S O L U T I O N S   M A N U A L

to accompany

E L E C T R I C I T Y   A N D   M A G N E T I S M

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CHAPTER 1

1.1 The cross product $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$. Since we want a unit vector, we just divide by the magnitude: $\hat{n} = \vec{A} \times \vec{B}/|\vec{A} \times \vec{B}|$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\hat{x} - 10\hat{y} + 30\hat{z}$$

$$|\vec{A} \times \vec{B}| = \sqrt{15^2 + 10^2 + 30^2} = 35$$

Hence, $\hat{n} = \dfrac{15\hat{x} - 10\hat{y} + 30\hat{z}}{35} = \dfrac{3\hat{x} - 2\hat{y} + 6\hat{z}}{7}$

1.2 The position vectors of these three points are $\vec{r}_1 = 2\hat{x} - \hat{y} + \hat{z}$, $\vec{r}_2 = 3\hat{x} + 2\hat{y} - \hat{z}$, and $\vec{r}_3 = -\hat{x} + 3\hat{y} + 2\hat{z}$. The position vector of an arbitrary point is $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. If all vectors lay in the plane, then the following triple cross product vanishes.

$$(\vec{r} - \vec{r}_1)\cdot(\vec{r}_2 - \vec{r}_1)\cdot(\vec{r}_3 - \vec{r}_1) = 0$$

which gives $11x + 5y + 13z - 30 = 0$.

1.3 The position vectors of these points and an arbitrary point are $\vec{r}_1 = 3\hat{x} + \hat{y} + 2\hat{z}$, $\vec{r}_2 = \hat{x} - 2\hat{y} - 4\hat{z}$, and $\vec{r} = x\hat{x} - y\hat{y} + z\hat{z}$. The equation of the plane is governed by the condition $(\vec{r} - \vec{r}_2)\cdot(\vec{r}_2 - \vec{r}_1) = 0$ which gives the equation $2x + 3y + 6z + 28 = 0$ for the plane.

1.4 **a)** $\nabla r^n = \hat{r}\, dr^n/dr = n\hat{r}r^{n-1} = nr^{n-2}\vec{r}$

b) $\nabla(\ln|\vec{r}|) = \hat{r}\, d\ln|\vec{r}|/dr = \hat{r}/r$

$\nabla(1/r) = \hat{r}\, d(1/r)/dr = -\hat{r}/r^2$

1.5 Consider the surface defined by f where $f(x,y,z) = 2xz^2 - 3xy - 4x - 7 = 0$. Recall that ∇f is normal to surface $f(x,y,z) = 0$, then

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} = (2z^2 - 3y - 4)\hat{x} - 3x\hat{y} + 4xz\hat{z}$$

At $(1,-1,2)$ we have: $\nabla f = (8 + 3 - 4)\hat{x} - 3\hat{y} + 8\hat{z} = 7\hat{x} - 3\hat{y} + 8\hat{z}$. The unit vector normal to the surface at this point is:

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{7\hat{x} - 3\hat{y} + 8\hat{z}}{\sqrt{7^2 + 3^2 + 8^2}} = \frac{7\hat{x} - 3\hat{y} + 8\hat{z}}{\sqrt{122}} .$$

1.6 By definition we have $d\Phi/ds_{max} = |\nabla\Phi|$. Now $\nabla\Phi = 2xyz^3\hat{x} + x^2z^3\hat{y} + 3x^2yz^2\hat{z}$, thus at $x = 2$, $y = 1$, $z = -1$ we have $\nabla\Phi = -4\hat{x} - 4\hat{y} + 12\hat{z}$. Hence, $d\Phi/ds$ is maximum along $(-4\hat{x} - 4\hat{y} + 12\hat{z})/\sqrt{4^2 + 4^2 + 12^2}$ direction and $|\nabla\Phi| = \sqrt{4^2 + 4^2 + 12^2} = \sqrt{176}$.

1.7 Using Eq. 1.57

$$\nabla\cdot(\vec{r}/r^3) = \nabla(1/r^3)\cdot\vec{r} + \nabla\cdot\vec{r}/r^3 = \frac{-3}{r^4}\hat{r}\cdot\vec{r} + \frac{3}{r^3} = 0$$

1.8 Using the vector identity given in Eq. (1.60) we have

$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \equiv 0$, since $\nabla\times\vec{A} = \nabla\times\vec{B} = 0$.

1.9 $\vec{A}$ is irrotational if $\nabla\times\vec{A}$ which gives

$\hat{x}(c + 1) - \hat{y}(a - 4) + \hat{z}(b - 2) = 0$. Therefore $a = 4$, $b = 2$, $c = -1$. Hence $\vec{A} = (x + 2y + 4z)\hat{x} + (2x - 3y - z)\hat{y} + (4x - y + 2z)\hat{z}$. Now $\vec{A} = \nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{x} + \frac{\partial\Phi}{\partial y}\hat{y} + \frac{\partial\Phi}{\partial z}\hat{z}$. Thus

$\frac{\partial\Phi}{\partial x} = x + 2y + 4z$. Partially integrate with respect to x:

$\Phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$. Now $\frac{\partial\Phi}{\partial y} = 2x - 3y - z$, thus partially integrating with respect to y gives

$\Phi = 2xy - 3y^2/2 - yz + g(x,z)$. Now $\frac{\partial\Phi}{\partial z} = 4x - y + 2z$, thus partially integrating with respect to z gives:

$\Phi = 4xz - yz + z^2 + h(x,y)$. Comparing the 3$\Phi$'s, we see that if $f(y,z) = -3y^2/2 + z^2 - yz$, $g(x,z) = x^2/2 + 4xz + z^2$, $h(x,y) = x^2/2 + 2xy - 3y^2/2$ then we get a Φ where $\nabla\Phi = \vec{A}$. That is

$\Phi = x^2/2 + 2xy + 4xz - 3y^2/2 + z^2 - yz$ + a constant.

1.10 $\nabla\cdot\vec{A} = 1 + 1 + a = 2 + a = 0$ which gives $a = -2$.

1.11 $\vec{E} = \vec{r}/r^2 = \hat{r}/r$ is conservative (spherical coordinates are implied) if $\vec{\nabla}\times\vec{E} = 0$. Note that $\vec{E} = E_r\hat{r}$.

$$\vec{\nabla} \times \vec{E} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} r & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ E_r & 0 & 0 \end{vmatrix}$$

$\vec{\nabla} \times \vec{E} = \frac{1}{r^2 \sin\theta}\left(r\hat{\theta}\ \partial E_r/\partial\phi - r\sin\theta\ \hat{\phi}\ \partial E_r/\partial\theta\right) = 0$. We can quickly see that any spherically symmetric (θ,ϕ independent) radial field is conservative. Thus $\nabla \times (\hat{r}/r) = 0$.

$\vec{E} = \hat{r}/r = -\nabla\Phi = -\left(\hat{r}\ \frac{\partial\Phi}{\partial r} + \hat{\theta}\ \frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\phi}\ \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)$.

Thus $1/r = -\ \partial\Phi/\partial r = -d\Phi/dr$. Or $-d\Phi = dr/r$ which integrates to $\Phi = -\ell nr + c$. Applying $\Phi(r = a) = 0$ gives $c = \ell na$; thus $\Phi = \ell na - \ell nr = -\ell n(r/a)$.

1.12 Notice that if we can find a scalar function $\Phi(x,y)$ such that $\vec{A} = \nabla\Phi$ as required in the second part of this problem, then it would be a necessary and sufficient condition that $\vec{A}$ be conservative. However, we will go ahead and explicitly show that $\nabla \times \vec{A} = 0$ to show that $\vec{A}$ is conservative:

$$\begin{aligned} \nabla \times \vec{A} = &+\hat{x}[\partial(3xz^2 - y)/\partial y - \partial(3x^2 - z)/\partial z] \\ &-\hat{y}[\partial(3xz^2 - y)/\partial x - \partial(6xy + z^3)/\partial z] \\ &+\hat{z}[\partial(3x^2 - z)/\partial x - \partial(6xy - z^3)/\partial y] \end{aligned}$$

$= \hat{x}(-1 - (-1)) - \hat{y}(3z^2 - (3z^2)) + \hat{z}(6x - (6x)) = 0$. Therefore $\nabla \times \vec{A} = 0 \rightarrow \vec{A}$ is conservative. Now we find Φ such that $\vec{A} = \nabla\Phi$.

$\vec{A} = (6xy + z^3)\hat{x} + (3x^2 - z)\hat{y} + (3xz^2 - y)\hat{z}$

$= \partial\Phi/\partial x\ \hat{x} + \partial\Phi/\partial y\ \hat{y} + \partial\Phi/\partial z\ \hat{z}$

Integrating $\partial\Phi/\partial x = 6xy + z^3$ gives $\Phi = 3x^2y + xz^3 + g(y,z)$.

Integrating $\partial\Phi/\partial y = 3x^2 - z$ gives $\Phi = 3x^2y - zy + h(x,z)$.

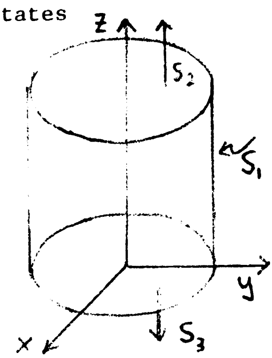
Integrating $\partial\Phi/\partial z = 3xz^2 - y$ gives $\Phi = xz^3 - zy + f(x,y)$.

By inspection of these three relations we can easily show that $\Phi(x,y,z) = 3x^2y + xz^3 - zy + C$.

1.13 $\nabla\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$. The surface integral $I = \oint_S \vec{r}\cdot\hat{n}da$ can be written in terms of a volume integral using the divergence theorem $I = \int_V \nabla\cdot\vec{r}\ dv = \int_V 3dv = 3V$ where V is the volume of the surface.

1.14 **a)** $\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 4 - 4y + 2z$.

b) From here on, it is easiest to switch to cylindrical coordinates where $\nabla\cdot\vec{A} = 4 - 4\rho\sin\phi + 2z$. In cylindrical coordinates, $x^2 + y^2 = 4$ is the same as $\rho = 2$. From the figure the unit vectors normal to S_1 is $\hat{\rho}$, normal to S_2 is $\hat{z}$, and normal to S_3 is $-\hat{z}$. The divergence theorem states



$$\int_V \nabla\cdot\vec{A}dv = \oint \vec{A}\cdot d\vec{a} = \int_V (4 - 4\rho\sin\phi + 2z)\rho d\rho d\phi dz$$

$$= \int_0^3 dz \int_0^{2\pi} d\phi \int_0^2 \rho d\rho(4 - 4\rho\sin\phi + 2z) = 84\pi$$

Now we calculate the surface integral

$\oint_S \vec{A}\cdot d\vec{a} = \int_{S_1} \vec{A}\cdot d\vec{a}_1 + \int_{S_2} \vec{A}\cdot d\vec{a}_2 + \int_{S_3} \vec{A}\cdot d\vec{a}_3$ using

$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$, and $\vec{A} = 4x\hat{x} - 2y^2\hat{y} + z^2\hat{z}$

$$\vec{A}\cdot d\vec{a}_1 = (4x\,\hat{x} - 2y^2\hat{y} + z^2\hat{z})\cdot(\hat{x}\cos\phi + \hat{y}\sin\phi)\rho d\phi dz$$

$$= (4\rho\cos^2\phi - 2\rho^2\sin^3\phi)\rho d\phi dz$$

This gives $\vec{A}\cdot d\vec{a}_1 = 8(\cos^2\phi - 2\sin^3\phi)d\phi dz$ at $\rho = 2$

$\vec{A}\cdot d\vec{a}_2 = (z^2\hat{z})\cdot(\hat{z})\rho d\phi d\rho$ which is equal to $9\rho d\rho d\phi$ at $z = 3$.

$\vec{A}\cdot d\vec{a}_3 = (z^2\hat{z})\cdot(-\hat{z})\rho d\phi d\rho$ which is zero at $z = 0$. Adding the three contributions we get

$$\oint_S \vec{A}\cdot d\vec{a} = 9\int_0^2 \rho d\rho \int_0^{2\pi} d\phi + 8\int_0^{2\pi} (\cos^2\phi - 2\sin^2\phi)d\phi \int_0^3 dz$$

which gives 84π, the same as $\int\nabla\cdot\vec{A}dv$.

1.15 a) We first find the unit normal. The cylindrical surface is described by $f(x,y) = x^2 + y^2 - 16 = 0$. We use $\hat{n} = \nabla f/|\nabla f|$. The gradient of f is $\nabla f = 2x\hat{x} + 2y\hat{y}$. The magnitude of the gradient is $|\nabla f| = \sqrt{x^2 + y^2}$. But everywhere on the surface we have $\sqrt{x^2 + y^2} = 4$, therefore $|\nabla f| = 8$. Hence $\hat{n} = (x\hat{x} + y\hat{y})/4$.

b) Substituting for $\vec{A}$ and $\hat{n}$ in $\hat{A}\cdot\hat{n}$ we get $\vec{A}\cdot\hat{n} = (xz + yx)/4$. Writing this in cylindrical coordinates we get

$(\rho z\cos\phi + \rho^2\sin\phi\cos\phi)/4$. Thus $\int\vec{A}\cdot\hat{n}da = \int\vec{A}\cdot\hat{n}\rho d\phi dz =$

$$\int_0^{\pi/4}\int_0^5 (z\cos\phi + 2\sin\phi\cos\phi)d\phi dz = 5(\frac{5}{\sqrt{2}} + 1)/2.$$

1.16 $\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$

$\nabla \times \vec{A} = \hat{x}(-2yz + 2yz) + \hat{y}(0 - 0) + \hat{z}(0 + 1) = \hat{z}$. Using

$\hat{n} = \hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, we have

$$(\nabla \times \vec{A})\cdot\hat{n}da = (\hat{z})\cdot(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)r^2\sin\theta d\theta d\phi$$
$$= \cos\theta\cdot d\phi d\theta\sin\theta$$

at the surface of the sphere ($r = 1$). Therefore

$$\int\nabla \times \vec{A}\cdot\hat{n}da = 2\pi\int_0^{\pi/2} \cos\theta\sin\theta d\theta = -2\pi \cos^2\theta/2 \Big|_0^{\pi/2} = \pi.$$ Now we

calculate $\oint_C \vec{A}\cdot d\vec{r}$. In this case $d\vec{r}$ is along a unit circle on the x,y plane and its direction is along $\hat{\phi}$. Therefore

$\vec{A}\cdot d\vec{r} = [(2x - y)\hat{x} - yz^2\hat{y} - y^2 z\hat{z}]\cdot[\hat{\phi}rd\phi]$. At $r = 1$ and using

spherical coordinates we have

$$\vec{A}\cdot d\vec{r} = [(2x-y)\hat{x} - yz^2\hat{y}]\cdot[-\hat{x}\sin\phi + \hat{y}\cos\phi]d\phi$$
$$= -(2\sin\theta\cos\phi - \sin\theta\sin\phi)\sin\phi - (\sin\theta\sin\phi)(\cos^2\theta)\cos\phi$$

Now, $\theta = \pi/2$ along C, therefore $\vec{A}\cdot d\vec{r} = -2 \cos\phi\sin\phi + \sin^2\phi$

$$\oint \vec{A}\cdot d\vec{r} = \int_0^{2\pi} (-2\cos\phi\sin\phi + \sin^2\phi)d\phi = \cos^2\phi \Big|_0^{\pi} + \pi = \pi$$

Thus $\int_S \vec{\nabla} \times \vec{A}\cdot\hat{n}da = \int_C \vec{A}\cdot d\vec{r}$.

1.17 **a)** $\int\hat{n}\ da = 0$. From problem 1.18 we have

$\int\nabla\Phi\ dv = \oint\Phi\ \hat{n}\ da$. Let Φ be a constant function such that

$\nabla\Phi = 0$, and we can take Φ outside the surface integral. Thus $0 = \Phi\int\hat{n}\, da$ or $\int\hat{n}\, da = 0$.

b) $\oint\vec{r} \times \hat{n}\, da = 0$. From problem 1.18 we have $\int\nabla \times \vec{B}\, dv = \oint\hat{n} \times \vec{B}\, da$. Taking $\vec{B} = \vec{r}$, the radius vector, then $\nabla \times \vec{r} = 0$ and we have $\oint\vec{r} \times \hat{n}\, da = 0$.

1.18 a) $\int\nabla \times \vec{B}\, dv = \int\hat{n} \times \vec{B}\, da$. Let $\vec{A} = \vec{B} \times \vec{C}$ in the divergence theorem where $\vec{C}$ is a constant vector. Then

$$\int\nabla\cdot(\vec{B} \times \vec{C}) = \oint(\vec{B} \times \vec{C})\cdot\hat{n}\, da$$

Since $\nabla\cdot(\vec{B} \times \vec{C}) = \vec{C}\cdot(\nabla \times \vec{B})$, and $(\vec{B} \times \vec{C})\cdot\hat{n} = \vec{C}\cdot(\hat{n} \times \vec{B})$ then $\vec{C}\cdot\int\nabla \times \vec{B}\, dv = \vec{C}\cdot\oint\hat{n} \times \vec{B}\, da$ or

$$\int\nabla \times \vec{B}\, dv = \oint\hat{n} \times \vec{B}\, da$$

b) $\int\nabla\Phi\, dv = \int\Phi\, \hat{n}\, da$. Let $\vec{A} = \Phi\vec{C}$ in the divergence theorem where $\vec{C}$ is a constant vector. Then

$$\int\nabla\cdot\Phi\vec{C}\, dv = \int\Phi\vec{C}\cdot\hat{n}\, da$$

Now we have $\nabla\cdot(\Phi\vec{C}) = \vec{C}\cdot\nabla\Phi$, then

$$\vec{C}\cdot\int\nabla\Phi\, dv = \vec{C}\cdot\int\Phi\, \hat{n}\, da \text{ or } \int\nabla\Phi\, dv = \int\Phi\, \hat{n}\, da$$

1.19 It is by far the easiest to solve these problems by using spherical coordinates, thereby taking advantage of the symmetries of these functions.

a) $\nabla^2 \ell nr = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\ell nr}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \times \frac{1}{r}\right) = \frac{1}{r^2}$

b) $\nabla^2 r^n = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dr^n}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(nr^2 \times r^{n-1}\right) = n(n + 1)r^{n-2}$

c) $\nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\left(\frac{1}{r}\right)}{dr}\right) = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{r^2}\right) = 0$

1.20 Given that $\Phi = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$

$$\vec{r}\cdot(\Phi\cdot\vec{r}) = (x\hat{x} + y\hat{y} + z\hat{z})\cdot[(\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z})\cdot(x\hat{x} + y\hat{y} + z\hat{z})]$$

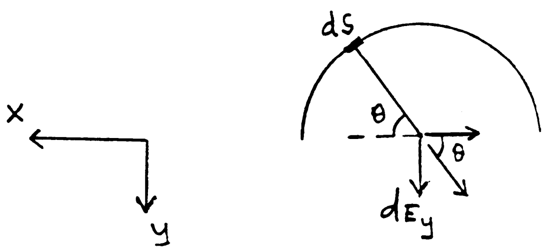
$= x^2 + y^2 + z^2 = (\vec{r}\cdot\Phi)\cdot\vec{r}$. There is no ambiguity in writing $\vec{r}\cdot\Phi\cdot\vec{r}$.

1.21 We use Eq. 1.76 with $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial x}$, and $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$; thus $\nabla\vec{r} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$.

CHAPTER 2

2.1 The total force is the vector sum of four individual forces of equal magnitudes but different directions. The magnitude of each force is $F = q_o q/(4\pi\varepsilon_o r^2)$ where $r^2 = 17$. Now only $F\cos\theta$ contributes, where θ is the angle each force makes with the normal to the plane of the charges. Thus $\vec{F}_{total} = 1.73\,\hat{z}$ N.

2.2 Consider a charge element $dq = \lambda ds$. We first notice that due to the symmetry of the problem, the x-component of the $\vec{E}$ field cancels, and $dE_y = (1/4\pi\varepsilon_o)\,dq'\sin\theta/R^2$. Substituting



$dq' = \lambda ds = \lambda R d\theta$, then $|\vec{E}| = (\lambda/4\pi\varepsilon_o R)\int_0^{\pi}\sin\theta d\theta =$
$-\lambda/(4\pi\varepsilon_o R)\cos\theta\,\Big|_0^{\pi} = \lambda/2\pi\varepsilon_o R$. But $\lambda = q/\pi R$, therefore $\vec{E} = q\hat{y}/2\pi^2\varepsilon_o R^2$.

2.3 $d\vec{E} = \dfrac{\lambda_o dz(2\hat{x} - z\hat{z})}{4\pi\varepsilon_o(4 + z^2)^{3/2}}$. Now the $\hat{z}$ integrals cancels, and

$$E = \int_{z=5}^{z=\infty} dE + \int_{z=-\infty}^{z=-5} dE = 2\int_{z=5}^{z=\infty} \frac{2\hat{x}\,\lambda_o dz}{4\pi\varepsilon_o(4 + z^2)^{3/2}} = 6.5 \times 10^8 \lambda_o \text{V/m}.$$

where λ_o is in C/m.

2.4 Using Coulomb's law we write $d\vec{E} = \frac{\lambda_o dz}{4\pi\varepsilon_o} \frac{R\hat{\rho} - z\hat{z}}{(R^2 + z^2)^{3/2}}$. Thus integrating from z = -d to z = d, makes the z component vanish because of symmetry, and thus

$$\vec{E} = \frac{\lambda_o R\hat{\rho}}{4\pi\varepsilon_o} \int_{-d}^{d} \frac{dz}{(R^2 + z^2)^{3/2}} = \frac{\lambda_o R\hat{\rho}}{4\pi\varepsilon_o R^2} \frac{z}{(R^2 + z^2)^{1/2}}\Bigg|_{-d}^{d}$$

$$\vec{E} = \frac{2\lambda_o d\ \hat{\rho}}{(4\pi\varepsilon_o R)(R^2 + d^2)^{1/2}}$$

2.5 From symmetry, $\vec{E}$ has a component only along the axis of the disk. Hence $E_z = \int dE\cos\theta = \frac{1}{4\pi\varepsilon_o} \int \frac{dq\ \cos\theta}{|\vec{r} - \vec{r}'|^2}$ where

$dq = \sigma da = \sigma\rho d\rho d\phi$. Hence $E_z = \frac{1}{4\pi\varepsilon_o} \int \frac{\sigma\rho d\rho d\phi}{(\rho^2 + z^2)} \cdot \frac{z}{\sqrt{\rho^2 + z^2}} =$

$$\frac{\sigma z}{4\pi\varepsilon_o} \cdot 2\pi \int_0^a \frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}} \text{ or } E_z = \frac{\sigma z}{2\varepsilon_o} \cdot \frac{-1}{(\rho^2 + z^2)^{1/2}}\Bigg|_0^a$$

$$E_z = \frac{\sigma}{2\varepsilon_o}\left[1 - \frac{z}{(a^2 + z^2)^{1/2}}\right]$$

2.6 By symmetry E has only a component in the z direction, the axis of the disk. Thus $dE_z = \frac{1}{4\pi\varepsilon_o} \frac{dq\ \cos\theta}{|\vec{r} - \vec{r}'|^2}$ where $dq = \sigma da$ and $da = 2\pi\rho d\rho$, thus

$$\vec{E}_z = \frac{1}{4\pi\varepsilon_o} \int_0^a \frac{\sigma\ \cos\theta(2\pi\rho)d\rho}{|\vec{r} - \vec{r}'|^2} = \frac{\sigma_o}{2\varepsilon_o} \int_0^a \frac{\cos\theta\ d\rho}{|\vec{r} - \vec{r}'|^2}.\quad \text{But}$$

$\rho = h\tan\theta$, $d\rho = h\sec^2\theta d\theta$, and $|\vec{r} - \vec{r}'| = h\sec\theta$ then

$$\vec{E}_z = \frac{\sigma_o}{2\epsilon_o h}\int_o^a \cos\theta d\theta = \frac{\sigma_o}{2\epsilon_o h}\sin\theta \Big|_o^{\rho=a} \text{ or } E_z = \frac{\sigma_o}{2\epsilon_o h}\left(\frac{\rho}{\sqrt{\rho^2 + h^2}}\right)\Big|_o^a =$$

$$\frac{\sigma_o}{2\epsilon_o h}\left(\frac{a}{\sqrt{a^2 + h^2}}\right).$$ Before discussing the field when $h >> a$,

note that the total charge on the disk is

$$q = \int \sigma da = \int_o^a \frac{\sigma_o}{\rho}(2\pi\rho)d\rho = 2\pi\sigma_o a$$

Now for $h >> a$, $(a^2 + h^2)^{-1/2} \approx 1/h$ thus

$|\vec{E}| \approx \sigma_o a/2\epsilon_o h^2 = q/4\pi\epsilon_o h^2$ which is the E field due to a point charge $q = 2\pi\sigma_o a$. Thus, far from the disk, its shape becomes unimportant and it looks like a point charge.

2.7 We use r' for the point of observation and r for the charge. $dq = \sigma\, dx\, dy$, $\vec{r}' = (0, 0, a)$, $\vec{r} = (x, y, 0)$

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\int_{x=0}^{2}\int_{y=0}^{2}\frac{(\vec{r} - \vec{r}')\sigma\, dx\, dy}{|\vec{r} - \vec{r}'|^3}.$$ Now $|\vec{r} - \vec{r}'| =$

$(x^2 + y^2 + 4)^{1/2}$. Thus

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\int_o^2 dx\int_o^2 dy\,\frac{(x\hat{x} + y\hat{y} - 2\hat{z})(2x(x^2 + y^2 + 4)^{3/2})}{(x^2 + y^2 + 4^2)^{3/2}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\int_o^2 dx\int_o^2 dy\; 2x(x\hat{x} + y\hat{y} - 2\hat{z}) = \frac{1}{4\pi\epsilon_o}\left(\frac{32}{3}\hat{x} + 8\hat{y} + 16\hat{z}\right).$$

2.8 By inspection we find that only the component along the axis of the disk survive. That is $E_z = \int dE \cos\theta =$

$$\frac{1}{4\pi\epsilon_o}\int\frac{dq}{(r^2 + h^2)}\cos\theta,$$ where $dq = \sigma da = \sigma_o \sin^2\phi\, \rho d\rho\, d\phi$

$$E_z = \frac{\sigma_o}{4\pi\varepsilon_o}\int_o^{2\pi} d\phi \int_o^a \frac{d\rho \sin^2\phi\rho d\phi}{(\rho^2 + h^2)} \cdot \frac{h}{\sqrt{\rho^2 + h^2}}$$

$$= \frac{\sigma_o h}{4\pi\varepsilon_o} \cdot \int_o^{2\pi} \sin^2\phi d\phi \int_o^a \frac{\rho d\rho}{(\rho^2 + h^2)^{3/2}} = \frac{\sigma_o h}{4\varepsilon_o}\left[\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}}\right]$$

2.9 a) Consider the $\sigma = \sigma'$ case. From Gauss' law we have $|E| = \sigma/\varepsilon_o$. Between the planes the fields substract, and outside they add. Thus

$\vec{E} = \sigma\hat{x}/\varepsilon_o \qquad x < -1$

$\vec{E} = 0 \qquad -1 < x < 1$

$\vec{E} = -\sigma\hat{x}/\varepsilon_o \qquad x > 1$

b) For the $\sigma' = -\sigma$ case, the fields add between the planes and subtract outside the planes

$\vec{E} = 0 \qquad x < -1$ and $x > 1$

$\vec{E} = \sigma\hat{x}/\varepsilon_o \qquad -1 < x < 1$

2.10 $\vec{E}_{sheet} = \frac{|\sigma|}{2\varepsilon_o}\begin{cases} -\hat{y} & \text{for } y > 2 \\ +\hat{y} & \text{for } y < 2 \end{cases}$

$\vec{E}_{line} = \frac{\lambda[(y + 1)\hat{y} + (z - 2)\hat{z}]}{2\pi\varepsilon_o((y + 1)^2 + (z - 2)^2}$. Thus check

$\vec{E}_{line} + \vec{E}_{sheet} = 0$. The $\hat{z}$ component can only be zero for $z = 2$. Hence for $z = 2$, $\vec{E}_{line} + \vec{E}_{sheet} =$

$\frac{\lambda(y + 1)\hat{y}}{2\pi\varepsilon_o(y + 1)^2} + \frac{\sigma}{2\varepsilon_o}\frac{\hat{y}(y - 2)}{|y - 2|} = 0$ requires $y + 1 = \frac{\pm\lambda}{\sigma\pi}$ or

$\left|\frac{\lambda}{\sigma\pi}\right| = \frac{4}{\pi}$. Thus $y + 1 = \pm\frac{4}{\pi}$ are solutions.

2.11 Let the Gaussian surface be a cylinder of length L, and radius r. $\vec{E}$ will be radial by symmetry. Then

$$\oint_s \vec{E}\cdot d\vec{a} = E\cdot(2\pi r \ell) = \frac{1}{\varepsilon_o} 2\pi \int_o^r 5re^{-2r}\cdot rdr \int_o^\ell dz$$

$$E(2\pi r\ell) = \frac{2\pi\ell}{\varepsilon_o} \int_o^r 5re^{-2r} rdr$$

$$= \frac{10\pi\ell}{\varepsilon_o} \left[\frac{r^2e^{-2r}}{-2} + \frac{e^{-2r}}{9}(-2r - 1)\right]_o^a .$$

$$E = \frac{5}{\varepsilon_o r}\left[\frac{1}{4} - \frac{e^{-2r}}{2}(r^2 + r + \frac{1}{2})\right]$$

2.12 a) For $r < R$ we have $\int \vec{E}\cdot d\vec{a} = \frac{\alpha}{\sigma_o} \int_o^r \frac{(4\pi r'^2)dr'}{r'^2}$ or $\int \vec{E}\cdot d\vec{a} = \frac{4\pi\alpha}{\varepsilon_o} \int_o^r dr' = \frac{4\pi\alpha}{\varepsilon_o} r = 4\pi r^2 E$, $\vec{E} = \frac{\alpha}{\varepsilon_o r}\hat{r}$.

b) For $r > R$, $\int \vec{E}\cdot d\vec{a} = \frac{\alpha}{\varepsilon_o} \int_o^R \frac{(4\pi r'^2)dr'}{r'^2} = 4\pi\alpha R/\varepsilon_o = 4\pi r^2 E$. Thus $E = \alpha R/\varepsilon_o r^2$.

2.13 Flux penetrating is equal to Q/ε_o where Q is inside S.

a) $(q_1 + q_2 + q_3)/\varepsilon_o = 1.1 \times 10^{-7}$ Coul/ε_o

b) $\frac{1}{\varepsilon_o}\int \sigma da = \frac{1}{\varepsilon_o}\int_o^{2\pi}\int_o^2 \frac{\sin^2\phi}{\rho}\rho d\rho d\phi = 2\pi$ Coul/ε_o

c) $\frac{1}{\varepsilon_o}\int \sigma da = \frac{1}{\varepsilon_o}\int_o^{2\pi} d\phi \sin\phi \int_o^2 \rho d\rho = 0$

d) $(20 - 20) \times 10^{-7}$ C/$\varepsilon_o = 0$

2.14 Since total flux is $\int \vec{E}\cdot d\vec{s} = \dfrac{Q_{enc}}{\varepsilon_o}$ then flux density is $\vec{D} = \varepsilon_o \vec{E} = q/4\pi r^2 = 9.55 \times 10^{-11} C/m^2$.

2.15 To find the charge densities, we use the differential form of Gauss' law: $\nabla\cdot\vec{E} = \rho/\varepsilon_o$

a) $\vec{E} = 10 \sin\theta\hat{r} + 2\cos\theta\hat{\theta}$

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(10r^2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(2\cos\theta\sin\theta) \text{ or}$$

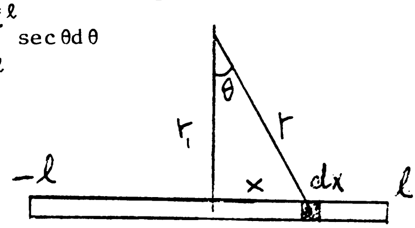
$$\rho = \frac{\varepsilon_o}{r\sin\theta}(18\sin^2\theta + 2\cos^2\theta).$$

b) $\nabla\cdot\vec{E} = \frac{1}{2}\alpha\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2 - a^2) = \alpha$. Thus charge density is $\rho = \alpha\varepsilon_o$ for $a < \rho < b$. Now $\nabla\cdot\vec{E} = \frac{1}{2}\alpha\frac{1}{\rho}\frac{\partial}{\partial\rho}(b^2 - a^2) = 0$ for $\rho > b$. Thus charge density $\rho = 0$ for $\rho > b$.

2.16 $\Phi = \dfrac{1}{4\pi\varepsilon_o}\displaystyle\int \dfrac{da}{|\vec{r} - \vec{r}\,'|}$. But $dq = \sigma da = \sigma r dr\theta$ so

$$\Phi = \frac{1}{4\pi\varepsilon_o}\int_{\rho_o/2}^{(\rho_o + W)/2}\int_0^{2\pi}\frac{r\sigma dr d\theta}{r} = \frac{\sigma W}{4\varepsilon_o}.$$ It doesn't depend on ρ_o.

2.17 a) $\Phi = \dfrac{\lambda}{4\pi\varepsilon_o}\displaystyle\int_{-\ell}^{\ell}\dfrac{dx}{r}$. Note that $x = r_1\tan\theta$, $dx = r_1\sec^2\theta d\theta$ and $r = r_1\sec\theta$, thus $\Phi = \dfrac{\lambda}{4\pi\varepsilon_o}\displaystyle\int_{-\ell}^{x=\ell}\sec\theta d\theta$



$$= \frac{\lambda}{2\pi\varepsilon_o}\ln|\sec\theta + \tan\theta|\Big|_{x=0}^{x=\ell} = \frac{\lambda}{2\pi\varepsilon_o}\ln\left|\frac{\sqrt{x^2 + r_1^2} + x}{r_1}\right|\Bigg|_{x=0}^{x=\ell}$$

$$= \frac{\lambda}{2\pi\varepsilon_o} \ln \left(\frac{\sqrt{\ell^2 + r_1^2} + \ell}{r_1}\right)$$

b) In the $\ell >> r_1$ limit we have $\Phi = -\frac{\lambda}{2\pi\varepsilon_o} \ln r_1$ + constant

c) For r_1, $r_2 << \ell$, we have $V_{12} = \Phi(r_1) - \Phi(r_2) =$ $\frac{\lambda}{2\pi\varepsilon_o} \ln \left|\frac{r_2}{r_1}\right|$. This is, as we would expect, the same as the potential difference of two points in the presence of an infinite line of charge.

2.18 a) We use $\vec{E} = -\nabla\Phi$ where $\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$ in spherical coordinates. Thus $\vec{E} = \hat{r}(2ar^{-3}\cos\theta + br^{-2}) + \hat{\theta}(ar^{-3}\sin\theta)$.

b) We use $\frac{\rho}{\varepsilon_o} = \nabla\cdot\vec{E}$. Consider first $\vec{r} \neq 0$.

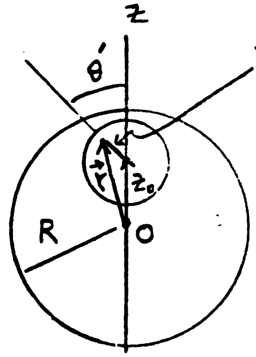
$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\theta)$$

$$\frac{\rho}{\varepsilon_o} = \frac{1}{r^2}\frac{\partial}{\partial r}(2ar^{-1}\cos\theta + b) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(ar^{-3}\sin^2\theta)$$

$$= -2ar^{-4}\cos\theta + 2ar^{-4}\cos\theta = 0$$

Thus there is no charge at $\vec{r} \neq 0$. We now inspect the equation as $\vec{r} \to 0$. Also we note that $\nabla\cdot\vec{r}/r^3 = -\nabla^2(1/r) = 4\pi\delta(\vec{r})$ where from example 1.4, the Dirac δ-function $\delta(\vec{r}) = 0$ for $r \neq 0$ and $\int\delta(\vec{r})d^3\vec{r} = 1$ if volume includes $\vec{r} = 0$. This would have been obvious if we noticed $\Phi = b/r$ is a point charge potential. We should also recognize $(a/r^2)\cos\theta$ as a dipole potential of dipole moment $4\pi\varepsilon_o a$, in the $\hat{z}$ direction and placed at $r = 0$. See example 2.9.

c) We use the following in one step: $\frac{\rho}{\varepsilon} = \vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\Phi) = -\nabla^2\Phi$. (See Ex. 3.20).



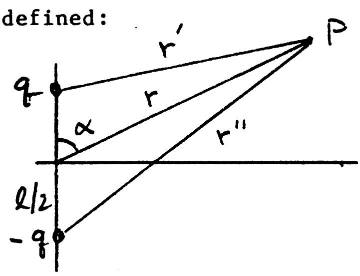
2.19 **a)** Using Gauss' law $\oint \vec{E}\cdot d\vec{a} = Q_{int}$ gives $\vec{E}_1 = \frac{\rho_o}{3\varepsilon_o}\vec{r}$.

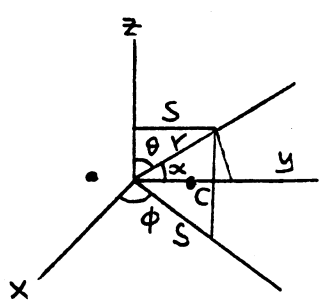
b) Using Gauss' law gives $\vec{E}_2 = \frac{\rho_o'}{3\varepsilon_o}\vec{r}'$

c) Take $\rho_o' = -\rho_o$ gives the cavity, thus $\vec{E}_2 = \frac{-\rho_o}{3\varepsilon_o}\vec{r}'$. Adding gives $\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_o}{3\varepsilon_o} z_o\hat{z}$.

d) $\Phi = -\int_0^{r'}\vec{E}\cdot d\vec{r} = \frac{-\rho_o}{3\varepsilon_o} z_o \int_0^{z'} dz = \frac{-\rho_o}{3\varepsilon_o} z_o z' = \frac{-\rho_o}{3\varepsilon_o} z_o r' \cos\theta'$. One can also find the potential by finding the potential due to the sphere and the cavity. (Same procedure used for $\vec{E}$.)

2.20 We notice that this problem has symmetry about the y axis, so it is natural to set up the problem as in example 2.9, with the axis redefined:





The potential at point P is clearly $\Phi_P = \frac{1}{4\pi\varepsilon_o}\{\frac{q}{r'} - \frac{q}{r''}\}$. Now, we note that

$$r' = \sqrt{r^2 + (\ell/2)^2 - r\ell\cos\alpha},\quad r'' = \sqrt{r^2 + (\ell/2)^2 + r\ell\cos\alpha}$$

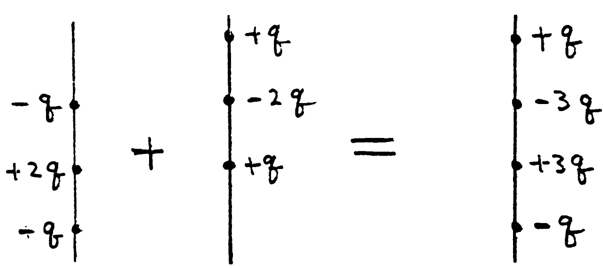
We are to assume $\ell << r$, then $\binom{r'}{r''} \approx r \mp \frac{\ell}{2}\cos\alpha$. Thus

$$\Phi_P \approx \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{r - \frac{\ell}{2}\cos\alpha} - \frac{q}{r + \frac{\ell}{2}\cos\alpha}\right] = \frac{1}{4\pi\varepsilon_o}\left[\frac{qr + \frac{q\ell}{2}\cos\alpha - qr + \frac{q\ell}{2}}{r^2 - \frac{\ell^2}{4}\cos^2\alpha}\right]$$

$$\Phi_P \approx \frac{1}{4\pi\varepsilon_o}\frac{q\ell\cos\alpha}{r^2}$$

as seen in example 2.9. We now need only to transpose this in terms of spherical coordinates, i.e. what is $\cos\alpha$ in terms of θ,ϕ? Note $\cos\alpha = \frac{c}{r} = \frac{c}{s} \times \frac{s}{r}$ but $\frac{s}{r} = \sin\theta$, $\frac{c}{s} = \cos(\frac{\pi}{2} - \phi) = \sin\phi$. Therefore $\cos\alpha = \sin\theta\sin\phi$, and $\Phi_P \approx \frac{1}{4\pi\varepsilon_o}\frac{q\ell\sin\theta\sin\phi}{r^2}$ in spherical coordinates.

2.21 We can superimpose the potentials of the four charges and expand assuming that $r >> a$ just as we did in Ex. 2.9. In here, however, we will use Eq. 2.69 just as we used it in Ex. 2.16.



