} \mathbf{A}$$

$$= \begin{pmatrix} 1000.0 \\ 1197.381672 \\ 1273.917422 \\ 1000.0 \end{pmatrix} \text{ when } \mathbf{A} = \mathbf{A}_1$$

or

$$= \begin{pmatrix} 1500.0 \\ 1796.072508 \\ 1910.876133 \\ 1500.0 \end{pmatrix} \text{ when } \mathbf{A} = \mathbf{A}_2$$

	$\ell^{(i)} = \lambda^{(i)}/\mu^{(i)}$	
i	$\mathbf{A}_1$	$\mathbf{A}_2$
1	50	75
2	60	90
3	255	382
4	10	15
expected number of simultaneous users	375	562

29. We approximate the job shop as a Jackson network with a single job type.

queue i	name
1	casting
2	planer
3	lathe
4	shaper
5	drill

$$\mathbf{R} = \begin{pmatrix} 0 & 0.54 & 0 & 0.46 & 0 \\ 0 & 0 & 0.46 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We obtain $\mathbf{R}$ by noticing that when a job departs a machine group that serves both job types, the probability it is a type 1 job is (historically) $460/(460+540) = 0.46$.

$$\mathbf{A} = \begin{pmatrix} 1/11 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

because all jobs arrive initially to the casting group, and the sample-mean interarrival-time gap was 11 minutes.

$$\begin{aligned} \lambda^{(1)} &= 1/11 \\ \lambda^{(4)} &= (0.54)\lambda^{(1)} \\ \lambda^{(5)} &= \lambda^{(4)} = (0.54)\lambda^{(1)} \\ \lambda^{(2)} &= (0.46)\lambda^{(1)} + \lambda^{(5)} = \lambda^{(1)} \\ \lambda^{(3)} &= (0.46)\lambda^{(2)} = (0.46)\lambda^{(1)} \end{aligned}$$

We approximate the expected service times as follows:

$$\begin{aligned} \frac{1}{\mu^{(1)}} &= (0.46)(125) + (0.54)(235) \approx 184.4 \\ \frac{1}{\mu^{(2)}} &= (0.46)(35) + (0.54)(30) \approx 32.3 \\ \frac{1}{\mu^{(3)}} &= 20 \\ \frac{1}{\mu^{(4)}} &= 250 \\ \frac{1}{\mu^{(5)}} &= 50 \end{aligned}$$

The utilizations are therefore $\rho_i = \lambda^{(i)}/(s_i\mu^{(i)})$.

i	s_i	ρ_i
1	19	0.88
2	4	0.75
3	3	0.27
4	16	0.77
5	5	0.50

The casting units are the most heavily utilized.

Modeling each group as an M/M/s queue we obtain

i	$\ell_q^{(i)}$	$w_q^{(i)}$ (minutes)
1	3.7	41.1
2	1.3	14.7
3	0.02	0.5
4	0.8	15.8
5	0.1	2.4

Clearly the longest delay occurs at the casting units. This appears to be the place to add capacity.

The expected flow times for each job type are approximated as

type 1: $41.1 + 125 + 14.7 + 35 + 0.5 + 20 = 263.3$ minutes

type 2: $41.1 + 235 + 15.8 + 250 + 2.4 + 50 = 639$ minutes

The expected WIP is

$$\begin{aligned}\sum_{i=1}^{5} \ell^{(i)} &= \sum_{i=1}^{5} \lambda^{(i)} w^{(i)} \\ &= \sum_{i=1}^{5} \lambda^{(i)} (w_q^{(i)} + 1/\mu^{(i)}) \\ &\approx 39 \text{ jobs}\end{aligned}$$

Comments: The approximation might be improved by adjusting for variability. The sample squared coefficient of variation of the interarrival times is

$$\widehat{\varepsilon}_a = \frac{(23.1)}{(11.0)^2} \approx 0.19 < 1$$

We might also expect $\varepsilon_s^{(i)} < 1$. These adjustments would reduce ℓ_q, w_q, flow times and WIP.

30. We have a network of two queues, one for the packer and one for the forklifts, denoted $i = 1$ and $i = 2$ respectively. Define a customer to be 1 case = 75 cans. Therefore, $1/\lambda^{(1)} = 150$ seconds = 2.5 minutes.