Thus we write (see Fig.) $\Phi^{(3)} = -\vec{\delta}^{(3)}\cdot\nabla\Phi^{(2)} = -a\hat{z}\cdot\nabla\Phi^{(2)}$, where $\Phi^{(2)} = \frac{qa^2}{4\pi\varepsilon_o r^3}[3\cos^2\theta - 1]$ (Ex. 2.16). Now

$$\nabla\Phi^{(2)} = \hat{r}\ [\frac{-3qa^2}{4\pi\varepsilon_o r^4}\ [3\ \cos^2\theta - 1]\ ] + \hat{\theta}\ [\frac{qa^2}{4\pi\varepsilon_o r^4}\ [-6\ \cos\theta\sin\theta]\ ]\ .$$

Using $\hat{z}\cdot\hat{r} = \cos\theta$, $\hat{z}\cdot\hat{\theta} = -\sin\theta$ (Eq. 1.3) we get

$$\Phi^{(3)} = \frac{3qa^2}{4\pi\varepsilon_o r^4}\ [3\ \cos\theta(3\ \cos^2\theta - 1) + 6\ \cos\theta(\cos^2\theta - 1)]$$

$$= \frac{6qa^3}{4\pi\varepsilon_o r^4}\ P_3(\cos\theta)\ .$$

2.22 We use the multipole expansion of Eq. 2.61 since we are interested in the potential at large distances. We note that $\Phi^{(2)} = 0$ and $\Phi^{(1)} = 0$. Thus because charge is in x-y plane $\Phi^{(2)} = \frac{1}{4\pi\varepsilon_o}\frac{1}{2r^3}\int[3(\hat{r}\cdot\vec{\rho}\,')^2 - \rho'^2]dq$. Let $\vec{r}$ be at $(r,\ \theta,\ 0)$; then $\hat{r}\cdot\vec{\rho}\,' = \rho'\sin\theta\cos\phi'$ and hence $3(\hat{r}\cdot\vec{\rho}\,')^2 - \rho'^2 =$ $[3\ \sin^2\theta\cos^2\phi' - 1]\rho'^2$. Now $dq = \frac{q}{2\pi\rho'}\ [\delta(\rho' - a) - \delta(\rho' - b)]\rho' d\rho' d\phi'$. Thus the integral becomes:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}[3\sin^2\theta\cos^2\phi' - 1]\frac{(\rho'^3\sin\theta')q}{2\pi\rho'}[\delta(\rho' - a) - \delta(\rho' - b)]d\rho' d\phi'$$

$$= \int_0^{2\pi}\ [3\ \sin^2\theta\cos^2\phi' - 1]\frac{a^3 q}{2\pi}\ d\phi' - \int_0^{2\pi}\ [3\ \sin^2\theta\cos^2\theta' - 1]\frac{b^2 q}{2\pi}\ d\phi'$$

$$= (a^2 - b^2)(\frac{3\ \sin^2\theta - 1}{2})\ \text{so}\ \Phi^{(2)} = \frac{q}{4\pi\varepsilon_o}\ \frac{(a^2 - b^2)}{2r^3}\ (\frac{3}{2}\ \sin^2\theta - 1)$$

$$\Phi^{(2)} = \frac{q}{4\pi\varepsilon_o}\ \frac{1}{2r^3}\ \frac{(b^2 - a^2)}{2}\ (3\ \cos^2\theta - 1)$$

This is the potential of a linear quadrupole where two charges of $-q$ each are at a distance of $\frac{1}{2}(a^2 - b^2)^{1/2}$ from the central charge of $2q$.

2.23 The dipole moment is defined as (taking the z axis to be the axis of symmetry) $\vec{p} = \int \vec{r}\,'dq$. Taking $dq = \sigma da = \sigma R^2 \sin\theta d\theta d\phi$ where R is the radius, and using $\vec{r}\,' = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$, give $\vec{p} = \sigma R^2 \int(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\sin\theta d\theta d\phi$. Only the $\hat{z}$ component survives, and we have

$$\vec{p} = -2\pi\sigma R^2\hat{z}\int_0^{\pi/2}\cos\theta\, d\cos\theta = \pi\sigma R^2\hat{z}.$$

2.24 We first calculate $\vec{E}$ and $\nabla\vec{E}$.

$\vec{E} = -\vec{\nabla}\Phi = -\hat{x}\frac{\partial\Phi}{\partial x} - \hat{y}\frac{\partial\Phi}{\partial y} - \hat{z}\frac{\partial\Phi}{\partial z} = -(\alpha_1 x + \alpha_2)\hat{x}$, $\nabla\vec{E} = -\alpha_1\hat{x}\hat{x}$, is a diadic.

a) $U = -\vec{p}\cdot\vec{E}$ is the potential energy of a dipole in an external field: $U = p_o\hat{x}\cdot(\alpha_1 x + \alpha_2)\hat{x} = p_o(\alpha_1 x + \alpha_2)$. At $x = 0$, $U = p_o\alpha_1$.

b) $\vec{f} = \vec{p}\cdot\nabla\vec{E} = (p_o\hat{x})\cdot(-\alpha_1\hat{x}\hat{x}) = -\alpha_1 p_o\hat{x} = 0$.

c) $\vec{\tau} = \vec{p}\times\vec{E} + \vec{r}\times\vec{f}$ where $\vec{r}\times\vec{f}$ is the usual torque due to the distance from the origin (or the point the torque is being measured around). Here, $\vec{r} = 0$ (at the origin) so $\vec{\tau} = \vec{p}\times\vec{E} = (p_o\hat{x})\times(-(\alpha_1 x + \alpha_2)\hat{x}) = 0$.

2.25 a) One can solve this by calclating $\vec{f} = -q\vec{E}_{dipole}$ or by $\vec{f} \cong (\vec{p}\cdot\nabla)\vec{E}_{charge}$. We will use the second for illustration of the method: $\vec{E}_{charge} = -\frac{q}{4\pi\varepsilon_o}\frac{(x\hat{x} + y\hat{y} + z\hat{z})}{(x^2 + y^2 + z^2)^{3/2}}$. Now take

$\vec{p} = p\hat{z}$ then $\vec{p}\cdot\vec{\nabla} = -\ p\dfrac{\partial}{\partial z}$ and hence $\vec{f} = -p\dfrac{\partial}{\partial z}\vec{E}_{charge}$

$$= -\frac{q}{4\pi\varepsilon_o}\left[\frac{3(x\hat{x} + y\hat{y} + z\hat{z})z}{(x^2 + y^2 + z^2)^{5/2}} - \frac{\hat{z}}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_o}\left[\frac{3\ pr\ \cos\theta\ \vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right]$$

$$= -\frac{q}{4\pi\varepsilon_o}\left[\frac{3\ \vec{p}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{p}}{r^3}\right] = -q\ \vec{E}_{dipole}$$

b) The couple is given by $\vec{\tau} = \vec{p} \times \vec{E} + \vec{r} \times \vec{f}$ according to Eq. 2.67.

$$\vec{\tau} = \vec{p} \times \left(\frac{-q}{4\pi\varepsilon_o}\frac{\vec{r}}{r^3}\right) + \vec{r} \times \vec{f} = -\frac{-2qp}{4\pi\varepsilon_o r^2}\hat{z} \times \hat{r} = \frac{-q\ p\ \sin\theta\ \hat{\phi}}{4\pi\varepsilon_o r^2}$$

2.26 Dipole $\vec{p}_1$ at the origin, dipole $\vec{p}_2$ at $\vec{r}$. This must be done carefully, expanding all products. If $\nabla\cdot\vec{E} = 0$ which is the case at $r\neq 0$, then $\vec{F} = (\vec{p}\cdot\nabla)\vec{E} = \nabla(\vec{p}\cdot\vec{E}) = \vec{p}\cdot(\nabla\vec{E})$.

a) The electric field of the dipole $\vec{p}_1$ at a point of observation $\vec{r}$ is given by Eq. 2.47:

$$-4\pi\varepsilon_o\vec{E}_1 = -3r^{-4}(\vec{p}_1\cdot\vec{r})\hat{r} + r^{-3}\vec{p}_1,$$ thus $\vec{p}_2\cdot\vec{E}_1$ is

$$4\pi\varepsilon_o\vec{p}_2\cdot\vec{E}_1 = 3r^{-3}(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r}) - r^{-3}(\vec{p}_1\cdot\vec{p}_2)$$

which is symmetric in $\vec{p}_1$, $\vec{p}_2$. Now

$$4\pi\varepsilon_o\nabla(\vec{p}_2\cdot\vec{E}_1) = 3\nabla(r^{-3})(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r}) + 3r^{-3}\nabla(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})$$
$$+ 3r^{-3}(\vec{p}_1\cdot\hat{r})\nabla(\vec{p}_2\cdot\hat{r}) - \nabla(r^{-3})(\vec{p}_1\cdot\vec{p}_2)$$

$$= 3\nabla(r^{-5})(\vec{p}_1\cdot\vec{r})(\vec{p}_2\cdot\vec{r}) + 3r^{-5}\nabla(\vec{p}_1\cdot\vec{r})(\vec{p}_2\cdot\vec{r})$$
$$+ 3r^{-5}(\vec{p}_1\cdot\vec{r})\nabla(\vec{p}_2\cdot\vec{r}) - \nabla(r^{-3})(\vec{p}_1\cdot\vec{p}_2)$$

$$\vec{F} = [-15(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\hat{r} + 3\vec{p}_1(\vec{p}_2\cdot\hat{r})$$
$$+ 3\vec{p}_2(\vec{p}_1\cdot\hat{r}) + 3(\vec{p}_1\cdot\vec{p}_2)\hat{r}]/4\pi\varepsilon_o r^4 .$$

b) In the form above, all terms are in the form $(\vec{p}_1\cdot\hat{r})$, $(\vec{p}_2\cdot\hat{r})$ or $(\vec{p}_1\cdot\vec{p}_2)$. All terms are proportional to p_1p_2. If $\hat{p}_1 = \hat{p}_2 = \hat{r}$ then $\vec{F} = \dfrac{-6p_1p_2\hat{r}}{4\pi\varepsilon_o r^4}$. The interdependence shows this is the maximum.

2.27 a) We first determine the $\vec{E}$ field of the two charges at the position of the dipole:

$$4\pi\varepsilon_o\vec{E} = \sum_i \frac{q_i\vec{r}_i}{r_i^3} = -q\left[\frac{(x-d)\hat{x} + z\hat{z}}{((x-d)^2 + z^2)^{3/2}} - \frac{(x+d)\hat{x} + z\hat{z}}{((x+d)^2 + z^2)^{3/2}}\right]$$

$$4\pi\varepsilon_o E_x = \frac{-q(x-d)}{((x-d)^2 + z^2)^{3/2}} - \frac{+q(x+d)}{((x+d)^2 + z^2)^{3/2}}$$

$$4\pi\varepsilon_o \frac{\partial E_x}{\partial z} = -\frac{6qdz}{(d^2 + z^2)^{5/2}} \text{ at } x = 0$$

$4\pi\varepsilon_o\ \partial E_x/\partial x = 0$ (by symmetry) at $x = 0$.

$$4\pi\varepsilon_o\ E_z = qz[((x-d)^2 + z^2)^{-3/2} - ((x+d)^2 + z^2)^{-3/2}]$$

$4\pi\varepsilon_o \partial E_z/\partial x = -6qdz(d^2 + z^2)^{-5/2}$ at $x = 0$. Note that $\partial E_z/\partial x =$ $\partial E_z/\partial z$. Thus

$\nabla\vec{E}(z = \ell) = 6qd\ell(d^2 + \ell^2)^{-5/2}(\hat{x}\hat{z} + \hat{z}\hat{x})/4\pi\varepsilon_o$ and

$$\vec{F} = \vec{p}\cdot\nabla\vec{E} = 6qd\ell(d^2 + \ell^2)^{-5/2}(p_x\hat{z} + p_z\hat{x})/4\pi\varepsilon_o$$

We use the results of problem 2.26. Taking $\hat{r} = \hat{z}$, $\hat{p}_1 = \hat{x}(\hat{p}_1\cdot\hat{r}) = 0$, then $\vec{F} = \dfrac{p_1}{4\pi\varepsilon_o\ell^4}[3\hat{x}(\hat{p}_2\cdot\hat{z})+3\hat{z}(\hat{p}_2\cdot\hat{x})]$. Equating the forces, we get: $\vec{p}_1 = 2qd(1 + d^2/\ell^2)^{-5/2}\hat{x}$.

2.28 We use Eq. 2.62.

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\int \frac{dq}{r'} + \vec{r}\cdot\int\frac{r'}{r'^3}\,dq + \frac{1}{2}\int\left[\frac{3(\vec{r}\cdot\vec{r}')^2}{r'^5} - \frac{r^2}{r'^3}\right]dq\right]$$

Since $r' = R =$ constant, then the first term gives $\Phi(\vec{r}) =$ $R\sigma/\varepsilon_o$. Because of symmetry one can show the other two integrals vanish. Thus $\Phi(\vec{r}) = R\sigma/\varepsilon_o$.

2.29 a) The monopole moment is equal to the total charge $= q$. The dipole moment is $\vec{p} = \int\vec{r}\cdot dq =$

$$\left(\int_{-z_o}^{z_o} \lambda z dz\right)\hat{z} = \frac{1}{2}\lambda z^2\Big|_{-z_o}^{z_o}\hat{z} = 0$$. The quadrupole moment =

$$Q_{zz} = \int\lambda(z'^2 - \frac{1}{3} r'^2)dz' = \frac{2}{3}\lambda \int_{-z_o}^{z_o} z'^2 dz' = \frac{4}{9}\lambda z_o^3 = \frac{2qz_o^2}{9}$$

b) The monopole moment is the total charge = q + q' = $2\pi R\lambda - 2\pi R\lambda = 0$. We note, firstly, that the point charge is added at $\vec{r}' = 0$ and thus makes <u>no</u> contribution to any moments higher than the monopole. Thus, the dipole and quadrupole moments are those calculated in Ex. 2.15

$$\vec{p} = \int\vec{r}dq = \int_0^{2\pi} R^2\lambda\hat{\rho}d\phi = 0$$

$$Q'_{xx} = \frac{1}{3}\lambda\int_0^{2\pi}(3x'^2 - r'^2)Rd\theta = \frac{1}{3}\lambda R\int_0^{2\pi}(3R^2\cos\phi - R^2)d\phi =$$

$$\frac{1}{3}\lambda R^3\int_0^{2\pi}[3(1 + \cos2\phi)/2 - 1]d\phi = qR^2/6$$

$Q'_{yy} = qR^2/6$, and $Q'_{zz} = -qR^2/3$.

2.30 We use Eq. 2.69, $\Phi^{(2)} = -\vec{\delta}^{(2)}\cdot\nabla\Phi^{(1)}$ and $\Phi^{(1)} = -q\, a\cos\theta/4\pi\varepsilon_o r^2$, and $\vec{\delta}^{(2)} = -a\hat{z}$. Thus

$$\Phi^{(2)} (a\cos\theta\hat{r} - a\sin\theta\hat{\theta})\cdot\left(\frac{2q\, a\cos\theta}{4\pi\varepsilon_o r^3}\hat{r} + \frac{q\, a\sin\theta}{4\pi\varepsilon_o r^3}\hat{\theta}\right)$$

$$\Phi^{(2)} = \frac{qa^2}{4\pi\varepsilon_o r^3}(3\cos^2\theta - 1).$$

CHAPTER 3

3.1 From the results of Ex. 3.1, and using $\theta_2 = \pi/2$, we have $\Phi = V\ \ell n(\tan\theta/2)/\ell n(\tan\theta_1/2)$. The charge density $\sigma = \varepsilon_o E_n$. Now $\vec{E} = -\hat{\theta}(d\Phi/d\theta)/r = -V\hat{\theta}/[r\sin\theta\,\ell n(\tan\theta_1/2)]$. Thus $\sigma = -\varepsilon_o V/r\,\ell n(\tan\theta_1/2)$ at $\theta = \pi/2$.

3.2 a) $\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$ which gives $\Phi = a\phi + b$ where the boundary conditions are: $\Phi(\phi = 0) = 0$ and $\Phi(\phi = \beta) = V$ thus $\Phi = \frac{V}{\beta}\phi$

b) From Gauss' law $\hat{n}\cdot\vec{E} = \sigma/\varepsilon_o$, but $\vec{E} = \frac{-1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi}$ or $\vec{E} = \frac{-V}{\beta\rho}\hat{\phi}$, thus $\sigma = \pm V/\beta\rho$. The total charge is $\int\sigma da$. Taking $da = h d\rho$ we get $Q = \frac{\pm V\varepsilon_o}{\beta}\int_{\rho_1}^{\rho_2}\frac{d\rho}{\rho}(h) = \frac{\pm V\varepsilon_o h}{\beta}\ell n\ \rho_2/\rho_1$.

3.3 From Ex. 3.4 we have

$$\Phi = V\ell n\left(\frac{\tan(\theta/2)}{\tan(\theta_2/2)}\right)/\ell n\left(\frac{\tan(\theta_1/2)}{\tan(\theta_2/2)}\right).$$ Now $-\vec{E} = +\vec{\nabla}\Phi = \frac{\hat{\theta}}{r}\frac{\partial\Phi}{\partial\theta}$ or

$$\vec{E} = \frac{-a\hat{\theta}}{r\sin\theta} = \frac{-V\hat{\theta}}{r\sin\theta\ \ell n\left(\frac{\tan(\theta_1/2)}{\tan(\theta_2/2)}\right)}$$

b) Inside the metal of the plates, $\vec{E} = 0$. At the surface, there is a surface charge density, σ, such that

$$\sigma = \frac{\pm\varepsilon_o V}{r\sin\theta\ \ell n\left(\frac{\tan(\theta_1/2)}{\tan(\theta_2/2)}\right)}$$

where θ stands for θ_1 or θ_2, and + is used for θ_1 and - is used for θ_2.

3.4 **a)** $\Phi_1(r,\theta) = \sum_{n=0} \{A_n r^n + B_n r^{-(n+1)}\} P_n(\cos\theta)$

$\Phi_2(r,\theta) = \sum_{n=0} \{A'_n r^n + B'_n r^{-(n+1)}\} P_n(\cos\theta)$

b) The following are boundary conditions that must be satisfied: 1) Φ is finite at $r = 0$ which gives $B_n = 0$ for $n > 0$

2) $\Phi \xrightarrow[r\to\infty]{} 0$ which requires $A'_n = 0$ for $n \geq 0$

3) $\Phi_1 = \Phi_2 = V_o \cos\theta$ at $r = R$ requires that $A_n = B_n = 0$ for $n \neq 1$, $A_1 R = V_o$, and $B_1 R^{-2} = V_o$. Therefore $\Phi_1(r,\theta) = \frac{V_o}{R} r \cos\theta$ $r < R$ and $\Phi_2(r,\theta) = V_o \left(\frac{R}{r}\right)^2 \cos\theta$ $r > R$.

c) $\vec{E}_1 = -\nabla\Phi_1 = \frac{-\partial\Phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}$ or , $\vec{E}_1 = \frac{-V_o}{R}\hat{z}$. Now

$\vec{E}_2 = -\nabla\Phi_2 = (V_o/R)\,(2\cos\theta\hat{r} + \sin\theta\,\hat{\theta})$. (Note that the tangential component is continuous). It cannot be conducting because there is a tangential component.

d) $E_{2n} - E_{1n} = \sigma/\varepsilon_o$. Now $E_{2n} = \frac{2V_o}{R}\cos\theta$ and $E_{1n} = \frac{-V_o}{R}\cos\theta$.

Thus $\sigma(\theta) = (3V_o\varepsilon_o/R)$

e) We can calculate it from the following expression: $\vec{p} = \int \sigma(\theta)\,\vec{r}\,da$ where $da = R^2 \sin\theta d\theta d\phi$ and $\vec{r} = R\hat{r}$. Alternatively we can read it off from the expression for the potential outside the sphere, since this potential is a dipole potential. Comparing with Eq. 2.44 we get $\vec{p} = 4\pi R^2 V_o\varepsilon_o\,\hat{z}$.

3.5 To get $\vec{E}$, just get $-\nabla\Phi$ where Φ is given in Eq. 3.35 -

$$\vec{E} = \hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\Phi\theta}$$

$$\vec{E} = -\hat{r}\left[\frac{-q}{4\pi\varepsilon_o r^2} + \frac{\delta q}{4\pi\varepsilon_o(R_2^3 - R_1^3)}\left(1 + \frac{2R_1^3}{r^3}\right)\cos\theta\right]$$

$$-\hat{\theta}\left[\frac{-\delta q}{4\pi\varepsilon_o(R_2^3 - R_1^3)}\left(1 - \frac{R_1^3}{r^3}\right)\sin\theta\right].$$

The charge density on the smaller sphere is $\sigma = \varepsilon_o E_r(R_1,\theta) = \frac{q}{4\pi}\left[\frac{1}{R_1^2} - \frac{3\delta\cos\theta}{(R_2^3 - R_1^3)}\right]$. The charge density on the outer sphere is calculated in a similar way: $\sigma = \frac{q}{4\pi}\left[\frac{-1}{R_2^2} - \frac{\delta(1 + R_1^3/R_2^3)}{(R_2^3 - R_1^3)}\cos\theta\right]$. For the total charge on the outer sphere we have $Q = \int\sigma(R_2,\theta)\,da = \int_{-1}^{1}\sigma(R_2,\theta)\,2\pi R_2^2 d(\cos\theta) = -q$.

3.6 We start by writing expansions for the potentials according to Eq. 3.28. For $r < R$: $\Phi_1 = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})\,P_n(\cos\theta)$. For $r > R$: $\Phi_2 = \sum_{n=0}^{\infty}(C_n r^n + D_n r^{-n-1})\,P_n(\cos\theta)$. Now $B_n = 0$ for all n since Φ_1 is finite at $r = 0$, and $C_n = 0$ for all n since Φ_2 is finite at $r = \infty$. Next we match boundary conditions:

$V_o\cos 2\theta = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$. Now $\cos(2\theta) = 2\cos^2(\theta) - 1$

$= \frac{4}{3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) - \frac{1}{3} = \frac{4}{3}P_2 - \frac{1}{3}P_o$. Thus

$A_o = -\frac{1}{3}V_o$, $A_2R^2 = \frac{4}{3}V_o$, $D_oR^{-1} = -\frac{1}{3}V_o$, $D_2R^{-3} = \frac{4}{3}V_o$ therefore

$\Phi_1 = -\frac{1}{3}V_o + \frac{4}{3}V_o r^2R^{-2}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$ and

$\Phi_2 = -\frac{1}{3}V_o r^{-1}R + \frac{4}{3}V_o r^{-3}R^3\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

b) $\vec{E} = -\nabla\Phi = -\hat{r}\frac{\partial}{\partial r} - \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}$ thus

$-\vec{E}_1 = \frac{4}{3}V_o 2rR^{-2}(\frac{3}{2}\cos^2\theta - 1)\hat{r} + \frac{4}{3}V_o r^2 R^{-2}(3\cos\theta\sin\theta)\hat{\theta}$ and

$-\vec{E}_2 = -\frac{1}{3}V_o(-r^{-2})R\hat{r} + \frac{4}{3}V_o(-3r^{-4})R^3(\frac{3}{2}\cos^2\theta - 1)\hat{r} +$

$\frac{4}{3}V_o r^{-4}R^3(3\cos\theta\sin\theta)\hat{\theta}$ and

$$\frac{\sigma(\theta)}{\varepsilon_o} = [\vec{E}_2(R) - \vec{E}_1(R)]\cdot\hat{r} = \varepsilon_o V_o R^{-1}\left(\frac{20}{3}\left(\frac{3}{2}\cos^2\theta - 1\right) - \frac{1}{3}\right).$$

3.7 Using the fact that the potential is finite at $r = 0$ and $r = \infty$ then we write the following for the potentials inside and outside the shell.

$\Phi_1 = \sum_n A_n r^n P_n(\cos\theta)$ and $\Phi_2 = \sum_n B_n r^{-(n+1)}P_n(\cos\theta)$. Now use the information given about the charge at the surface, $E_{2n} - E_{1n}|_{r=R}$

$= \frac{\sigma_o(\cos\theta - 1)^2}{\varepsilon_o}$ where $E_{in} = \frac{-\partial\Phi_i}{\partial r}$. Note

$(\cos\theta - 1)^2 = \frac{2}{3}(P_2(\cos\theta) - 3P_1(\cos\theta) + 2P_0(\cos\theta))$. Thus

$$\sum_n (n+1)B_n R^{-(n+2)}P_n(\cos\theta) + \sum_n nA_n R^{(n-1)}P_n(\cos\theta) =$$

$2\sigma_o/3\varepsilon_o \times (P_2 - 3P_1 + 2P_0)$. This gives $3B_2R^{-4} + 2A_2R = 2\sigma_o/3\varepsilon_o$, $2B_1R^{-3} + A_1 = -\frac{2\sigma_o}{\varepsilon_o}$, $B_0R^{-2} = \frac{4\sigma_o}{3\varepsilon_o}$, and $(n+1)B_nR^{-(n+2)} = -nA_nR^{n-1}$ for $n > 2$. We also have the boundary conditions $\Phi_1|_R = \Phi_2|_R$ which gives

$$\sum_n A_n R^n P_n(\cos\theta) = \sum_n B_n R^{-(n+1)} P_n(\cos\theta) \text{ or } A_n = B_n R^{-(2n+1)}.$$

Solving these and the previous relations simultaneously gives for the constants: $B_o = \frac{4\sigma_o}{3\varepsilon_o} R^2$, $B_1 = \frac{-2\sigma_o}{3\varepsilon_o} R^3$,

$B_2 = \frac{2\sigma_o}{15\varepsilon_o} R^4$, $A_o = \frac{4\sigma_o}{3\varepsilon_o} R$, $A_1 = \frac{-2\sigma_o}{3\varepsilon_o}$, $A_2 = \frac{2\sigma_o}{15\varepsilon_o} R^{-1}$, and $A_n = B_n = 0$ for $n > 2$. Thus

$$\Phi_1 = \frac{2\sigma_o R}{3\varepsilon_o}\left(2 - \left(\frac{r}{R}\right)\cos\theta + \frac{1}{5}\left(\frac{r}{R}\right)^2 \frac{1}{2}(3\cos^2\theta - 1)\right),$$

$$\Phi_2 = \frac{2\sigma_o R}{3\varepsilon_o}\left(2\left(\frac{R}{r}\right) - \left(\frac{R}{r}\right)^2 \cos\theta + \frac{1}{5}\left(\frac{R}{r}\right)^3 \frac{1}{2}(3\cos^2\theta - 1)\right).$$

3.8 Since the potential should be finite at $\rho = 0$ and at $\rho \to \infty$, then we write the following expansions for Φ inside and outside cylinder: $\Phi_1(\rho,\phi) = \sum_{m=0}^{\infty} \rho^m (C_m \cos m\phi + D_m \sin m\phi)$ and $\Phi_2(\rho,\phi) = \sum_{m=0}^{\infty} \rho^{-m}(E_m \cos m\phi + F_m \sin m\phi)$. We solve for the coefficients by applying the boundary conditions. First the potential is continuous at $\rho = \rho_o$, that is $\Phi_1 = \Phi_2$,

$$\sum_{m=0}^{\infty} \rho_o^m (C_m \cos m\phi + D_m \sin m\phi) = \sum_{m=0}^{\infty} \frac{1}{\rho_o^m}(E_m \cos m\phi + F_m \sin m\phi).$$

Secondly the boundary condition $E_{2n} - E_{1n} = \sigma/\varepsilon_o$ at $\rho = \rho_o$ gives

$$\sum_{m=0}^{\infty} m \frac{1}{\rho_o^{m+1}}(E_m \cos m\phi + F_m \sin m\phi) + \sum_{m=0}^{\infty} m\, \rho_o^{m-1}(C_m \cos m\phi + D_m \sin m\phi) = \frac{\sigma_o}{\varepsilon_o}\cos 3\phi.$$

Note that on RHS of this relation, one has $\cos 3\phi$ that is $m = 3$ only and nothing else, thus $E_m = C_m = 0$

for $m \neq 3$ and $F_m = D_m = 0$ for all m. Solving relations from both boundary conditions give $E_3 = \sigma_o \rho_o{}^4/6\varepsilon_o$ and $C_3 = \sigma_o/6\varepsilon_o\rho_o{}^2$. Thus

$$\Phi_1 = \frac{\sigma_o \rho^3}{6\varepsilon_o \rho_o^2} \cos 3\phi, \quad \Phi_2 = \frac{\sigma_o \rho_o^4}{6\varepsilon_o \rho^3} \cos 3\phi$$

$$\vec{E}_1 = -\frac{\sigma_o \rho^2}{2\varepsilon_o \rho_o^2} (\hat{\rho} \cos 3\phi - \hat{\phi} \sin 3\phi)$$

$$\vec{E}_2 = \frac{\sigma_o \rho_o^4}{2\varepsilon_o \rho^4} (\hat{\rho} \cos 3\phi + \hat{\phi} \sin 3\phi)$$

3.9 Since there are no charges, $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$. Now separate variables by taking $\Phi(y,z) = Y(y)Z(z)$ which when substituted gives $\frac{d^2 Z}{dz^2}/Z = -\frac{d^2 Y}{dy^2}/Y$. Both sides of last equation must equal a constant thus $\frac{1}{Z}\frac{d^2 Z}{dz^2} = +\alpha^2 = \frac{-1}{Y}\frac{d^2 Y}{dy^2}$ or $\frac{d^2 Z}{dz^2} = +\alpha^2 Z$, $\frac{d^2 Y}{dy^2}\ \alpha^2 y = 0$ which gives $Z = A \cosh \alpha z + B \sinh \alpha z$, and $Y = D \cos \alpha y + C \sin \alpha y$. The boundary conditions imply: $Z(o) = 0$ gives $Z = B \cdot \sinh \alpha z$, $Y(o) = 0$ gives $Y = C \sin \alpha y$, $Y(y_o) = 0$ gives $Y = C \sin n\pi y/y_o$ where $n = 1,2,\ldots$ or $\alpha = n\pi/y_o$. The most general solution is $\Phi(y,z) = \sum_n A_n \sinh(n\pi z/y_o) \sin(n\pi y/y_o)$. To get A_n we use the information about the fourth side, that is $\Phi(y,z_o) = V_o$ or $V_o = \sum_n A_n \sinh(n\pi z_o/y_o) \sin(n\pi y/y_o)$. Multiply by $\sin(m\pi y/y_o)$ and integrate from o to y_o.

$$V_o \int_o^{y_o} \sin\frac{m\pi y}{y_o}\,dy = \sum_n A_n \sinh\left(\frac{n\pi z_o}{y_o}\right) \int_o^{y_o} \sin\frac{n\pi y}{y_o} \sin\frac{m\pi y}{y_o}\,dy.$$

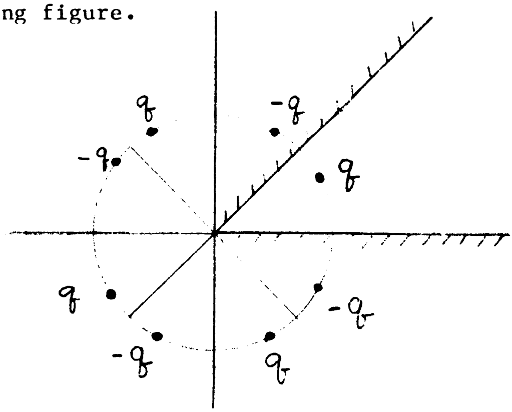
Now $\int_o^{y_o} \sin\frac{n\pi y}{y_o} \sin\frac{m\pi y}{y_o}\,dy = \delta mn\,\frac{y_o}{2}$, and

$$\int_o^{y_o} \sin\frac{m\pi y}{y_o}\,dy = 2y_o/m\pi \text{ (m odd)}.$$

Thus $A_n = \left\{\frac{2V_o y_o}{n\pi}\cdot\frac{2}{y_o}\cdot\frac{1}{\sinh\left(\frac{n\pi z_o}{y_o}\right)}\right.$, n odd; therefore

$$\Phi(y,z) = \frac{4V_o}{\pi} \sum_{n\text{ odd}} \frac{1}{n \sinh\left(\frac{n\pi z_o}{y_o}\right)} \sinh\left(\frac{n\pi z}{y_o}\right) \sin\left(\frac{n\pi y}{y_o}\right).$$

3.10 The angle between the plates is 45° = 180°/4. Thus we need 2n − 1 = 7 image charges. They are distributed as shown in accompanying figure.



3.11 We use the method of images in this problem. We replace the sphere by placing an effective charge q', a distance ξ' away where $\xi' = 2R^2/\ell$ and $q' = -2Rq/\ell$. Taking the charge to be on the z axis, then $\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_o}\left\{\frac{q}{|\vec{r} - \vec{\xi}|} + \frac{q'}{|\vec{r} - \vec{\xi}'|}\right\}$ where $\vec{\xi} = \ell\,\hat{z}/2$, $\vec{\xi}' = \xi'\hat{z}$. Explicitly:

$$\Phi = \frac{1}{4\pi\varepsilon_o}\left\{\frac{q}{(r^2 + (\ell/2)^2 - \ell r\cos\theta)^{1/2}} - \frac{2Rq/\ell}{(r^2 + (\frac{2R^2}{\ell})^2 - \frac{4R^2}{\ell}\ r\ \cos\theta)^{1/2}}\right\}.$$

We now determine $\sigma(\theta)$. Now

$\vec{E}\cdot\hat{n}| = \frac{-\partial\Phi}{\partial r}$ gives $\vec{E}\cdot(-\hat{n}) = \frac{+\partial\Phi}{\Phi r} = \frac{\sigma}{\varepsilon_o}$ at $r = R$ which gives

$$\sigma(\theta) = \frac{-q}{4\pi R^2}\left\{\frac{1 - (\frac{\ell}{2R})^2}{(1 + (\frac{\ell}{2R})^2 - (\frac{\ell}{R})\ \cos\theta)^{3/2}}\right\}.$$

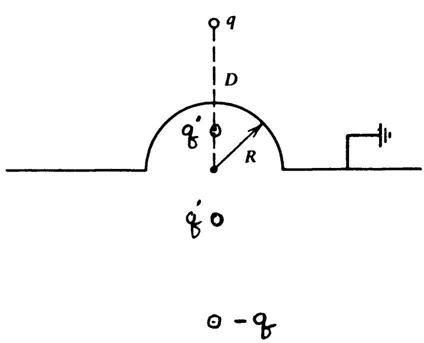
Now for $\ell << R$, we can neglect terms of order $(\frac{\ell}{2R})^2$. Thus

$\sigma(\theta) = \frac{-q}{4\pi R^2}\ \left\{\frac{1}{(1 - (\frac{\ell}{R})\cos\theta)^{3/2}}\right\}$. The force between the charge, q, and the sphere is just the force between the charge, q, and its image, $q' = -2Rq/\ell$. From general results

$$|\vec{F}| = \frac{1}{4\pi\varepsilon_o}\ \frac{qq'}{|\vec{\xi}' - \vec{\xi}|^2} = \frac{1}{4\pi\varepsilon_o}\ \frac{-2Rq^2/\ell}{((\frac{\ell}{2})^2 + (\frac{2R^2}{\ell})^2 - 2R^2)}$$ and

$$|\vec{F}| \approx \frac{-q^2}{4\pi\varepsilon_o}\ (\frac{\ell}{2R^3}) \qquad \text{for } \ell << R.$$

3.12



a) We want to find the image charges which will make the potential on the boss and plate zero. To make $\Phi = 0$ on the boss surface, we must have an image charge $q' = -qR/D$ placed at $y = R^2/D$, as shown above. To make $\Phi = 0$ on the flat part, we must have two image charges $-q'$ at $-y$ (below the plate) and $-q$ at $-D$ (below the plate) as shown.

b) The potential on the side of the plate is

$$\Phi = \frac{q}{4\pi\varepsilon_o|\vec{r} - D\hat{z}|} - \frac{qR/D}{4\pi\varepsilon_o|\vec{r} - (R\ /D)\hat{z}|} + \frac{qR/D}{4\pi\varepsilon_o|\vec{r} + (R^2/D)\hat{z}|} - \frac{-q}{4\pi\varepsilon_o|\vec{r} + D\hat{z}|}$$

or explicitly

$$\Phi(r,\theta) = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{(r^2 + D^2 - 2rD\cos\theta)^{1/2}} - \frac{R/D}{(r^2 + R^4/D^2 - (2rR^2/D)\cos\theta)^{1/2}}\right.$$

$$\left.\frac{-1}{(r^2 + D^2 + 2rD\cos\theta)^{1/2}} + \frac{R/D}{(r^2 + R^4/D^2 + (2rR^2/D)\cos\theta)^{1/2}}\right]$$

c) Charge induced on boss is given by $\vec{E}\cdot\hat{n} = \sigma/\varepsilon_o$ and $Q = \int_{boss} \sigma da = -\varepsilon_o \int \nabla\Phi\cdot\hat{n}dA$ is the total charge. Now, on boss, $\vec{E}$ must be purely radial, thus $Q = -2\pi\varepsilon_o R^2 \int_0^{\pi/2} \frac{\partial\Phi}{\partial r}\Big|_{r=R} \sin\theta d\theta$. Evaluating $\frac{\partial\Phi}{\partial r}\Big|_{r=R}$ and carrying out the integration gives

$$Q = -q\left[1 - \frac{D^2 - R^2}{D\sqrt{D^2 + R^2}}\right]$$

d) Force between the charge and the plate is just the force between q and the three image charges:

$$F = \frac{q^2}{4\pi\varepsilon_o}\left[\frac{-R/D}{(D - R^2/D)^2} + \frac{R/D}{(D + R^2/D)^2} - \frac{1}{(2D)^2}\right].$$

3.13 a) We use the results of the example in the book on the method of images on a sphere. We need an image dipole located at R^2/Z_o and of moment $-pR^3/Z_o^3$. The distance between the dipole and image is $(Z_o - R^2/Z_o)$. Note that $\vec{p}_1 \cdot \hat{r} = 0$, $\vec{p}_2 \cdot \hat{r} = 0$, thus using the results of problem 2.26 we get

$$\vec{f} = \frac{3(\vec{p}_1 \cdot \vec{p}_2)}{4\pi\varepsilon_o r^4}\hat{r} = \frac{-p_o^2(R^3/Z_o^3)\hat{r}}{4\pi\varepsilon_o\left(Z_o - \frac{R^2}{Z_o}\right)^4} \text{ (toward the sphere).}$$

b) $\vec{E} = q\vec{r}/4\pi\varepsilon r^3$ is the electric field from a point charge at $r = 0$; thus $\nabla\vec{E} = (q/4\pi\varepsilon_o)(r^{-3}\nabla(\vec{r}) + \vec{r}\nabla(r^{-3}))$. Now $\nabla(\vec{r}) = \overleftrightarrow{1} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$, $\nabla(r^{-3}) = -3r^{-4}\hat{r} = -3r^{-3}\vec{r}$, thus $\nabla\vec{E} = (q/4\pi\varepsilon_o)(-3r^{-3}\hat{r}\hat{r} + r^{-3}\overleftrightarrow{1})$. Now $r = Z_o$, and $(\vec{p}\cdot\hat{r}) = 0$, then $\vec{f} = \vec{p}\cdot\nabla\vec{E} = q\vec{p}/4\pi\varepsilon_o Z_o^3$ (see problem 2.25). This force is added to the force in part a to give the total force. Note that the force on a point charge due to a dipole is obvious, and there must be an equal and opposite forces.

3.14 The torque acting is $\vec{\tau} = \vec{r} \times \vec{F} + \vec{p} \times \vec{E}$ where $\vec{F}$ is the force acting on it and $\vec{E}$ is the electric field acting on it. Taking $\vec{E}$ to be that of the image dipole, we find that it is along $\vec{p}$. Moreover $\vec{F}$ (from the previous problem 13.13) is along $\vec{r}$, thus $\vec{\tau} = 0$.

13.15 Using the results of Ex. 3.14, and taking $\alpha = 0$, we find that the image dipole needed is $\vec{p}' = \hat{z}(R_o/Z_o)^3\, p$ and located at $b = R_o^2/Z_o$. This is an addition to a charge of $R_o p/Z_o^2$ located in the same place. Using procedures similar to those used in problems 2.25 and 2.26, we find that the force and torque are

$$\vec{F} = -\frac{p^2 R_o Z_o}{2\pi\varepsilon_o (Z_o^2 - R_o^2)^4} (Z_o^2 + 2R_o^2)\,\hat{z}.$$ The torque here is zero

because $\vec{r}$, $\vec{F}$, $\vec{E}$ and $\vec{p}$ are along each other. In general for any angle α between dipole and the line joining the center of the sphere and the dipole we have

$$F = -\frac{p^2 R_o Z_o}{4\pi\varepsilon_o} \frac{[(2Z_o^2 + R_o^2)\cos^2\alpha + 3R_o^2)]}{(Z_o^2 - R_o^2)^4}$$

$$\tau = -\frac{p^2 R_o Z_o^2 \sin 2\alpha}{3\pi\varepsilon_o (Z_o^2 - R_o^2)^3}$$

3.16 We use the method of images. We need to find the location of the line of charge and its image. Given $x_o = d$, and the radius of the cylinder R. Thus the line charges are located at $\pm a$ where $a = (R^2 + x_o^2)^{1/2}$. Thus the potential

$$\Phi = \frac{\lambda}{2\pi\varepsilon_o} \ln(\rho - /\rho +) = \frac{\lambda}{4\pi\varepsilon_o} \ln \left(\frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}\right).$$ Now the charge density is $\sigma = \varepsilon_o E_n$ evaluated at $x = 0$, thus

$$\sigma(y) = \varepsilon_o (\partial\Phi/\partial x)|_{x=0} = \frac{-\lambda}{4\pi} \frac{\partial}{\partial x} \left(\ln \left(\frac{y^2 + (x + a)^2}{y^2 + (x - a)^2}\right)\right)\Big|_{x=0}$$

which gives $\sigma = \frac{-\lambda}{\pi} \left(\frac{a}{y^2 + a^2}\right)$

3.17 We use the method of images. Since the cylinders are identical, then we have symmetry. The locations of the lines are at $x = \pm a$ where $a = ((\Delta/2)^2 + R^2)^{1/2}$. The potential is given by

$$\Phi = \frac{\lambda}{4\pi\varepsilon_o} \ln\left[\frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}\right]$$

a) The charge density is $\sigma = -\varepsilon_o \nabla\Phi$ evaluated at the surface $(x - \Delta/2)^2 + y^2 = R^2$. This gives after some algebra:

$$\sigma = \pm \frac{b^2\lambda}{[(x^2 - y^2 - b^2)^2 + 4x^2y^2]^{1/2}}$$

where $b = 2R^4/(\Delta^2\sqrt{\Delta^2 - 4R^2})$.

b) The force between the cylinders will be that between the image line charges. Thus from Chapter 2:

$$\frac{dF}{d\ell} = \frac{-\lambda^2}{4\pi\varepsilon_o[(\frac{\Delta}{2})^2 + R^2]^{1/2}}$$

3.18 We will solve the problem by solving for the potentials directly. In the region outside the cylinder there is no charge, so the potential satisfies Laplace's Eq.: $\nabla^2\Phi_2 = 0$. By symmetry, the potential has only ρ dependence:

$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\Phi}{\partial\rho}) = 0$ which gives $\Phi_2 = A\ln\rho + B$. Φ_1, the potential inside the cylinder satisfies Poisson's Eq.:

$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\Phi}{\partial\rho}) = \frac{-\alpha}{\varepsilon_o}$ which after two integrations gives

$\Phi_1 = \frac{-\alpha}{4\varepsilon_o}\rho^2 + A'\ln\rho + B'$. Since the potential at ∞ is not equal to zero, then we take it to be zero at $\rho = 0$. The requirement that $\Phi_1(0) = 0$ requires $A' = B' = 0$. Applying Gauss' law on a cylinder that encloses the charge distribution evaulates A: $Q = \int \alpha dv = \int_0^{\rho_o} \alpha\ 2\pi\rho\ell d\rho = \pi\alpha\ell\rho_o^2 = -A\varepsilon_o 2\pi\rho\ell/\rho$ which

gives $A = -\alpha\rho_o^2/2\varepsilon_o$. Thus $\Phi_2 = \dfrac{-\alpha\rho_o^2}{2\varepsilon_o} \ell n\rho + B$. Applying the

boundary condition $\Phi_1 = \Phi_2$ at $\rho = \rho_o$ gives

$$B = \frac{-\alpha}{4\varepsilon_o}\rho_o^2 + \frac{2\rho_o^2}{2\varepsilon_o}\ell n\rho_o \text{ and hence } \Phi_1(\rho) = \frac{-\alpha}{4\varepsilon_o}\rho^2,$$

$$\Phi_2(\rho) = \frac{-\alpha\rho_o^2}{4\varepsilon_o}\{2\ell n(\rho/\rho_o) + 1\}.$$

3.19 For $r < R_1$ and $r > R_2$, $\nabla^2\Phi = 0$ which gives $\Phi_1 = A/r + b$, and $\Phi_3 = C/r + D$. For $R_1 < r < R_2$, $\nabla\Phi_2 = 1/r^2$ or $\dfrac{d}{dr}(r^2 \dfrac{d\Phi_2}{dr}) = -\rho/\varepsilon_o = -\beta/\varepsilon_o r$ which after two integrations gives $\Phi_2(r) = \dfrac{-\beta r}{2\varepsilon_o} - \dfrac{E}{r} + F$. For $r < R_1$, Φ_1 must be finite everywhere, thus $A = 0$. For $r > R_2$, $\Phi_3 \to 0$ at ∞ thus $D = 0$. Now continuity of Φ at R_1 and R_2 gives

$B = \dfrac{-\beta R_1}{2\varepsilon_o} - \dfrac{E}{R_1} + F$, $\dfrac{C}{R_2} = \dfrac{-\beta R_2}{2\varepsilon_o} - \dfrac{E}{R_2} + F$. We can get the constant

C by Gauss' law, or by realizing that C/r should be the potential if all the charge was concentrated at the center of the shell. Thus $C = (\beta/2\varepsilon_o)(R_2^2 - R_1^2)$. Applying Gauss' law inside the shell gives $\int \vec{E}_2 \cdot \hat{n}\ da = [\dfrac{+\beta}{2\varepsilon_o} - \dfrac{E}{r^2}] \cdot 4\pi r^2 = \dfrac{\int \rho dv}{\varepsilon_o}$ or

$E = \frac{\beta}{2\varepsilon_o} R_1^2$, and hence $F = \beta R_2/\varepsilon_o$, and $B = \beta(R_2 - R_1)/\varepsilon_o$.
Therefore $\Phi_1 = (\beta/\varepsilon_o)(R_2 - R_1)$,

$$\Phi_2 = -\frac{\beta r}{2\varepsilon_o} - \frac{\beta}{2\varepsilon_o}\frac{R_1^2}{r} + \frac{\beta R_2}{\varepsilon_o}, \quad \Phi_3 = \frac{\beta}{2\varepsilon_o}\frac{(R_2^2 - R_1^2)}{r}.$$

3.20 See part (b) of Ex. 2.12. It deals with a uniformly charged sphere using Gauss' law to get $\vec{E}$ and then integrating to get Φ. The direct boundary value or integrational method can be used.

3.21 The potentials in the $z < -z_o$, $z > z_o$ and $-z_o < z < z_o$, regions satisfy the equations $\frac{d^2\Phi_1}{dz^2} = 0$, $\frac{d^2\Phi_2}{dz^2} = 0$, $\frac{d^2\Phi_0}{dz^2} = -\rho/\varepsilon_o$. The boundary conditions are $\Phi_o = \Phi_1$, $E_o = E_1$ at $z = -z_o$, $\Phi_0 = \Phi_2$, $E_0 = E_2$ at $z = z_o$, $\Phi_1 \propto z$, $E_1 \rightarrow$ constant as $z \rightarrow -\infty$, $\Phi_2 \propto z$, $E_2 \rightarrow$ constant as $z \rightarrow +\infty$. $E_o(o) = 0$ at $z = 0$. Now integrating the equation for Φ_o gives

$\Phi_o = \frac{4\rho_o z_o^2}{\pi^2 \varepsilon_o} \cos(\pi z/2z_o) + A_o z + B_o$ where A and B are constants. $E_o = \frac{-d\Phi_o}{dz} = [\frac{2\rho_o z_o}{\pi\varepsilon_o}\sin(\pi z/2z_o) - A]\hat{z}$. Since $E_o(o) = 0$, then $A_o = 0$. Thus $\Phi_o = \frac{4\rho_o z_o^2}{\pi^2\varepsilon_o}\cos(\pi z/2z_o) + B_o$ and

$E_o = \frac{2\rho_o z_o}{\pi\varepsilon_o}\sin(\pi z/2z_o)\hat{z}$. Now $\Phi_2 = C_1 z + C_2$ and $E_2 = -C_1$. But $E_o = E_2$ at $z = z_o$ gives $E_2 = -C_1 = \frac{2\rho_o z_o}{\pi\varepsilon_o}$. Now $\Phi_2 = \Phi_0$ gives

$$B_o = \frac{-2\rho_o z_o^2}{\pi\varepsilon_o} + C_2 \quad \text{and} \quad \Phi_2 = \frac{-2\rho_o z_o}{\pi\varepsilon_o} z + \frac{2\rho_o z_o^2}{\pi\varepsilon_o} + B_o.$$

3.22 We guess a solution of the form $\Phi = A \sin a_1 x$ $\sin a_2 y \sin a_2 z$. Substituting in Poisson's equation $\nabla^2\Phi = \frac{-\rho_o}{\varepsilon_o} \sin ax \sin a_2 y \sin a_3 z$ gives $A(a_1^2 + a_2^2 + a_3^2) = \rho_o/\varepsilon_o$

or $A = \frac{\rho_o}{\varepsilon_o} \frac{1}{(a_1^2 + a_2^2 + a_3^2)}$ and

$$\Phi = \frac{\rho_o}{\varepsilon_o} \frac{1}{(a_1^2 + a_2^2 + a_3^2)} \sin a_1 x \sin a_2 y \sin a_3 z.$$

3.23 We can write the volume charge distrubtion as $\rho(x,y,z) = \sigma_o \cos(a_1 x) \cos(a_2 y)\, \delta(z)$. We solve Poisson's equation $\nabla^2\Phi = \rho/\varepsilon_o$ or

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{\sigma_o}{\varepsilon_o} \cos(a_1 x) \cos(a_2 y)\, \delta(z).$$ Assume

$\Phi(x,y,z) = F(z) \cos(a_1 x) \cos(a_2 y)$. If we can find an $F(z)$ and Poisson's equation is satisfied, then, by uniqueness theorem, we get our Φ. Plug Φ into the differential equation to get

$$-a_1^2 \cos a_1 x \cos a_2 y\, F(z) - a_2^2 \cos(a_1 x) \cos(a_2 y)\, F(z) + \frac{d^2F}{dz^2} \cos a_1 x \cos a_2 y = \frac{\sigma_o}{\varepsilon_o} \cos(a_1 x) \cos(a_2 y)\, \delta(z).$$

For $z \neq 0$, then we get $-(a_1^2 + a_2^2)\, F(z) + \frac{d^2F}{dz^2} = 0$ or

$F = A\, e^{-\sqrt{a_1^2 + a_2^2}\, z}$ for $z > 0$ and

$F = A\, e^{+\sqrt{a_1^2 + a_2^2}\, z}$ for $z < 0$.

Now evaluate A. At $z = 0$, $\left[\frac{d\Phi}{dz}\Big|_{z=0+} - \frac{d\Phi}{dz}\Big|_{z=0-}\right] = \frac{-\sigma}{\varepsilon_o}$ or

$$2A\sqrt{a_1^2 + a_2^2}\cos a_1x \cos a_2y = \frac{+\sigma_o}{\varepsilon_o}\cos a_1x\cos a_2y.$$

Thus $A = \dfrac{\sigma_o}{2\varepsilon_o\sqrt{a_1^2 + a_2^2}}$ and

$$\Phi(x,y,z) = \frac{\sigma_o}{2\varepsilon_o\sqrt{a_1^2 + a_2^2}}\, e^{-\sqrt{a_1^2 + a_2^2}\,z}\cos a_1x\cos a_2y,\ z > 0.$$

$$\Phi(x,y,z) = \frac{\sigma_o}{2\varepsilon_o\sqrt{a_1^2 + a_2^2}}\, e^{+\sqrt{a_1^2 + a_2^2}\,z}\cos a_1x\cos a_2y,\ z < 0.$$

3.24 We use $\nabla^2\Phi = -\rho/\varepsilon_o$ with $\Phi(r) = (1/4\pi\varepsilon_o)e^{-\alpha r^3}$ and $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$. Thus $\rho = -\varepsilon_o\nabla^2\Phi = \frac{3\alpha r}{4\pi}(4 - 3\alpha r^3)e^{-\alpha r^3}$.

3.25 Now $\Phi = \frac{q}{4\pi\varepsilon_o}(\frac{1}{r} + \frac{1}{a})\exp(\frac{-2r}{a}) + \frac{q}{4\pi\varepsilon_o r} - \frac{q}{4\pi\varepsilon_o r}$

$= \frac{q}{4\pi\varepsilon_o r} + \frac{q}{4\pi\varepsilon_o}\left[(\frac{1}{a})e^{-2r/a} + \frac{e^{-2r/a}-1}{r}\right]$. Using $\nabla^2\Phi = -\rho/\varepsilon_o$

gives $\rho = -\varepsilon_o\nabla^2\Phi = (-q\nabla^2/4\pi)\left[\frac{1}{r} + \frac{1}{a}e^{-2r/a} + \frac{e^{-2r/a} - 1}{r}\right] =$

$\frac{-q}{4\pi}\left[\nabla^2(\frac{1}{r}) + \nabla^2(\frac{e^{-2r/a} - 1}{r})\right]\frac{-q}{4\pi a}\nabla^2(e^{-2r/a})$. The first two terms

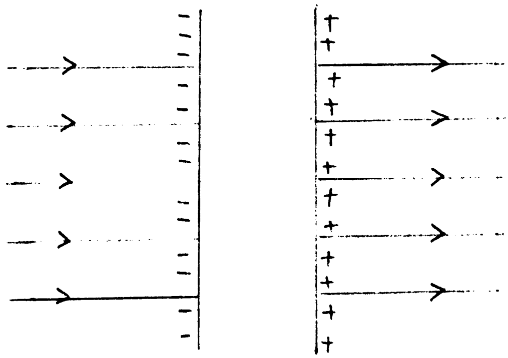
give $q\delta(r) - (q/4\pi)(2/a)^2 e^{-2r/a}/r$ (see Ex. 3.20). The last terms give $\nabla^2(e^{-2r/a}) = (-4/a)e^{-2r/a}[1/r - 1/a]$. Thus

$\rho(r) = q\,\delta(r) - (q/\pi a^3)e^{-2r/a}$.

3.26 **a)** We use $d^2\Phi/dz^2 = 0$ to find that $\Phi_1 = A_1 z + B_1$ and $\Phi_2 = A_2 z + B_2$. Now at z = 0 we have $\Phi_1 = 0$ and at z = a we have $\Phi_2 = 0$, therefore $\Phi_1 = A_1 z$ and $\Phi_2 = A_2(z - a)$. Now as $z \to -\infty$ and as $z \to \infty$ we have $\vec{E} = E_o\hat{z}$. Therefore $\Phi_1 = -E_o z$ and $\Phi_2 = -E_o(z - a)$.

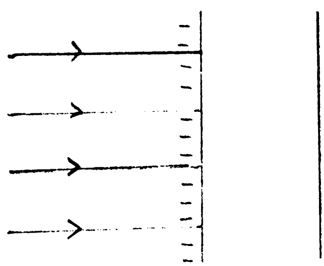
b) $\sigma(z = 0) = -\varepsilon_o\hat{z}\cdot\vec{E}(z = 0) = -\varepsilon_o E_o$, $\sigma(z = a) = \varepsilon_o\hat{z}\cdot\vec{E}(z = a) = \varepsilon_o E_o$

c)



d) $\sigma(z = 0) = -\varepsilon_o E_o$, $\sigma(z = a) = 0$. Moreover $\vec{E} = 0$ in the $z > a$ region as predicted by Gauss' law since $\sigma(z = a) = 0$. Thus $\vec{E} = E_o\hat{z}$ in the $z < 0$ region as predicted by Gauss' law.

e)



f) Before grounding we have

$$\frac{d\vec{F}}{da} = \frac{1}{2}\varepsilon_o E^2\hat{n} = -\frac{1}{2}\varepsilon_o E_o^2\hat{z} + \frac{1}{2}\varepsilon_o E_o^2\hat{z} \equiv 0.$$ After grounding we

have $\frac{d\vec{F}}{da} = \frac{1}{2} \varepsilon_o E^2 \hat{n} = - \frac{1}{2} \varepsilon_o E_o^2 \hat{z} + 0 = - \frac{1}{2} \varepsilon_o E_o^2 \hat{z}$. The factor of $1/2$ has not been yet explained. It is explained in Chapter 4 as due to self fields (Eq. 117).

CHAPTER 4

4.1 We are given that $\vec{P} = P\hat{z}$

a) We know $\rho = -\vec{\nabla}\cdot\vec{P}$, $\sigma = \hat{n}\cdot\vec{P}$ so $\rho = -\vec{\nabla}\cdot P\hat{z} = 0$. Now on upper surface $\hat{n} = \hat{r}$, thus $\sigma = \hat{r}\cdot\hat{P} = P\cos\theta$. On lower surface $\hat{n} = -\hat{z}$ thus $\sigma = -P$.

b) $q = \int_s \sigma da + \int_v \rho dv = P\int_u \cos\theta\, da - P\int_L da$. Now

$$P\int_u \cos\theta da = PR^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta\sin\theta d\theta = \pi R^2 P, \text{ and}$$

$P\int_L da = \pi R^2 P$. Therefore $q = \pi R^2 P - \pi R^2 P = 0$.

c) Since the material is neutral then net charge = 0.

d) By integration: $\vec{p} = \int \vec{r}\, dq = \int_L \vec{r} dq + \int_u \vec{r} dq$. Let $\vec{\rho}$ be the projection of $\vec{r}$ on x, y plane, i.e. $\vec{r} = r\cos\theta\,\hat{z} + r\sin\theta\,\hat{\rho} = \vec{z} + \vec{\rho}$. Now on upper surface we have $dq = P\cos\theta\, R^2 d\phi\, \sin\theta d\theta$ while on lower surface we have $dq = -P\rho d\rho d\phi$. Thus

$$\vec{p} = \int_0^{\pi} R\sin\theta d\theta \int_0^{2\pi} d\phi P\cos\theta(\cos\theta\hat{z} + \sin\theta\hat{\rho}) - \int_0^{2\pi} d\phi \int_0^{R} d\rho\, P\rho^2\hat{\rho}.$$

By symmetry we see that contribution proportional to $\hat{\rho}$ vanish

$$\vec{p} = -2\pi R^2\hat{z} \int_0^{\pi} P\cos^2\theta\sin\theta d\theta = \frac{2\pi}{3} R^3 P\,\hat{z}$$

Now by definition of $\vec{P}$: $\vec{p} = \int \vec{P}\, dv = P\hat{z}V = \frac{2\pi}{3} R^3 P\hat{z}$. We see the two answers are equal.

4.2 Dipole moment: $\vec{p} = P\int dv = P\pi R^2 L/2$.

Polarization charge densities are: $\sigma_p = \vec{P}\cdot\hat{n} = -P_o$ for rectangular surface. $\sigma_p = \vec{P}\cdot\hat{n} = P_o\hat{x}\cdot\hat{\rho} = P_o\cos\phi$ for curved surface. $\sigma_p = \vec{P}\cdot\hat{n} = 0$ for top and bottom. $\rho_p = -\vec{\nabla}\cdot\vec{P} = 0$.

4.3 Given $x^2 + y^2 = R^2$ and $\vec{P} = (ax^2 + b + cy + a)x\hat{x} + py\hat{y}$

Now $\sigma = \vec{P}\cdot\hat{n}$, $\hat{n} = \frac{\nabla f}{|\nabla f|}$ with $f(x,y) = 0$ is the equation of surface. $\nabla f = 2x\hat{x} + 2y\hat{y}$ and $|\nabla f| = 2(x^2 + y^2)^{1/2} = 2R$

$$\text{Thus } \hat{n} = \frac{2x\hat{x} + 2y\hat{y}}{2(x^2 + y^2)^{1/2}} = \frac{x\hat{x} + y\hat{y}}{R}$$

Thus we have $\sigma = \vec{P}\cdot\hat{n} = \left((ax^2 + b + cy + a)x\hat{x} + py\hat{y}\right)\cdot \frac{(x\hat{x} + y\hat{y})}{R}$

$\sigma_p = (ax^4 + bx^2 + cyx^2 + x^2 + pxy\ /R$ and

$\rho_p = -\vec{\nabla}\cdot\vec{P} - -(3ax^2 + b + cy + a)$.

4.4 **a)** Using Gauss' law in the region $\rho < \rho_1$ gives $\vec{D} = \vec{E} = 0$. In the region $\rho_1 < \rho < \rho_2$ it gives

$\vec{D} = \frac{\alpha}{2}(\rho - \frac{\rho_1^2}{\rho})\hat{\rho}$ and $\vec{E} = \frac{\alpha}{2\varepsilon_o}(\rho - \frac{\rho_1^2}{\rho})\hat{\rho}$. In the region $\rho_2 < \rho < \rho_3$

$\vec{D} = \frac{\alpha}{2\rho}(\rho_2^2 - \rho_1^2)\hat{\rho}$ and $\vec{E} = \frac{\vec{D}}{K\varepsilon_o} = \frac{\alpha}{2K\rho\varepsilon_o}(\rho_2^2 - \rho_1^2)\hat{\rho}$

Finally in $\rho > \rho_3$ region we have $\vec{E} = \frac{\vec{D}}{\varepsilon_o} = \frac{\alpha}{2\rho\varepsilon_o}(\rho_2^2 - \rho_1^2)\hat{\rho}$

b) $\vec{P} = 0$ for $\rho > \rho_3$ and for $\rho < \rho_2$. But $\vec{P} = (1 - 1/K)\times \frac{\alpha}{2\rho}(\rho_2^2 - \rho_1^2)\hat{\rho}$ for $\rho_2 < \rho < \rho_3$. From the polarization we get the polarization charge. The volume density

$\rho_p = -\nabla\cdot\vec{P} = \frac{1}{\rho}\frac{d}{d\rho}(\rho P) = 0$, while the surface density is

$\sigma_p = \hat{n}\cdot\vec{P}$ giving $\sigma_p = -(1 - \frac{1}{K})\frac{\alpha}{2\rho_2}(\rho_2^2 - \rho_1^2)$ at $\rho = \rho_2$ and

$\sigma_p = (1 - 1/K)\frac{\alpha}{2\rho_3}(\rho_2^2 - \rho_1^2)$ at $\rho = \rho_3$. Finally $\sigma_p = 0$ for all

other surfaces.

c) $V = \int_o^{\rho_3} \vec{E}(r)\cdot d\vec{r} = \int_{\rho_1}^{\rho_2} E_1 dr + \int_{\rho_2}^{\rho_3} E_2 dr$

$$= \frac{\alpha}{2\varepsilon_o}[(\rho_2^2 - \rho_1^2)/2 + \ell n\frac{\rho_2}{\rho_1}((\rho_2^2 - \rho_1^2)/K - \rho_1^2)]$$

4.5 a) Take the z-axis perpendicular to the dielectric interface, and the origin of coordinate system at the center of the spheres. From symmetry we have $\vec{E} = E_o\hat{r}$. Thus the electric field at the interfaces is purely tangential to it, hence, equating the electric fields at the boundary of the dielectric gives $E_v(r) = E_d(r) = E(r)$. Applying Gauss' law to a spherical surface gives

$$\int\vec{D}\cdot\vec{da} = q = r^2\int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\phi\varepsilon(\phi)|E| = 2\pi r^2(\varepsilon_o + \varepsilon)E$$

$$\vec{E} = \frac{q\hat{r}}{2\pi(\varepsilon_o + \varepsilon)r^2}$$

b) We use $\sigma_f = \vec{D}\cdot\hat{n}$. This gives the following for the vacuum and dielectric regions: $\sigma_f = \frac{q}{2\pi a^2}(\frac{\varepsilon_o}{\varepsilon + \varepsilon_o})$,

$\sigma_f = \frac{q}{2\pi a^2}(\frac{\varepsilon}{\varepsilon + \varepsilon_o})$

c) We use $\vec{P} = \vec{D} - \varepsilon_o\vec{E}$. Thus at $r = a$ we have

$$\vec{P} = \frac{q\hat{r}}{2\pi a^2}(\varepsilon - \varepsilon_o)/(\varepsilon + \varepsilon_o) \text{ and}$$

$$\sigma_p = \vec{P}\cdot\hat{n} = -\frac{q\hat{r}}{2\pi a^2}(\varepsilon - \varepsilon_o)(\varepsilon + \varepsilon_o).$$

4.6 **a)** There are three regions in all of which the electric field is normal to the plates (say along x). We use Gauss in all three cases. In the vacuum next to σ_1 plate, we have $\int\vec{D}\cdot d\vec{a} = \int \sigma da$ or $\vec{D} = \sigma_1\hat{x}$. In the charged slab we have $h < x < h + t$: $\int\vec{D}\cdot d\vec{a} = \int\sigma da + \int_h^x(\rho dx)\, da$ or $\vec{D} = (\sigma_1 + \rho(x - h))\hat{x}$.

In vacuum above slab we have $\int\vec{D}\cdot d\vec{a} = \int\sigma da + \int_h^{h+t} \rho dx da$ or $\vec{D} = (\sigma_1 + t\rho)\hat{x}$.

b) 1) On the top plate we have $\sigma_f = \vec{D}\cdot\hat{n} = -(\sigma_1 + t\rho)$

2) In the dielectric we have $\vec{P} = \vec{D} - \varepsilon_o\vec{E} = \vec{D}\left(\frac{K - 1}{K}\right)$ or $\vec{P} = \left(\frac{K - 1}{K}\right)(\sigma_1 + \rho(x - h))\hat{x}$

i) Volume polarization $\rho_p = -\nabla\cdot\vec{P} = -\frac{K - 1}{K}\rho$

ii) On the lower surface $(x = h)$ we have $\sigma_p = \vec{P}\cdot\hat{n} = -\frac{K - 1}{K}\sigma_1$

iii) On the upper surface $(x = h + t)$ we have $\sigma_p = \frac{K - 1}{K}(\sigma_1 + \rho t)$.

4.7 We use Gauss' law for $\vec{D}$.

a) $\int\vec{D}\cdot d\vec{a} = q$ which gives $\vec{D} = q\hat{r}/(4\pi\varepsilon_o r^2)$

b) $\vec{P} = \vec{D}\left(1 - \frac{\varepsilon_o}{\varepsilon}\right) = \frac{q}{4\pi r^2}(1 - c + \alpha r)\hat{r}$. Thus

$$\rho_p = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{d}{dr}(r^2 P) = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{q}{4\pi}(1 - c + \alpha r)\right) \text{ or}$$
$$\rho_p = -q\alpha/4\pi r^2.$$

4.8 The differential form of Gauss' law gives $\nabla\cdot\vec{D} = \rho_f = 0$. Therefore $dD/dx = 0$ or $\vec{D} = C\hat{x}$ and $\vec{E} = \vec{D}/\varepsilon_o K = C\hat{x}/\varepsilon_o K$ where C is a constant. But we know

$$V = -\int_0^d E dx = \frac{-C}{\varepsilon_o K}\int_0^d \exp(-\alpha x)dx = \frac{+C}{\alpha\varepsilon_o K}(\exp(-\alpha x)\Big|_0^d =$$

$\frac{+C}{\alpha\varepsilon_o K}(\exp(-\alpha d)-1)$. Thus $C = -V\alpha\varepsilon_o K/(1-\exp(-\alpha d))$ and

$\vec{E} = -V\ \alpha\ \hat{x}\ \exp(-\alpha x)/(1-\exp(-\alpha d))$.

4.9 a) From Gauss' law $\vec{E} = \frac{-q}{2\pi\rho\varepsilon_o}\hat{\rho}$

b) If we let $\varepsilon = \alpha/\rho$ with α constant we see that $\vec{E}$ will be constant.

4.10 a) Since there are no free charge at x = 0 then $\vec{D}_1\cdot\hat{n} = D_2\cdot\hat{n}$ or $D_{1x} = D_{2x}$ which gives $D_{2x} = 1.5$, and hence $E_{2x} = .6/\varepsilon_o$. Also continuity of tangential component of $\vec{E}$ gives $\vec{E}_1\cdot\hat{t} = \vec{E}_2\cdot\hat{t}$ or $E_{1y} = E_{2y}$, and $E_{1z} = E_{2z}$.

Thus $E_{2y} = -4/\varepsilon_o$, and $E_{2z} = 6/\varepsilon_o$. Thus

$\vec{E}_2 = (1/\varepsilon_o)(.6\hat{x} - 4\hat{y} + 6\hat{z})C/m^2$.

b) $\vec{D}_1 = 1.5\hat{x} - 2\hat{y} + 3\hat{z}\ C/m^2$. $\vec{D}_1$ projected onto yz plane gives $-2\hat{y} + 3\hat{z}$. Thus $\cos\theta_1 = |-2\hat{y} + 3\hat{z}|/|\vec{D}_1|$ which gives $\theta_1 =$

$$\cos^{-1}\left(\frac{\sqrt{13}}{\sqrt{2.25+13}}\right) = 22.6^0$$

Since $\vec{D}_2 = 1.5\hat{x} - 10\hat{y} + 15\hat{z} = -10\hat{y} + 15\hat{z}$ so

$$\theta_2 = \cos^{-1}\left(\frac{\sqrt{335}}{\sqrt{335+2.25}}\right) = 4.7^0$$

4.11 Assume there are no free charge at the interface. The parallel $\vec{E}$ component is continuous thus $E_1 \sin\theta_1 = E_2 \sin\theta_2$. The normal $\vec{D}$ is continuous thus $\varepsilon_o E_1 K_1 \cos\theta_1 = \varepsilon_o E_2 K_2 \cos\theta_2$ where θ_1, θ_2 are the angles the field makes with the normal to the interface in regions 1 and 2. These two equations give $K_1 \tan\theta_2 = K_2 \tan\theta_1$.

4.12 a) Let us take the field and potential in the region $0 < x < 1$ to be E_1 and Φ_1 and the field and potential in the region $1 < x < 2$ to be E_2 and Φ_2. The potential difference $V = -\int \vec{E}\cdot d\vec{r}$ gives $-V = E_1(1) + E_2(1)$. Moreover the continuity of the perpendicular component of $\vec{D}$ gives $E_1 K_1 = E_2 K_2$. Solving for E_1 and E_2 simultaneously gives $-V = E_1(1 + K_1/K_2)$. Thus $\Phi_1 = -xE_1 = xVK_2/(K_1+K_2)$. Now $\Phi_1(x = 1) = -VK_2/(K_2+K_1)$ and $\Phi_2 = \Phi_1(x = 1) + (x - 1)E_2 = [xVK_1 + V(K_2 - K_1)]/(K_1 + K_2)$.

b) At $x = 0$ we have $\vec{E}_1 = -VK_2\hat{x}/(K_2 + K_1)$ and $\vec{D} = K_1 \varepsilon_o \vec{E}_1 = -VK_1K_2\varepsilon_o\hat{x}/(K_1 + K_2)$. Thus the free charge $= \hat{n}\cdot\vec{D} = \sigma_f = -VK_1K_2\varepsilon_o/(K_2 + K_1)$, and the bound charge $= \hat{n}\cdot\vec{P} = \hat{n}\cdot(\vec{D}_1 - \varepsilon_o\vec{E}_1) = \sigma_b = VK_2\varepsilon_o(K_1 - 1)/(K_2 + K_1)$.

c) The bound charge is $\sigma_b = \hat{x}\cdot(\vec{P}_1 - \vec{P}_2) = \hat{x}\cdot(\vec{D}_1 - \varepsilon_o\vec{D}_1 - \vec{D}_2 + \varepsilon_o\vec{E}_2) = \varepsilon_o V(K_2 - K_1)/(K_2 + K_1)$.

4.13 $\Phi_1 = Ar\theta$, $\Phi_2 = Aa^2\theta/r$. Now $\vec{P}_i = \vec{D}_i - \varepsilon_o\vec{E}_i = (\varepsilon_i - \varepsilon_o)\vec{E}_i$. Also $\vec{E}_i = -\nabla\Phi_i$ thus $\vec{E}_1 = -\nabla\Phi_1 = -(\frac{d\Phi}{dr}\hat{r} + \frac{1}{r}\frac{d\Phi}{d\theta}\hat{\theta}) = -A(\theta\hat{r} + \hat{\theta})$. Therefore

$\vec{P}_1 = A(\varepsilon_o - \varepsilon_1)(\hat{\theta r} + \hat{\theta})$. Similarly

$\vec{P}_2 = A(\varepsilon_o - \varepsilon_2)(-\theta(a/r)^2\hat{r} + (a/r)^2\hat{\theta})$. Thus

$\rho_1 = -\nabla\cdot\vec{P}_1 = -A(\varepsilon_o - \varepsilon_1)(\frac{1}{r^2}\frac{d}{dr}(r^2\theta) + (1/r\sin\theta)\frac{d}{d\theta}(\sin\theta))$

$= -A(\varepsilon_o - \varepsilon_1)(2\theta/r + \cot\theta/r)$.

$\rho_2 = -\nabla\cdot\vec{P}_2 = -A(\varepsilon_o - \varepsilon_2)(a^2/r^3)\cot\theta$. The surface polarization charge densities at r = a, and r = b are $\sigma_1 = \vec{P}_1\cdot\hat{r} - \vec{P}_2\cdot\hat{r} = A(\varepsilon_o - \varepsilon_1)\theta + A(\varepsilon_o - \varepsilon_2)$, and $\sigma_2 = \vec{P}_2\cdot\hat{r} = -A\theta(\varepsilon_o - \varepsilon_2)(a/b)^2$.

The free charge density at r = a is $\sigma_f = D_{2n} - D_{1n}$ so $\sigma_f = [\varepsilon_2(d\Phi_2/dr)_{r=a} - \varepsilon_1(d\Phi_1/dr)_{r=a}] = A\theta(\varepsilon_1 + \varepsilon_2)$.

4.14 We have for a solution from Eq. 3.65

$\Phi = C\ln\rho + D + \sum_{m\neq 0}^{\infty}(E_m\rho^{-m} + F_m\rho^m)(A_m\sin m\phi + B_m\cos m\phi)$. We have three regions $\Phi_1(\rho)$ in $\rho < a$, $\Phi_2(\rho)$ in $a < \rho < b$, and $\Phi_3(\rho)$ in $\rho > b$. We also have the boundary conditions:

1) $\vec{E} = E_o\hat{x}$ gives $\Phi_3(\infty) = -E_o x = -E_o\rho\cos\phi$

2) At $\rho = b$ we have $\Phi_2(b) = \Phi_3(b)$ and $D_2(b)\cdot\hat{\rho} = D_3(b)\cdot\hat{\rho}$

3) At $\rho = a$ we have $\Phi_2(a) = \Phi_1(a) = V_o$ = constant

Thus we write the following expansions for Φ: $\Phi_1 = V_o$,

$\Phi_2 = V_o + (\frac{A}{\rho} + B\rho)\cos\phi$, $\Phi_3 = V_o + (\frac{C}{\rho} - E_o\rho)\cos\phi$. We now apply these conditions. Conditions two and three give

$\varepsilon(B - A/b^2) = \varepsilon_o(E_o + A/b^2)$, $(Bb + A/b) = (C/b - E_ob)$,

$V_o = (A/a + Ba)\cos\phi + V_o$ or $A/a = -Ba$. Solving these equations simultaneously give

$B = 2E_o b^2/(a^2(1 - k) - b^2(k + 1))$

$A = -2E_o b^2 a^2/(a^2(1 - k) - b^2(k + 1))$

$C = -E_o b^2(a^2 - b^2 + k(a^2 + b^2))/(a^2(1 - k) + b^2(k + 1))$

b) $\sigma_f(\phi) = \vec{D}\cdot\hat{n}$. From (a) $\sigma_f = -\varepsilon(B - A/a^2)\cos\phi = -2\varepsilon B\cos\phi$ with B given in (a).

4.15 a) First consider the case $\varepsilon_1 > \varepsilon_2$. The electric field will exert a torque on the cylinder due to the induced polarization: the cylinder will orientate itself to create zero net torque. The two such positions are with the axis of the cylinder aligned (1) and perpendicular (2) to the field. In (1) the cylinder will be in a stable equilibrium. In (2), the cylinder will be in unstable equilibrium. For a thin disk, its axis will be perpendicular to the field in a stable equilibrium.
b) For the case $\varepsilon_1 < \varepsilon_2$, we have stable equilibrium when the axis of the cylinder is perpendicular to the field, and the axis of the thin disk parallel to the field.

4.16 a) Since the polarization of the material is permanent, then the permeability of the unpolarized material is irrelevant. The field inside the cavity is dictated by the polarization only. We can use the polarization charge technique. Now $\rho_p = -\nabla\cdot\vec{p} = 0$, $\sigma_p = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r} = P\cos\theta$ on the surface of the cavity. There are no other charges because the material is infinite. The electric field corresponding to this charge is uniform inside the cavity and it is equal to $\vec{E} = \vec{P}/3\varepsilon_o$. [See Exs. 4.2, 2.17, and 3.6.]

b) When the polarization is not permanent, then we have to solve the problem in an external field with the permitivity ε of the material becoming relevant. We have already solved a similar problem in Ex. 4.8. In this example we have a dielectric sphere of permitivity ε_2 place in an infinite medium of permitivity ε_1 and with an electric field $\vec{E} = E_o\hat{z}$. Now we take $\varepsilon_2 = \varepsilon_o$, $\varepsilon_1 = \varepsilon$, and $E_o = P/(\varepsilon - \varepsilon_o)$. Thus from Eq. 4.86 we have

$$\vec{E}_{sphere} = E_o\hat{z} - \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon} E_o\hat{z} = \frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} E_o\hat{z}$$

changing this result to the present case gives

$$\vec{E}_{hole} = \frac{3\varepsilon_o \vec{P}}{(\varepsilon_1 + 2\varepsilon_o)(\varepsilon - \varepsilon_o)}$$

4.17 a) See Ex. 3.5 to get $\vec{F} = -(1/4\pi\varepsilon_o)q^2/4d^2$.

b) The potential is $\Phi(x,y,z) = (1/4\pi\varepsilon_o)\{q/(x - d)^2 + y^2 + z^2)^{1/2} - q/((x + d)^2 + y^2 + z^2)^{1/2}\}$. Thus

$\sigma = \varepsilon_o E_n = -(\partial\Phi/\partial x)|_{x=0} =$

$-(q/4\pi)\{(x - d)/((x - d)^2 + y^2 + z^2)^{3/2} +$

$(x + d)/((x + d)^2 + y^2 + z^2)^{3/2}\}|_{x=0} =$

$-(2qd/4\pi)\{1/(d^2 + y^2 + z^2)^{3/2}\} = -(qd/2\pi)/(\rho^2 + d^2)^{3/2}$. Now

$F = \int \frac{\sigma^2}{2\varepsilon_0} da$ where $da = 2\pi\rho d\rho$, thus

$F = \dfrac{q^2 d^2}{8\pi^2 \varepsilon_o} \int \dfrac{2\pi\rho d\rho}{(\rho^2 + d^2)^3} = \dfrac{q^2 d^2}{4\pi\varepsilon_o} \int_0^\infty \dfrac{\rho d\rho}{(\rho^2 + d^3)^3}$. We do the integration by letting $u = \rho^2 + d^2$, hence $du = 2\rho d\rho$ and the force becomes $\dfrac{q^2 d^2}{4\pi\varepsilon_o} \int_0^\infty \dfrac{du}{2u^3} = \dfrac{q^2 d^2}{16\pi\varepsilon_o} \dfrac{1}{(\rho^2 + d^2)^2} \Big|_0^\infty = \dfrac{-q^2}{16\pi\varepsilon_o d^2} \hat{x}$

which is the same as in (a).

4.18 Because the potential has to be finite at $r = 0$ and $r = \infty$, and because the normal component of P on the surface of the sphere $P_n = P_o \cos\theta$ involves $\cos\theta$ only, we keep only the $\cos\theta$ terms in the expansion of the potential inside and outside Φ_1 and Φ_2:

$$\Phi_1 = A_1 r\cos\theta, \quad \Phi_2 = \frac{B_1}{r^2}\cos\theta$$

Taking $\Phi_1 = \Phi_2$ at $r = R$ gives $A_1 R = B_1/R^2$. Matching $D_{1n} = D_{2n}$ at $r = R$, we get

$$(\varepsilon_o \vec{E}_1 + \vec{P})\cdot\hat{n} = \varepsilon_o \vec{E}_2 \cdot \hat{n} \text{ or } -(A_1 + P_o/\varepsilon_o) = 2B_1/R^3.$$

Solving for A_1 and B_1 we get $A_1 = P_o/3\varepsilon_o$, $B_1 = R^3 P_o/3\varepsilon_o$, hence

$$\Phi_1 = P_o r\cos\theta/3\varepsilon_o, \quad \Phi_2 = R^3 P_o \cos\theta/3\varepsilon_o r^2.$$

4.19 We use expansions for the potentials of the form of Eq. 3.65. The potential inside the cylinder Φ_1 has to be finite at $\rho = 0$, and the potential outside the cylinder Φ_2 has to vanish at $\rho = \infty$. Moreover because at $\rho = a$, the polarization is $P_o \cos\phi$, then we will keep the $\cos\phi$ terms only. Thus

$\Phi_1 = A\rho \cos n\phi, \quad \Phi_2 = A' \cos\phi/\rho$

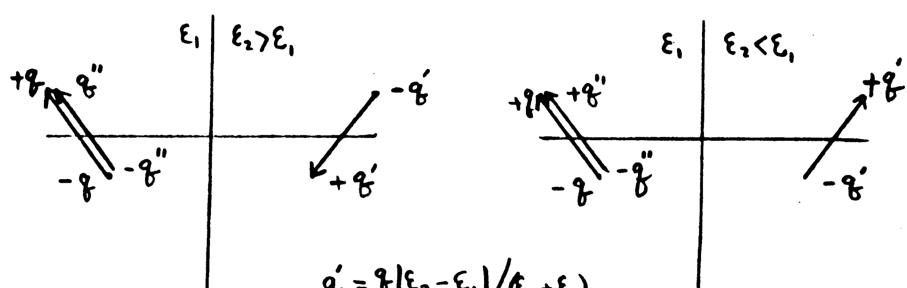
Taking $\Phi_1 = \Phi_2$ at $\rho = a$ gives $A_1 a = A'/a$. Matching $D_{1n} = D_{2n}$ at $\rho = a$ gives $(\varepsilon_o \vec{E}_1 + \vec{P}_1)\cdot\hat{n} = \varepsilon_o \vec{E}_2\cdot\hat{n}$ or $-(\varepsilon_o A + P_o) = \varepsilon_o A'/a^2$. Solving for A and A' gives: $A = -P_o/2\varepsilon_o$ and $A' = -a^2 P_o/2\varepsilon_o$, and hence

$$\Phi_1 = -\frac{P_o \rho}{2\varepsilon_o}\cos\phi, \quad \Phi_2 = -\frac{a^2 P_o}{2\varepsilon_o \rho}\cos\phi$$

The fields are

$$\vec{E}_1 = \frac{P_o \hat{x}}{2\varepsilon_o}, \quad \vec{E}_2 = -\frac{a^2 P_o}{2\varepsilon_o \rho^2}[\hat{\rho}\cos\phi + \hat{\phi}\sin\phi]$$

4.20 We use the results of section 4.7 to introduce the charge q' and q" as images of q, and the charges -q' and -q" as images of the charge -q, as shown in the figure.



$$q' = q|\varepsilon_2 - \varepsilon_1|/(\varepsilon_1 + \varepsilon_2)$$
$$q'' = 2\varepsilon_2 q/(\varepsilon_1 + \varepsilon_2)$$

Thus the images are: an image dipole $\vec{p}'$ in material ε_2 at distance d, and an image dipole $\vec{p}''$ in material ε_1 at the location of the real dipole. The moments of the image dipoles are

$$p' = -\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} p, \quad p'' = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} p.$$

4.21 a) $\langle\vec{E}\rangle = \frac{1}{\Delta v} \int dz'\ dy'\ dx'(\hat{x}\ x'^n)$

$$= \frac{1}{\Delta x} \int_x^{x+\Delta x} \hat{x}\ x'^n dx' = \frac{1}{\Delta x} \hat{x} \left[\frac{x'^{n+1}}{n+1}\right]_x^{x+\Delta x}$$

$$= \frac{\hat{x}}{\Delta x} \left[\frac{(x + \Delta x)^{n+1} - x^{n+1}}{n+1}\right] .$$

b) $\frac{\partial\langle\vec{E}\rangle}{\partial x} = \frac{\hat{x}}{\Delta x} [(x + \Delta x)^n - x^n]$.

c) $\langle\frac{\partial\vec{E}}{\partial x}\rangle = \langle\hat{x}\ nx^{n-1}\rangle$, or

$$\langle\frac{\partial\vec{E}}{\partial x}\rangle = \frac{1}{\Delta x} \int_x^{x+\Delta x} \hat{x}\ n\ x'^{n-1} dx'$$

$= \frac{\hat{x}}{\Delta x} \cdot n \left[\frac{x^n}{n}\right]_x^{x+\Delta x} = \frac{\hat{x}}{\Delta x} [(x + \Delta x)^n - x^n] = \frac{\partial\langle\vec{E}\rangle}{\partial x}$. The operations of averaging and differentiation are indeed interchangeable.