    Clearly, $\lambda^{(2)} = \lambda^{(1)}$, $\frac{1}{\mu^{(1)}} = 2$ minutes, $\frac{1}{\mu^{(2)}} = 3 + 1 = 4$ minutes.

    If s_2 is the number of forklifts, then just to keep up we must have

$$\rho_2 = \frac{\lambda^{(2)}}{s_2 \mu^{(2)}} = \frac{4}{s_2(2.5)} < 1$$

    Therefore, at least $s_2 = 2$ forklifts are required.

    To approximate $\ell_q^{(2)}$ and $w^{(2)}$ we need to go further. Notice that

$$\begin{aligned}
\varepsilon_a^{(1)} &= 0 \\
\varepsilon_s^{(1)} &= \frac{\frac{(3-1)^2}{12}}{2^2} = \frac{1}{12} \\
\varepsilon_d^{(1)} &= \varepsilon_a^{(1)} \quad \text{(from } (A5)\text{)} \\
\varepsilon_a^{(2)} &= \varepsilon_d^{(1)} = 0 \text{ (since all departures from the packer go to the forklifts)} \\
\varepsilon_s^{(2)} &= \frac{1^2}{(3+1)} = 1/4
\end{aligned}$$

    Therefore, for the forklifts we can approximate $w_q^{(2)}$ as

$$\begin{aligned}
w_q^{(2)} &\approx w_q(\lambda^{(2)}, \varepsilon_a^{(2)}, \mu^{(2)}, \varepsilon_s^{(2)}, s_2) \\
&= \left(\frac{\varepsilon_a^{(2)} + \varepsilon_s^{(2)}}{2}\right) w_q(\lambda^{(2)}, \mu^{(2)}, s_2) \\
&= (1/8) w_q(1/2.5, 1/4, s_2) \\
w^{(2)} &= w_q^{(2)} + 1/\mu^{(2)}
\end{aligned}$$

    and

$$\ell_q^{(2)} = \lambda^{(2)} w_q^{(2)} = \frac{w_q^{(2)}}{2.4}$$

s_2	$w_q^{(2)}$	$\ell_q^{(2)}$	$w^{(2)}$
2	0.89	0.35	4.89
3	0.10	0.04	4.10

It appears that there will be very little queueing even with the minimum of 2 forklifts.

31. We first approximate the drive-up window as an M/M/s/4 queue with $\lambda_1 = 1/2$ customer/minute and $\mu_1 = 1/1.8$ customer/minute. (This approximation is rough because the true service-time distribution is more nearly normal, and $\varepsilon_s < 1$.)

s	π_4
1	0.16
2	0.03

π_4 is the probability that an arriving car will find the queue full and thus have to park. Adding a window reduces this dramatically.

The rate at which customers are turned away from the drive-up window is $\lambda_1\pi_4 = (1/2)\pi_4$. Therefore, the overall arrival rate into the bank is $\lambda_2 = 1 + (1/2)\,\pi_4$ customers per minute. This process will **not** be a Poisson process, but we approximate it as one. As a first cut we model the tellers inside the bank as an M/M/s_2 queue with $\mu_2 = 1/1.4$ customer/minute.

(s_1, s_2)	λ_2	w_q (minutes)
(1, 2)	1.08	1.9
(2, 2)	1.02	1.5
(1, 3)	1.08	0.2

The bank can now decide which improvement in performance is more valuable.

Comment: The approximation for the inside tellers can be improved by using the GI/G/s adjustment. Clearly, $\varepsilon_s = 1.0/(1.4)^2 \approx 0.5$. The exact value of ε_a can also be computed, but requires tools not used in this book, so set $\varepsilon_a = 1$ for a Poisson process.

32. No answer provided.

33. No answer provided.

34. No answer provided.

35. No answer provided.

Chapter 9

Topics in Simulation of Stochastic Processes

1. Solve $\widehat{\sigma}/\sqrt{k} \leq 20$ which gives $k = (110/20)^2 \approx 31$ replications.

2–7. No answer provided.

8. To obtain a rough-cut model we will (a) use a 0.5 probability that an item joins the queue of each inspector, rather than selecting the shortest queue, and (b) treat all processing times as exponentially distributed (later we will refine (b)).

   - Current System

station j	name	a_{0j}	$\mu^{(j)}$
1	repair 1	0.125	0.167
2	repair 2	0.125	0.167
3	inspect	0	0.333

$$\mathbf{R} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.1 & 0.1 & 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 0.125 \\ 0.125 \\ 0 \end{pmatrix}$$

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \\ \lambda^{(3)} \end{pmatrix} = [(\mathbf{I} - \mathbf{R})']^{-1}\mathbf{A} = \begin{pmatrix} 0.156 \\ 0.156 \\ 0.313 \end{pmatrix}$$

The utilization of the technicians is

$$\rho^{(j)} = \frac{\lambda^{(j)}}{\mu^{(j)}} \approx 0.93 \qquad \text{for } j = 1, 2$$

Treating each station as an M/M/1 queue the expected flow time is

$$\left(w_q^{(1)} + \frac{1}{\mu^{(1)}} + w_q^{(3)} + \frac{1}{\mu^{(3)}} \right) (1.25)$$

$$\approx (85 + 6 + 47 + 3)(1.25) \approx 141 \text{ minutes}$$

since $w_q^{(1)} = w_q^{(2)}$ and the expected number of cycles through the system is $1/0.8 = 1.25$ cycles.

• Proposed System

The repair technicians now become a single station with $\lambda = \lambda^{(1)} + \lambda^{(2)} = 0.313$ and $\mu = 0.167$. The utilization is $\rho = \frac{\lambda}{2\mu} \approx 0.93$, so it is unchanged.

Treating the repair station as an M/M/2 queue, the expected flow time is

$$\begin{aligned} & \left(w_q + \frac{1}{\mu} + w_q^{(3)} + \frac{1}{\mu^{(3)}} \right) (1.25) \\ \approx \ & (43 + 6 + 47 + 3)(1.25) \\ \approx \ & 124 \text{ minutes} \end{aligned}$$

which is a dramatic reduction.

• Refinement

We can refine the approximation by using the ideas in Section 8.10. Notice that

j	$\varepsilon_s^{(j)}$
1	0.0625
2	0.0625
3	0.0370

Since the arrival process is Poisson, $\varepsilon_a^{(j)} = 1$ throughout. Therefore, the preceding values of $w_q^{(j)}$ can be modified by the factor $(1 + \varepsilon_s^{(j)})/2$.

9. To obtain a rough-cut model, first note that the registration function is not really relevant to the issue of additional bed versus additional doctor, so we will ignore it. We also (a) ignore the interaction of beds and doctors, (b) treat treatment time as exponentially distributed, and (c) develop a composite patient and ignore patient type and priority.

   - Additional doctor.

   With an additional doctor there is a one-to-one correspondence between bed and doctor.

   $\lambda = 1/20$ patient/minute

   $1/\mu = 72(0.15) + 25(0.85) \approx 32$ minutes

   Using an M/M/3 model

   $\ell_q = 0.3$ patients waiting for a doctor and a bed

   $w_q = 6.3$ minutes wait for a doctor and a bed

   - Additional bed.

   In terms of time to wait to see a doctor, the beds are not a constraint. Using an M/M/2 model

   $\ell_q = 2.8$ patients waiting to see a doctor

   $w_q = 57$ minutes to see a doctor

   The expected number of patients in the system is

   $$\ell = \lambda w = \lambda(w_q + 1/\mu) \approx 4.45$$

   Thus, with 4 beds the expected number waiting for a bed is $4.45 - 4 = 0.45$, which is larger than the 0.3 patients when adding another doctor.

   It appears that adding a doctor will be better in terms of time to see a doctor and time to reach a bed.

10. To obtain a rough-cut model we will replace all random variables by their expected values and treat it as deterministic.

    Suppose the company is open 8 hours per day. Then the period of interest is $90(8) = 720$ hours.

    The number of lost sales during 1 day is (8 hours) (1 customer/hour) (6 bags/customer) = 48 bags.

    If the company orders s bags, they will run out in

    $$\frac{s \text{ bags}}{6 \text{ bags/hour}} = s/6 \text{ hours}$$

so a complete cycle is approximately s/6 + 8 hours.