4.22 a) Since $\vec{f} = x^2\hat{x}$ then $\vec{E} = x^2\hat{x}$ and $\vec{D} = \varepsilon\vec{E}$ since q = 1 Coulomb. At one face of the differential cube, we have, $\vec{D} = \varepsilon\vec{E} = \varepsilon x^2\hat{x}$, and at the opposite face we have $\vec{D} = \varepsilon[\vec{E} + (\partial\vec{E}/\partial x]\Delta x$. Thus the net flux of

$\vec{D} = \varepsilon \frac{\partial \vec{E}}{\partial x} \Delta x \cdot \Delta y \Delta z = 2x\varepsilon \, \Delta x \Delta y \Delta z$. This net flux must be equated to the total charge inside the cube, that is $\rho_f \, \Delta x \Delta y \Delta z = 2x\varepsilon \, \Delta x \Delta y \Delta z$ or $\rho = 2x\varepsilon \cdot C/m^3$.

b) $\rho_f = \nabla \cdot \vec{D}$. Thus $\rho_f = \varepsilon \partial x^2 / \partial x = 2\varepsilon x \; C/m^3$.

4.23 a) Because of the symmetry, the fields and the potentials will depend on x only. Thus $\nabla \cdot \vec{D} = \rho$ and $\nabla \cdot \vec{D} = 0$ give $\vec{D}_1 = (\rho x + b)\hat{x}$ for $x < d$, $\vec{D}_2 = f\hat{x}$ for $d < x < 2d$. The potentials given by $d\Phi/dx = -E$ give

$$\Phi_1 = \frac{-\rho x^2 - 2bx + 2c}{2\varepsilon}, \quad \Phi_2 = -\frac{fx + g}{\varepsilon_o}$$

Apply the boundary conditions: $\Phi_1(0) = 0 = \Phi_2(2d)$, $\Phi_1(d) = \Phi_2(d)$, $D_1(d) = D_2(d)$. Thus $c = 0$, $b = -\frac{1}{2}\rho d - \rho d\varepsilon/2(\varepsilon + \varepsilon_o)$, $f = -\rho d\varepsilon_o/2\varepsilon$, and $g = -fd$. Thus

$$\Phi_1 = -\frac{\rho x^2}{2\varepsilon} + \frac{\rho d(2\varepsilon + \varepsilon_o)}{2\varepsilon(\varepsilon + \varepsilon_o)} x, \quad \Phi_2 = \frac{\rho d(2d - x)}{2(\varepsilon + \varepsilon_o)}$$

$D_1 = \rho x - \rho d(2\varepsilon + \varepsilon_o)/2(\varepsilon + \varepsilon_o)$, $D_2 = -\rho x d\varepsilon_o/2\varepsilon$.

b) $\sigma_1 = \vec{D} \cdot \hat{x}$ evaluated at $x = 0$. Thus $\sigma_1 = b$.

Force/Area $= d\vec{F}/da = \sigma_1^2 \hat{x}/2\varepsilon = b^2 \hat{x}/2\varepsilon$.

CHAPTER 5

5.1 Given $\rho = 1\ g/cm^3$, $K = 4$, and molecular weight = 59. Thus $N = 6.02 \times 10^{23}/59.07 = 1.02 \times 10^{22}$ molecules/cm^3. From Eq. 5.5 we have $\alpha = \frac{3\varepsilon_o}{N}\ \frac{K-1}{K+2}$. Thus $\alpha = 1.3 \times 10^{-39}\ coul^2 m/N_t$ where N_t is the Newton, the unit of force.

5.2 a) Taking $\vec{E}$ along the z axis, then using Eq. 5.23

$\langle p_z \rangle / p = \langle \cos\theta \rangle = \frac{1}{3}\ pE/kT = 1.3 \times 10^{-6}$.

b) $\vec{p} = N \langle p_z \rangle V\hat{z} = 7 \times 10^{-12}$ C/m, $\vec{P} = N \langle p_z \rangle\ \hat{z} = 7 \times 10^{-7}\ C/m^2$

c) $\langle \cos\theta \rangle = 2.6 \times 10^{-3}$ double the average dipole moment, thus we need $T = 150°$ K.

d) If saturation is achieved, then $\langle p_z \rangle = p$, thus $\hat{p}$ of the block is $N\ pV = 4.95 \times 10^{-5}$ C/m

5.3 a) $\vec{E}_{dipole} = (p/4\pi R^3)\{2\cos\theta\hat{r} + \sin\theta\ \hat{\theta}\}$. The induced dipole is $\vec{p}_i = \alpha\vec{E} = (p\alpha/4\pi\varepsilon_o R^3)\ \{2\cos\theta\hat{r} + \sin\theta\hat{\theta}\}$

b) $U = -\vec{p}\cdot\vec{E} = -\frac{p\alpha}{4\pi\varepsilon_o R^3}[\{2\cos\theta\hat{r} + \sin\theta\hat{\theta}\}] \cdot \frac{p}{4\pi\varepsilon_o R^3}\{2\cos\theta\hat{r} + \sin\theta\hat{\theta}\}$

$$= -\alpha\left(\frac{p}{4\pi\varepsilon_o R^3}\right)^2 (4\cos^2\theta + \sin^2\theta) = -\alpha(p/4\pi\varepsilon_o R^3)^2(1 + 3\cos^2\theta)$$

c) $\langle U\rangle = \frac{\int U(\theta)d\Omega}{\int d\Omega} = -\frac{p^2\alpha(2\pi)/(4\pi\varepsilon_o R^3)^2 \int_o^\pi (1 + 3\cos^2\theta)\sin\theta d\theta}{(4\pi)}$

$$= -\alpha p^2/8\pi^2\varepsilon_o^2 R^6$$

5.4 a) Let θ_1 and θ_2 be the angles p_1 and p_2 make with a line joining the molecules. Then we know the interaction energy $U = (-1/4\pi\varepsilon_o R^3)(3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r}) - \vec{p}_1\cdot\vec{p}_2)$. Now

$\hat{p}_1\cdot\hat{r} = \cos\theta_1$, $\hat{p}_2\cdot\hat{r} = \cos\theta_2$,

$\hat{p}_1\cdot\hat{p}_2 = \sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2 + \sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2 + \cos\theta_1\cos\theta_2 = \sin\theta_1\sin\theta_2(\cos(\phi_1 - \phi_2)) + \cos\theta_1\cos\theta_2$.

$U = (-p_1p_2/4\pi\varepsilon_o R^3)(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2(\cos(\phi_1 - \phi_2))$.

b) $P(\theta_1,\theta_2,\phi_1,\phi_2) = \frac{\exp(-U/kT)}{\int_o^\pi \sin\theta_1' d\theta_1' \int_o^\pi \sin\theta_2' d\theta_2' \int_o^{2\pi} d\phi_1 \int_o^{2\pi} d\phi_2 \exp(-U/kT)}$

c) Let $\phi = \phi_1 - \phi_2$

$$\langle U\rangle_T = \frac{\int_o^\pi \sin\theta_1 d\theta_1 \int_o^\pi \sin\theta_2 d\theta_2 \int_o^{2\pi} d\phi\, U \exp(-U/kT)}{\int_o^\pi \sin\theta_1 d\theta_1 \int_o^\pi \sin\theta_2 d\theta_2 \int_o^{2\pi} d\phi \exp(-U/kT)}$$

Taking $U/kT \ll 1$ then $\exp(-U/kT) \approx 1 - U/kT$. Let us call Denominator = D and Numerator = N of the above integral

$$D = \int_o^\pi \sin\theta_1 d\theta_1 \int_o^\pi \sin\theta_2 d\theta_2 \int_o^{2\pi} d\phi(1 + p_1p_2/(4\pi\varepsilon_o R^3 kT) \times (2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\phi))$$

Note that $\int_o^{2\pi} \cos\phi d\phi = 0$, and $\int_o^\pi \sin\theta \cos\theta d\theta = 0$, then $D = 8\pi$

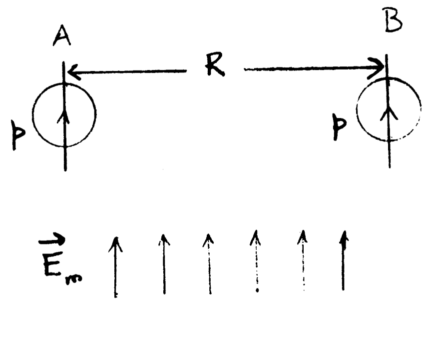
$N = \int_o^\pi \sin\theta_1 d\theta_1 \int_o^\pi \sin\theta_2 d\theta_2 \int_o^{2\pi} d\phi[U - U^2/kT]$. Note that we have already calculated the first term in this integral, so we need to calculate the second term. Remember that U^2 is proportional to $4\cos^2\theta_1\cos^2\theta_2 - 4\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\cos\phi +$ $\sin^2\theta_1\sin^2\theta_2\cos^2\phi$. The middle term goes to zero since $\int_0^{2\pi}\cos\phi d\phi = 0$, but $\int_0^{\pi}\cos^2\theta\sin\theta d\theta = 2/3$, and $\int\sin^2\theta d\theta = 4/3$, therefore $N = 4\cdot 2/3\cdot 2\pi + 4/3\cdot 4/3\cdot\pi = 48\pi/9 = 10\pi/3$ so

$$\langle U\rangle = \frac{16\pi/3}{8\pi}\cdot\frac{p_1^2 p_2^2}{16\pi^2\varepsilon_o^2 R^6 kT} = \frac{p_1^2 p_2^2}{24\pi^2\varepsilon_o^2 R^6 kT}$$

5.5 **a)** Using Eq. 5.17, we write $p_o = 2\alpha\, E_o/[1-(2\alpha/4\pi\varepsilon_o R_o^3)]$. Solving for $2\alpha = p_o/[4\pi\varepsilon_o R^3 p_o + E_o]$.

b) The molecular polarizability for all R is given by Eq. 5.17 $\alpha' = 2\alpha/[1-(2\alpha/4\pi\varepsilon_o R^3)]$.

5.6 Consider the figure where it shows the atoms and the external field. From Eq. 5.14 $\vec{p} = \alpha(\vec{E}_m + \vec{E}')$ where $\vec{E}'$ is an



additional electric field produced by each atom at the site of the other. Using Eq. 2.46 for $\theta = 90°$ we get $\vec{E}'(\theta = 90°) = -p\hat{z}/4\pi\varepsilon_o R^3$. Thus $\vec{E}' = -\alpha(\vec{E}_m + \vec{E}')/4\pi\varepsilon_o R^3$. Solving for $\vec{E}'$, and hence for $\vec{p}$ and $\vec{\alpha}$ gives

$$\vec{E}' = -\frac{\alpha \vec{E}_m}{4\pi\varepsilon_o R^3[1 + \alpha/4\pi\varepsilon_o R^3]}, \quad \vec{p} = \frac{-\alpha \vec{E}_m}{1 + \alpha/4\pi\varepsilon_o R^3}$$

$$\alpha' = 2\alpha/(1 + \alpha/4\pi\varepsilon_o R^3)$$

5.7 Using Eqs. 5.9 and 5.21 we get $\alpha \rightarrow \alpha' + \alpha =$ $4\pi\varepsilon_o R^3 + p_o(\coth \eta - \frac{1}{\eta})$ where $\eta = p_o E_o/kT$. At high temperatures we have $\alpha = 4\pi\varepsilon_o R^3 + p_o{}^2/3kT$. The average dipole moment is $\langle p \rangle = (4\pi\varepsilon_o R^3 + p_o{}^2/3kT)E_o$.

5.8 Using Eq. 5.36 we write $K - 1 = 3/\delta(T-T_c)$. Since K is very large, then $K - 1 \simeq K$, and we have $1/K = \delta(T-T_c)$. The points (1/K, T): (.0035, 133) and (.0105, 153) define the line and we get $\delta = 1.05 \times 10^{-3}$ and $T_c = 123°K$.

5.9 The ferroelectric condition is given by Eq. 5.34: $N\alpha/3\varepsilon_o = 1$. But from Eq. 5.5 we have $N\alpha/3\varepsilon_o = (K - 1)/(K + 2) = .00096$ which is different from 1; thus it is not ferroelectric. For the liquid we have $N\alpha/3\varepsilon_o = .367$ which is not ferroelectric. In this case $K = 2.74$.

5.10 In Eq. 4.4 we have $\sigma_p = -(K-1)\sigma_f/K$. For ferroelectric materials $K \gg 1$, hence $\sigma_p \simeq \sigma_f$.

CHAPTER 6

6.1 See example 6.1.

6.2 $\Phi(r) = A\exp(-\alpha r^3)$.

a) From Poisson equation we have $\nabla^2\Phi = -\rho/\varepsilon_o$ or $\rho = -\varepsilon_o\nabla^2\Phi$.

Note Φ has only radial dependence, thus

$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\Phi}{dr})$. Now $\frac{d\Phi}{dr} = -3\alpha r^2 A\exp(-\alpha r^3)$, and

$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\Phi}{dr}) = \frac{1}{r^2}\frac{d}{dr}(-3\alpha r^4 A\exp(-\alpha r^3))$

$= A/r^2\ (-12\alpha r^3\exp(-\alpha r^3) + 9\alpha^2 r^6\exp(-\alpha r^3))$

$= 3A\alpha r\exp(-\alpha r^3)(3\alpha r^3-4)$. Thus the density is

$\rho = 3A\varepsilon_o\alpha r(4-3\alpha r^3)\exp(-\alpha r^3)$.

b) $V = (1/2)\int\rho\Phi dv = (3/2)\ A^2\varepsilon_o\alpha\int r\exp(-2\alpha r^3)(4-3\alpha r^3)\ dv$

$= 4\pi\ (3/2)\ A^2\varepsilon_o\alpha\int\exp(-2\alpha r^3)(4-3\alpha r^3)r^3 dr$.

We know $\int_0^\infty x^{n-1}\exp(-x)\ dx = \Gamma(n)$ with $\Gamma(n)$ the gamma function

$\Gamma(n+1) = n\Gamma(n)$, and $\Gamma(n+1) = n!$, if n is a positive integer.

Let $x = 2\alpha r^3$ and put the integral in the form of the Γ integral

we get $V = \frac{4\pi\varepsilon_o A^2}{2^{4/3}\alpha^{1/3}}\ \Gamma(4/3)$.

6.3 We use Eq. 6.9: $\Phi_j = \sum_{k=1}^{3} P_{jk}Q_k$, where the coefficients of potential are: $P_{jj} = 1/4\pi\varepsilon_o a$, $P_{jk} = 1/4\pi\varepsilon_o\ell$ for $j \neq k$ since the spheres are identical and they are at the same distance from each other. Grounding sphere 1 changes the charge on it but does not affect the charge on the other two. Thus $\Phi_1 = 0 =$

$(q_1/a) + (q/\ell) + (q/\ell)$. Solving for q_1 gives $q_1 = -2qa/\ell$. Now, ungrounding sphere 1 and grounding 2 gives $\Phi_2 = 0 = (q_1/\ell) + (q_2/a) + (q/\ell)$ which gives $q_2 = -\frac{qa}{\ell}(1 - \frac{2a}{\ell})$. Now ungrounding 2, and grounding 3 gives $\Phi_2 = 0 = (q_1/\ell) + q_2/\ell) + (q_3/a)$. This solves for $q_3 = \frac{qa^2}{\ell^2}(3 - \frac{2a}{\ell})$. The "until equilibrium" refers to allowing the charge on each sphere to adjust to the external potential. You do not have to keep grounding spheres.

6.4 Since the force between the spheres is given by $F = -\partial U/\partial r$ where U is the electrostatic energy of the system, and r is the distance between them, then we need to calculate U as a function of r. We use the coefficients of potential to solve the problem. We use Eqs. 6.8 and 6.12-6.13.

$U_1 = 1/2 \sum_j Q_j \Phi_j$, $\Phi_1 = P_{11}Q_1 + P_{12}Q_2$, $\Phi_2 = P_{21}Q_1 + P_{22}Q_2$ where $P_{11} = P_{22}$, and $P_{21} = P_{12}$ because of the symmetry. Now when $Q = q_1$, $Q_2 = 0$, and $\Phi_1 = V$. These equations give $V = P_{11}q_1$ and $\Phi_2 = P_{21}q_1 = P_{12}q_1$ which can be solved for $P_{11} = V/q_1$. The energy of the system is $U = 1/2\, Q_1^2\, P_{11}$, and hence from the force $F = 1/2\, q_1^2 \left(\frac{dP_{11}}{dr}\right)$ we get $dP_{11}/dr = 2F/q_1^2$. In the second operation, we have $\Phi_1 = P_{11}q_1 + P_{12}q_2$, and $V = P_{12}q_1 + P_{11}q_2$ which gives $P_{12} = V(q_1 - q_2)/q_1^2$, and hence $U_2 = 1/2\, P_{11}(q_1^2 + q_2^2) + P_{12}q_1q_2$. From the force between the spheres we can calculate dP_{12}/dr as follows:

$$-F = 1/2\,(q_1^2 + q_2^2)\,\frac{dP_{11}}{dr} + q_1q_2\,\frac{dP_{12}}{dr} \text{ or}$$

$dP_{12}/dr = -F(2q_1^2 + q_2^2)/q_1^3 q_2$.

Now that we know P_{11}, P_{12} and their derivatives with r we can calculate the properties of the last operation. In the last operation sphere 1 is grounded, thus $\Phi_1 = 0$, while the charge on the second does not change, therefore $0 = P_{11}Q_1 + P_{12}q_2$ or $Q_1 = -q_2(q_1 - q_2)/q_1$. The force between them can be calculated from $F = -\partial U_3/\partial r$ where $U_3 = \frac{1}{2} P_{11}(Q_1^2 + q_2^2) + P_{12}Q_1 q_2$, or $F = q_2(2q_1^2 - q_2^2)\ F/q_1^3$.

6.5 a) See Ex. 6.8. By symmetry $P_{11} = P_{22} = P_{33} = P_{44}$. Now $\Phi_1 = \dfrac{q}{4\pi\varepsilon_o a} = P_{11}q$. thus $P_{11} = 1/4\pi\varepsilon_o a$. By Symmetry: $P_{12} = P_{21} = P_{14} = P_{41} = P_{34} = P_{43} = 1/4\pi\varepsilon_o \ell = P_{23} = P_{32}$. Also by symmetry: $P_{13} = P_{31} = P_{42} = P_{24} = 1/4\pi\varepsilon_o(\sqrt{2}\,\ell)$.

c) (1) When we connect 1 and 2 then the charge merely splits because of symmetry, that is $q_1 = q/2$, $q_2 = q/2$. (2) When we connect 1 and 3, then because spheres 1 and 3 are symmetric with respect to sphere 2, the charge splits equally $q_1 = q/4$, and $q_3 = q/4$. (3) Similarly there is symmetry in the last operation and we get $q_1 = q/8$ and $q_4 = q/8$. Thus the final charges are $q/8$, $q/2$, $q/2$, $q/8$.

6.6 a) See example 6.6.

b) From above we have $C = 4\pi\varepsilon_o ab/(b - a)$. Now if $b - a = d \ll a,b$ then $C \approx 4\pi\varepsilon_o a^2/d = A\varepsilon_o/d$ with $A = 4\pi a^3$ the surface area of the sphere.

6.7 Let us call the fields in the regions $r < a$, $a < r < b$, $b < r < c$ and $r > c$ as $\vec{E}_1$, $\vec{E}_2$, $\vec{E}_3$ and $\vec{E}_4$ respectively.

a) $\vec{E}_1 = 0$, $\vec{E}_2 = (Q/4\pi\varepsilon_o r^2)\,\hat{r}$, $\vec{E}_3 = 0$, $\vec{E}_4 = (Q/4\pi\varepsilon_o r^2)\hat{r}$

b) $\Phi = -\int_\infty^a \vec{E}\cdot d\vec{r} = -\int_\infty^c \vec{E}_4\cdot d\vec{r} - \int_c^b \vec{E}_3\cdot d\vec{r} - \int_b^a \vec{E}_2\cdot d\vec{r}$

$= \dfrac{Q}{4\pi\varepsilon_o c} + \dfrac{Q}{4\pi\varepsilon_o}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$

c) $C = Q/V = \left(\dfrac{1}{4\pi\varepsilon_o}\left(\dfrac{1}{c} + \dfrac{1}{a} - \dfrac{1}{b}\right)\right)^{-1}$

d) We know that capacitors in series add inversely

$\dfrac{1}{C_T} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$ with $C_1 = 4\pi\varepsilon_o c$ and $C_2 = 4\pi\varepsilon_o/(\frac{1}{a} - \frac{1}{b})$.

6.8 **a)** $U = \frac{1}{2}CV^2 = \frac{1}{2}(\varepsilon_o A/d)\,V^2$.

b) Treat the system as two capacitors in series. Let the sheet of metal be a distance b away from one side of the capacitor, then $1/C_T = 1/C_1 + 1/C_2$ with $C_1 = A\varepsilon_o/b$ and $C_2 = A\varepsilon_o/(d - b - a)$. Thus $C_T = A\varepsilon_o/(d - a)$. Now during the operation the charge stays constant. Thus $U = Q^2/2C_i - Q^2/2C_f$. Using $C_i = A\varepsilon_o/d$ and $C_f = A\varepsilon_o/(d-a)$ gives $U = A\varepsilon_o a\, V^2/d^2$.

c) Since Q = constant, then $Q = C_iV_i = C_fV_f$ which gives $V_f = C_iV/C_f = V(d - a)/d$.

6.9 We use the results of Ex. 3.5.

$\Phi = \dfrac{q}{4\pi\varepsilon_o}\left(\dfrac{1}{r} - \dfrac{1}{R_2}\right) + \dfrac{\delta q}{4\pi\varepsilon_o(R_2^3 - R_1^3)}\left(r - \dfrac{R_1^3}{r^2}\right)\cos\theta$ to first order in δ. Now it is important to realize that this result was derived from the boundary condition $\Phi|_{\text{sphere }\#2} = 0$. (Indeed, if you plug $r = R_2 + \delta \cos\theta$ in the above result, after much algebra you will get $\Phi_2 = 0$). Therefore we have

$$\Phi_1 = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \quad \Phi_2 = 0,$$

$$\Phi_1 - \Phi_2 = \frac{q}{4\pi\varepsilon_o R_1 R_2}(R_2 - R_1) = \Delta V,$$

$C = \frac{q}{\Delta V} = \frac{4\pi\varepsilon_o R_1 R_2}{R_2 - R_1}$ which is the capacitance of a concentric spherical conductor. Thus, there is no change in the capacitance in the non-concentric case to first order in δ. For a larger δ, we have to include higher order terms to Φ. The next term is the $P_2(\cos\theta)$ dependence which would be proportional to δ^2. In this case the potential difference will depend on δ^2, thus giving a change in C. THe next correction is found to be

$$\Delta C = \delta^2 \frac{R_1^2 R_2^2}{(R_2 - R_2)^2 (R_2^3 - R_1^3)}.$$

6.10 **a)** $W = QV/2 = Q^2/8\pi\varepsilon R$.

b) Since $\vec{E} = q/4\pi\varepsilon r^2 \,\hat{r}$ and $\vec{D} = q/4\pi r^2 \,\hat{r}$, then

$W = \frac{1}{2} \int \vec{E}\cdot\vec{D}\, dv = \frac{\varepsilon Q^2}{4\pi} \int_R^\infty \frac{1}{r^2}\, dr = \frac{Q^2}{8\pi\varepsilon R}$ which is the same as calculated in (a).

c) $W = \frac{Q^2}{8\pi\varepsilon R}$. Thus $\Delta W = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{R} - \frac{1}{R'}\right) = \frac{Q^2}{8\pi\varepsilon}(\Delta R/RR')$.

d) The energy goes into mechanical work. Consider the sphere at a radius r. The force acting on an element of surface area is $d\vec{F} = \sigma^2 da\, \hat{n}/2\varepsilon$ where $\sigma = Q/4\pi r^2$. The total pressure acting on the surface is then $F_r = \int dF = (1/2\varepsilon) \frac{Q^2}{(4\pi r^2)} \times 4\pi r^2 = \frac{Q^2}{8\pi\varepsilon r^2}$.

The mechanical work is $W_m = \int_R^{R'} F_r dr = \frac{Q^2}{8\pi\varepsilon}\int_R^{R'} \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{R} - \frac{1}{R'}\right)$.

6.11 a) The large conducting sphere is grounded but has charge Q_1, thus $\Phi_1 = P_{11}Q_1 + P_{12}Q_2 = 0$ where P_{11} is the "self" coefficient of potential $1/4\pi\varepsilon_o R_1$. Due to symmetry we have $P_{12} = P_{21} = 1/4\pi\varepsilon_o d$. Now $Q_2 = q$, thus $P_{11}Q_1 = -P_{12}Q_2$ which gives $Q = -qR_1/d$.

b) If the sphere is then neutral, $Q_1 = 0$, and hence $\Phi_1 = P_{11}Q_1 + P_{12}Q_2 = q/4\pi\varepsilon_o d$.

c) From Ex. 3.52, the image method gives the same results as above.

6.12 The field at r inside the sphere is found to be equivalent to all charge in the region inside a sphere of radius r, all at the origin. This is only true for spherical symmetry (no θ, ϕ dep.) $\vec{E}_1(r) = \frac{\hat{r}\, q(<r)}{4\pi\varepsilon_o r^2} = \frac{Qr^3}{R^3}\frac{\hat{r}}{4\pi\varepsilon_o r^2} = \frac{Qr}{4\pi\varepsilon_o R^3}\hat{r}$ for $r \le R$.

For $r > R$, consider all charge at the origin. Thus $\vec{E}_2(r) = \frac{Q}{4\pi\varepsilon_o r^2}$ for $r \ge R$. Now

$$U = \frac{\varepsilon_o}{2}\int_{\substack{\text{all}\\ \text{space}}} E^2\, dv \text{ or } U = \frac{\varepsilon_o}{2}\int_o^R E_1^2(r)\, 4\pi r^2 dr +$$

$$\frac{\varepsilon_o}{2}\int_R^\infty E_2^2(r)\, 4\pi r^2 dr = \frac{Q^2}{8\pi\varepsilon_o R^6}\int_o^R r^4 dr + \frac{Q^2}{8\pi\varepsilon_o}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_o R^6} \times \frac{R^5}{5} + \frac{Q^2}{8\pi\varepsilon_o} \times \frac{1}{R} = \frac{3Q^2}{20\pi\varepsilon_o R}.$$

6.13 $\Phi = Ae^{-\alpha r^3}$, $\vec{E} = -\nabla\Phi = 3A\alpha r^2 e^{-\alpha r^3}\,\hat{r}$. Thus

$$W = (1/2)\int \varepsilon_o E^2\, dv = 18\pi\, A^2\alpha^2\varepsilon_o 4\pi \int e^{-2\alpha r^3} r^2\, dr$$

$$= \frac{4\pi\varepsilon_o A^2}{2^{4/3}\alpha^{1/3}}\,\Gamma\,(4/3) \text{ as in 6.2.}$$

6.14 1) In the first case the outer shell is neutral. Because the outershell is neutral; then from Gauss' law there will be $\vec{E} = \frac{q\,\hat{r}}{4\pi\varepsilon_o r^2}$ around each sphere, including the outside outer sphere. With charge + q on the inner sphere (outside surface), a charge −q will appear on the inner surface of the outer sphere and +q on the outer surface of the outer sphere. Thus from Eq. 6.8 the energy of the system is

$U_{int} = (q_1^2 + q_2^2)/8\pi\varepsilon_o r_1$ independent of r_2. 2) When the outer spheres are connected, we find because of symmetry, that the charges on the very outer surfaces of the outer spheres equalize, while the charges on their inner surfaces or the charges on the inner sphere are unaffected. Thus $q_1' = q_2' = q' + (q_1 + q_2)/2$, corresponding to a flow of charge $\pm(q_1 - q_2)/2$.

Now $U_{final} = (q_1^2 + q_2^2)/(8\pi\varepsilon_o)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{2\,q'^2}{8\pi\varepsilon_o r_2}$

$$\Delta U = -\frac{(q_1 - q_2)^2}{16\pi\varepsilon_o r_2} = \frac{-2\,(q_1 - q_2)^2}{4}\,\frac{1}{8\pi\varepsilon_o r_2}.$$ Thus the energy change comes from the charge that flowed.

6.15 We are given that $\Phi = -(6\phi/\pi)\, V_o$.

a) $\vec{E} = -\nabla\Phi = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\,\hat{\phi} = \frac{6V_o}{\rho\pi}\,\hat{\phi}$ $u = (1/2)\,\varepsilon_o E^2 = 18\varepsilon_o V_o^2/\rho^2\pi^2$

b) $U = \int u\,dv = 18\varepsilon_o V_o^2/\pi^2 \int_o^{\pi/6} d\phi \int_o^1 dz \int_{.1}^{.6} d\rho/\rho = 3\varepsilon_o(\ln 6)V_o^2/\pi.$

6.16 a) $U = \frac{1}{2}\left(\frac{q_1q_2}{4\pi\varepsilon_o r_{12}} + \frac{q_2q_1}{4\pi\varepsilon_o r_{21}}\right).$

But $r_{12} = r_{21} = d$, thus $U = q_1q_2/4\pi\varepsilon_o d$.

b) Let us take the z axis to be along the line joining the charges $U = \frac{\varepsilon_o}{2}\int E^2 dv$. But $\vec{E} = \frac{q_1\hat{r}}{4\pi\varepsilon_o r^2} + \frac{q_2(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_o|\vec{r} - \vec{r}\,'|^3}$.

$$U = \frac{1}{8\pi}\int\left(\frac{q_1^2}{r^4} + \frac{q_2^2}{|\vec{r} - d\hat{z}|^2}\right)dv + \frac{q_1q_2}{16\pi^2\varepsilon_o}\int\frac{(\vec{r} - \vec{r}\,')\cdot\hat{r}\,dv}{r^2|\vec{r} - \vec{r}\,'|^3}.$$

The first two terms constitute the self energy, that is the energy needed to assemble the charges themselves. This is a divergent quantity. The last term, on the other hand, constitute the interaction energy. This integral can be evaluated easily if one recognizes that $dv = da\,dr$ where $da = r^2\sin\theta\,d\theta\,d\phi$ and $d\vec{a} = \hat{r}\,da$. (See Eqs. 1.22 and 1.27). Thus $U_{int} = \frac{q_1q_2}{16\pi^2\varepsilon_o}\int\frac{dr}{r^2}\int\frac{(\vec{r} - \vec{r}\,')\cdot d\vec{a}}{|\vec{r} - \vec{r}\,'|^3}$. Now the area integral is just 4π since the integrand is $d\Omega$. (See Eqs. 1.23 and 2.23.) Thus $U_{int} = \frac{q_1q_2}{4\pi\varepsilon_o}\int_\infty^d \frac{dr}{r^2} = -\frac{q_1q_2}{4\pi\varepsilon_o d}$. Thus the difference between the two methods is the self energy and the minus sign. We have a sign difference because what we just calculated is the energy stored.

6.17 $V = 3x^2 + 4y^2$, $\vec{E} = -\nabla V = (6x\hat{x} + 8y\hat{y})$.

$U = \int u\,dv$ where $u = \varepsilon_o E^2/2 = \varepsilon_o(18x^2 + 32y^2)$. Thus

$$U = \varepsilon_o \int_0^1 dz \int_0^1 dx \int_0^1 dy\ (18x^2 + 32y^2) = 50\ \varepsilon_o/3J$$

6.18 a) First we use $U = (1/2) \int_s \sigma\,\Phi da$ where σ = surface charge density $= Q/4\pi R^2$, but $Q = CV = 4\pi\varepsilon_o RV_o$ thus $\sigma = \varepsilon_o V_o/R$ and therefore $U = 2\pi R\varepsilon_o V_o^2$.

b) Now for $r < R$ we have $\vec{E} = 0$, and for $r > R$ we have $\vec{E} = V_o R/r^2$. Thus

$$U = (\varepsilon_o/2) \int E^2 dv = (\varepsilon_o/2) \int_R^\infty \frac{V_o^2 R^2}{r^4} dv = 2\pi R\varepsilon_o V_o^2.$$

6.19 a) From Problem 3.2 we have $\Phi = \frac{V}{\beta}\phi$.

b) From Problem 3.2 we have at plates $\phi = 0$, and $\phi = \beta$.

$Q = (-V\varepsilon_o h/\beta)\ln \rho_2/\rho_1$, $\sigma = -V\varepsilon_o/\beta\rho$, $Q = (V\varepsilon_o h/\beta)\ln \rho_2/\rho_1$,

$\sigma = V\varepsilon_o/\beta\rho$.

c) $C = \frac{Q}{\Delta V} = (h\varepsilon_o/\beta)\ln(\rho_2/\rho_1)$.

d) $U = \frac{1}{2} CV^2 = (h\varepsilon_o V^2/2\beta)\ln(\rho_2/\rho_1)$. Consider a virtual rotation of one plate in $\hat{\phi}$ direction, then

$$\vec{\tau} = \frac{\partial U}{\partial \beta}\hat{z} = (-h\varepsilon_o V^2/2\beta^2)\ln(\rho_2/\rho_1)\hat{z}.$$

6.20 From the results of problem 4.6 which were arrived at using Gauss' law, we find that the electric field in the slab using $|\vec{F}| = \int E dq$ where $dq = A\rho dx$, $E = [\sigma_1 + \rho(x-h)]/\varepsilon$, thus

$$|\vec{F}| = \frac{\rho A}{\varepsilon} \int_h^{h+t} (\sigma_1 + \rho(x - h))dx = \frac{\rho A}{\varepsilon} (\sigma x + \frac{1}{2}\rho x^2) \Big|_h^{h+t}$$

$$\frac{F}{A} = \frac{\rho}{\varepsilon}(\sigma_1 t + \frac{1}{2}\rho t^2)$$

6.21 **a)** The stable configuration is when the rod is aligned with the external field. The unstable one is when the rod is perpendicular to the field. A force arises because a dipole moment is induced in each of the conductors which exerts a force on the other dipole. From Ex. 3.7, the potential field and the moment due to the presence of one sphere are:

$$\Phi = \frac{E_o a^3}{r^2}\cos\theta,\ \vec{E} = \frac{p}{4\pi\varepsilon_o r^3}\{2\cos\theta\hat{r} + \sin\theta\hat{\theta}\},\ \vec{p} = 4\pi\varepsilon_o a^3 E_o \hat{z}.$$

Because $\ell >> a$, then one takes this field to represent the field near each of the two spheres. The force of one dipole on each sphere (we are only concerned with the force in the $\hat{\theta}$ direction) is $F_\theta = (p\cdot\nabla)E_\theta = p\frac{\partial E_\theta}{\partial z} = \frac{p^2}{4\pi\varepsilon_o}\frac{\partial}{\partial z}\left(\frac{\rho}{r^4}\right)$ or $F_\theta = \frac{p^2}{2\pi\varepsilon_o \ell^4}(\sin 2\theta)$.

The total torque is just: $|\vec{\tau}| = \frac{p^2}{2\pi\varepsilon_o \ell^3}(\sin 2\theta)$.

c) The work necessary is just $W = \int\vec{\tau}\cdot d\vec{\theta}$ or

$$W = \frac{p^2}{2\pi\varepsilon_o \ell^3}\int_o^{\pi/2}\sin 2\theta d\theta = -8\pi\varepsilon_o a^6 E_o^2/\ell^3 .$$

6.22 We use the notations of Fig. 3.13. The potential energy of the dipole is $U = -\vec{p}\cdot\vec{E}$ where $\vec{E}$ is the electric field of its image. Now

$$\vec{E} = \frac{p_o}{4\pi\varepsilon_o(2d)^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) = \frac{p_o}{4\pi\varepsilon_o(2d)^3}(2\cos\theta\hat{z} - \sin\theta\hat{x}).$$ But

$p = p_o\cos\theta\hat{z} + p_o\sin\theta\hat{x}$. Thus

$$U = -\frac{p_o^2}{4\pi\varepsilon_o(2d)^3}(2\cos^2\theta - \sin^2\theta) = -\frac{p_o^2}{4\pi\varepsilon_o(2d)^3}(3\cos^2\theta - 1).$$ Thus

the energy needed is $W = -U$.

6.23 a) We replace the sphere by an image charge q' located at Z_o where $q' = \frac{-bq}{r}$, and $Z_o = \frac{b^2}{r}$ as given by the method of images. Thus the force F between the two charges is

$$F = \frac{q'q}{4\pi\varepsilon_o(r - Z_o)^2} = \frac{-bq^2/r}{4\pi\varepsilon_o\left(r - \frac{b^2}{r}\right)^2} = \frac{-bq^2 r}{4\pi\varepsilon_o(r^2 - b^2)^2}.$$

$$W = \int_\infty^r \vec{F}(r')\cdot d\vec{r}' = \frac{bq^2}{4\pi\varepsilon_o}\int_\infty^r \frac{r'dr'}{(r'^2 - b^2)^2}$$ or

$$W = \frac{-bq^2}{8\pi\varepsilon_o}\left\{\frac{1}{r'^2 - b^2}\right\}\Bigg|_\infty^r = \frac{-bq^2}{8\pi\varepsilon_o(r^2 - b^2)}.$$

b) Yes as position of real charge changes, the induced charge on the sphere changes. In other words, electrons flow to or from ground to change the induced charge on the sphere.

c) If we isolate the sphere and place a charge Q on it, it is like placing a charge $Q - q'$ at the center of the sphere (see Eq. 3.116). Thus the force becomes:

$$F = \frac{-bq^2 r}{4\pi\varepsilon_o(r^2 - b^2)^2} + \frac{q(Q - q')}{4\pi\varepsilon_o r^2} = \frac{-bq^2 r}{4\pi\varepsilon_o(r^2 - b^2)^2} + \frac{q(Q + \frac{bq}{r})}{4\pi\varepsilon_o r^2}.$$

$$W = \int_\infty^r \vec{F}\cdot d\vec{r} = \frac{-bq^2}{8\pi\varepsilon_o(r^2 - b^2)} + \int_\infty^r \frac{-q}{4\pi\varepsilon_o}\left(\frac{Q}{r^2} + \frac{bq}{r^3}\right)dr.$$

$$W = \frac{-bq^2}{8\pi\varepsilon_o(r^2 - b^2)} + \frac{q}{4\pi\varepsilon_o}\left(\frac{Q}{r} + \frac{bq}{2r^2}\right).$$

d) The difference is clearly $\Delta W = \frac{q}{4\pi\varepsilon_o}\left(\frac{Q}{r} + \frac{bq}{2r^2}\right)$. This is just the work done in moving the point charge against the charged sphere.

6.24 The energy density is given by Eq. 6.61: $u = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{2}\vec{E}\cdot(\varepsilon_o\vec{E} + \vec{P})$. From problem 4.19 the potential and the field inside and outside the cylinder

$$\Phi_1 = (P_o/2\varepsilon_o)\rho\cos\phi, \quad \vec{E}_1 = -P_o\hat{x}/2\varepsilon_o$$

$$\Phi_2 = \frac{P_o}{2\varepsilon_o}\frac{\rho_o^2}{\rho}\cos\phi, \quad \vec{E}_2 = \frac{P_o}{2\varepsilon_o}\frac{\rho_o^2}{\rho^2}(\cos\phi\hat{\rho} + \sin\phi\hat{\phi}).$$

Thus the energy density outside the cylinder is $u_o = P_o^2\rho_o^2/8\varepsilon_o\rho^4$. For the region inside the cylinder we have $u_i = -P_o^2/2\varepsilon_o$.

CHAPTER 7

7.1 Using Eq. 7.4, $\vec{J} = \rho\langle\vec{v}\rangle$, we get $I/A = ne\langle\vec{v}\rangle$. Thus

$$\langle\vec{v}\rangle = \frac{I}{Ane} = 3.1 \text{ cm/s}.$$

7.2 Now equipotentials are spherical shells, and the field $\vec{E}$ and the current $\vec{J}$ are radial. Thus using $d\ell = dr$, and $A = 4\pi r^2$ we get:

$$R = \int_a^b \frac{dr}{\sigma_c 4\pi r^2} = \frac{1}{4\pi\sigma_c}\left(\frac{1}{a} - \frac{1}{b}\right).$$

7.3 $R = \int_a^c \frac{dr}{4\pi r^2 \sigma_{c_1}} + \int_c^b \frac{dr}{4\pi r^2 \sigma_{c_2}}$

$$R = \frac{1}{4\pi\sigma_{c_1}}\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{1}{4\pi\sigma_{c_2}}\left(\frac{1}{c} - \frac{1}{b}\right)$$

7.4 $R = \int_a^b \frac{d\rho}{\sigma_c 2\pi\rho\ell} = \frac{1}{2\pi\sigma_c\ell}\,\ell n\rho\,\Big|_a^b = \frac{1}{2\pi\sigma_c\ell}\,\ell n\left(\frac{b}{a}\right)$

7.5 a) Using $\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ and $\vec{J} = \sigma_c\vec{E}$, we get $\sigma_c\nabla\cdot\vec{E} = -\frac{\partial\rho}{\partial t}$.

Integrating over the volume of the capacitor we get

$$\sigma_c\int\nabla\cdot\vec{E}dv = -\frac{\partial}{\partial t}\int\rho dv = -\frac{dQ}{dt}.$$

By the divergence theorem we have $\int\nabla\cdot\vec{E}dv = \int\vec{E}\cdot d\vec{a}$, and

$$\sigma_c\int\vec{E}\cdot d\vec{a} = -\frac{dQ}{dt}.$$

Now for a parallel plate capacitor, $\vec{E} = Q/K\varepsilon_o A$, then

$$\sigma_c \frac{Q}{K\varepsilon_o A} A = -\frac{dQ}{dt} \quad \text{or} \quad \frac{dQ}{dt} = -\frac{\sigma_c Q}{K\varepsilon_o} .$$

The solution is $Q(t) = Q_o e^{-(\sigma_c/K\varepsilon_o)t}$.

b) The rate at which heat is generated is

$$P = \frac{dW}{dt} = \int \vec{J}\cdot\vec{E}dv$$

$$P = \int_v \vec{J}\cdot\vec{E}dv = \int \sigma_c E^2 dv = \frac{\sigma_c Q^2 V}{(K\varepsilon_o A)^2}$$

Total heat $= \dfrac{\sigma_c dQ_o^2}{(K\varepsilon_o)^2 A} \displaystyle\int_o^\infty e^{-2(\sigma_c/K\varepsilon_o)t} dt = \dfrac{Q_o^2 d}{2AK\varepsilon_o}$. Originally, the

energy is $Q^2/2C = Q_o^2/2K\varepsilon_o(A/d)$. Thus the total heat produced is equal to the original energy.

c) $\tau = \dfrac{K\varepsilon_o}{\sigma_c} = 3.8 \times 10^{-24}$sec.

7.6 We will use the boundary conditions: the normal $\vec{J}$ is continuous $J_{n1} = J_{n2}$, and the tangential E is continuous: or $\dfrac{J_{t1}}{\sigma_{1c}} = \dfrac{J_{t2}}{\sigma_{2c}}$. But $J_{n1} = J_1\cos\theta_1$, $J_{n2} = J_2\cos\theta_2$, $J_{t1} = J_1\sin\theta_1 = J_{n1}\tan\theta$, $J_{t2} = J_2\sin\theta_2 = J_{n2}\tan\theta_2$. Then using $J_{n1} = J_{n2}$ we get $J_{n1}\tan\theta_1/\sigma_{1c} = J_{n2}\tan\theta_2/\sigma_{2c}$ or $\sigma_{1c}\tan\theta_2 = \sigma_{2c}\tan\theta_1$.

7.7 a) From Eq. 7.38 with no external electromotive fields we have: $\nabla\cdot(\sigma_c\nabla\Phi) = (\nabla\sigma_c)\cdot(\nabla\Phi) + \sigma_c(\nabla^2\Phi)$. Note that the current of the E field and the potential depend only on x. Using

$\nabla\Phi = \hat{x}\frac{d\Phi}{dx}$, $\vec{\nabla}\sigma_c = a\hat{x}$ and $(\nabla^2\Phi) = \frac{d^2\Phi}{dx^2}$, we get

$$\alpha(a + x)\frac{d^2\Phi}{dx^2} + \alpha\frac{d\Phi}{dx} = 0.$$

b) $E_x = -\frac{d\Phi}{dx}$. Thus $(a + x)\frac{dE_x}{dx} + E_x = 0$ or $\frac{dE_x}{E_x} = -\frac{dx}{a + x}$ which integrates to $\ln E_x = -\ln(a + x) + \text{const}$ or $E_x = A/(a + x)$.

$$\Phi_a - \Phi_o = -\int_a^o E_x dx = A\int_o^a \frac{dx}{a + x} = A\int_a^{2a}\frac{dx}{x} = A\ln\left(\frac{2a}{a}\right) = A\ln 2$$

c) Since $\vec{J} = \sigma_c\vec{E} = \alpha(a + x)\frac{A\hat{x}}{a + x} = \alpha A\hat{x}$, then $\vec{I} = \vec{J}a^2 = \alpha Aa^2\hat{x}$.

and $R = \frac{V}{I} = \frac{A\ln 2}{\alpha Aa^2} = \frac{\ln 2}{\alpha\, a^2}$

7.8 This problem is similar to Ex. 7-11 except that both spheres are completely immersed in the infinite medium. We first use Eqs. 7.58-7.59. Now Eqs. 7.60-7.62 become

$$\Phi_1 = \frac{1}{4\pi\sigma_c a_1} I_1 + \frac{1}{4\pi\sigma_c \ell} I_2,\quad \Phi_2 = \frac{1}{4\pi\sigma_c a_2} I_2 + \frac{1}{4\pi\sigma_c \ell} I_1$$

Using $I_1 = I = -I_2$, we get:

$$\Phi_1 - \Phi_2 = \left(\frac{1}{4\pi\sigma_c a_1} + \frac{1}{4\pi\sigma_c a_2}\right)I + \frac{1}{4\pi\sigma_c \ell} I$$

Thus $R = \frac{1}{4\pi\sigma_c}\left(\frac{1}{a_1} + \frac{1}{a_2} - \frac{2}{\ell}\right)$.

7.9 This problem can be solved by introducing two image currents across the boundary such that the J_n across the boundary vanishes. Thus the image of $I_1 = I$ is taken to equal to I and the image $I_2 = -I$ is taken to be equal to $-I$. Now using a similar procedure to that of Ex. 7.11 we write

$$\Phi_1 = \frac{I}{4\pi\sigma_c}\left(\frac{1}{a_1} - \frac{1}{\ell} + \frac{1}{2b} - \frac{1}{\sqrt{\ell^2 + 4b^2}}\right)$$

$$\Phi_2 = \frac{I}{4\pi\sigma_c}\left(\frac{1}{\ell} - \frac{1}{a_2} - \frac{1}{2b} + \frac{1}{\sqrt{\ell^2 + 4b^2}}\right)$$

$$\frac{\Phi_1 - \Phi_2}{I} = R' = R + \frac{1}{4\pi\sigma_c}\left(\frac{1}{b} - \frac{1}{\sqrt{\ell^2 + 4b^2}}\right)$$

7.10 From Eq. 7.36 we have $\varepsilon/R\sigma_c$.

7.11 Using the expression $R = \int d\ell/\sigma_c A$ given in problem 7.2, we can find the resistance between the plates

$$R = \int \frac{dz}{A\sigma_c} = \frac{d}{A\sigma_c}$$

Alternatively we can use Eq. 7.36 $RC = \varepsilon/\sigma_c$ to arrive at the same result.

7.12 Using Eq. 7.72 we have $\sigma_c = nq^2\tau/m$. Thus $\tau = \sigma_c m/nq^2 = 2.1 \times 10^{-11}$sec.

7.13 The potential of the ith conductor is given by Eq. 7.57:

$\Phi_i = \sum_k R_{ik}I_k$. But the heat generated due to the current leaving the ith conductor is $Q_i = \Phi_i I_i$. Thus the total heat is $\sum_i Q_i = \sum_{i,k} R_{ik}I_iI_k$.

7.14 We use the equations developed for the circuit of Fig. 7.16: $-I_3R_3 + I_4R_4 = 0$, $I_2R_2 - I_1R_1 = 0$ and $-\varepsilon_1 - I_1R_1 - I_4R_4 = 0$. Along with these equations we have $I_2 = I_3 + I_5$, $I_3 = I_6 - I_4$, $I_4 = I_1 + I_5$ and $I_6 = I_1 + I_2$. Solving these equations for I_5 we get:

$$I_5 = \frac{-\varepsilon_1(R_1/R_2 - R_4/R_3)}{(1 + R_4/R_3)(R_1 + R_4) + R_4(R_1/R_2 + R_4/R_3)}$$

Thus $I_5 = 0$ when $R_1/R_2 = R_4/R_3$.

7.15 Differentiate I_5 with respect to R_3 using the result of problem 7.14. Thus

$$S = CR_3 \frac{dI_5}{dR_3} = \frac{(\text{bottom})d(\text{top})/dR_3 - (\text{top})\ d(\text{bottom})/dR_3}{(\text{bottom})^2}\Bigg|_{\text{balance}}$$

Since the top = 0 at balance, then

$$S = CR_3 \frac{d(\text{top})/dR_3}{(\text{bottom})} = \frac{C\varepsilon_1}{R_1 + R_2 + R_3 + R_4}$$

7.16 a) When we take ε_2 as dead, we replace it by a resistanceless wire. Thus the resultant resistance is $R = 1/2\Omega$, and the power delivered is $P_1 = \varepsilon_1^2/R = 200$ Watt.

b) We replace ε_1 by a simple wire, thus one of the 1Ω

resistances gets shorted giving R = 1Ω and hence $P_2 = \varepsilon_2^2/R$ = 100 Watt.

c) The actual currents in the circuit can be determined from Kirchhoff's law. Assume current I_1 and I_2 to flow clockwise in the loop on the left and on the right respectively. Thus $10 - (I_1 - I_2) = 0$ and $10 - I_1 + 2I_2 = 0$. Subtracting the equations gives $I_2 = 0$, and $I_1 = 10$ Amps. Thus total power delivered is $\varepsilon_1 I_1 + \varepsilon_2 I_2 = 100$ Watt which is different from the sum of the individual ones in a and b.

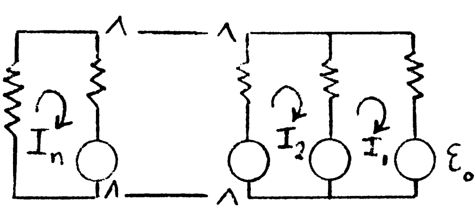
7.17 **a)** Using Kirchhoff's law for the series case we get $\sum\varepsilon_i - \sum R_i I - RI = 0$ or $n\varepsilon_o - nR_I - RI = 0$ giving $I = n\varepsilon_o/(R + nR_I)$.

b) We use Kirchhoff's law. Assume current $I_1, I_2, \ldots I_n$ to flow in the loops as shown. Set the equations for each loop, for example we have

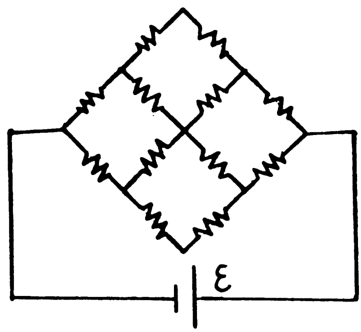
$(-2I_1 + I_2)R_I = 0$, $(-2I_2 + I_3 + I_1)R_I = 0$,

$(-2I_3 + I_4 + I_2)r = 0, \ldots, -(R_I + R)I_n + R_I I_{n-1} + \varepsilon_o = 0$

Solve for I_n. Take n = 4 as an example and then generalize.



7.18 We use Kirchhoff's laws to set the equations for circulating currents similar to the analysis of Fig. 7.16. We determine the current I through the source ε, then $R = \varepsilon/I$. This gives $R = 3r/2$ for $n = 2$, $R = 13r/7$ for $n = 3$ and $R = 47r/22$ for $n = 4$.



CHAPTER 8

8.1 The equation of motion for a particle of mass m and charge q in a region with $\vec{E}$ and $\vec{B} \neq 0$ is given by Eq. 8.2 $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = m\vec{a}$. Note that the magnetic force $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$, and hence for orbital radius r we have $mv^2/r = qvB \sin\theta$. The smallest B needed is for $\sin\theta = 1$; thus $B = mv/qr$ which gives $B = 5.69 \times 10^3$ Tesla.

8.2 To get $\vec{B}$ we use Ampere's law $\oint \vec{B}\cdot d\vec{\ell} = \mu_o \int \vec{J}\cdot d\vec{a}$. For $\rho < \rho_o$ we get:

$$\oint \vec{B}\cdot d\vec{\ell} = B(2\pi\rho) = \mu_o \int_0^{2\pi} \int_0^{\rho} e^{-2\rho'} \rho' d\rho' d\phi'$$

$$B = \frac{2\pi\mu_o}{2\pi\rho} \frac{e^{-2\rho'}}{-2} (\rho' + \frac{1}{2}) \Big|_0^{\rho} = \frac{\mu_o}{4\rho}[1 - 2\rho e^{-2\rho} - e^{-2\rho}].$$

For $\rho > \rho_o$ we have

$$B(2\pi\rho) = \mu_o \int_0^{\rho_o} e^{-2\rho'} \rho' d\rho' \int_0^{2\pi} d\phi = \frac{\mu_o}{4\rho}[1 - e^{-2\rho_o} - 2\rho_o e^{-2\rho_o}].$$

8.3 We will first find the field produced by the sheet. The procedure is the same used in Ex. 8.6. We calculate contributions from opposite sides of the sheet, then total up to give a horizontal component. Hence $\vec{B} = -B\hat{x}$

$$B = \int (dB_1 + dB_2)\cos\phi = 2\int_0^{W/2} \frac{\mu_o K_o}{2\pi} \frac{dx}{(x^2+h^2)^{1/2}} \frac{h}{(x^2+h^2)^{1/2}} =$$

$\frac{\mu_o K_o h}{\pi} \frac{1}{h} \tan^{-1}(\frac{x}{h}) \Big|_0^{W/2} = -\frac{\mu_o K_o}{\pi} \tan^{-1}(\frac{W}{2h})\hat{x}$. Now $\vec{F} = I\vec{\ell} \times \vec{B}$. Thus

$\frac{\vec{F}}{\ell} = -\frac{I\mu_o K_o}{\pi} \tan^{-1}(\frac{W}{2h})\hat{y}$. It is an attractive force. When W becomes large, then $\vec{F}/\ell = -I\mu_o K_o \hat{y}/2$.

8.4 a) $\vec{B} = \frac{100\mu_o}{\rho} \{\frac{4\rho_o^2}{\pi^2} \sin\frac{\pi\rho}{2\rho_o} - \frac{2\rho\rho_o}{\pi} \cos\frac{\pi\rho}{2\rho_o}\}\hat{\phi}$, $\nabla \times \vec{B} = \mu_o\vec{J}$.

In cylindrical coordinates: $\nabla \times \vec{B} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \rho(B_\phi) & 0 \end{vmatrix}$

$$\nabla \times \vec{B} = z(\frac{100\mu_o}{\rho}[\frac{\partial}{\partial\rho}(\frac{4\rho_o^2}{\pi^2} \sin\frac{\pi\rho}{2\rho_o} - \frac{2\rho\rho_o}{\pi} \cos\frac{\pi\rho}{2\rho_o})])$$

$\vec{J} = 100 \sin\frac{\pi\rho}{2\rho_o}\ \hat{z}$

b) $I = 100\int_A \sin\frac{\pi\rho}{2\rho_o}\, da$, $da = 2\pi\rho d\rho$, $I = 200\pi\int_0^{\rho_o} \rho\sin\frac{\pi\rho}{2\rho_o}\, d\rho$

$$I = 200\pi\{\frac{1}{(\frac{\pi}{2\rho_o})^2}\sin\frac{\pi\rho}{2\rho_o} - \frac{\rho}{(\frac{\pi}{2\rho_o})}\cos\frac{\pi\rho}{2\rho_o}\}\Big|_0^R = 200\pi(\frac{4\rho_o^2}{\pi^2})$$

$I = \frac{800\rho_o^2}{\pi}$ Amps

8.5 Using $\vec{B} = \frac{\mu_o I}{2\pi\rho}\hat{\phi}$, and $\vec{\nabla}\cdot\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z}$, and noting that $\vec{B}$ only has a ϕ component and ρ dependence, so $\vec{\nabla}\cdot\vec{B} = 0$ and

$$\vec{\nabla} \times \vec{B} = \hat{\rho}(\frac{1}{\rho}\frac{\partial B_z}{\partial\phi} - \frac{\partial B_\phi}{\partial z}) + \hat{\phi}(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho}) + \frac{\hat{z}}{\rho}(\frac{\partial}{\partial\rho}(\rho B_\phi) - \frac{2B_\rho}{\partial\phi})$$

The only term we can use is $\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\phi)$ where $\rho B_\phi = \mu_o I/2\pi$ and $\frac{\partial}{\partial\rho}(\mu_o I/2\pi) = 0$. Now, check $\rho = 0$ carefully:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\phi) = \frac{B_\phi}{\rho} + \frac{\partial B_\phi}{\partial\rho} = \frac{\mu_o I}{2\pi\rho^2} - \frac{\mu_o I}{2\pi\rho^2} .$$

For $\rho \to 0$ we have $(\infty - \infty)$ which is infinite which indicate the presence of a filamentary current at $\rho = 0$.

8.6 Let's consider the contribution to the force on the diagonal wire due to current I_2. $\vec{B}_1 = \frac{\mu_o I_2}{2\pi x}\hat{z}$. The force on the diagonal wire is given by $\vec{F} = I_1\int d\vec{\ell} \times \vec{B}$ where $d\vec{\ell} = \hat{x}dx + \hat{y}dy$. But $y = x + c$, then $dy = dx$, and $d\vec{\ell} = dx\hat{x} + dx\hat{y}$. Thus

$$\vec{F} = \frac{\mu_o I_1 I_2}{2\pi}\int_{x_o}^{x_o+a} \frac{(dx\hat{x} + dx\hat{y})}{x} x\hat{z} = \frac{\mu_o I_1 I_2}{2\pi}\int_{x_o}^{x_o+a} \frac{dx}{x}(\hat{x} - \hat{y})$$

$$\vec{F} = \frac{\mu_o I_1 I_2}{2\pi} \ell n(\frac{x_o + a}{x_o})(\hat{x} - \hat{y}).$$

8.7 From Ex. 8.9 we find that the vector potential of a filamentary current along z is $\vec{A}_1 = -\mu_o I\hat{z}\ \ell n\ \rho_1/2\pi$. For a filamentary current pointing along $-z$ we have $\vec{A}_2 = \mu_o I\hat{z}\ \ell n\ \rho_2/2\pi$ where ρ_1 and ρ_2 are measured from the currents. Thus at any point in space

$$\vec{A} = \vec{A}_1 + \vec{A}_2 = \mu_o I\ \ell n\ (\rho_2/\rho_1)/2\pi.$$

8.8 **a)** $\vec{A}_1 = -By\hat{x}$: $\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -By & 0 & 0 \end{vmatrix} = B\hat{z}$

$\nabla\cdot\vec{A}_1 = (\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z})\cdot(-By\hat{x}) = 0$

$\vec{A}_2 = Bx\hat{y}$: then $\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & Bx & 0 \end{vmatrix} = B\hat{z}$

and $\nabla\cdot\vec{A}_2 = 0$. $\vec{A}_3 = -\frac{1}{2}\,\vec{r} \times \vec{B}$: We may choose the B-field along any axis without any loss of generality. Take $\vec{B} = B\hat{z}$. Thus $-\frac{1}{2}\,\vec{r} \times \vec{B} = -\frac{1}{2}(By)\hat{x} + \frac{1}{2}(Bx)\hat{y}$. Hence $\nabla \times \vec{A}_3 = -\frac{1}{2}(-B - B)\hat{z} = B\hat{z}$. Moreover we have $\nabla\cdot\vec{A}_3 = 0$.

b) To show that $\vec{A}_1 - \vec{A}_2$ is the gradient of a function ψ, it suffices to show that $\nabla \times (\vec{A}_1 - \vec{A}_2) = 0$. Taking $\vec{A}_1 - \vec{A}_2 = -B(y\hat{x} + x\hat{y})$, then

$\nabla \times (\vec{A}_1 - \vec{A}_2) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -By & -Bx & 0 \end{vmatrix} = -B + B = 0$. Hence

$\vec{A}_1 - \vec{A}_2 = \nabla\psi$. It is easily seen that $\psi = -Bxy + \text{constant}$.

8.9 **a)** Ampere's law gives $\oint_c \vec{B}\cdot d\vec{\ell} = \mu_o I$, where c is a circle of radius ρ around the wire. Thus we get $B(2\pi\rho) = \mu_o I_1$ or $\vec{B} = \mu_o I_1\hat{\phi}/2\pi\rho$.

b) $F = \int \vec{B}\cdot d\vec{a} = \int_o^t \int_r^{r_o} \frac{\mu_o I_1}{2\pi}\frac{1}{\rho}\, d\rho\, dz = \frac{\mu_o I_1}{2\pi} t\, \ell n(\frac{r_o}{r})$

c) We use Eq. 8.46: $\int_s \vec{B}\cdot d\vec{a} = F = \oint_c \vec{A}\cdot d\vec{\ell}$ where c is the loop and s is the surface bounded by c, and $\vec{A}$ is in the z direction, and depends only on ρ. Thus $F = [A(r) - A(r_o)]t$, and hence

$\vec{A}(r) - \vec{A}(r_o) = \frac{\mu_o I_1}{2\pi} \ln(\frac{r_o}{r})\hat{z}$

d) The force on ab is $d\vec{F} = I_2 d\vec{\ell} \times \vec{B} = I_2 d\ell B(-\hat{z})$ or

$$\vec{F} = -\hat{z}\, I_2 \int_r^{r_o} \frac{\mu_o I_1}{2\pi\rho}\, d\rho = - \frac{-\mu_o I_1 I_2}{2\pi} \ln(\frac{r_o}{r})\hat{z}$$

The force on bc is $d\vec{F} = I_2 d\vec{\ell} \times \vec{B} = I_2 d\ell B(-\hat{y})$ or

$$\vec{F} = -\hat{y} I_2 \int_0^t \frac{\mu_o I_1}{2\pi r}\, dz = - \frac{-\mu_o I_1 I_2}{2\pi r}\, t\, \hat{y}$$

8.10 a) The magnetic field in the solenoid is $\vec{B} = \mu_o In\hat{z}$. The flux $F = BA = \mu_o Inxy$.

b) Using $\oint \vec{A} \cdot d\vec{\ell} = F$ and taking $\vec{A}_1 = A(x)\hat{y}$, we get $A_1(x)y - A_1(0)y = A_1(x)y = \mu_o Inxy$ which gives $A_1(x) = \mu_o Inx$ and $\vec{A}_1 = \mu_o Inx\hat{y}$.

c) Take $\vec{A}_2 = A_2(y)\hat{x}$. Then we have $-A_2(y)x + A_2(0)x = -A_2(y)x = \mu_o Inxy$ which gives $A_2(y) = -\mu_o Iny$ and $\vec{A}_2 = -\mu_o Iny\hat{x}$.

d) Take $\vec{A}_3 = (\vec{A}_1 + \vec{A}_2)/2 = -\mu_o In(y\hat{x} - x\hat{y})/2$. Check $\nabla \cdot \vec{A}_3 = 0 = -\mu_o In(\partial y/\partial x - \partial x/\partial y) = 0$. It is clear that the potentials are as plotted.

8.11 We expect $\vec{A}$ to be along z. Thus $\nabla \times \vec{A} = -\hat{\phi}\partial A_z/\partial\rho$. Now using $\vec{B} = \frac{\mu_o I}{2\pi\rho}\hat{\phi}$ we get from $\nabla \times \vec{A} = \vec{B}$: $-\partial A_z/\partial\rho = \mu_o I/2\pi\rho$ which integrates to $\vec{A} = - \frac{-\mu_o I \ell n\rho}{2\pi}\hat{z}$.

8.12 **a)** From example 8.12 and taking the point of observation at the center of the loop that is z = 0, we get: $\vec{B} = \frac{\mu_o I}{2r}\hat{z}$

where r is the radius of the loop.

b) Integrate loops from a to b, dI = INdr

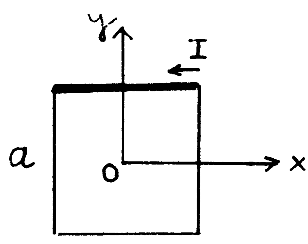
$$\vec{B} = \frac{\mu_o \hat{z} NI}{2}\int_a^b \frac{dr}{r} = \frac{\mu_o \hat{z}NI}{2}\ell n\, r\Big|_a^b = \frac{\mu_o \hat{z}NI}{2}\ell n\left(\frac{b}{a}\right)$$

Check for b ≈ a: $\ell n\left(\frac{b}{a}\right) \approx \ell n\left(1 + \frac{b-a}{a}\right) \approx \frac{b-a}{a}$, thus

$\vec{B} = \mu_o \hat{z}NI(b - a)/2a$. Now check for b − a << a: $b - a = \frac{1}{N}$,

thus $\vec{B} = \mu_o \hat{z}I/2a$.

8.13 From symmetry we have for each side $\vec{B} = B\hat{z}$. We use



$$d\vec{B} = \frac{\mu_o I}{4\pi}\frac{d\vec{\ell} \times \vec{\xi}}{\xi^3}.$$ For the side shown $\xi = -x\hat{x} - (a/2)\hat{y}$,

$d\vec{\ell} = -\hat{x}dx$, thus

$$\vec{B}_{side} = \frac{\mu_o I}{8\pi}\int_{-a/2}^{a/2} \frac{adx}{(x^2 + a^2/4)^{3/2}}.$$ Using x = au/2 and dx = adu/2

$$\vec{B}_{side} = \frac{\mu_o I\hat{z}}{\pi a}\int_0^1 \frac{du}{(\mu^2 + 1)^{3/2}} = \frac{\sqrt{2}\,\mu_o I\hat{z}}{2\pi a}$$

$$\vec{B}_{square} = 4\vec{B}_{side} = \frac{2\sqrt{2}\,\mu_o I\hat{z}}{\pi a}.$$

8.14 From problem 8.12 we have for a loop of radius b: $\vec{B}_o = \frac{\mu_o I}{2b}\hat{z}$. B for 2 wires from 0 to ∞ is the same as a wire from $-\infty$ to ∞, $\vec{B} = \frac{\mu_o I}{2\pi b}\hat{z}$. Thus the total field is

$$\vec{B} = \frac{\mu_o I}{4b}\hat{z} + \frac{\mu_o I}{2\pi b}\hat{z} = \frac{\mu_o I\hat{z}}{b}\left(\frac{1}{4} + \frac{1}{2\pi}\right).$$

8.15 Consider a band of loops at θ in the range dθ, thus the current in Rdθ × IN/πR = INdθ/π. From Ex. 8.12 the field on the axis of a loop of radius r and current I is $dB = \mu_o I r^2/2\xi^2$ where ξ is the distance of any current element from the point of observation. Taking $r = R\sin\theta$, $\xi = R$ and integrate we get

$$B = \int \frac{\mu_o I}{2R}\sin^2\theta\left(\frac{Nd\theta}{\pi}\right) = \frac{\mu_o NI}{2\pi R}\int_0^{\pi}\sin^2\theta d\theta$$

$$B = \frac{\mu_o NI}{2\pi R}\left[\frac{1}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi}, \qquad \vec{B} = \frac{\mu_o IN}{4R}\hat{z}.$$

8.16 Consider the result of Ex. 8.15 about the B field on the axis of a spinning disc:

$$B = \frac{1}{2}\mu_o \rho\omega t\left\{\frac{r^2 + 2z^2}{\sqrt{r^2 + z^2}} - 2|z|\right\}$$

where t is the thickness of the disc, and r is its radius.

We consider the above spinning sphere as a collection of spinning discs with variable radii r and thickness t = dz. Since we want the field at the center of the sphere, then for all of the discs we have $r^2 + z^2 = R^2$. Thus

$$B = \mu_o \rho\omega \int_o^R \{\frac{R^2 + z^2}{R} - 2z\} dz\ \hat{z}$$

$$\vec{B} = \mu_o \rho\omega\{Rz \cdot \frac{z^3}{3R} - z^2\} \Big|_o^R \hat{z} = \frac{1}{3} \mu_o \rho\omega R^2\ \hat{z}\ .$$

8.17 From Ex. 8.12, the field <u>on</u> the axis of a current carrying loop:

$$\vec{B} = \frac{\mu_o I R^2}{2(z^2 + R^2)^{3/2}}\ \hat{z}$$

If we venture slightly from the $\hat{z}$ axis, we expect $\vec{B} = \vec{B}(\rho,z)$, i.e. we will introduce a $\vec{B}$ field in the ρ direction. But $\vec{\nabla}\cdot\vec{B} = 0$ gives $\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0$ or

$\frac{\partial B_z}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho)$. If we are not too far off the axis, then:

$B_z \approx \frac{\mu_o I R^2}{2(z^2 + R^2)^{3/2}}$, hence $\frac{\partial B_z}{\partial z} = \frac{-3\mu_o I R^2 z}{2(z^2 + R^2)^{5/2}}$, therefore

$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) = \frac{3\mu_o I R^2 z}{2(z^2 + R^2)^{5/2}}$ or $\rho B_\rho = \int \frac{3\mu_o I R^2 z}{2(z^2 + R^2)^{5/2}} \rho d\rho =$

$\frac{3\mu_o I R^2 z \rho^2}{4(z^2 + R^2)^{5/2}} + C$, $B_\rho = \frac{3\mu_o I R^2 z\rho}{4(z^2 + R^2)^{5/2}} + \frac{C}{\rho}$. But at $\rho = 0$ we

require $B_\rho = 0$, thus $C = 0$ and $B_\rho = \frac{3\mu_o I R^2 z\rho}{4(z^2 + R^2)^{5/2}}$

$$\vec{B}(\rho,z) = \frac{3\mu_o I R^2 z\rho}{4(z^2 + R^2)^{5/2}}\ \hat{\rho} + \frac{\mu_o I R^2}{2(z^2 + R^2)^{3/2}}\ \hat{z}\ .$$

8.18 From example 8.12, the field on the z axis, from a single coil is $\vec{B}(z) = \dfrac{\mu_o IR^2\hat{z}}{2(z^2 + R^2)^{3/2}}$. Now

$$\frac{\partial B}{\partial z} = \frac{\mu_o}{2} R^2 I(z^2 + R^2)^{-5/2}(-\frac{3}{2})(2z)\hat{z}$$

$$\frac{\partial^2 B}{\partial z^2} = -\frac{3}{2}\mu_o IR^2(z^2 + R^2)^{-7/2}[-5z^2 + (z^2 + R^2)]\hat{z}.$$

For coils at $z = \pm R/2$ we add the two terms. Since $\partial B/\partial z$ is an odd function of z, then $\sum_{z=\pm\frac{R}{2}} \frac{\partial B}{\partial z} = 0$. Now

$$\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=\frac{R}{2}} = -\frac{3}{2}\mu_o IR^2(z^2 + R^2)^{-7/2}[-5\frac{R^2}{4} + (\frac{R^2}{4} + R^2)]\hat{z} = 0$$

$$\vec{B} = \mu_o IR^2(\frac{R^2}{4} + R^2)^{-3/2}\hat{z} = \frac{\mu_o I}{R}(\frac{5}{4})^{-3/2}\hat{z}.$$

8.19 **a)** $\vec{B} = -\nabla\Phi_m = -(B_o/\mu_o)\ \frac{z}{b}\hat{x} + \hat{z}(1 + \frac{x}{b})$. We must make sure that $\nabla\cdot\vec{B} = 0$ and $\nabla\times\vec{B} = 0$ for the potential to be reasonable. Both are clearly satisfied.

b) $F = \nabla(\vec{m}\cdot\vec{B})$ as given by Eq. 8.107. Now $\vec{m} = \pi a^2 I\hat{z} = \pi a^2 fe\hat{z}$ where f is the frequency of revolution which is equal to $v/2\pi a$: The speed for a circular orbit is $v = (e^2/4\pi\varepsilon_o a)^{1/2}$.

Thus $\vec{F} = -\dfrac{B_o m}{\mu_o}\nabla(1 + \frac{x}{b}) = -\dfrac{B_o m}{\mu_o b}\hat{x}$.

8.20 From Eq. 8.97 we get for a small current loop:

$\vec{A} = \dfrac{\mu_o I}{4\pi}(\pi a^2\hat{z})\times(\dfrac{\vec{\xi}}{\xi^3})$. At large distances $\vec{\xi} = (\vec{r} - \vec{r}') = \vec{r}$. Thus

$$\vec{A} = \frac{\mu_o I}{4\pi} \pi a^2 \left(\frac{\hat{z} \times \hat{r}}{r^2}\right).$$ Using $\hat{z} \times \hat{r} = \hat{\phi}\sin\theta$ we get:

$$\vec{A} = \frac{\mu_o I \ \pi a^2 \sin\theta}{4\pi r^2} \hat{\phi} .$$

8.21 a) The magnetic moment for each turn is $\vec{m}_n = I\vec{A}$ where $\vec{A} = \pi R^2 \hat{z}$. Hence $\vec{m} = NI\pi R^2 \hat{z}$.

b) Force $= \nabla(\vec{m}\cdot\vec{B})$ as given in Eq. 8.107

$$\vec{F} = \nabla\left(NI\pi R^2 \hat{z} \cdot \frac{B_o}{\sqrt{2}} (\hat{x} + \hat{y})\right) = 0.$$

c) Torque $= \vec{m}\times\vec{B}$ as given in Eq. 8.109

$$\vec{\tau} = \frac{NI\pi R^2 B_o}{\sqrt{2}} \hat{z} \times (\hat{x} + \hat{y}) = \frac{NI\pi R^2 B_o}{\sqrt{2}} (\hat{y} - \hat{x}).$$

8.22 a) The current is taken counter clockwise. Force due to a magnetic field on a current I is given by $\vec{F} = I \int_c d\vec{\ell} \times \vec{B}$, where c is the path of current. We apply this to each side: obviously $\vec{F}_{AC} = 0$. For AB, $d\vec{\ell} = dy(-\hat{y})$ and $\vec{F}_{AB} = I|AB|B\ (-\hat{y} \times \hat{x})$ so $\vec{F}_{AB} = +1.98\ \hat{z}$Nt. Now $\vec{F}_{BC} = IB(0.3/0.5)\hat{y} \times \hat{x} = 1.98\hat{z}$Nt. See that net force = 0 on a loop in a uniform magnetic field.

b) Dipole moment: for a current loop $\vec{m} = I\vec{A}$: for counter-clockwise sense of current, looking from top, points upward: so $\vec{m} = 6 \times \frac{1}{2}(0.3)(0.4)\hat{z} = .36\hat{z}\text{A}\cdot\text{m}^2$. The torque on a dipole $= \vec{\tau} = \vec{m} \times \vec{B} = .396\hat{y}\text{N}\cdot\text{m}$.

8.23 **a)** $\vec{m}$ = superposition of 2 dipole moments $= \vec{m}_x + \vec{m}_z$

$$\vec{m}_x = \frac{I\pi a^2}{2}\hat{x},\ \vec{m}_z = \frac{I\pi a^2}{2}\hat{z}$$

$$\vec{m} = \frac{I\pi a^2}{2}(\hat{x} + \hat{z}) = \pi(10^{-4})(\hat{x} + \hat{z})\ \text{Amp-m}^2$$

b) Since $\sqrt{3^2 + 4^2}$ meters >> 1 cm, the loops are effectively dipoles with the net dipole moment given in (a). The field due to a dipole is $\vec{B} = \frac{\mu_o}{4\pi r^3}[3(\hat{r}\cdot\vec{m})\hat{r} - \vec{m}]$. Since the dipole is at origin, $\vec{r} = 3\hat{x} + 4\hat{y}$, and $r = 5$m then

$$\vec{B} = \frac{\mu_o}{4\pi r^3}\{3[\frac{(3\hat{x} + 4\hat{y})}{5}\cdot(\pi \times 10^{-4})(\hat{x} + \hat{z})]\frac{(3\hat{x} + 4\hat{y})}{5} -$$

$$(\pi \times 10^{-4}(\hat{x} + \hat{z})\}$$

$$\vec{B} = \frac{\pi}{5^3} \times 10^{-11}[(\frac{27}{25} - 1)\hat{x} + \frac{36}{25}\hat{y} - \hat{z}]\ \text{Tesla}$$

8.24 Since $\vec{\tau} = \vec{m} \times \vec{B}$ then $|\vec{m}| = \frac{|\vec{\tau}|}{B} = 1.96 \times 10^{-24}$ amps m^2. Now $I = \frac{e\omega}{2\pi}$ and $I = \frac{m}{\pi r^2}$, then $\omega = \frac{2m}{er^2} = 2 \times 10^{16}\ \text{sec}^{-1}$.

8.25 **a)** By example 8.19, the magnetic dipole moment of a charged spinning disc of thickness t is $\vec{m} = \frac{\pi\rho t\vec{\omega}R^4}{4}$. Now, we replace t by dz and r^2 by $R^2 - z^2$, then

$$|\vec{m}|_{sphere} = \frac{2\pi\rho\omega}{4}\int_o^R (R^2 - z^2)^2 dz$$

$$m_{sphere} = \frac{\pi\rho\omega}{2}\int_o^R (R^4 - 2R^2z^2 + z^4)dz \text{ or } \vec{m}_{sphere} = \frac{4\pi\rho\omega R^5}{15}\hat{z}.$$

b) Now, consider the sphere as a "collection" of current loops with a magnetic moment $dm = AdI$ where A is the area of the loop, and dI is the current in the loop. Now $dI = dq/T$, $dq = \sigma(2\pi r)Rd\theta$, and $T = 2\pi r/v$. Thus $dI = \frac{\sigma(2\pi r)Rd\theta v}{2\pi r} = \sigma Rvd\theta$. Taking $v = \omega r$, $dI = \sigma R\omega r d\theta$. Now $r = R\sin\theta$, thus $dI = \sigma R^2\omega\sin\theta d\theta$, and $dm = (\pi r^2)(\sigma R^2\omega\sin\theta d\theta) = \sigma\pi R^4\omega\sin^3\theta d\theta$,

$$m = \sigma\pi R^4\omega \int_0^\pi \sin^3\theta d\theta = -\sigma\pi R^4\omega \int_0^\pi d\cos\theta(1 - \cos^2\theta),$$

$$\vec{m} = \sigma\pi R^4\omega\{-1/3(\cos\theta)(\sin^2\theta + 2)\Big|_0^\pi\}\hat{z} = \frac{4}{3}\sigma\omega\pi R^4\hat{z}.$$

CHAPTER 9

9.1 $\vec{m} = (a_1y^2 + b_1)\,\hat{y} + a_2x^2\,\hat{x}$

a) The volume pole density is $\rho m = -\nabla\cdot\vec{m} = -2a_1y - 2a_2x$. The surface density is $\sigma_m = \vec{m}\cdot\hat{n}$, but $\hat{n} = \nabla f/|\nabla f|$ where f is the equation of the sphere $= x^2 + y^2 + z^2 = r^2$. Using $\nabla f = 2x\hat{x} + 2y\hat{y} + 2z\hat{z}$, $|\nabla f| = 2r$, then $\hat{n} = x\hat{x} + y\hat{y} + z\hat{z}/r$ and $\sigma_m = (1/r)[a_2x^3 + (a_1y^2 + b_1)y]$.

b) $$\vec{J}_m = \nabla \times \vec{m} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2x^2 & (a_1y^2 + b_1) & 0 \end{vmatrix} = 0.$$

$$\vec{K}_m = \vec{m} \times \hat{n} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_2x^2 & (a_1y^2 + b_1) & 0 \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{vmatrix}$$

$\vec{K}_m = \hat{x}(z/r)(a_1y^2 + b_1) - \hat{y}(z/r)a_2x^2 + \hat{z}[(y/r)\ a_2x^2 - (x/r)(a_1y^2 + b_1)]$.