The number of orders for the summer is

$$\frac{720}{s/6+8}$$

The expected number of lost bags is

$$\left(\frac{720}{s/6+8}\right)(48)$$

And the average stock level is

$$\frac{1/2(s/6)s}{s/6+8}$$

11. To obtain a rough-cut model we will (a) ignore the interaction of loading bays and forklifts, and (b) treat all service times as exponentially distributed (later we refine (b)).

   • Adding a fork lift.

   We first treat the number of bays as unlimited.

   $\lambda = 6.5$ trucks/hour

   $1/\mu = 15$ minutes $= 1/4$ hour

   Based on an M/M/3 model

   $w_q = 0.05$ hours $= 3$ minutes

   $\ell_q = 0.3$ trucks

   $\rho = \frac{\lambda}{3\mu} = \frac{6.5}{12} \approx 0.54$ utilization

   Now treating the bays as servers we use an M/M/4 model with

   $\lambda = 6.5$

   $1/\mu = 15 + 15 = 30$ minutes $= 1/2$ hour

   $w_q = 0.41$ hour $= 25$ minutes

   $\ell_q = 2.7$ trucks in the lot

   Approximate total time: $0.41 + 0.05 + 1/2 = 0.96$ hour ≈ 58 minutes.

   • Adding a bay

   Again start by treating the bays as unlimited, and the forklifts as an M/M/2 with

   $\lambda = 6.5$

   $1/\mu = 1/4$

$w_q = 0.49$ hours

$\ell_q = 3.2$ trucks

$\rho = \frac{\lambda}{2\mu} = 0.81$ utilization

Next treat the bays as servers in an M/M/5 model with

$\lambda = 6.5$

$1/\mu = 1/2$

$w_q = 0.09$ hours $= 5.4$ minutes

$\ell_q = 0.6$ trucks in the lot

Approximate total time: $0.09 + 0.49 + 1/2 = 1.08$ hours ≈ 65 minutes.

Adding a bay looks better, but only by about 7 minutes. The values of w_q and ℓ_q can be refined by using the ideas in Section 8.10. Notice that

	ε_s
forklift	0.01
bay	$\frac{(1.5^2+3(5)^2)}{(15+15)^2} \approx 0.09$

12. For $n = 1, \mathbf{p}^{(1)} = \mathbf{p}'\ \mathbf{P} = \boldsymbol{\pi}'\ \mathbf{P} = \boldsymbol{\pi}'$ by Equation (6.21).

    Suppose the result is true for all $n \leq k$, for some $k > 1$. Then

$$\begin{aligned} \mathbf{p}^{(k+1)} &= \mathbf{p}'\mathbf{P}^{k+1} \\ &= (\mathbf{p}'\ \mathbf{P}^k)\mathbf{P} \\ &= \boldsymbol{\pi}'\ \mathbf{P} \\ &= \boldsymbol{\pi}' \end{aligned}$$

    Therefore, the result is true for all n by induction.

13. No answer provided.

14. No answer provided.

15. $$\widehat{\sigma}_{XZ} = \frac{1}{k-1}\left(\sum_{i=1}^{k} X_i Z_i - \frac{\left(\sum_{j=1}^{k} X_j\right)\left(\sum_{h=1}^{k} Z_h\right)}{k}\right)$$

    In this example (using Table 9.2, columns 2 and 4)

    $k = 10$

$\sum_{i=1}^{k} X_i Z_i = 644.99$
$\sum_{j=1}^{k} X_j = 80.3$
$\sum_{h=1}^{k} Z_h = 80.3$
$\widehat{\sigma}_{XZ} \approx 0.02$

16.

$$
\begin{aligned}
\sum_{i=1}^{k}(V_i - \bar{V})^2 &= \sum_{i=1}^{k}(X_i - Z_i - (\bar{X} - \bar{Z}))^2 \\
&= \sum_{i=1}^{k}((X_i - \bar{X}) - (Z_i - \bar{Z}))^2 \\
&= \sum_{i=1}^{k}\Big\{(X_i - \bar{X})^2 - 2(X_i - \bar{X})(Z_i - \bar{Z}) \\
&\qquad +(Z_i - \bar{Z})^2\Big\} \\
&= \sum_{i=1}^{k}(X_i - \bar{X})^2 + \sum_{i=1}^{k}(Z_i - \bar{Z})^2 \\
&\qquad - 2\sum_{i=1}^{k}(X_i - \bar{X})(Z_i - \bar{Z}) \\
&= (k-1)(\widehat{\sigma}_X^2 + \widehat{\sigma}_Z^2 - 2\widehat{\sigma}_{XZ})
\end{aligned}
$$

17. There are mostly disadvantages.

   • Unless we understand the bias quite well, we have no idea what overload approximately compensates the underload. Therefore, the bias may still be quite significant, and we will still have to study it.