9.2 This is the magnetic version of problem 4.2. Given $\vec{M} = M_o\hat{y}$ where M_o = constant.

a) $\rho_m = -\vec{\nabla}\cdot\vec{M} = 0$. The surface density $\sigma_m = \hat{M}\cdot\hat{n}$ is $M_o\hat{z}\cdot\hat{y} = 0$ on top side, $-M_o\hat{z}\cdot\hat{y} = 0$ on the bottom side, $M_o\vec{y}\cdot\hat{y} = M_o$ on front face, and $M_o\hat{y}\cdot\hat{\rho} = M_o\sin\phi$ on curved surface.

b) $q_m = \int_v \rho_m dv + \int_s \sigma_m da = \int_{-L/2}^{L/2} dz \int_{-R}^{R} dx\, M_o + \int_{\pi}^{2\pi} Rd\phi \int_{-L/2}^{L/2} dzM_o \sin\phi$

$2LM_o - 2LRM_o = 0$. Note: The region is $\pi \leqslant \phi \leqslant 2\pi$ by convention.

c) No magnetic monopoles have been detected, although there are theories that predict them, and a Stanford experiment may have detected one. We do not know for certain that they do exist.

d) $\vec{m} = \int_v \vec{r}\rho(\vec{r})dv + \int_s \vec{r}\sigma(\vec{r})da = \int_o^R d\rho \int_{\pi}^{2\pi} Rd\phi(\rho\hat{\rho} + {}^1\!/_2\,\hat{z})\sigma_m^{top} +$

$\int_o^R d\rho \int_{\pi}^{2\pi} Rd\phi(\rho\hat{\rho} - \frac{L}{2}\hat{z})\,\sigma_m^{bottom} + \int_{-L/2}^{L/2} dz \int_{-R}^{R} dx(z\hat{z} + x\hat{x})\sigma_m^{face} +$

$\int_{-L/2}^{L/2} dz \int_{\pi}^{2\pi} Rd\phi(z\hat{z} + R\hat{\rho})\sigma_m^{curve} = 0 + 0 + 0 + LR^2M_o \int_{\pi}^{2\pi} (\hat{x}\cos\phi +$

$\hat{y}\sin\phi)M_o \sin\phi d\phi = \frac{\pi R^2 LM_o}{2} = M_o V.$

9.3 a) Magnetic pole densities $\rho_m = -\vec{\nabla}\cdot\vec{M} = 0$. The surface density on flat faces is $\sigma_m = \hat{M}\cdot\hat{n} = 0$. For circular edge, we have $\hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$, thus $\sigma_m = M_o \cos\theta$.

b) The magnetic current density is $\vec{J}_m = \vec{\nabla} \times \vec{M} = 0$. The surface current density on the flat surfaces are $\vec{K}_m = M_o\hat{x} \times \hat{z} = -M_o\hat{y}$ on the upper surface, and $\vec{K}_m = M_o\hat{x} \times (-\hat{z}) = M_o\hat{y}$ on the lower surface. The surface current density for circular edge is $K_m = M_o\hat{x} \times (\cos\theta\hat{x} + \sin\theta\hat{y}) = M_o \sin\theta\hat{z}$.

c) $\vec{B} = \frac{\mu_o}{4\pi} \int \frac{\sigma_m \vec{\xi}}{\xi^3} da + M\mu_o$. Now for $R >> T$;

$\vec{\xi} \approx -R\hat{\rho}$, thus $\vec{B} = -\frac{\mu_o}{4\pi}\int_o^{2\pi} \frac{M_o\cos\theta}{R^3} R^2(\cos\theta\hat{x} + \sin\theta\hat{y})d\theta \int_o^T dz + \mu_o M$.

$\vec{B} = \vec{B} = -\frac{\mu_o T}{4\pi}\frac{M_o\hat{x}}{R}\int_o^{2\pi}\cos^2\theta\, d\theta = -\frac{\mu_o T\vec{M}}{4R} + \mu_o\vec{M}$. Since $\vec{B}-\mu_o\vec{M} = \vec{H}$, then $\vec{H} = -\mu_o TM/4R$.

9.4 We use the method of magnetic pole density to find Φ_m along axis. Now $\rho_m = 0$ and $\sigma_m = M_o$ at $z = \ell$, and $-M_o$ at $z = 0$. Thus

$$\Phi_m = \frac{1}{4\pi}\int\frac{\sigma_m}{\xi}da = -\frac{M_o}{4\pi}[-\int_o^{2\pi}d\phi\{\int_o^a\frac{\rho d\rho}{\sqrt{z^2+\rho^2}} + \int_o^a\frac{\rho d\rho}{\sqrt{(z-\ell)^2+\rho^2}}\}]$$

$$\Phi_m = \frac{M_o}{2}\{\Big|_{z^2}^{a^2+z^2}\sqrt{\rho} - \Big|_{(z-\ell)^2}^{a^2+(z-\ell)^2}\sqrt{\rho}\}$$

$$\Phi_m = \frac{M_o}{2}[\sqrt{a^2+z^2} - \sqrt{a^2+(z-\ell)^2} - |z| + |z-\ell|].$$

9.5 Let us consider the potential inside the sphere first:

b) From 9.16 $\Phi_m = \frac{-1}{4\pi}\int\frac{\rho_m}{\xi}dv - \frac{1}{4\pi}\int\frac{\sigma_m}{\xi}da$ where $\rho_m = -\nabla\cdot\vec{M}$, $\sigma_m = \vec{M}\cdot\hat{n}$ and $\vec{M} = M_1(r)\hat{r} + \vec{M}_o$. Using $\vec{M} = (\frac{M_1}{r})\vec{r} + \vec{M}_o$ and Eq. 1.57 we can show that $\rho_m = -2M_1(r)/r - \dot{M}_1(r)$. Moreover $\sigma_m = M_1(r) + \vec{M}_o\cdot\hat{r}$. We will evaluate the volume integral which we will call I_v first. Using Eq. 2.59 we get:

$I_v = -\{\frac{1}{r}\int_o^r\rho_m r^2\,dr + \int_r^R\rho_m r\,dr\}$. Substituting for ρ_m the volume integral becomes

$$I_v = \frac{2}{r}\int_o^r M_1(r)r\,dr + \frac{1}{r}\int_o^r\dot{M}_1(r)r^2dr + 2\int_r^R M_1(r)dr + \int_r^R M_1(r)r\,dr.$$

We now perform an integration by parts on the second and fourth terms.

$$I_v = \frac{2}{r}\int_0^r M_1(r)r\,dr + \frac{1}{r}M_1(r)r^2\Big|_0^r - \frac{1}{r}\int_0^r M_1(r)(2r)dr +$$

$$2\int_r^R M_1(r)\,dr + M_1(r)r\Big|_r^R - \int_r^R -\dot{M}_1(r)dr = M_1(r)r + M_1(R)R - M_1(r)r +$$

$\int_r^R M_1(r)dr = M_1(R)R + \int_r^R M_1(r)dr.$ Now, look at the surface integral I_s. First we note that from Ex. 9.7 the contribution to the potential from $\vec{M} = \vec{M}_o$ is given by $\Phi_m = (1/3)\,\vec{M}_o\cdot\vec{r}$. The contribution of the constant density M(R) is first from analogy with electric charge problems, a constant potential -MR. Thus

$$\Phi_m = M_1(R)R + \int_r^R M_1(r)dr + \frac{1}{3}\vec{M}_o\cdot\vec{r} - M_1(R)R.$$

$$\Phi_m = \int_r^R M_1(r)dr + \frac{1}{3}\vec{M}_o\cdot\vec{r}.$$

a) Outside the sphere, the potential satisfies the following:

$\Phi_2 = \sum_{n=0} A_n r^{-(n+1)}P_n(\cos\theta)$. But $\Phi_1 = \Phi_2$ at $r = R$. Thus

$\int_r^R M_1(r)dr + \frac{1}{3}M_oR\cos\theta = A_1R^{-2}\cos\theta$. Thus $A_1 = \frac{1}{3}M_oR^3$ and $\Phi_2 = (1/3)\,M_oR^3\cos\theta/r^2$. Since Φ_2 is independent of $M_1(R)$, then the field outside of the sphere must also be independent of $M_1(R)$, as $\vec{B}_2 = -\mu_o\nabla\Phi_2$.

9.6 We can easily show that the fields in this situation are given by $\vec{H}_1 = \frac{2\mu_2}{\mu_1 + \mu_2}\vec{H}_o$, $\vec{H}_2 = \frac{2\mu_1}{\mu_1 + \mu_2}\vec{H}_o$, by showing that these fields satisfy Laplace's equation and the boundary conditions. By uniqueness, therefore, these would be the only solutions.

1) By example 9.4 $\vec{H}_1$ and $\vec{H}_2$ satisfy Laplace's equation.

2) There is no field parallel to the interface, so clearly $(\vec{H}_{2t} - \vec{H}_{1t}) = \vec{K}_{normal}$ is satisfied.

3) $B_{1n} = B_{2n} \rightarrow \mu_1 H_1 = \mu_2 H_2$, $\frac{2\mu_1\mu_2}{\mu_1 + \mu_2} H_o = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} H_o$ is satisfied. Thus $\vec{H}_1$ and $\vec{H}_2$ above give the field equations, as they satisfy all the requisite conditions.

9.7 **a)** Find $\vec{B}$ using Ampere's law

i) For $0 < \rho < \rho_1$, $\oint \vec{B}\cdot d\vec{\ell} = 0$ thus $\vec{B} = 0$.

ii) For $\rho_1 < \rho < \rho_2$, $\oint \vec{B}\cdot d\vec{\ell} = \mu_o I_{enc} = \frac{\mu_o I(\rho^2 - \rho_1^2)}{\rho_2^2 - \rho_1^2}$ or

$$\vec{B} = \frac{\mu_o I}{2\pi\rho} \frac{\rho^2 - \rho_1^2}{\rho_2^2 - \rho_1^2} \hat{\phi}$$

iii) For $\rho > \rho_2$, we have a field as if all the current is at $\rho = 0$: $\vec{B} = \frac{\mu_o I}{2\pi\rho}\hat{\phi}$

b) By symmetry we have $\vec{A} = A_z\hat{z}$ and $A_z = A(\rho)$ only. Thus we solve $\nabla^2 A_z = -\mu_o \vec{J}$.

i) For $\rho < \rho_1$: $\nabla^2 A_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}A_z = 0$; thus $A_z = C\,\ell n\,\rho + D$. Now A_z must be finite at $\rho = 0$; therefore $C = 0$. Moreover we take the potential to be zero so we set $D = 0$. Thus $\vec{A} = 0$ for $\rho < \rho_1$.

ii) For $\rho_1 < \rho < \rho_2$, we have $J = I/(\pi(\rho_2^2 - \rho_1^2))$

$\frac{\partial}{\partial\rho} \rho \frac{\partial}{\partial\rho} A_z = I\rho/[\pi(\rho_2^2 - \rho_1^2)]$ or

$A_z = -\frac{\mu_o I}{4\pi} \frac{\rho^2}{\rho_2^2 - \rho_1^2} + C_o \ln\rho + D_o$. Match the boundary

conditions on H_t at ρ_1; $\vec{H}_t = \frac{I}{2\pi} \frac{\rho}{\rho_2^2 - \rho_1^2} \hat{\phi} - \frac{C_o}{\mu_o \rho} \hat{\phi}$ then at ρ_1 $\vec{H}_t$

should vanish, thus $\frac{I}{2\pi} \frac{\rho_1}{\rho_2^2 - \rho_1^2} - \frac{C_o}{\rho_2} = 0$; $C_o = \frac{I\mu_o}{2\pi} \frac{\rho_1^2}{\rho_2^2 - \rho_1^2}$.

Also A_z should vanish at $\rho = \rho_1$ since A_z for $\rho < \rho_1$ is taken zero. Thus

$$\vec{A} = [-\frac{\mu_o I}{4\pi} \frac{(\rho^2 - \rho_1^2)}{\rho_2^2 - \rho_1^2} + \frac{\mu_o I}{2\pi} \frac{\rho_1^2}{\rho_2^2 - \rho_1^2} \ln\frac{\rho}{\rho_1}]\hat{z}$$

and $\vec{B} = [\frac{\mu_o I}{2\pi\rho} \frac{\rho^2 - \rho_1^2}{\rho_2^2 - \rho_1^2}]\hat{\phi}$ as expected from part (a).

iii) for $\rho > \rho_2$: $A_z = C_1 \ln\rho + D_1$; $\vec{B} = -\frac{C_1}{\rho}\hat{\phi}$; but from Ampere's law we have

$$\frac{\mu_o I}{2\pi\rho_2} = -\frac{C_1}{\rho_2} \text{ or } C_1 = -\frac{\mu_o I}{2\pi} \text{ and } \vec{A} = -\frac{\mu_o I\hat{z}}{2\pi} \ln\rho + D_1\hat{z}$$

The constant D_1 can be determined by matching $\vec{A}$ at $\rho = \rho_2$.

$$D_1 = -\frac{\mu_o I}{4\pi} + \frac{\mu_o I}{2\pi} \frac{\rho_1^2}{\rho_2^2 - \rho_1^2} \ln\rho_2/\rho_1 + \frac{\mu_o I}{2\pi} \ln\rho_2$$

9.8 For a uniformly magnetized sphere we have $\vec{m} = \int \vec{M} dv = \frac{4}{3}\pi a^3 \vec{M}$, $\vec{B} = \mu_o(\vec{M} + \vec{H})$. But $\vec{\tau} = \vec{m} \times \vec{B}$, thus

$\vec{\tau} = \frac{4}{3}\pi a^3 \mu_o \vec{M} \times (\vec{M} + \vec{H}) = \frac{4}{3}\pi a^3 \mu_o \{\vec{M} \times \vec{M} + \vec{M} \times \vec{H}\}$

$\tau = \frac{4}{3}\pi a^3 \mu_o MH \sin\alpha$ where α is the angle between $\vec{M}$ and $\vec{H}$.

9.9 Let us label the three regions $r < R_1$, $R_1 < r < R_2$ and $r > R_2$ by 1, 2, and 3, with potentials Φ_1, Φ_2, and Φ_3 respectively. Then $\Phi_1 = \sum_{n=0} D_n r^n P_n(\cos\theta)$. Note that $\Phi_1 \to 0$ as $r \to 0$. $\Phi_2 = \sum_{n=0} [B_n r^n + C_n r^{-(n+1)}] P_n(\cos\theta)$. $\Phi_3 = -B_o r\cos\theta + \sum_{n=0} A_n r^{-(n+1)} P_n(\cos\theta)$. Now we have the following conditions:

1) $\Phi_3 = \Phi_2$ at $r = R_2$

2) $\Phi_2 = \Phi_1$ at $r = R_1$

3) $\mu_1 \frac{\partial\Phi_2}{\partial r} = \mu_2 \frac{\partial\Phi_3}{\partial r}$ at $r = R_2$

4) $\mu_1 \frac{\partial\Phi_2}{\partial r} = \mu_2 \frac{\partial\Phi_1}{\partial r}$ at $r = R_1$

Because of orthogonality of $P_n(\cos\theta)$, only the $n = 1$ term is involved (due to the magnetic field $\Phi = -H_o r\cos\theta$ for large r). Thus $\Phi_1 = D_1 r\cos\theta$, $\Phi_2 = B_1 r\cos\theta + C_1 r^{-2}\cos\theta$ and $\Phi_3 = -H_o r\cos\theta + A_1 r^{-2}\cos\theta$ where $H_o = B_o/\mu_2$.

From (1) $-B_o R_2 + \frac{A_1}{R_2^2} = B_1 R_2 + \frac{C_1}{R_2^2}$ or $-B_1 R_2^3 - C_1 + A_1 = H_o R_2^3$.

From (2) $B_1 R_1 + \frac{C_1}{R_1^2} = D_1 R_1$ or $-D_1 R_1^3 + B_1 R_1^3 + C_1 = 0$.

From (3) $2\mu_2 A_1 + \mu_1 B_1 R_2^3 - 2C_1 \mu_1 = -\mu_2 R_2^3 H_o$.

From (4) $\mu_1 B_1 - \frac{2C_1\mu_1}{R_1^3} = D_1\mu_2$ or $\mu_1 R_1^3 B_1 - 2\mu_1 C_1 - D_1\mu_2 R_1^3 = 0$.

Solving these equations for the constants, we get the following for D_1 and hence for Φ_1 and B_1:

$$D_1 = \frac{-9\mu_1\mu_2H_o}{(2\mu_2 + \mu_1)(\mu_2 + 2\mu_1) - 2(\frac{R_1}{R_2})^3(\mu_2 - \mu_1)}$$

$$\Phi_1 = \frac{-\ 9\mu_1\mu_2H_oz}{\{(2\mu_2 + \mu_1)(\mu_2 + 2\mu_1) - 2(\frac{R_1}{R_2})^3(\mu_2 - \mu_1)^2\}}$$

$$\vec{H}_1 = \frac{9\mu_1\mu_2H_o\hat{z}}{\{(2\mu_2 + \mu_2)(\mu_2 + 2\mu_1) - 2(\frac{R_1}{R_2})^3(\mu_2 - \mu_1)^2\}}$$

For $\mu_1 >> \mu_2$, we rewrite the expression:

$$\vec{H}_1 = \frac{9(\frac{\mu_2}{\mu_1})H_o\hat{z}}{\{(2\frac{\mu_2}{\mu_1} + 1)(\frac{\mu_2}{\mu_1} + 2) - 2(\frac{R_1}{R_2})^3(\frac{\mu_2}{\mu_1} - 1)^2\}}$$

giving $\hat{H}_1 = 0$. Thus, for a material of high permeability, the H field in the cavity is expelled. Thus, the shell acts like a shield of the magnetic field.

9.10 We take $\vec{H} = H_o\hat{y}$, with a potential $-H_oy = -H_o\rho\sin\phi$. We should now see that only n = 1, $\sin(\phi)$ terms can enter the Φ's so we rewrite our Φ's in the cavity, in the shell, and outside in the form: $\Phi_1 = A\rho\sin\phi$, $\Phi_2 = (B\rho + C\rho^{-1})\sin\phi$,

$\Phi_3 = (-H_o\rho + D\rho^{-1})\sin\phi$. The corresponding $\vec{H}$ fields are:

$$\vec{H}_1 = -A\hat{\rho}\sin\phi - A\hat{\phi}\cos\phi,\ \vec{H}_2 = -\hat{\rho}(B - C\rho^{-2})\sin\phi - \hat{\phi}(B + C\rho^{-2})\cos\phi,$$

$$\vec{H}_3 = -\hat{\rho}(-H_o - D\rho^{-2})\sin\phi - \hat{\phi}(-H_o + D\rho^{-2})\cos\phi.$$

We now require either Φ or H_ϕ to be continuous at $\rho = a,b$; thus

$$A = B + Ca^{-2},\ B + Cb^{-2} = -H_o + D\,b^{-2}$$

We now require B_ρ to be continuous. Note that $B_1 = \mu_o H_1$, $B_2 = \mu H_2$ and $B_3 = \mu_o H_3$. Thus we get at a and b: $\mu B - \mu Ca^{-2} = \mu_o A, \mu B - \mu Cb^{-2} = -\mu_o H_o - \mu_o Db^{-2}$. These four equations must be solved for A, B, C, D in terms of H_o, a, b, μ, μ_o giving

$$B = \frac{-2\mu_o H_o b^2(\mu + \mu_o)}{(\mu + \mu_o)^2 b^2 - (\mu - \mu_o)^2 a^2},\ C = \frac{a^2(\mu - \mu_o)}{(\mu + \mu_o)}B,$$

$$A = \frac{2\mu C}{a^2(\mu - \mu_o)},\ D = C - Bb^2\left(\frac{\mu - \mu_o}{\mu + \mu_o}\right).$$

In the material we have $\vec{B}_2 = \mu\vec{H}_2 = -\mu\nabla\Phi_2 = -\hat{\rho}\mu(B - C\rho^{-2})\sin\phi - \hat{\phi}\mu(B + C\rho^{-2})\cos\phi$

$$= \frac{2\mu\mu_o H_o b^2\left[\left((\mu+\mu_o) - \frac{a^2}{\rho^2}(\mu-\mu_o)\right)\hat{\rho}\sin\phi + \left((\mu+\mu_o) + \frac{a^2}{\rho^2}(\mu-\mu_o)\right)\hat{\phi}\cos\phi\right]}{(\mu + \mu_o)^2 b^2 - (\mu - \mu_o)a^2}$$

9.11 We have an infinitely long cylinder polarized such that $\vec{M} = M_o\hat{x}$. Note Φ will have only $\cos\phi$ dependence because of matching boundary conditions with $\vec{M}\cdot\hat{n} = M_o \cos\phi$. Thus the potential inside and outside the cylinder are: $\Phi_1 = A_1\rho$, $\cos\phi + C_1 \frac{\cos\phi}{\rho}$ (We take $C_1 = 0$ because Φ_1 is finite as $\rho \to 0$) $\Phi_2 = A_2\rho \cos\phi + C_2 \frac{\cos\phi}{\rho}$ (we take $A_2 = 0$ because Φ_2 is finite as $\rho \to \infty$). Now at $\rho = \rho_o$ we have $\Phi_1 = \Phi_2$ therefore $A_1 = C_2/\rho_o^2$. Also at ρ/ρ_o we have $B_{1n} = B_{2n}$. But $B_{1n} = \mu_o\partial\Phi/\partial\rho + \mu_o\vec{M}\cdot\hat{\rho}$ then $-\mu_o A_1 + \mu_o M_o = \mu_2\, C_2/\rho_o^2 =$ which gives $A_1 = \mu_o M_o/(\mu_2+\mu_o)$; $C_2 = \mu_o M_o \rho_o^2/(\mu_o + \mu_2)$

$$\vec{B}_1 = -\frac{\mu_o^2 M_o}{\mu_o + \mu_2}(\cos\phi\ \hat{\rho} - \sin\phi\ \hat{\phi}) + \mu_o M_o \hat{x} = \frac{\mu_2\mu_o M_o}{\mu_o + \mu_2}\ \hat{x}$$

$$\vec{B}_2 = \frac{\mu_o\mu_2 M_o\rho_o^2}{\mu_o + \mu_2}\left[\frac{\cos\phi}{\rho^2}\ \hat{\rho} + \frac{\sin\phi}{\rho^2}\ \hat{\phi}\right]$$

Note we did not use μ_1; there is no need to give this parameter. In fact, once the cylinder is magnetized, then it does not make any sense to introduce μ_1.

9.12 Because of the linear nature of the electromagnetic field equations, we can consider the field as a superposition, of the field induced by the external magnetic field, and the field due to the current in the wire. Consider first the effect of the external field. The magnetic potentials inside and outside the cylinder due to the external B field, are

$$\Phi_1 = A_1\rho\cos\phi \text{ and } \Phi_2 = -H_o\rho\cos\phi + \frac{B_1}{\rho}\cos\phi$$

The boundary conditions at $\rho = \rho_o$ are $\Phi_1 = \Phi_2$, and $B_{1n} = B_{2n}$ or $-\mu_1 \frac{\partial \Phi_1}{\partial \rho} = -\mu_2 \frac{\partial \Phi_2}{\partial \rho}$ which give $A_1\rho_o = -H_o\rho_o + B_1/\rho_o$ and $-\mu_1 A_1 = \mu_2 H_o + \mu_2 B_1/\rho_o^2$. Solving gives

$$B_1 = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} H_o \rho_o^2 \,, \quad A_1 = \frac{-2\mu_2}{\mu_1 + \mu_2}, \quad \Phi_1 = \frac{-2\mu_2}{\mu_1 + \mu_2} H_o \rho \cos\phi$$

$$\Phi_2 = -H_o \rho \cos\phi + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \frac{\rho_o^2}{\rho} H_o \cos\phi$$

$$\vec{H}_1 = \frac{2\mu_2 H_o}{\mu_1 + \mu_2} \cos\phi \hat{\rho} - \frac{2\mu_2 H_o}{\mu_1 + \mu_2} \sin\phi \hat{\phi}$$

$$\vec{H}_2 = \left(1 + \frac{(\mu_1 - \mu_2)}{\mu_1 + \mu_2} \frac{\rho_o^2}{\rho^2}\right) H_o \cos\phi \hat{\rho} - \left(1 - \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)} \frac{\rho_o^2}{\rho^2}\right) H_o \sin\phi \hat{\phi}$$

Now, the magnetic field due to the current is simply calculated using Ampere's law:

$$\int \vec{H} \cdot d\vec{\ell} = I \text{ or } \vec{H}_1 = \frac{I\rho}{2\pi\rho_o^2} \hat{\phi} \text{ and } \vec{H}_2 = \frac{I}{2\pi\rho} \hat{\phi}.$$

The field inside and outside the wire is, therefore, given as the sum of the two fields:

$$\vec{H}_1 = \frac{2\mu_2 H_o}{\mu_1 + \mu_2} \cos\phi \hat{\rho} + \left(\frac{I\rho}{2\pi\rho_o^2} - \frac{2\mu_2 H_o}{\mu_1 + \mu_2} \sin\phi\right) \hat{\phi}$$

$$\vec{H}_2 = \left(1 + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \frac{\rho_o^2}{\rho^2}\right) H_o \cos\phi \hat{\rho} + \left[\frac{I}{2\pi\rho} - \left(1 - \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \frac{\rho_o^2}{\rho^2}\right) H_o \sin\phi\right] \hat{\phi}$$

9.13 a) Small spherical cavity. Take $\vec{M} = M_o\hat{z}$ and $\vec{H} = H_o\hat{z}$. From boundary conditions at $r = \infty$ we conclude that only terms of order $\cos\theta$ survive; thus the potential inside and outside the cavity take the forms:

$$\Phi_1 = \frac{A_1}{r^2}\cos\theta + A_2 r\cos\theta, \quad \Phi_2 = -H_o r\cos\theta + \frac{C_1}{r^2}\cos\theta$$

Moreover, since Φ_1 must be finite at $r = 0$, then $A_1 = 0$. Now we match the boundary conditions at $r = R$:

i) $\quad \Phi_1(R) = \Phi_2(R)$ or

$\quad A_2R = -H_oR + C_1/R^2$ or $A_2 = -H_o + C_1/R^3$

ii) $\quad B_{1n}(R) = B_{2n}(R)$. But $\vec{B}_1 = -\mu_o\nabla\Phi_o + \mu_o\vec{M}$, thus

$B_{1n} = -A_2\mu_o\cos\theta$; $B_{2n} = \mu_oH_o\cos\theta + (2\mu_oC_1/R^3)\cos\theta + \mu_oM_o\cos\theta$,

hence $-A_2\mu_o = \mu_oH_o + 2\mu_oA_2 + 2\mu_oH_o + \mu_oM_o$

$-3A_2 = 3H_o + M_o$ or $A_2 = -H_o - M_o/3$, thus

$\Phi_1(\vec{r}) = -(H_o + \frac{M_o}{3})r\cos\theta = -(H_o + \frac{M_o}{3})z, \quad \vec{B}_1(\vec{r}) = (H_o + \frac{M_o}{3})\hat{z}$

b) For a cylinder with axis parallel to $\vec{M}$, we have $\vec{H}_t$ is continuous. Thus $\vec{H}_{1t} = \vec{H}_{2t} = H_o\hat{z}$. Therefore $\vec{B}_1 = \mu_oH_o\hat{z}$.

c) For a cylinder with axis transverse to $\vec{M} = M_o\hat{x}$, then matching at $r = \infty$ requires that only $\cos\phi$ terms are kept.

$\Phi_1 = (A_1/\rho)\cos\phi + A_2\rho\cos\phi$, $\Phi_2 = (A_3/\rho)\cos\phi - H_o\rho\cos\phi$. Since Φ_1 is finite as $\rho \to 0$, then $A_1 = 0$. Moreover matching boundary conditions at $\rho = \rho_o$ gives

i) $\quad \Phi_1(\rho_o) = \Phi_2(\rho_o)$ or $A_2\rho_o = A_3/\rho_o - H_o\rho_o$ or $A_3/\rho_o^2 = A_2 + H_o$.

ii) $\quad B_{1n}(\rho_o) = B_{2n}(\rho_o)$. But $\vec{B} = -\mu_o\nabla\Phi + \mu_o\vec{M}_o$, then

$-A_2 = A_3/\rho_o^2 + H_o + M_o$. Thus $A_2 = -H_o - M_o/2$, and

$\Phi_1 = -(H_o + M_o/2)\rho\cos\phi$, $\vec{B}_1 = \mu_o(H_o + M_o/2)\hat{x}$.

9.14 The correct boundary conditions at the boundary are first $B_{1n} = B_{2n}$ or $B_1\cos\theta = B_2\cos\theta_2$ where θ_1 and θ_2 are the angles between B_1 and B_2 and the normal to the interface, and the condition $H_{1t} = H_{2t}$ or $B_1\sin\theta_1/\mu_1 = B_2\sin\theta_2/\mu_2$. Eliminating B_1 and B_2 from these two conditions by dividing then we get

$\tan\theta_1/\mu_1 = \tan\theta_2/\mu_2$ or $\tan\theta_1/\tan\theta_2 = \mu_1/\mu_2$.

9.15 **a)** We use $\vec{m} = \frac{1}{2}\int \vec{r} \times \vec{J}\, dv$ where $\vec{J} = \frac{q\omega}{4\pi R}\,\delta(r - R)\sin\theta\hat{\phi}$

$$\vec{m} = \frac{q\omega}{8\pi R}\int r(\hat{r} \times \hat{\phi})\sin\theta\ \delta(r - R)r^2 dr\ \sin\theta\ d\theta d\phi$$

3

$\vec{m} = -\frac{-q\omega R^3}{8\pi R}\int\hat{\theta}\ \sin^2\theta\ d\theta\ d\phi$. Now only the z component is nonzero by symmetry. Thus $\hat{\theta} = \hat{\rho}\cos\theta - \hat{z}\sin\theta$, hence

$$\vec{m} = \frac{q\omega R^2}{4}\int_0^{\pi}\sin^3\theta\ \hat{z}\ d\theta = \frac{q\omega R^2}{3}\ \hat{z}.$$

b) To use $\vec{m} = \int\vec{m}\ dv$ we need to find the effective magnetization. $\vec{K}_m = \vec{M} \times \hat{n} = \vec{M} \times \hat{r} = (q\omega/4\pi R)\sin\theta\hat{\phi}$. Thus

$\vec{M} = \frac{q\omega}{4\pi R}\ \hat{z}$ (see Eq. 9.20).

$$\vec{m} = \int\vec{M}\ dv = \frac{q\omega\hat{z}}{4\pi R}\int dv = \frac{q\omega}{4\pi R}\ \hat{z}\ \frac{4}{3}\ \pi R^3 = \frac{q\omega R^2}{3}\ \hat{z}.$$

c) Use form of vector potential (Eq. 9.108) $\vec{A}_2 = (\mu_o q\omega R^2/12\pi r^2)$ $\sin\theta\hat{\phi}$. But $\vec{A}_{\text{mag dipole}} = \frac{\mu_o}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3} = \frac{\mu_o}{4\pi}\frac{m}{r^2}\sin\theta\hat{\phi}$. By comparing $\vec{A}_2$ and $\vec{A}_{\text{mag dipole}}$, we read $\vec{m} = \frac{q\omega R^2}{3}\ \hat{z}$.

9.16 This problem is solved using the method of images as done in Ex. 9.9. We use the notations of the figure of this example. The force on the wire I is just the force exerted by the image current I_1. Thus from Eq. 8.28

$$\frac{dF}{d\ell} = \mu_o I\, I_1/2\pi(2d).$$ Since $I_1 = I(\mu - \mu_o)/(\mu + \mu_o)$, then

$$\frac{dF}{d\ell} = \mu_o\left(\frac{\mu - \mu_o}{\mu + \mu_o}\right)\frac{I^2}{4\pi d}$$

9.17 Taking $\mu \gg \mu_o$ in problem 9.16 gives the required answer.

9.18 This problem ($\mu_1 \gg \mu_2$) is similar to a line of charge and a conducting cylinder, which was solved using the method of images (see Figs. 3.23 and 3.24). We therefore use the method of images. Here, however, we need two image currents: one is located inside the cylinder and it is equal to I. The position of the image current can be determined in terms of a and b using the relation $x_< x_> = R^2$ where $x_<$ and $x_>$ are the distances of the image and the original currents from the center of the cylinder, and R is the radius of the cylinder. (See discussion on page 111). Thus $x_< = a^2/b$. The distance between the two currents is therefore $D = b - a^2/b = (b^2 - a^2)/b$. The second image current is $-I$ and located at the axis of the cylinder. Thus the force exerted on the current is equal to the force exerted on it by the two image currents:

$$\frac{dF}{d\ell} = \frac{\mu_2 I_1 I}{2\pi\ D} - \frac{\mu_2 I^2}{2\pi b} = \frac{\mu_2 b I^2}{2\pi(b^2 - a^2)} - \frac{\mu_2 I^2}{2\pi b} = \frac{\mu_2 a^2 I^2}{2\pi b(b^2 - a^2)} .$$

To prove that these are correct values of the currents we satisfy the boundary conditions as follows: taking the origin at the axis of the cylinder, then we have I at $x = -b$, I_1 at $x = -d$, and I_2 at $x = 0$.

Using the boundary condition H tangential continuous, and because $\vec{B} = \mu\vec{H}$ is finite, then $H_1 \to 0$ as $\mu_1 \to \infty$. Now the continuity of B normal implies $\vec{B}$ must be normal outside the cylinder. The vector potential $\vec{A} \propto I\hat{z}\ \ell n((x + b)^2 + y^2) + I_1\hat{z}\ \ell n((x + d)^2 + y^2) + I_2\hat{z}\ \ell n(x^2 + y^2)$. But $\vec{B} = \vec{\nabla} \times \vec{A} =$

$$(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x})A_z,\ \hat{\phi} = \hat{y}\cos\phi - \hat{x}\sin\phi = \hat{y}\frac{x}{\rho} - \hat{x}\frac{y}{\rho},$$

$$\frac{\partial A_z}{\partial x} = \frac{2I(x + b)}{(x + b)^2 + y^2} + \frac{2I_1(x + d)}{(x + d)^2 + y^2} + \frac{2I_2 x}{x^2 + y^2}, \text{ and}$$

$$\frac{\partial A_z}{\partial y} = \frac{2Iy}{(x + b)^2 + y^2} + \frac{2I_1 y}{(x + d)^2 + y^2} + \frac{2I_2 y}{x^2 + y^2} \text{ then}$$

$\vec{B}_{tangential} = 0$ or $\vec{B}\cdot\hat{\phi} = 0$ gives

$$\vec{B}\cdot\hat{\phi}\alpha\ \frac{2I[(x + b)x + y^2}{(x + b)^2 + y^2} + \frac{2I_1[(x + d)x + y^2}{(x + d)^2 + y^2} + \frac{2I_2[x^2 + y^2]}{x^2 + y^2} = 0.$$

This should be true at any (x,y). We will evaluate at $x = -a$, $+a$, and 0 to get three conditions respectively:

$$\frac{aI}{a-b} + \frac{aI_1}{a-d} + I_2 = 0, \quad \frac{aI}{a+b} + \frac{aI_1}{a+d} + I_2 = 0$$

$$\frac{Ia^2}{a^2+b^2} + \frac{I_1 a^2}{a^2+d^2} + I_2 = 0$$

Solving these three equations for I_1 and I_2 and d gives exactly the above solutions. In fact if we remove the condition $\mu_1 \gg \mu_2$, then the off axis image current is $I_1 = I(\mu_1 - \mu_2)/(\mu_1 + \mu_2)$ and the on axis image current is $-I_1$, and the force becomes

$$\frac{dF}{d\ell} = \frac{\mu_2 a^2 I^2}{2\pi b} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} .$$

9.19 Just like the previous problem we use the method of images. The images are the same, one outside the cylinder and one on the axis. However the force is now given only by the current outside the cylinder. Thus we get:

$$\frac{dF}{d\ell} = \frac{\mu_1 b I^2}{2\pi(a^2 - b^2)} .$$

9.20 We attempt to find the force between the dipole, and its image. The magnetic field, due to the image dipole, at the point of the original dipole is

$$\vec{B} = \frac{\mu_o m}{4\pi\varepsilon_o r^3} \{2\cos\theta\hat{r} + \sin\theta\hat{\theta}\}.$$

The force on the "real" dipole due to a magnetic field is given by $\vec{F} = (\vec{m}\cdot\nabla)\vec{B}$. Now noting that

$$\vec{m}\cdot\nabla = (m\cos\theta\hat{r} + m\sin\theta\hat{\theta})\cdot(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}) = m\cos\theta\frac{\partial}{\partial r} + \frac{m}{r}\sin\theta\frac{\partial}{\partial\theta}$$

(and noting that $\partial\hat{r}/\partial\theta = \hat{\theta}$, $\partial\hat{\theta}/\partial\theta = -\hat{r}$, $\partial\hat{\theta}/\partial r = 0$, and $\partial\hat{r}/\partial r = 0$, see Table 1.1), then $\vec{F} = \dfrac{3m^2\hat{r}}{64\pi\varepsilon_o z_o^4}(1 + \cos^2\theta)$. The torque acting is $\vec{\tau} = \vec{m}\times\vec{B}$ which can be easily shown to be as given in 9.120.

9.21 For the bar on the right, we have an analogy with a solenoid: $\vec{B} = \mu_o M(\cos\theta_1 - \cos\theta_2)$ where

$$\cos\theta_1 = \frac{L + \ell_g/2}{((L + \ell_g/2)^2 + (\rho_o/2)^2)^{1/2}} \approx \frac{L}{(L^2 + \frac{\rho_o^2}{4})^{1/2}} \approx 1 \quad \text{and}$$

$$\cos\theta_2 = \frac{\ell_g/2}{((\ell_g/2)^2 + (\rho_o/2)^2)^{1/2}} \approx \frac{\ell_g}{\rho_o} \approx 0$$

$\vec{B}_{axis} \cong \mu_o\vec{M}\{\dfrac{L}{(L^2 + \rho_o^2/4)^{1/2}} - \dfrac{\ell_g}{\rho_o}\} \approx \mu_o\vec{M}$. For the bar on the left we have a similar result. Thus $\vec{B} = 2\mu_o\vec{M}$ and $\vec{H} = 2\vec{M}$.

9.22 We use the notations of Fig. 9.18. Thus we write $NI - H_1\ell_1 = H_2\ell_2 = H_3\ell_3$, $F_1 = F_2 + F_3$ or $B_1 = B_2 + B_3$. Thus we make the following table using the above H equations and the magnetization curve.

H_1	H_2	H_3	B_1	B_2	B_3	$B_1 - B_2 - B_3$
4000	3125	1250	.7	.76	.44	-.5
4500	1875	750	.72	.54	.31	-.13
4650	1500	600	.73	.485	.26	-.015
4750	1250	500	.73	.44	.22	+.07
5000	625	250	.74	.265	.11	+.365

Thus it is clear that the relation $B_1 = B_2 + B_3$ is satisfied for $H_1 \simeq 4675$. This can be seen by plotting $B_1 - B_2 - B_3$ as a function of H_1 and extrapolating for a zero value. Thus the operating point of the circuit is $B_1 \simeq .73$, $B_2 \simeq .48$, and $B_3 \simeq .25$T.

9.23 Using the notation of Fig. 9.21 we have

$$B_m = -\frac{\ell_m H_m}{RA_m}, \quad R = \frac{\ell_g}{\mu_o A_g} + \frac{\ell_s}{\mu_s A_s}$$

For $\ell_g = 1$ cm, and since $\mu_s >> \mu_o$, then $R \simeq \ell_g/\mu_o A_g$. Thus

$$B_m = -\mu_o \frac{\ell_m}{\ell_g} \frac{A_g}{A_m} H_m = -10\mu_o H_m$$

Plotting this equation on the magnetization curve of Alnico 5 (given in Fig. 9.8c) gives the operating point of the magnet at the intersection of the two curves.

9.24 We follow the procedure described in Ex. 9.13. Let the fields (H,B) in materials one and two be (H_1, B_1) and (H_2, B_2). Thus according to Ex. 9.13 we have

$H_1\ell_1 + H_2\ell_2 = NI$ or $H_2 = NI/\ell_2 - H_1\ell_1/\ell_2$

$B_1A_1 = B_2A_2$ or $B_2 = B_1A_1/A_2$

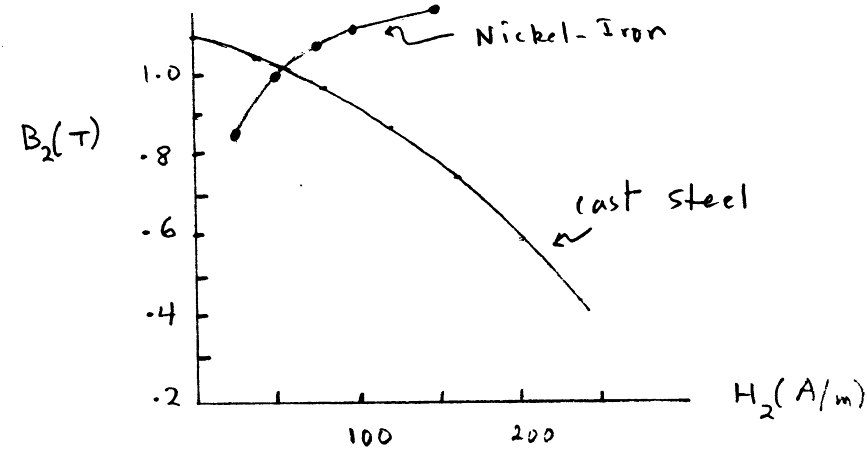
These two relations have to be consistent with the $H_2 - B_2$ magnetization curve, and $H_1 - B_1$ magnetization curve of the materials. Thus from Fig. 9.8 for cast steel, we can select the following pairs (H_1, B_1): (200, .33), (250, .44), (300, .55), (350, .65), (400, .73), (450, .78), and (500, .83). We substitute these values in the top two equations to generate pairs of (H_2, B_2). For example the (200, .33) pair gives (240, .44) as follows:

$H_2 = 40/.1 - 200 \times .08/.1 = 400 - 160 = 240$

$B_2 = .33 \times 3 \times 10^{-4}/2.25 \times 10^{-4} = .44.$

Continuing we get the following pairs: (240, .44), (200, .59), (160, .73), (120, .87), (80, .97), (40, 1.04), (0, 1.11). We now plot these points on the magnetization curve of material 2 of Fig. 9.8 as shown in the figure. The crossing of the two curves gives the operating point of the system: $B_2 = 1.01$ T, $B_1 = .76$T, and flux $= B_2A_2 = B_1A_1 = 2.28 \times 10^{-4}$ Tm^2.

9.25 The toroids (circular cross section) have a $\vec{B}$ field inside ($\hat{\phi}$ direction). We want to find an $\vec{A}$ satisfying $\vec{\nabla} \times \vec{A} = \vec{B}$ on the (z) axis. In Ex. 8.12 and in problem (8.18) we solved the $\vec{B}$ field on the z axis for a current loop, satisfying $\vec{\nabla} \times \vec{B} = \mu_o\vec{J}$



and found $\vec{B}(z) = \dfrac{\mu_o I R^2 \hat{z}}{2(z^2 + R^2)^{3/2}}$ where the loop is of radius R, placed at z = 0 with a current I = current density times cross sectional area. For a toroid with $\vec{B}$ = flux density, giving

a) $F = \text{flux} = \pi(\frac{t}{2})^2 B$ let $\mu_o J \rightarrow B$ and $\vec{B} \rightarrow \vec{A}$ then

$$\vec{A}(z) = \frac{F\, R^2 \hat{z}}{2(z^2 + R^2)^{3/2}} = \frac{\pi(\frac{t}{2})^2 B R^2 \hat{z}}{2(z^2 + R^2)^{3/2}}$$

b) For two toroids, at $z = \pm d$

$$\vec{A}(z) = \frac{\pi t^2 B R^2 \hat{z}}{8} \left(\frac{1}{((z - d)^2 + R^2)^{3/2}} + \frac{1}{((z + d)^2 + R^2)^{3/2}}\right)$$

9.26 a) Using Ampere's law we get: $\oint \vec{H} \cdot d\vec{\ell} = 2\pi\rho H(\rho) = N_o I$, or $\vec{H}(\rho) = (N_o I/2\pi\rho)\hat{\phi}$. Now we need $H(\rho) \geq H_o$ in order to saturate the Ferrite, hence we need at minimum, enough H at $\rho = b$ to saturate that is $H(b) = H_o = N_{o\ min} I/2\pi b = N_{o\ min}(1)/2\pi b$ or $N_{o\ min} = 2\pi b H_o$ at I = 1 amp.

b) Now for I = 0, and again by symmetry, Ampere's law gives $\oint \vec{H}\cdot d\vec{\ell} = 2\pi\rho H(\rho) = N_o I = 0$ or $H = 0$. But $\vec{B} = \mu_o(\vec{H} + \vec{M})$, hence $\mu_o\vec{M} = \hat{\phi}B_o$.

c) The energy per unit volume is (relevant to Chapter 12)

$$u = \oint d\vec{B}\cdot\vec{H} = \oint dB\left(\frac{N_o I}{2\pi\rho}\right) = \left(\frac{N_o I}{2\pi\rho}\right) 4B_o = \frac{2N_o B_o}{\pi\rho}$$

i.e., the change in B with H is neglible now, from part (a), $N_o = 2\pi b H_o$, hence, $u = 4H_o B_o b/\rho$, i.e. H varies with ρ, but $B = B_o$ = constant, and

$$u = \int u dv = 4H_o B_o b \int_a^b \frac{1}{\rho}\,(2\pi\rho c)d\rho = 8\pi H_o B_o bc(b - a) \text{ per cycle.}$$

CHAPTER 10

10.1 a) $\vec{m} = I\vec{a}$, $I = \frac{-e}{T} = \frac{-e\omega}{2\pi}$, $\vec{a} = \pi\rho^2\hat{z}$, thus

$$\vec{m} = \left(\frac{-e\omega}{2\pi}\right)\pi\rho^2\hat{z} = \frac{-e\omega\rho^2}{2}\hat{z}$$

b) In the presence of an external $\vec{B}$ field ($\vec{B} = B\hat{z}$) we have from Eq. 10.9: $\Delta\vec{m} = -\frac{1}{4}\frac{e^2\rho^2}{m_e}\vec{B}$, therefore $\vec{m} = \frac{-e\rho^2}{2}\left(\omega + \frac{eB}{2m_e}\right)\hat{z}$

c) $\chi_m = -\frac{1}{6}\frac{\mu_o \ell^2 N\langle r^2\rangle}{m_e}$ (from Eq. 10.13). Thus

$$\langle r^2\rangle = \frac{-6\chi_m m_e}{\mu_o e^2 N} = 6.1 \times 10^{-21} m^2 \text{ or } \langle r\rangle \sim 7.8 \times 10^{-11} \text{ m.}$$

10.2 a) Since the 2 electrons are moving in the same orbit and speed but in opposite directions, then their angular momenta $\vec{L}$ will be opposite and will cancel out. Hence $\vec{m} = 0$ since $\vec{m}$ is proportional to $\vec{L}$.

b) For each electron we have, from Eq. 10.9, for the change in $\vec{m}$: $\Delta\vec{m} = -\frac{1}{4}\frac{e^2\rho^2}{m_e}\vec{B}$. For two electrons we have twice as much moment: $\Delta\vec{m} = -\frac{1}{2}\frac{e^2\rho^2}{m}\vec{B}$.

10.3 The probablity P of being aligned with the field is

$$P = \frac{e^{mB/kT}}{e^{mB/kT} + e^{-mB/kT}}.$$ If 75% atoms are aligned, then

$$P = .75 = \frac{e^{mB/kT}}{e^{mB/kT} + e^{-mB/kT}}.$$ We now solve for T

$$.75\,\frac{(e^{mB/kT}+e^{-mB/kT})}{e^{mB/kT}} = 1 \text{ or } \frac{4}{3} = e^{-2mB/kT} + 1 \text{ thus}$$

$$\frac{1}{3} = e^{-2mB/kT} \text{ which gives } \frac{2mB}{kT} = \ell n3. \quad \text{Thus } T = 3.67^0K.$$

10.4 The results can be obtained from problem 10.3. Note that the previous problem we had $2mB/kT = \ell n3$, then $T = \frac{2mB}{k\ell n3}$ which gives 5.6×10^{-3}°K.

10.5 **a)** Since $\eta << 1$, then from Eq. 10.18 we have

$$\chi_{para} = \frac{Nm_o^2\mu_o}{3kT} = 3.3 \times 10^{-3}.$$

b) Since $\vec{M} = \chi\vec{H}$ and $\vec{H} = \frac{\vec{B}}{\mu_o}$, then $\vec{M} = \frac{\chi\vec{B}}{\mu_o} = (2626)\ (0.1) =$ 262.6 A/m

$m = \int MdV = M(\text{volume of bar}) = 2.63 \times 10^{-3}\ A/m^2$

c) At saturation, $M = Nm_o = (10^{23}\text{atoms}/m^3)\ (1.8 \times 10^{-23}A/m^2) =$ 1.8×10^6 A/mm $= \int MdV = M(vol) = 18\ A/m^2$.

10.6 From the Curie-Weiss Law, we have $\chi = \frac{C}{T - T_c}$, therefore

$$\frac{1}{\chi} = \frac{T - T_c}{C} \quad \text{and} \quad \frac{5.82 \times 10^3 g/cc}{1.35 \times 10^4 g/cc} = \frac{600 - T_c}{1000 - T_c}$$

which gives $T_c \approx 297$°K.

10.7 The saturation magnetization is $M = mN = 8.44 \times 10^5$ A/m. From Eq. 10.46 we have $\gamma = \frac{3kT_cN}{\mu_o Ms^2} = 2656.$

10.8 From problem 10.7 we have for Ni, $N = 9.1 \times 10^{28}$ atoms/m^3, and $\gamma = 2656$. From Eq. 10.45,

$$\gamma = \frac{2\alpha n}{\mu_o \beta^2 N} \text{ or } \alpha n = \frac{\gamma\mu_o\beta^2 N}{2} = \frac{\gamma\mu_o N}{2}\left(\frac{ge\hbar}{2m}\right)^2$$ which gives

$\alpha n = 5.2 \times 10^{-20}$ joules.

CHAPTER 11

11.1 We use Eq. 11.5, for a loop of radius $\rho < \rho_o$ and for a loop of radius $\rho > \rho_o$. $\oint_c \vec{E}\cdot d\vec{\ell} = \frac{-d}{dt}\int \vec{B}\cdot d\vec{a}$ or

$E_\phi(2\pi\rho) = \frac{-d}{dt} B(\pi\rho^2)$ giving $|E_\phi| = \frac{1}{2}\rho^2 \frac{dB}{dt} = \frac{1}{2}\rho\,(.1)$ volts/m,

$|E_\phi| = .05\rho$ for $\rho \le \rho_o$. For $\rho > \rho_o$ we have $\nabla \times \vec{E} = 0$ or

$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi) = 0$, thus $E_\phi = \frac{\text{const}}{\rho}$. Now at $\rho = \rho_o$, E must be continuous, therefore $\frac{\text{const}}{\rho_o} = .05\,\rho_o$ giving const $= .05\,\rho_o^{\,2}$.

Thus $|E_\phi| = .05\rho_o^2/\rho$ for $\rho > \rho_o$.

11.2 The flux in the coil is $F = \int \vec{B}_1\cdot d\vec{a} = \mu_o n_1 I\pi r^2 N_2$

$\mathcal{E} = -\frac{dF}{dt} = -\mu_o n_1 N_2 \pi r^2 \frac{dI}{dt} = -7.9 \times 10^{-5}(3 + 4t)$ volts, and at

$t = 2$ seconds $i = \frac{\mathcal{E}}{R} = -\frac{(7.9 \times 10^{-5})(11)}{.15\Omega} = 5.8 \times 10^{-3}$A.

11.3 **a)** We use $\oint \vec{A}\cdot d\vec{\ell} = F = \int \vec{B}\cdot d\vec{a}$. For $\rho \le R$, $F = \pi B_o(t)\rho^2$, thus $A_\phi = \rho B_o(t)/2$. For $\rho > R$, we have $F = \pi B_o R^2$, thus $A_\phi = R^2 B_o/2\rho$.

b) $\vec{E}^i = -oA/\partial\vec{A}/\partial t$ which gives

$E = E_\phi = -\frac{1}{2}\rho\frac{dB_o}{dt}$ for $\rho \le R$ and $-\frac{1}{2}\frac{R^2}{\rho}\frac{dB_o}{dt}$ for $\rho > R$.

c) The current density in the disk is $\vec{J} = \sigma_c\vec{E} = \sigma_c\hat{\phi}E_\phi$.

d) The power dissipated is calculated from the power density J^2/σ_c. Thus

$P = \int (J^2/\sigma_c)dv = \int \sigma_c E^2 dv = \frac{\delta\sigma_c}{2\pi}\left(\frac{\pi R^2}{2}\cdot\frac{dB_o}{dt}\right)^2\left[1 + 4\,\ell n\,\frac{a}{R}\right]$.

11.4 Since $\vec{B} = B_o\hat{z}$ and $\vec{v} = \vec{v}_o\hat{x}$, then $\vec{E}_i = \vec{v} \times \vec{B} = -v_oB_o\hat{y}$ and $\mathcal{E} = \int(\vec{v} \times \vec{B})\cdot d\vec{r} = \int(-v_oB_o)\hat{y}\cdot(dx\hat{x} + dy\hat{y}) = -2v_oB_oR.$

11.5 The wire (long) has current I flowing. Use cylindrical coordinates to find B of the wire $\vec{B}(\rho) = \frac{\mu_o I\hat{\phi}}{2\pi\rho}$. We can calculate the flux, find the change, or, use $\mathcal{E} = \oint(\vec{v}\times\vec{B})\cdot d\vec{\ell}$ which is equivalent. Now $\vec{v} \times \vec{B} = \frac{v\mu_o I\hat{z}}{2\pi\rho}$. Since we have only a z-component, only sides parallel to the z axis contribute:

$$\mathcal{E} = \hat{z}\cdot\int_o^b(\vec{v} \times \vec{B}(\ell))dz + \hat{z}\cdot\int_b^o(\vec{v} \times \vec{B}(\ell + a))dz$$

$$\mathcal{E} = \frac{v\mu_o Ib}{2\pi}\left(\frac{1}{\ell} - \frac{1}{\ell + a}\right) = \frac{v\mu_o Ib}{2\pi}\left(\frac{a}{\ell^2 + \ell a}\right)$$

If the loop is stationary, and the current changes,

$$\mathcal{E} = \frac{dF}{dt} \text{ where } F = \int_o^b dz \int_\ell^{\ell+a} d\rho \,\frac{\mu_o I_1}{2\pi\rho} = \frac{\mu_o Ib}{2\pi}\,\ell n\left(1 + \frac{a}{\ell}\right)$$

$$\mathcal{E} = \frac{dF}{dt} = \frac{\mu_o b}{2\pi}\,\ell n\left(1 + \frac{a}{\ell}\right)\frac{dI}{dt} \text{ and } \mathcal{E}_2 = M\,\frac{dI_1}{dt}$$

$$\text{Thus } M = \frac{\mu_o b}{2\pi}\,\ell n\left(1 + \frac{a}{\ell}\right) = \text{mutual inductance}$$

11.6 a) We use Eq. 11.21 $\mathcal{E} = \oint_{loop}(\vec{v} \times \vec{B})\cdot d\vec{\ell}$. Now $\vec{v} = \vec{\rho} \times \vec{\omega} = \rho\omega\phi$, $\vec{v} \times \vec{B} = -\omega\rho B\hat{z}$, thus

$\mathcal{E} = \oint(\vec{v} \times \vec{B})\cdot d\ell = -\omega\ell\hat{z}\,(.03\text{ m x }.25\text{T} - .05\text{m x }.8\text{T}) =$ .85 volts. Thus the current in the loop is $\frac{\mathcal{E}}{R} = I = 4.25$ amps.

b) $\frac{\mathcal{E}^2}{R} = I^2R = I\mathcal{E} = 3.62$ watts

c) $\vec{F}$ = force $= q\vec{v} \times \vec{B} = \oint I d\vec{\ell} \times \vec{B}$, and

$\vec{\tau} = \oint_{loop} I\vec{\rho} \times [\phi\vec{\ell} \times \vec{B}) = -I\,\ell\hat{z}(\rho_1 B_1 - \rho_2 B_2) = IV = 3.62$ watts,

same as in (b).

11.7 a) Stationary loop with $A = 25\ \text{cm}^2$. Flux $= \vec{B}\cdot\vec{A} = BA$,

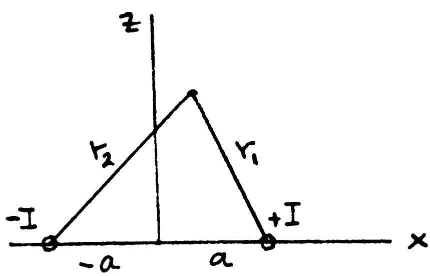
thus $\frac{dF}{dt} = \mathcal{E} = A\frac{dB}{dt} + B\frac{dA}{dt} = A\frac{dB}{dt}$ (stationary loop)

$\mathcal{E} = A\frac{dB}{dt} = AB_o\omega\cos(\omega t) = 7.5\cos\omega t$

b) $F = AB$, $\mathcal{E} = \frac{dF}{dt} = A\frac{dB}{dt} + B\frac{dA}{dt}$. Use $A = \ell x = \ell vt$, thus

$\frac{dA}{dt} = \ell v$, and $F = \ell vtB_o\sin\omega t$. Thus

$\mathcal{E} = \frac{dF}{dt} = B_o\ell v[\sin(\omega t) + \omega t\cos(\omega t)]$. .



11.8 a) Using the formula $M = \frac{\mu_o}{4\pi}\int\int \frac{d\vec{\ell}_1\cdot d\vec{\ell}_2}{|\vec{r}_1 - \vec{r}_2|}$ we get

$$M_{sm} = \frac{\mu_o}{4\pi}\int_o^1 d\ell_m \int_{-L}^{L} d\ell_1 \left[\frac{1}{(r_1^2 + \ell_1^2)^{1/2}} - \frac{1}{(r_2^2 + \ell_2^2)^{1/2}}\right]$$

where m refers to the moving wire, and s refers to the stationary wire.

$$= \frac{\mu_o}{4\pi} \int d\ell_m \log\{\frac{L + \sqrt{r_1^2 + L^2}}{-L + \sqrt{r_1^2 + L^2}} \times \frac{-L + \sqrt{r_2^2 + L^2}}{L + \sqrt{r_2^2 + L^2}}\}$$

Expanding as L becomes large we get:

$$\frac{M_{1m}}{\ell} = \frac{\mu_o}{4\pi} \int d\ell_m \log(r_2/r_1)^2 = \frac{\mu_o}{2\pi} \log r_2/r_1$$

b) The induced emf per unit length

$$\frac{\mathcal{E}}{\ell} = \frac{-IdM_{sm}}{dt} = \frac{\mu_o I}{2\pi}\left[\frac{\dot{x}(x - a) + \dot{z}z}{(x - a)^2 + z^2} - \frac{\dot{x}(x + a) + \dot{z}z}{(x + a)^2 + z^2}\right]$$

At the origin we have $x = z = 0$, and we have $\vec{v} = v_o\hat{x}$, thus $\frac{\mathcal{E}}{\ell} = -\mu_o I\, v_o/\pi a$.

11.9 Using the dipole approximation where r >> size of loop, loop 1 has area A_1, current I_1, and using Eq. (8.87), we have $\vec{m}_1 = I_1 A_1(\hat{x} \cos\omega t + \hat{y} \sin \omega t)$. The field produced by loop 1 at the location of the second loop is calculated from 8.98, and using $\vec{r} = \hat{y}r$

$$\vec{B}_2(\vec{r}) = \frac{3\mu_o(\hat{r}\cdot\vec{m}_1)\hat{r} - \vec{m}_1}{4\pi r^3} \quad \text{thus}$$

$$\vec{B}_2(r) = \frac{3(\hat{y}\cdot\vec{m}_1)\hat{y} - \vec{m}_1}{4\pi r^3}\mu_o = \frac{\mu_o I_1 A_1}{4\pi r^3}(2\hat{y} \sin(\omega t) - \hat{x} \cos(\omega t)$$

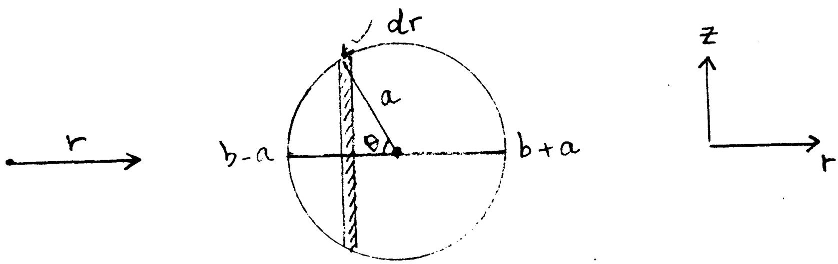
But the area $\vec{A}_2 = A_2\hat{x}$, then $\vec{B}_2(r)\cdot\vec{A}_2 = \frac{\mu_o I_1 A_1 A_2}{4\pi r^3}(-\hat{x}\cos(\omega t)) = F_2$. Thus

$$\mathcal{E}_2 = \frac{dF_2}{dt} = \frac{\mu_o I_1 A_1 A_2}{4\pi r^3}\,\omega \sin(\omega t) \text{ and}$$

$$I_2 = \frac{\mathcal{E}_2}{R} = \frac{\mu_o I_1 A_1 A_2 \omega \sin(\omega t)}{4\pi r^3}$$

11.10 Using Ampere's Law we get $\vec{B} = \mu_o IN\hat{\phi}/2\pi\rho$. The flux through N loops is $F = N\int\vec{B}\cdot d\vec{a} = (\mu_o IN^2/2\pi)\int a d\rho/\rho$. Thus $F = \mu_o IN^2 a\ \ell n(\rho_2/\rho_1)/2\pi$, hence the self inductance is $L = \mu_o N^2 a\ \ell n(\rho_2/\rho_1)/2\pi$.

11.11 We know that for a toroidal solenoid:



$\vec{B} = \mu_o IN\hat{\phi}/2\pi r$ and $F = \int\vec{B}\cdot d\vec{a}$, thus the flux is

$$F = \frac{\mu_o In}{2\pi}\int\frac{1}{r}(2zdr) \text{ where } z = a\sin\theta, \text{ or}$$

$F = = \frac{\mu_o IN}{2\pi}\int\frac{2a\sin\theta}{r}\,dr$. Now $r = b - a\cos\theta$, $dr = a\sin\theta d\theta$, hence

$$F = \frac{\mu_o INa^2}{\pi} \int_o^{\pi} \frac{\sin^2\theta d\theta}{b - a\cos\theta} .$$ From Integral Tables:

$$\int_o^{\pi} \frac{\sin^2\theta d\theta}{b - a\cos\theta} = \frac{\pi b}{a^2}(1 - \sqrt{1 - a^2/b^2}).$$ Thus

$$F = b\mu_o IN(1 - \sqrt{1 - a^2/b^2}) = \mu_o IN(b - \sqrt{b^2 - a^2})$$ and

$$L = \frac{NF}{I} = \mu_o N^2(b - \sqrt{b^2 - a^2})$$ and $$L/\text{length} = \frac{\mu_o N^2}{2\pi b}(b - \sqrt{b^2 - a^2})$$

11.12 Let currents I_o and $-I_o$ flow along the inner and outer conductors respectively. Take the z axis along current I_o. Then Ampere's Law gives $\vec{H} = I_o\hat{\phi}/2\pi\rho$ and $\vec{B} = \mu\vec{H}$ for the inside region and 0 for the outside region. The flux linking the two conductors is that crossing a plane of constant ϕ. Thus

$$F = \int \vec{B}\cdot\vec{da} = \frac{\mu I_o}{2\pi}\int \frac{d\rho dz}{\rho} = \frac{\mu I_o \ell}{2\pi} \ln\frac{b}{a} .$$

Thus $L/\ell = (\mu/2\pi)\ \ln(b/a)$.

11.13 The difference between this problem and problem 11.12 is that the inner conductor is solid and the current is uniformly distributed. For the region between the conductors the fields and the flux are identical to those in 11.12. An extra flux due to the fields inside the inner conductor has to be accounted for. The B field for the region resulting from $J = I_o/\pi a^2$ is $B = \hat{\phi}\mu_o I_o \rho/2\pi a^2$. Thus the extra flux is

$$F = \int \vec{B}\cdot d\vec{a} = \ell\mu_o I_o/4\pi.$$

Thus $L/\ell = \mu_o/4\pi + (\mu/2\pi)\,\ell n(b/a)$.

11.14 This problem will be solved using the method of images. This procedure is very similar to the electrostatic case discussed in Chapter 3 and in Examples 6.7 and 9.9. We need to find the two currents I and -I that will produce constant potential on both conductors. Referring to Example 6.7 and to its figure 6.4, we will use the notations in the figure. We find that the currents are located at a and -a where

$$R = \frac{2ma}{m^2 - 1}, \quad x_o = \frac{m^2 + 1}{m^2 - 1}\, a, \quad m = \frac{x_o}{R} + \left[\left(\frac{x_o}{R}\right)^2 - 1\right]^{1/2}$$

The B field in the region between the cylinder and the plate is produced by

$$\vec{B} = \frac{\mu_o I}{2\pi\rho_+}\,\hat{\phi}_+ - \frac{\mu_o I}{2\pi\rho_-}\,\hat{\phi}_-$$

where $(\rho_+, \hat{\phi}_+)$ and $(\rho_-, \hat{\phi}_-)$ are the distance and unit vector in the $\hat{\phi}$ direction with respect to origins located on the I and -I currents (Example 9.9). Now the flux linking the two conductors can be calculated by the flux penetrating the x-z plane.

$$F = \int \vec{B}\cdot\vec{da} = \int_{a-(x_o-R)}^{a} \frac{\mu_o I}{2\pi\rho_+}\, d\rho_+\, dz + \int_{a}^{(a+x_o-R)} \frac{\mu_o I}{2\pi\rho_-}\, d\rho_- dz$$

$$F = \int_{a-(x_o-R)}^{a+x_o-R} \frac{\mu_o I}{2\pi\rho_+} d\rho_+ dz, \text{ or } F/\ell = \frac{\mu_o I}{2\pi} \ln m$$

Changing to the notation of the present problem, we replace x_o by d and R by a and we get:

$$L/\ell = \frac{\mu_o}{2\pi} \ln\{\frac{d}{a} + [(\frac{d}{a})^2 - 1]^{1/2}\} = \frac{\mu_o}{2\pi} \cosh^{-1} \frac{d}{a}$$

Now if d >> a, then $[(d/a)^2 - 1]^{1/2} \approx d/a$, and

$$L/\ell = \frac{\mu_o}{2\pi} \ln(2d/a) .$$

11.15 In this problem there is twice as much flux linking the two conductors. Moreover the distance to the symmetry plane is d/2. Thus we get from the previous problem

$$L/\ell = \frac{\mu_o}{\pi} \cosh^{-1} \frac{d}{2a}$$

11.16 Consider a loop with a radius a, resistance R, inductance L, in a magnetic field $\mu_o H_o \hat{y}$, and take $\vec{\omega} = \omega\hat{z}$.

a) Flux $= -\pi a^2 \mu_o H_o \sin(\omega t)$. Thus

$\mathcal{E} = -\frac{dF}{dt} = \pi a^2 \mu_o H_o \omega \cos(\omega t)$ or using complex notations $\pi a^2 \mu_o H_o \omega e^{i\omega t}$. Kirchoff's loop law gives $\mathcal{E} = IR + i\omega LI$ which

gives $I = \frac{\mathcal{E}}{R + i\omega L} = \frac{R - i\omega L}{R^2 + \omega^2 L^2} \pi a^2 \mu_o H_o \omega e^{i\omega t}$

b) $\vec{\tau} = \vec{m} \times \vec{B}$ and $\vec{m} = I\vec{A} = I\ (\hat{x} \cos(\omega t) + \hat{y} \sin(\omega t))\pi a^2$

$$I = \frac{R\cos(\omega t) + \omega L\sin(\omega t)}{R^2 + \omega^2 L^2}\,\pi a^2 \mu_o H_o \omega.$$ Thus $\vec{\tau} = I\vec{A} \times \vec{B} =$

$$\frac{R\cos^2(\omega t) + \omega L\sin(\omega t)\cos(\omega t)}{R^2 + \omega^2 L^2}\,\pi^2 a^4 \mu_o^2 H_o^2 \omega\hat{z}$$

c) $$P = \vec{\tau}\cdot\vec{\omega} = \frac{R\cos^2(\omega t) + \omega L\sin(2\omega t)/2}{R^2 + \omega^2 L^2}\,\pi^2 a^4 \mu_o^2 H_o^2 \omega^2.$$

$$\langle P_{in}\rangle = \frac{R\pi^2 a^4 \mu_o^2 H_o^2 \omega^2}{2(R^2 + \omega^2 L^2)},\quad \langle P\rangle_{resistor} = \tfrac{1}{2}\,\mathrm{Re}(II^*R) =$$

$$\frac{(R - i\omega L)(R + i\omega L)R}{2(R^2 + \omega^2 L^2)^2}\,\pi^2 a^4 \mu_o^2 H_o^2 \omega^2 = \frac{R}{2(R^2 + \omega^2 L^2)}\,\pi^2 a^4 \mu_o^2 H_o^2 \omega^2$$

$$\langle P_{in}\rangle = \tfrac{1}{2}\,\mathrm{Re}(IV^*) = \frac{R}{2(R^2 + \omega^2 L^2)}\,\pi^2 a^4 \mu_o^2 H_o^2 \omega^2.$$

11.17 a) We use the dipole approximation when the loops are far away from each other. The B field due to loop 1 at the site of loop 2 is normal to this loop and is equal to $B_{12} = \frac{2\mu_o m_1}{4\pi h^3}$, where $m_1 = \pi R_1^2 I_1$. The flux through loop 2 is then $F_{21} = B_{12}A_2 = \pi\mu_o R_1^2 R_2^2 I_1/2$ which gives a mutual inductance $M = F_{21}/I_1 = \pi\mu_o R_1^2 R_2^2/2$.

b) First by using $\phi_{12} = 2\phi - \pi$, then one can write Eq. 11.37 as

$$M = \frac{-\mu_o k\sqrt{R_1 R_2}}{2}\int_{-\pi/2}^{\pi/2} \frac{\cos 2\phi\, d\phi}{(1 - k^2\sin^2\phi)^{1/2}}.$$ For $R_1 \approx R_2 \approx R$ and

$h \ll R$, k becomes nearly one and the integrand diverges when ϕ approaches $\pi/2$. Thus we use a limiting procedure and define $\delta = \pi/2 - \phi$:

$$M \simeq \mu_o R \int_0^{\pi/2} \frac{\cos 2\delta \, d\delta}{(1 - k^2\cos^2\delta)^{1/2}} = \int_0^{\delta'} \frac{d\delta}{(1 - k^2 + k^2\delta^2)^{1/2}} + \int_{\delta'}^{\pi/2} \frac{\cos 2\delta \, d\delta}{\sin\delta}$$

where we chose $1 - k^2 << \delta'^2 << 1$. Integrating gives

$$M = \mu_o R \left[\ell n\left(\sqrt{\frac{4\delta'^2}{1 - k^2}}\right) - \ell n\left(\frac{\delta'}{2}\right) - 2\right] \simeq \mu_o R\left[\ell n\left(\frac{4}{\sqrt{1 - k^2}}\right) - 2\right]$$

Using $(1 - k^2)^{1/2} \simeq h/2R$, then we have $M \simeq \mu_o R\left[\ell n\left(\frac{8R}{h}\right) -2\right]$.

11.18 a) The voltage drop across each one of them is V and it is related to L and M as follows $V = L_1 dI_1/dt + M dI_2/dt$, $V = L_2 dI_2/dt + M dI_1/dt$ where I_1 and I_2 are the current through them. Eliminating I_2 from the first equation and I_1 from the second we get, $V(L_2 - M) = (L_1L_2 - M^2)\, dI_1/dt$ and $V(L_1 - M) = (L_1L_2 - M^2)\, dI_2/dt$. Adding the two equations and using $I = I_1 + I_2$, then $V = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}\frac{dI}{dt}$.

b) The effective inductance is defined via $V = L_{eff}\, dI/dt$ which gives $L_{eff} = (L_1L_2 - M^2)/(L_1 + L_2 - 2M)$.

CHAPTER 12

12.1 a) From Eq. 12.5, the energy stored is $U = \frac{1}{2} LI^2 = \frac{1}{2} L(\frac{E}{R})^2 = \frac{1}{2} L\frac{E^2}{R^2} = 62.5$ Joules

b) From Eq. 12.6, $U = \frac{1}{2} I\Phi$, thus $\Phi = \frac{2U}{I} = \frac{2U}{E/R} = 25$ Webers.

12.2 a) Take the z axis along the I current, then from Ampere's law we have $\vec{B} = \mu_o I\hat{\phi}/2\pi\rho$ between the two conductors. The magnetic energy density is $u = B^2/2\mu_o = \mu_o I^2/8\pi\rho^2$. The total energy per unit length is

$$U/\ell = \frac{\mu_o I^2}{\ell 8\pi^2} \int \frac{\rho d\rho dz d\phi}{\rho^2} = \frac{\mu_o I^2}{4\pi} \ell n \frac{b}{a}$$

b) $U = \frac{1}{2} LI^2$, then $L/\ell = \mu_o \ell n(b/a)/2\pi$.

12.3 a) Using Ampere's law, inside conductor: we get $\int \vec{B}\cdot d\vec{\ell} = \mu_o I_i = B(2\pi\rho) = \mu_o \pi\rho^2 I/\pi a^2$ therefore $\vec{B} = \frac{\mu_o \rho I}{2\pi a^2}\hat{\phi}$

Energy stored $= \int \frac{1}{2}\frac{B^2}{\mu_o} dv$, Energy/length =

$$\frac{1}{2\mu_o}\int B^2 \rho d\rho d\phi = \frac{1}{2\mu_o}\int_o^a d\rho \int_o^{2\pi} d\phi\rho \frac{\mu_o^2 \rho^2 I^2}{\pi^2 4a^4} = \frac{\mu_o I^2}{16\pi}.$$

b) Now, for inductor $U = \frac{1}{2} LI^2$. Thus energy/length

$= \frac{U}{\ell} = \frac{1}{2}\frac{L}{\ell} I^2 = \frac{\mu_o I^2}{16\pi}$. Thus the inductance per unit length L/ℓ is $\frac{L}{\ell} = \frac{\mu_o}{8\pi}$.

12.4 **a)** Ampere's law gives $\vec{B} = \mu_o IN\hat{\phi}/2\pi\rho$.

b) The magnetic energy density $u = B^2/2\mu_o = \mu_o I^2N^2/8\pi^2\rho^2$.

c) The total magnetic energy is

$U = \int u dv = \int u\rho d\rho d\phi dz = \mu_o I^2N^2/(8\pi^2)a \times 2\pi \int \frac{d\rho}{\rho}$

$U = (\mu_o I^2N^2a/4\pi)\ell n\rho_2/\rho_1$. But $U = \frac{1}{2} LI^2$, then

$L = (\mu_o N^2a/2\pi)\ell n(\rho_2/\rho_1)$.

12.5 **a)** From Eq. 12.37 we have $W = \oint \vec{H}\cdot d\vec{B}$ which is the area of the hysterisis loop. From the figure we find it is $\sim .2$ J/m^3. But the volume is 20 cm^3, then the loss is 4×10^{-6} Joules.

b) Power loss is $W/\Delta t = 2.4 \times 10^{-4}$ Watt.

c) We need to find the current in the loop.

Using $\oint \vec{H}\cdot d\vec{\ell} = NI$, and taking 20 A/m for the maximum H, then we get $20 \times .2 = 100\ I_{max}$ which gives $I_{max} = 4 \times 10^{-2}$ A. Since $L = NF/I$ where F is the flux through each loop, then

$L = 100 \times .01/(4 \times 10^{-2}) = 25$ H.

d) If we double the current, H will double, but B and the flux will not because the system is at saturation, therefore $L = NF/I$ will decrease.

12.6 The force can be calculated by finding the change in the magnetic energy of the system when the rod is virtually moved a small distance Δx. Because the flux is constant, then we use $U = \int (B^2/2\mu)dv$. Thus

$$U_2 - U_1 = \frac{B_o^2}{2}\left(\frac{1}{\mu} - \frac{1}{\mu_o}\right)A\Delta x = -\frac{B_o^2 A\Delta x}{2\mu\mu_o}(\mu - \mu_o).$$ But $\mu = (1 + \chi_m)\mu_o$, thus $\Delta U/\Delta x = F = -B_o^2 A\chi_m/2\mu_o(1 + \chi_m)$ an attractive force.

12.7 **a)** Using Ampere's law we get $B = \mu_o \frac{N}{\ell} I$ along axis, thus

$$U = \int u dv = \frac{1}{2\mu_o}\int B^2 dv$$

$$= \frac{1}{2\mu_o}\int_o^R \frac{\mu_o^2 N^2 I^2}{\ell^2} RdR \int_o^L dz \int_o^{2\pi} d\phi = \frac{\mu_o}{2}\frac{N^2 I^2}{\ell}\pi R^2$$

$$\vec{F} = \frac{\partial U}{\partial R}\hat{\rho} = \frac{\mu_o N^2 I^2 \pi R}{\ell}\hat{\rho}.$$ Thus $\frac{\vec{F}}{2\pi RN} = \frac{\mu_o NI^2}{2\ell}\hat{\rho}$.

b) When flux remains constant instead of current we have $F = -\frac{\partial U}{\partial R}$. To keep the flux constant, the force must be in the opposite direction $\frac{\vec{F}}{\text{turn}} = \frac{-\mu_o NI^2}{2\ell}\hat{\rho}$.

12.8 The total energy of the system is the energy of interaction of both magnets with the external B field plus the interaction energy of the magnets with each other: $U = U_1 + U_2 + U_{12}$ where $U_1 = -\vec{m}_i\cdot\vec{B} = -2m\mu_o H\cos\theta_1$, $U_2 = -\vec{m}_2\cdot\vec{B} = -3m\mu_o H\cos\theta_2$. Now to find the U_{12} interaction energy we consider the B field due to one $\vec{B}_1 = \frac{\mu_o}{4\pi}\frac{1}{r^3}\{3(\vec{m}_i\cdot\hat{r})\hat{r} - \vec{m}_1\}$. At the position of the second magnet we have $\hat{r} = \hat{x}$ and $r = d$, thus $\vec{B}_1 = \frac{\mu_o}{2\pi}\frac{m}{d^3}\{2\sin\theta_1\hat{x} - \cos\theta_1\hat{z}\}$. Since $U_{12} = -\vec{m}_2\cdot\vec{B}_1$, and $\vec{m}_2 = 3m\sin\theta_2\hat{x} + 3m\cos\theta_2\hat{z}$, then

$$U_{12} = \frac{-3\mu_o m^2}{2\pi d^3}\{2\sin\theta_1 \sin\theta_2 - \cos\theta_1 \cos\theta_2\}, \text{ and}$$

$$U = -2m\mu_o H\cos\theta_1 - 3m\mu_o H\cos\theta_2 - \frac{3\mu_o m^2}{2\pi d^3}\{2\sin\theta_1 \sin\theta_2 - \cos\theta_1 \cos\theta_2\}$$

To show that $\theta_1 = \theta_2 = 0$ are equilibrium positions we minimize the energy:

$$\frac{\partial U}{\partial \theta_1} = 2m\mu_o H\sin\theta_1 - \frac{3\mu_o m^2}{2\pi d^3}\{2\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2\} = 0$$

$$\frac{\partial U}{\partial \theta_2} = 3m\mu_o H\sin\theta_2 - \frac{3\mu_o m^2}{2\pi d^3}\{2\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2\} = 0$$

These relations are satisfied if $\theta_1 = \theta_2 = 0$, thus proving that such is a position of equilibrium. We can determine the condition that this equilibrium is stable $\theta_1 = \theta_2 = 0$:

$$\frac{\partial^2 U}{\partial \theta_1^2} = 2m\mu_o H - \frac{3\mu_o m^2}{2\pi d^3} \text{ requiring } H > \frac{3m}{4\pi d^3}$$

$$\frac{\partial^2 U}{\partial \theta_2^2}\Big| = 3m\mu_o H - \frac{3\mu_o m^2}{2\pi d^3} \text{ requiring } H > \frac{m}{2\pi d^3}.$$ Thus for stable equilibrium we should have $H > \frac{3m}{4\pi d^3}$.

12.9 a) Vector potential of infinite wire is

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\frac{I\,dz}{\sqrt{\rho^2+z^2}} = \frac{\mu_o I\,\hat{z}}{2\pi}\ln\rho.$$

b) From $U = \frac{1}{2}\int \vec{J}\cdot\vec{A}dv$ we get $U = \frac{1}{2}\int I\vec{A}\cdot d\vec{\ell} =$ $\frac{1}{2}\sum_{i\neq j}\int I_1\vec{A}_j\cdot d\vec{\ell}_i$ where $I_i d\ell_i$ is the i'th current, and, $\vec{A}_j$ is the vector potential due to the i'th current. The energy of one current in one vector potential is $U = \int I\vec{A}\cdot d\vec{\ell}$, for example, moving a current from $\rho = \infty$. So, $U = \frac{1}{2}\int I_1\vec{A}_2\cdot d\vec{\ell}_1 +$ $\frac{1}{2}\int I_2\vec{A}_1\cdot d\vec{\ell}_2 = \int I_1\vec{A}_2\cdot d\vec{\ell}_1 = \int_{-\infty}^{\infty} I_1\frac{\mu_o I_2\ln\rho}{2\pi}dz$. For a wire of length L, $U = \int_0^L \frac{\mu_o I_1 I_2 \ln\rho dz}{2\pi} = \frac{\mu_o I_1 I_2 L}{2\pi}\ln\rho$, $\frac{U}{L} = \frac{\mu_o I_1 I_2}{2\pi}\ln\rho.$

c) $\frac{F}{L} = \frac{d}{d\rho}\frac{U}{L} = \frac{\mu_o I_1 I_2}{2\pi\rho} = \frac{\mu_o I_1 I_2}{2\pi R}$ at $\rho = R$.

12.11 We use the following expressions for the moments: $\vec{m}_1 = I_1\,\pi a^2\hat{z}$ and $\vec{m}_2 = I_2\pi a^2\hat{z}$. In the case $\ell \gg a$, $\ell \gg b$, consider a dipole field due to loop 2.

$$\vec{B}_2(\vec{r}) = \mu_o\frac{3(\hat{r}\cdot\vec{m}_2)\hat{r} - \vec{m}_2}{4\pi r^3}.$$ At loop 1 we have $\vec{r} = \ell\hat{z}$, hence $\vec{B}_{21}(\ell) = 2\mu_o m_2\hat{z}/(4\pi\ell^3) = \mu_o I_2\pi b^2\hat{z}/(2\pi\ell^3)$. The magnetic energy of loop 1 in loop 2's field is $U = -\vec{B}_{21}\cdot\vec{m}_1 =$

$\frac{-\mu_o I_1 I_2\pi a^2 b^2}{2\ell^3}$. Thus $\vec{f} = \frac{\partial U}{\partial\ell}\hat{z} = \frac{3\mu_o\pi I_1 I_2 a^2 b^2}{2\ell^4}\hat{z}$. If I_1 and I_2 are in the same sense then the force is repulsive, whereas it is attractive if they are in the opposite sense.

12.10 **a)** From Ampere's law $\vec{H} = NI_o\hat{z}/L_o$, and hence $\vec{B} = \mu_o NI_o\hat{z}/L_o$. The magnetic energy density is $u = B^2/2\mu_o = \mu_o N^2 I_o^2/2L_o^2$, and the total energy is $U = uV = u \times \pi R^2 L_o = \pi\mu_o N^2 I^2{}_o R^2/2L_o$.

b) Since the flux remains constant, then if L_o changes to L_1, I_o has to change to $I_1 = I_o L_1/L_o$.

c) To determine the tension we need to write the energy in terms of B directly since it is kept constant: $U = \pi B^2 R^2 L_o/2\mu_o$, thus $T = -dU/dL_o = -\pi B^2 R^2/2\mu_o$.