   • When we attempt to study the bias, we must either study the underload and overload individually, which is twice as much work, or study them together, in which case the combined bias process may have very unusual behavior.

   An advantage is that we might be able to bound the bias by looking at convergence from above and below.

18. If $\widehat{\theta}$ is an estimator of θ, then the bias of $\widehat{\theta}$ is $\mathrm{E}[\widehat{\theta} - \theta]$ where the expectation is with respect to the distribution of $\widehat{\theta}$.

   Let S_0 be the initial state, as in the Markov chain. Then, if we sample the initial state from the steady-state distribution, we proved

$$\mathrm{E}[\widehat{\theta}] = \sum_{x \in \mathcal{M}} \mathrm{E}[\widehat{\theta} \mid S_0 = x] \Pr\{S_0 = x\} = \theta \qquad (1)$$

When we fix the initial state, the expectation of interest is

$$\mathrm{E}[\widehat{\theta} \mid S_0 = 1]$$

a **conditional expectation**, only 1 term of (1).

Remember that "expectation" is a mathematical averaging process over the possible outcomes. When we sample the initial state, all initial states are possible. When we fix the initial state, only 1 initial state is possible. Therefore, the **probability distribution** of the estimator changes, even though the results of the simulation (outcome) may be the same.

19. $\boldsymbol{\pi} \approx \begin{pmatrix} 0.429 \\ 0.333 \\ 0.238 \end{pmatrix}$

Therefore

$$\begin{aligned} \Pr\{S = 3\} &= \pi_3 \approx 0.238 \\ \mathrm{E}[S] &= \pi_1 + 2\pi_2 + 3\pi_3 \approx 1.809 \\ \mathrm{Var}[S] &= \sum_{j=1}^{3}(j - 1.809)^2 \pi_j \approx 0.630 \end{aligned}$$

A 1% relative error for $\Pr\{S = 3\}$ implies that

$$\frac{\mid 0.238 - p_{13}^{(n)} \mid}{0.238} \leq 0.01$$

Therefore, $\mid 0.238 - p_{13}^{(n)} \mid \leq 0.002$.

A 1% relative error for $\mathrm{E}[S]$ implies that

$$\frac{\mid 1.809 - \mathrm{E}[S_n \mid S_0 = 1] \mid}{1.809} \leq 0.01$$

Therefore, $\mid 1.809 - \mathrm{E}[S_n \mid S_0 = 1] \mid \leq 0.018$ where

$$\mathrm{E}[S_n \mid S_0 = 1] = \sum_{j=1}^{3} j \; p_{1j}^{(n)}$$

A 1% relative error for $\mathrm{Var}[S]$ implies that

$$\frac{\mid 0.630 - \mathrm{Var}[S_n \mid S_0 = 1] \mid}{0.063} \leq 0.01$$

Therefore, $| \, 0.630 - \text{Var}[S_n \mid S_0 = 1] \, | \leq 0.006$ where

$$\text{Var}[S_n \mid S_0 = 1] = \sum_{j=1}^{3} (j - \text{E}[S_n \mid S_0 = 1])^2 p_{1j}^{(n)}$$

n	$p_{13}^{(n)}$	$\text{E}[S_n \mid S_0 = 1]$	$\text{Var}[S_n \mid S_0 = 1]$
0	0	1.0	0
1	0.2	1.5	0.65
2	0.25	1.67	0.721
3	0.258	1.735	0.711
4	0.256	1.764	0.691
5	0.251	1.780	0.674
6	0.248	1.789	0.662
7	0.245	$\boxed{1.796}$	0.652
8	0.243	1.800	0.646
9	0.241	1.803	0.641
10	$\boxed{0.240}$	1.805	0.638
11	.240	1.806	$\boxed{0.636}$

They do not all converge at the same time.

20. No answer provided.

21. No answer provided.