12.12 Consider moving the rod a distance dx from the field H_1 to H_2. This is equivalent to taking a piece dx from field H_1 to field H_2. Before moving piece of rod, the energy stored in the small piece of volume Adx, in the field H_1 is $U_1 = \frac{1}{2}\mu H_1{}^2 Adx$. After moving the piece, there is a vacuum, and the energy $U_f = \frac{1}{2}\mu_o H_1{}^2 Adx$. Thus, by removing the lower piece, the energy charges: $\Delta U_{bottom} = \frac{1}{2}(\mu - \mu_o)H_1{}^{2Adx}$. Similarly, by placing the piece in the field H_2 the energy changes: $\Delta U_{top} = +\frac{1}{2}(\mu - \mu_o)H_2{}^2 Adx$. Total change $\Delta U_{total} = \frac{1}{2}(\mu - \mu_o)(H_2{}^2 - H_1{}^2 Adx$. Now

$$F = \frac{dU}{dx} = \frac{1}{2}(\mu - \mu_o)(H_2^2 - H_1^2)A = \frac{1}{2}(\mu_o X)(H_2^2 - H_1^2)A.$$

12.13 $M = \frac{\mu_o}{4\pi}\oint_{c1}\oint_{c2}\frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{r}_1 - \vec{r}_2|}$, $U = MI_1I_2$. $\vec{F}_2 = -\vec{F}_1 = \nabla_2 U =$

$\frac{\mu_o}{4\pi}\nabla_2 \oint_{c1}\oint_{c2} \frac{d\vec{\ell}_1\cdot d\vec{\ell}_2}{|\vec{r}_1-\vec{r}_2|}$. Now, it is almost obvious that this equals $\frac{\mu_o}{4\pi}\oint_{c1}\oint_{c2}\nabla_2 \frac{d\vec{\ell}_1\cdot d\vec{\ell}_2}{|\vec{r}_1-\vec{r}_2|}$. But $\nabla_r \frac{1}{|\vec{r}-\vec{r}_o|} = \frac{\vec{r}_o-\vec{r}}{|\vec{r}-\vec{r}_o|^3}$ then

$$\vec{F} = \frac{\mu_o}{4\pi}\oint_{c1}\oint_{c2} \frac{d\vec{\ell}_1\cdot d\vec{\ell}_2(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}.$$ Now compare to Biot Savart law

$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\oint \frac{I_1 d\ell_1 x(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$

$$\vec{F}_2 = \oint I_2 d\vec{\ell}_2 x\vec{B}_1 = \frac{\mu_o}{4\pi}\oint_{c1}\oint_{c2} \frac{I_1 I_2 d\vec{\ell}_2 x(d\vec{\ell}_1 x((\vec{r}_2-\vec{r}_1))}{|\vec{r}_2-\vec{r}_1|^3}.$$ But

we can show that only a piece of the cross product contributes, that is $d\vec{\ell}_2 \times (d\vec{\ell}_1 \times (\vec{r}_2-\vec{r}_1))$ can be replaced by $d\vec{\ell}_1\cdot d\vec{\ell}_2(\vec{r}_1-\vec{r}_2)$ as shown below. Use $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ so $d\vec{\ell}_2 \times (d\vec{\ell}_1 \times (\vec{r}_2-\vec{r}_1)) = (d\vec{\ell}_2\cdot(\vec{r}_2-\vec{r}_1))d\vec{\ell}_1 - (d\vec{\ell}_2\cdot d\vec{\ell}_1)(\vec{r}_2-\vec{r}_1)$.

Now if $\oint_{c1}\oint_{c2} \frac{(d\vec{\ell}_2\cdot(\vec{r}_2-\vec{r}_1))d\vec{\ell}_1}{|\vec{r}_2-\vec{r}_1|^3} = 0$ then it is right:

$$\oint_{c2} d\vec{\ell}_2\cdot[\oint_{c1}\frac{(\vec{r}_2-\vec{r}_1)d\vec{\ell}_1}{|\vec{r}_2-\vec{r}_1|^3}] = \int_{s2} da\ \hat{n}\cdot\nabla_2 x[\oint_{c1}\frac{(\vec{r}_2-\vec{r}_1)d\vec{\ell}_1}{|\vec{r}_2-\vec{r}_1|^3}]$$

$$= \int_{s2} da\ \hat{n}\cdot\oint_{c1}\nabla x \frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3} d\vec{\ell}_1 = 0.$$

12.14 In Eq. 12.7, ΔX is the virtual gap, but here ΔX is a real gap, and we want to see what happens if it changes. (Use X here, instead). But we can use the same method. Now we have $H_m L + 2H_g X = NI$, $\mu H_m = \mu_o H_g$.

Thus $H_m(\ell + \frac{2\mu}{\mu_o} X) = NI$ or $H_m = \frac{NI}{\ell + 2\mu X/\mu_o}$.

$$U = \tfrac{1}{2}\int \mu\, H_m^2\, dv + \tfrac{1}{2}\int \mu_o H_g^{\,2} dv = \frac{\mu\mu_o N^2 I^2 A}{2(\mu_o L + 2\mu X)}$$

Now, see what happens when we change X:

$$F = \frac{dU}{dX}\bigg|_I = \frac{\mu\mu_o N^2 I^2 A\ (-2\mu)}{2(\mu_o \ell + 2\mu X)^2} = \frac{-\mu^2 \mu_o N^2 I^2 A}{(\mu_o \ell + 2\mu X)^2}$$

CHAPTER 13

13.1 **a)** $V = RI + L\, dI/dt$. For $t < 0$, $V = 0$, $I = 0$ and for $t > T$, $V = 0$.

b) For $0 < t < T$ $\quad L\, dI/dt + RI = V_o \quad I = A + Be^{-\alpha t}$ A, B, α are constants. Put this into the equation $(-B\alpha L + RB)e^{-\alpha t} + RA = V_o$. For all t, $A = V_o/R$, $\alpha = R/L$. At $t = 0$, $I = 0 \rightarrow$ $B = -A = -V_o/R$, $I = (V_o/R)\ (1 - \exp(-Rt/L))$.

c) $V_L = L\, dI/dt$, $dI/dt = (V_o/L)\exp(-Rt/L)$

$V_L = V_o \exp(-Rt/L)$.

d) At $t = T$, $I_T = (V_o/R)(1 - \exp(-RT/L))$.

For $t > T$, $RI + L\, dI/dt = 0$, $I = I_T \exp(-R(t - T)/L)$

$V_L = L\, dI/dt = (-RI_T)\exp(-R(t - T)/L)$

$\quad = -V_o(1 - \exp(-RT/L))\exp(-R(t - T)/L)$.

13.2 The transient solution of $L\, dI/dt + RI = V$ is $I = I_o \exp(-Rt/L)$ as usual. This is the solution to the homogeneous equation. The inhomogeneous equation $L\, dI/dt + RI = V_o\exp(-Rt/L)$ is a special case. The particular solution must be different from the solution to the homogeneous equation. Using variation of parameters, we write $I(t) = u(t)\exp(-Rt/L)$ where $u(t)$ is a function of t, and not a constant. We now get $L\, du/dt - Ru + Ru = V_o$ after factoring $\exp(-Rt/L)$. Thus $du/dt = V_o/L$ or $u(t) = V_o t/L + I_o$ where I_o is our transient solution. Thus $I = (V_o t/L + I_o)\exp(-Rt/L)$. Many degenerate differential equations have solutions with an extra power of t multiplying the original solution.

13.3 We write Kirchhoff's loop equations $I_1 + I_2 = dQ/dt$, $Q/C + LdI_2/dt = 0$, $RI_1 - LdI_2/dt = 0$. Differentiating the second equation with respect to time, and using the resulting equation along with the third equation to eliminate I_1 and Q from the first equation: $\frac{d^2I_2}{dt^2} + \frac{1}{CR}\frac{dI_2}{dt} + \frac{1}{CL}I_2 = 0$. The solution of this equation with $L = 2R^2C$ is $I_2 = e^{-kt}(A \cos kt + B \sin kt)$ where $k = 1/2\,RC$. Because at $t = 0$, $I_2 = 0$, then $A = 0$, and $I_2 = B\,e^{-kt} \sin kt$. From the second equation: $Q = -kCLe^{-kt}\,B(-\sin kt + \cos kt)$. Using $Q = Q_o$ at $t = 0$, then we get the required answer.

13.4 We use results of Ex. 13.2. **a)** The time constant of the circuit is $\tau = RC = 10^{-4}$sec. Using Eq. 13.17 we get $V_c = V_o/2$. **b)** Using Eq. 13.19, $V_R = RC\,dE/dt = 10^{-2}V_o$ where V_o is the maximum voltage in one cycle of the source.

13.5 $dq/dt = I = 86.6\, q_o \exp(-5t) \sin(86.6t)$ is a solution to $L\, d^2q/dt^2 + R\, dq/dt + q/c = 0$, $\omega = 86.6\ s^{-1}$ with solution

$$q(t) = q_o \exp(-\alpha t)[\cos(\omega t) + \frac{\alpha}{\omega}\sin(\omega t)]$$

$$I(t) = -q_o \exp(-\alpha t)\left(\frac{\alpha^2 + \omega^2}{\omega}\right)\sin(\omega t)$$

$$\omega^2 = \frac{1}{\omega C} - \frac{R^2}{4L^2} \qquad \alpha = \frac{R}{2L} \qquad Q = \frac{\omega L}{R} = \frac{\omega}{2\alpha}$$

a) $\omega = 86.6\ s^{-1}$ $\alpha = 5\ s^{-1}$ $Q = 8.66$

b) $R = 0.1\ \Omega$ $L = \frac{R}{2\alpha} = 0.01$ Henry

c) Energy $= U = \frac{1}{2} LI^2$ $\frac{\Delta U}{U} = 1 - \frac{I^2(t)}{I^2(t+T)}$ $T = \frac{2\pi}{\omega}$

$I(t + T) = I(t) \exp(-\alpha T) = I(t) \exp(-2\pi\alpha/\omega)$

$\frac{\Delta U}{U} = 1 - \exp(-4\pi\alpha/\omega)$

13.6 $I_o = V_o/R_1$ after the switch is opened L

$dI/dt + (R_1 + R_2)I = 0$ has solution $I = I_o \exp(-(R_1 + R_2)t/L)$.

Thus $V = (R_1 + R_2)I_o \exp(-(R_1 + R_2)t/L)$

$V_{max} = (R_1 + R_2)I_o = V_o(R_1 + R_2)/R_1$.

a) $V_o = 20$ Volts $R_1 = 100\ \Omega$ $V_{max} = 100$ Volts

$R_1 + R_2 = 5R_1$ $R_2 = 4R_1 = 400\ \Omega$

b) $dI(0)/dt = -(R_1 + R_2)I_o/L$ $L = 10$ Henry, thus

$dI(0)/dt = -500\Omega \times 200$ mA/10 Henry $= 10\ A\ s^{-1}$

13.7 $V = IR + L\ dI/dt$, $V = IR + i\omega LI$ where $i\omega L$ is the impedance of the inductance. Substituting in this equation $I = I_o e^{i\omega t}$, and $V = V_o e^{i\omega t}$, gives $V_o = I_oR + i\omega LI_o$. This has the solution $I_o = \frac{V_o}{R + i\omega L}$.

a) $I_o = V_o \frac{R - i\omega L}{R^2 + \omega^2 L^2}$, $\phi = \tan^{-1}(R/\omega L)$.

b) They are in series, the current is the same, $\Delta\phi = 0$.

c) $V_R = I_oR$ $V_L = i\omega I_o$ $\Delta\phi = \frac{\pi}{2} = 90^0$

13.8 $V = V_o\, e^{i\omega t}$ $\qquad I = I_o\, e^{i\omega t}$

$V = R_1 I - iI/\omega C + i\omega L I + R_2 I$

$$I_o = \frac{V_o}{(R_1 + R_2) + i\omega L - i/\omega C} = V_o \frac{(R_1 + R_2)I - i\omega L + i/\omega C}{(R_1 + R_2)^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$\left|\frac{I_o}{V_o}\right| = [(R_1 + R_2)^2 + (\omega L - 1/\omega C)^2]^{-1/2}$$

$$\frac{d}{d\omega}\left|\frac{I_o}{V_o}\right| = -\frac{1}{2}\left|\frac{I_o}{V_o}\right|^{-3/2} 2(\omega L - 1/\omega C)\,(L + 1/\omega^2 C) = 0$$

which gives $\omega^2 = 1/LC$ or ω = infinity.

a) maximum for $\omega^2 = 1/LC$ and minimum for $\omega = \infty$.

b) $I_{max} = \dfrac{V_o}{R_1 + R_2}$

c) $|I_o| = \frac{1}{2} I_{max}$ $\qquad |I_o|^{-2} = 4\, I_{max}^{-2}$

$$(R_1 + R_2)^2 + (\omega L - 1/\omega C)^2 = 4(R_1 + R_2)^2$$

$$\omega^2 L^2 - 2L/C + 1/\omega^2 C^2 = 3(R_1 + R_2)^2$$

$$\omega^2 = \frac{3(R_1 + R_2)^2 + \frac{2L}{C} \pm (9(R_1 + R_2)^4 + 6(R_1 + R_2)^2\, L/C)^{1/2}}{2L^2}.$$

13.9 The impedances to be used are $i\omega L$, R, $1/i\omega C_1$, and $1/i\omega C_2$. The loops on the left and on the right have the following inpedances: $z_1 = \dfrac{1}{i/i\omega L + i\omega C_1} = i\omega L/(1 - \omega^2 LC_1)$, $z_2 = \dfrac{1}{1/R + i\omega C_2} = R/(1 + i\omega RC_2)$. The total impedance is

$z = z_1 + z_2$. Taking the imaginary part of z and equating it to zero gives $\omega^2 = (C_2 - L/R)/LC_2(C_1 + C_2)$.

13.10 a) $\frac{1}{z} = i\omega C + \frac{1}{R + i\omega L}$.

b) $\frac{1}{z_C} = i\omega C_o \frac{\varepsilon}{\varepsilon_o} = i\omega C_o \left[1 - \frac{\omega_p^2}{\omega^2 - i\gamma\omega}\right]$

$= i\omega C_o + \frac{1}{i\omega/(C_o\omega_p^2) + \gamma/(C_o\omega_p^2)}$. Thus by comparison:

$C = C_o$, $L = \frac{1}{C_o\omega_p^2}$, and $R = \gamma L = \frac{\gamma}{C_o\omega_p^2}$.

13.11 a) $I_1 - \frac{V_1}{R_1} + I_2 + \frac{V_2 - V_1}{R_2} = 0 \qquad I_2 + \frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} = 0$

b) $V_2 = \frac{I_1R_1R_2 - I_2R_2^2}{(R_2 + R_3)(R_1 + R_2)/R_3 - R_1}$, $\quad V_1 = V_2\left(\frac{R_2 + R_3}{R_3}\right) + I_2R_2$

Now for $I_1 = 2A$, $I_2 = 1A$, $R_1 = 1\Omega$, $R_2 = 2\Omega$, $R_3 = 3\Omega$ we have $V_1 = 2$ volts and $V_2 =$ zero volts.

c) $I_{R3} = 0$, $I_{R2} = 1A$, and $I_{R1} = 2A$.

13.12 $\frac{1}{z_C} = i\omega C + \frac{1}{R}$, thus $z_C = \frac{R - i\omega R^2 C}{1 + \omega^2 R^2 C^2}$

$z = i\omega L + Z_C = \frac{R - i\omega R^2 C + i\omega L + i\omega^3 R^2 C^2 L}{1 + \omega^2 R^2 C^2}$

$|z|^2 = \frac{R^2 + \omega^2(L + \omega^2 R^2 C^2 L - R^2 C)^2}{(1 + \omega^2 R^2 C^2)^2}$.

For large R we have $|z|^2 = \dfrac{1 + \omega^2 R^2 C^2 (\omega^2 CL - 1)^2}{\omega^4 R^2 C^4}$ which

peaks at $\omega^2 CL = 1$, with a value of $|z| = \dfrac{1}{\omega^2 RC^2} = \dfrac{L}{RC}$.

13.13 $y_C = i\omega C$, $y_R = 1/R$. $y_{RC} = i\omega C + 1/R$

$y_L = -i/\omega L$, $y = y_L y_{RC}/(y_L + y_{RC})$

$$\frac{V_2}{V_o} = \frac{y}{y_{RC}} = \frac{y_L}{y_L + y_{RC}} = \frac{-i/\omega L}{i\omega C + 1/R - i\omega L}$$

$= \dfrac{R}{-\omega^2 LCR + i\omega L - R}$. Thus $\phi = \tan^{-1}\left(\dfrac{\omega L/R}{1 + \omega^2 LC}\right)$.

13.14 The heat loss is simply $Q = \frac{1}{2}\,\mathrm{Re}VI^*$. Since $I = V_o e^{i\omega t}/z$,

then $Q = \frac{1}{2} V_o^2/Z$ where $Z = |z|$, or $Q = \dfrac{1}{2}\,\dfrac{\gamma\omega_p^2}{\omega^2 + \gamma^2}\,C_o V_o^2$. The instantaneous energy stored is the sum of the energy stored in the capacitence and the inductor which is $W = \frac{1}{2} C_o V^2 + \frac{1}{2} L_o I^2$. The time average gives

$$\langle W\rangle = \tfrac{1}{4}(C_o V_o^2 + L_o Io^2) = \tfrac{1}{4}\left(1 + \frac{\omega_p^2}{\omega^2 + \gamma^2}\right) C_o V_o^2.$$

13.15 Use $E = E_o e^{i\omega t}$: $E_o = R_1 I_1 + i\omega L_1 I_1 + i\omega M I_2$ and $0 = R_2 I_2 + i\omega L_2 I_2 + i\omega M I_1$ or $E_o - (R_1 + i\omega L_1) I_1 = i\omega M I_2$, $(R_2 + i\omega L_2) I_2 = -i\omega M I_1$.

Thus $E_o = (R_1 + i\omega L_1 + \omega^2 M^2/(R + i\omega L_2))\, I_1$ or

$$I_1 = \frac{(R_2 + i\omega L_2)\mathcal{E}_o}{(R_1 + i\omega L_1)(R_2 + i\omega L_2) + \omega^2 M^2}, \quad I_2 = \frac{-i\omega M}{R_2 + i\omega L_2} I_1$$

$$I_2 = \frac{-i\omega M \mathcal{E}_o}{(R_1 + i\omega L_1)(R_2 + i\omega L_2) + \omega^2 M^2}$$

$$\frac{P_2}{P_1} = \frac{R_2 |I_2|^2}{R_1 |I_1|^2} = \frac{\omega^2 M^2 R_2}{(R_2^2 + \omega^2 L_2^2) R_1}$$

$$\frac{d}{dR_2}\left(\frac{P_2}{P_1}\right) = 0 = \omega^2 M^2 (R_2^2 + \omega^2 L_2^2) R_1 - (\omega^2 M^2 R_2)(2R_2) R_1$$

$$R_2^2 + \omega^2 L_2^2 - 2R_2^2 = 0 \rightarrow \omega^2 L_2^2 = R_2^2 \text{ and } R_2 = \omega L_2 .$$

13.16 $Z_c = -i/\omega C$. Considering two voltage dividers,

$$\frac{R_a}{R_a + R_b} = \frac{-i/\omega C_s}{(-i/\omega C_s) + (-i/\omega C_x)} = \frac{C_x}{C_x + C_s} \quad \text{which gives}$$

$$C_s = \frac{C_x R_b}{R_a} \qquad \text{or} \qquad C_x = \frac{C_s R_a}{R_b}$$

13.17 $\dfrac{R_3}{R_4} = \dfrac{R_L + i\omega L \frac{-i}{\omega C}}{R_2}$. Equate real and imaginary parts

$$\omega L - \frac{1}{\omega C} = 0 \text{ and } \frac{R_3}{R_4} = \frac{R_L}{R_2} \quad \text{or} \quad R_L R_4 = R_2 R_3 .$$

13.18 $(i\omega C + 1/R_4)\, R_3 = (i\omega L + R_L)/R_2$

$i\omega C\, R_2 R_3 R_4 + R_2 R_3 = i\omega L R_4 + R_L R_4$. Equating real and imaginary parts give $C R_2 R_3 = L$ and $R_2 R_3 = R_L R_4$

$$Q = \frac{\omega L}{R_L} = \frac{\omega C R_2 R_3}{R_2 R_3 / R_4} = \omega \; CR_4 .$$

13.19 **a)** Kirchhoff's loop laws for phasors give the following three equations for the three small loops (assume three current loops I_1, I_2 and I_3 flowing clockwise in these three loops from left to right) $\hat{V} = i\omega L_1 \hat{I}_1 \frac{-i}{\omega C_1} (\hat{I}_1 - \hat{I}_2)$,

$0 = \frac{-i}{\omega C_1} (\hat{I}_2 - \hat{I}_3 + \hat{I}_2 - \hat{I}_1) \frac{-i}{\omega C} \hat{I}_2$, $0 = i\omega L_1 \hat{I}_3 \frac{-i}{\omega C_1} (\hat{I}_3 - \hat{I}_2)$

The determinant of this set of equations is

$$\begin{vmatrix} (i\omega L_1 - i/\omega C_1) & i/\omega C_1 & 0 \\ i/\omega C_1 & -2i/\omega C_1 - i/\omega C & i/\omega C_1 \\ 0 & i/\omega C_1 & i\omega L_1 - i/\omega C_1 \end{vmatrix} = 0$$

$(i\omega L_1 - i/\omega C_1)[(-2i/\omega C_1 - i/\omega C)(i\omega L_1 - i/\omega C_1) + 1/\omega^2 C_1{}^2]$
$-i/(\omega C_1)[i/(\omega C_1)(i\omega L_1 - i/\omega C_1)] = 0$. This gives $\omega_1^2 = 1/L_1 C_1$, $\omega_2^2 = 1/L_1(C_1 + 2C)$.

b) When there is no coupling between the two outer loops ($C = 0$), then $\omega_1 = \omega_2 = 1/L_1 C_1$. When there is very tight coupling C is large then $\omega_1 = 1/L_1 C_1$.

13.20 We can use a similar procedure to the one used in problem 13.19 to show that $\omega_1^2 = 1/L_1 C_1$ and $\omega_2^2 = \frac{2}{LC_1} + \frac{1}{L_1 C_1}$.

13.21 We write three mesh Kirchhoff's equations using phaser notations:

$i\omega L(\hat{I}_3 - \hat{I}_1) + R(\hat{I}_3 - \hat{I}_1) = \hat{E}_o$

$$(i\omega L - i/\omega C)\hat{I}_1 - i\omega L\hat{I}_3 + R(\hat{I}_1 - \hat{I}_2) = 0$$

$$R(3\,\hat{I}_2 - \hat{I}_1 - \hat{I}_3) = 0.$$

Solving these equations we get:

$$\hat{I}_1 = \frac{\hat{E}_o(R + 3i\omega L)}{R^2 + 3L/C + 2iR(\omega L - 1/\omega C)} = \frac{\hat{E}_o(R + 3i\omega L)}{R^2 + 3\,L/C}$$

$$\hat{I}_2 = \frac{\hat{E}_o(R + 2i\omega L - i/\omega C)}{R^2 + 3\,L/C + 2iR(\omega L - 1/\omega C)} = \frac{\hat{E}_o(R + i\omega L)}{R^2 + 3\,L/C}$$

Then $I_1 - I_2 = 2i\,E_o e^{i\omega t}/[\omega(3L + CR^2)]$ with an amplitude as required.

b) $I_1 - I_2 = |I_1 - I_2|\,e^{i\pi/2}$. Thus the phase difference is $\pi/2$.

13.22 a) Series: $z = R + i\omega L - i/\omega C = R + i\,(\omega L - 1/\omega C)$

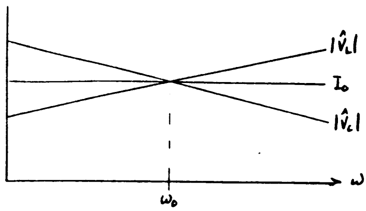
$$y = \frac{1}{z} = \frac{R - i(\omega L - 1/\omega C)}{R^2 + (\omega L - 1/\omega C)^2}$$

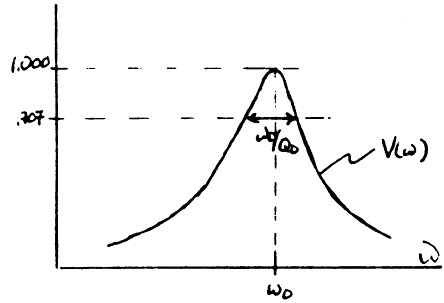
Parallel: $y = i\omega C + \dfrac{R - i\omega L}{R^2 + \omega^2 L^2} = \dfrac{R - i\omega L + i\omega CR^2 + i\omega^3 CL^2}{R^2 + \omega^2 L^2}$

$$z = 1/y = \frac{R^2 + \omega^2 L^2}{R - i\omega L + i\omega CR^2 + i\omega^3 CL^2}\,.$$

b) $V = Iz$, $|V| = I|z|$ for constant I.

Series: $|z|^2 = R^2 + (\omega L - 1/\omega C)^2 = R^2 + \omega^2 L^2 + 2L/C + 1/\omega^2 C^2.$





c) parallel $|z|^2 = \dfrac{(R^2 + \omega^2 L^2)^2}{R^2 + (-\omega L + \omega CR^2 + \omega^3 CL^2)^2}$

factoring twice: $|z|^2 = \dfrac{R^2 + \omega^2 L^2}{(\omega^2 CL - 1)^2 + \omega^2 C^2 R^2}$

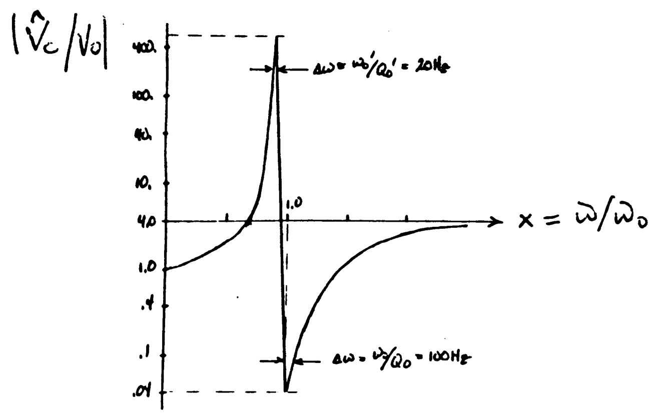
We now find the maximum:

$$\frac{d}{d\omega}|z|^2 = 2\omega L^2(\omega^4 C^2 L^2 - 2\omega^2 CL + 1 + \omega^2 C^2 R^2)$$

$$- (R^2 + \omega^2 L^2)[2(\omega^2 CL - 1)(2\omega CL) + 2\omega C^2 R^2]$$

$$= -2\omega C(2\omega^2 CL^2 - 2L + CR^2) = 0$$

which gives $\omega_o^2 = \dfrac{2L - CR^2}{2CL^2}$. This reduces for R = 0 to $\omega_o^2 = \dfrac{1}{CL}$.



13.23 **a)** Using the condition for resonance $\omega_o^2 C_1 L = 1$, we get $C_1 = 10^{-6}$F.

b) $Q_o = \omega_o L/R$, thus $Q_o = 100$. At this frequency the impedance $Z = (Q_o^2 + 1)\,R \simeq 10^4\Omega$ (see Eq. 13.97). Since $Q_o > 10$ and $Q_o \gg 1$, then it is a simultaneous phase resonance and antiresonance.

c) $\frac{1}{z} = i\omega C_1 + 1/(1/R + i\omega L)$. Thus at $\omega/\omega_o = .9$, we get $z \simeq 25(1 + 18i)\Omega$. This impedance is inductive with a reactance equal to 450 Ω.

d) To annul the reactance of the parallel circuit with a capacitor we take $1/\omega C_2 = 450$ which gives $C_2 = .25 \times 10^{-6}$ F.

e) $|V_c/V_o| = \left| \frac{i/i\omega C_2}{z + 1/i\omega C_2} \right.$ where z is the impedance of the A-B terminal. This gives $4 \times 10^{-2}\Omega$ at the resonance frequency $\omega_o = 10^4$/s. At $\omega = .9\,\omega_o$ we get 450 Ω.

13.24 **a)** $$z = \frac{i\omega L + R/i\omega C}{R + 1/i\omega C} = \frac{R + i\omega[L - R^2C(1 - \omega^2 LC)]}{1 + \omega^2 R^2 C^2}$$

b) $z = Z\,e^{i\phi}$ where $Z = \sqrt{\frac{R^2(1 - 2\omega^2 LC)}{1 + \omega^2 R^2 C^2} + \omega^2 L^2}$ and

$\tan\phi = [L - R^2C(1 - \omega^2 LC)]/R$.

c) Taking $\phi = 0$ gives the resonance frequencies $\omega_o^2 = 1/LC - 1/R^2C^2$.

d) The average power is $\langle P\rangle = (\text{Re}z)I_s^2$, where $I_s = V_o Z$ is the amplitude of the current through the source. The source voltage is taken of the form $V = V_o e^{i\omega t}$. Thus $\langle P\rangle = RI_s^2/(1 + \omega^2R^2C^2)$.

e) We need to calculate the current in the resistor. First let us calculate the impedance of the circuit and the current in the inductor and hence in the source for $\omega^2LC = 1$.

$z_o = (R + i\omega L)/(1 + \omega^2R^2C^2)$, $I = V_o e^{i\omega t}/z_o$. $V_R = V - V_L = V - i\omega LI = V(1 - i\omega t/z_o) = -iVR\omega C$. Thus $I_R = V_R/R = -iV\omega C$. Thus the average power $\langle P \rangle = \frac{1}{2} R\, I_R^* I_R = \frac{1}{2} R\, V_o^2\, \omega^2 C^2$.

13.25 $z_1 = i\omega L - \dfrac{i}{\omega C}$, $z_2 = \dfrac{1}{i\omega C - i/\omega L}$

b) From example 13.7, $\cosh(\pm\gamma) = 1 + z_1/2z_2 = 1 - (\omega^2LC - 1)^2/2\omega^2LC$. Then we have the following frequency range: $0 \leq (\omega^2LC - 1)^2/\omega^2LC \leq 4$.

CHAPTER 14

14.1 **a)** Gauss' law between the plates where $\hat{z}$ is normal to them gives $\vec{D} = \sigma\hat{z} = \frac{q_o}{A}\sin(\omega t)\hat{z}$, thus $\vec{J}_D = \frac{\partial\vec{D}}{\partial t} = \frac{q_o\omega}{A}\cos(\omega t)\hat{z}$.

The displacement current $I_D = \vec{J}_D \cdot \vec{A} = q_o\omega\cos(\omega t)$.

The conduction current $I_c = dq/dt = q_o\omega\cos(\omega t)$. Observe that the two are equal.

b) Since I_D is in the $\hat{z}$ direction then $\vec{B} = B_o\hat{\phi}$.

c)

$$\nabla x(B_o\hat{\phi}) = (\mu q_o\omega/A)\cos(\omega t)\hat{z} \rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_o) = (\mu q_o\omega/A)\cos(\omega t)\hat{z}$$

$B_o = \mu(\frac{\rho}{2} + \frac{c}{\rho})\frac{q_o\omega}{A}\cos(\omega t)$. But $B_o \neq \infty$ at $\rho = 0$, then $C = 0$;

therefore $\vec{B} = \frac{\mu q_o\omega}{2\ A}\rho\cos(\omega t)\hat{\phi}$.

14.2 We are given $V = 250\sin(377t)$ volts. (See Ex. 6.5)

$I_c = dQ/dt$ where $Q = C\ V$. We know the capacitance of the coaxial cylinders from Eq. 6.36 is

$$C = \frac{2\pi L\varepsilon}{\ln(b/a)} = \frac{(2\pi)(.5m)(6.7 \times 8.85 \times 10^{-12}F/m)}{\ln(.6/.5)}.$$

Thus $C = 1.0 \times 10^{-9}F$. Therefore $I_c = dQ/dt = 9.6 \times 10^{-5}$

$\cos(377t)$. $I_D = \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{a} = \varepsilon \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$. But inside the capacitor $\vec{E} = Q/(2\pi\varepsilon\rho L)$. Since $Q/L = (2\pi\varepsilon V/\ln(b/a)$, then $=$ $\vec{E}\ V/\rho\ln(b/a))\hat{\rho}$. Now $dE/dt = 250/(\rho\ln(b/a))377\cos(377t)$. Note:

$\int da = 2\pi L\rho$ then $I_D = \dfrac{250(377)\cos 377t\ 2\pi L\rho}{\rho\ln(b/a)} = (9.6 \times 10^{-5})$ $\cos(377t)$. We see that $I_D = I_c$.

14.3 We are given $V = 150\sin(500t)$ volts. (See Ex. 6.6.) $I_c = dQ/dt$ where $Q = 4\pi\varepsilon V\ ba/(b - a)$, therefore $I_c = 7.1 \times 10^{-5}$

$(\frac{ba}{b - a})\cos(500t)$. Now the displacement current is

$I_D = \varepsilon \int \frac{d\vec{E}}{dt} \cdot d\vec{a}$. But $E = \dfrac{Q}{4\pi\varepsilon r^2}$ then $\dfrac{dE}{dt} = \dfrac{I_c}{4\pi\varepsilon r^2}$,

so $I_D = I_c/(4\pi r^4) \int da = I_c/(4\pi r^2)4\pi r^2 = I_c$, so $I_D = I_c$.

14.4 From Eq. 14.17 we have $\nabla^2 \binom{\vec{E}}{\vec{B}} - \mu\varepsilon \frac{\partial 2}{\partial t^2} \binom{\vec{E}}{\vec{B}} = 0$

a) $E = (Ac)\exp(x-ct)$, $B = A\exp(x-ct)$. We use a single notation.

$$\frac{\partial^2}{\partial x^2} \binom{E}{B} = \binom{c}{1} A\exp(x-ct), \quad \frac{\partial^2}{\partial t^2} \binom{E}{B} = \binom{c^3}{c^2} A\exp(x-ct)$$

Using $\mu\varepsilon = 1/c^2$ we get

$$\nabla^2 \binom{E}{B} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \binom{E}{B} = \binom{c}{1} A\exp(x-ct) - \binom{c}{1} A\exp(x-ct) = 0.$$

b) $E = (Ac)\ln(x + ct)$, $B = -A\ \ln(x + ct)$

$$\frac{\partial}{\partial x}\binom{E}{B} = \binom{Ac}{-A}\frac{1}{x + ct} \text{ and } \frac{\partial^2}{\partial x^2}\binom{E}{B} = A\binom{-c}{1}\frac{1}{(x - ct)^2}$$

$$\frac{\partial}{\partial t}\binom{E}{B} = \binom{Ac^2}{-Ac}\frac{1}{x + ct} \rightarrow \frac{\partial^2}{\partial t^2}\binom{E}{B} = A\binom{-c^3}{c^2}\frac{1}{(x + ct)^2}$$

$$\text{so } \nabla^2\binom{E}{B} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\binom{E}{B} = A\binom{-c}{1}\frac{1}{(x + ct)^2} - \binom{-c}{1}\frac{1}{(x + ct)^2} = 0$$

14.5 a) We are given $\vec{E} = E_o\hat{y}\sin(\omega t - kz)$. For free space we have $\varepsilon = \varepsilon_o$. So $\vec{D} = \varepsilon\vec{E} = \varepsilon_o\vec{E} = \varepsilon_o E_o\hat{y}\sin(\omega t - kz)$. We integrate Maxwell's Equation

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}, \text{ where } \nabla \times \vec{E} = -\hat{x}\frac{dE_y}{dz} = kE_o\hat{x}\cos(\omega t - kz):$$

$$\vec{B} = \int^t \nabla \times \vec{E}\, dt = -\frac{k}{\omega}E_o\hat{x}\sin(\omega t - kz) = -\frac{E_o}{c}\hat{x}\sin(\omega t - kz)$$

$$\vec{H} = \frac{\vec{B}}{\mu_o} = -\frac{E_o}{\mu_o c}\hat{x}\sin(\omega t - kz) = -\varepsilon_o E_o c\hat{x}\sin(\omega t - kz).$$

b) We are given $\vec{H} = H_o\hat{x}\exp(-i(\omega t + kz))$. Integrating Maxwell's Equation $\nabla \times \vec{H} = \vec{J}_f + \frac{d\vec{D}}{dt}$ where $\nabla \times \vec{H} = ik\hat{y}\,H_o\exp(-i(\omega t + kz))$ and $\vec{J}_f = 0$ we get:

$$\vec{D} = \int^t \vec{\nabla} \times \vec{H}\, dt = -\frac{k}{\omega}\hat{y}\,H_o\exp(-i(\omega t + kz))$$

$$\text{so } \vec{E} = \frac{\vec{D}}{\varepsilon_o} = \frac{k}{\varepsilon_o\omega}\hat{y}\,H_o\exp(-i(\omega t + kz)).$$

14.6 We are given K = 10, and K_M = 1000. From Eq. 14.23 we have $v = c/(KK_M)^{1/2}$. Therefore $v = 3 \times 10^8$m/s, and $\lambda = v/f = 3 \times 10^{-2}$m.

14.7 We have $\vec{E} = E_o\hat{x}\exp(i(kz - \omega t))$ thus $\vec{B} = (E_o/c)\hat{y}\exp(i(kz - \omega t)$.

a) Real fields are

$$\vec{E}_r = E_o\hat{x}\cos(kz - \omega t), \ \vec{B}_r = (E_o/c)\hat{y}\cos(kz - \omega t)$$

$$u = \frac{1}{2}\varepsilon_o|E_r|^2 + \frac{1}{2}\frac{1}{\mu_o}|B_r|^2 = \frac{1}{2}\varepsilon_o E_o^2\cos^2(kz - \omega t) +$$

$(1/2)\mu_o c^2 E_o^2\cos^2(kz - \omega t)$ which adds to $u = \varepsilon_o E_o{}^2\cos^2(kz - \omega t)$.

$$\vec{S} = \vec{E} \times \vec{B}/\mu_o = E_o^2/(\mu_o c)\cos^2(kz - \omega t)\hat{z}.$$

Now $\vec{v} = c\hat{z}$ thus $\vec{S} = \dfrac{E_o^2}{\mu_o c^2}(c\hat{z})\cos^2(kz - \omega t) = u\vec{v}$.

b) One period: $\langle\vec{S}\rangle = \dfrac{E_o^2\hat{z}}{\mu_o c}\langle\cos^2(kz - \omega t)\rangle = \dfrac{E_o^2\hat{z}}{\mu_o c}\left(\frac{1}{2}\right)$.

Over infinite time: $\langle\vec{S}\rangle = \dfrac{E_o^2\hat{z}}{\mu_o c}\lim_{T\to\infty}\dfrac{1}{T}\displaystyle\int_0^T \cos^2(kz - \omega t)dt$

$$= \frac{E_o^2\hat{z}}{\mu_o c}\lim_{T\to\infty}\left[\frac{1}{\omega T}\left(\frac{\omega T}{2} + \frac{\sin 2\omega t}{4}\right)\right] = \frac{E_o^2\hat{z}}{\mu_o c}\left(\frac{1}{2}\right).$$

14.8 $\vec{E} = E_o \hat{x}\exp(i(kz - \omega t) + E'_o \hat{x}\exp(i(kz - \omega t + \phi)$. The real field is $\vec{E} = E_o \hat{x}\cos(kz - \omega t) + E'_o \hat{x}\cos(kz - \omega t + \phi)$. The real $\vec{B}$ field is $\vec{B} = \frac{E_o}{c}\hat{y}\cos(kz - \omega t) + \frac{E'_o}{c}\hat{y}\cos(kz - \omega t + \phi)$. Using $u = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2} B^2/\mu_o$ we get

$u = (\frac{1}{2}\varepsilon_o + {}^1\!/_2\, \mu_o c^2)\ [E_o^2\cos^2(kz-\omega t) + E'^2_o\cos^2(kz-\omega t+\phi)$
$+ 2\, E_o E'_o \cos(kz-\omega t)\cos(kz-\omega t+\phi)]$. Note: $\cos(\alpha + \beta) = \cos\alpha\cos\beta$
$- \sin\alpha\sin\beta$, $\cos^2\phi = (1+\cos 2\phi)/2$ and $\sin^2\phi = (1-\cos 2\phi)/2$, thus
$u = \varepsilon_o[E_o^2\cos^2(kz-\omega t) + E'^2_o(\cos(kz-\omega t)\cos\phi - \sin(kz-\omega t)\sin\phi)^2$
$+ 2\, E_o E'_o\cos(kz-\omega t)\ (\cos(kz-\omega t)\cos\phi - \sin(kz-\omega t)\sin\phi)]$
$= (\varepsilon_o/2)[E_o^2 + E_o^2\cos(2kz-2\omega t) + E'^2_o + E'^2_o\cos(2kz-2\omega t)(\cos 2\phi -$
$E'^2_o\sin(2kz-2\omega t)\sin 2\phi + E_o E'_o\cos\phi + E_o E'_o\cos(2kz-2\omega t)\cos\phi -$
$E_o E'_o\sin(2kz-2\omega t)\sin\phi]$
$= (\varepsilon_o/2)[E_o^2 + E'^2_o + E_o E'_o\cos\phi + (E_o^2 + E'^2_o\cos 2\phi + E_o E'_o\cos\phi)$
$\cos(2kz-2\omega t) - (E'^2_o\sin 2\phi + E_o E'_o\sin\phi)\ \sin(2kz-2\omega t)]$
= a constant energy field + a wave field with velocity $c = \omega/k$.

$\langle\vec{S}\rangle = \frac{1}{2}\,\mathrm{Re}(\vec{E}^* \times \vec{H}) - \frac{\hat{z}}{2\mu_o c}\,[E_o\exp(-i(kz-\omega t)) + E'_o\exp(-i(kz-\omega t))]$

$\times\ [E_o\exp(i(kz - \omega t)) + E'_o\exp(i(kz - \omega t))]$

$= \frac{\hat{z}}{2\mu_o c}\,(E_o^2 + E'^2_o + 2E_o E'_o\cos\phi)$.

Upon comparing with 14.7 we note that in general u, $\vec{S}$, $\langle\vec{S}\rangle$ are not equal to the sums of the separate waves.

14.9 $\vec{E} = E_o\hat{x}\exp(i(kz - \omega t)) + E'_o\,\hat{y}\exp(i(kz - \omega t + \phi))$

$\vec{H} = H_o\hat{y}\exp(i(kz - \omega t)) + H'_o\,\hat{x}\exp(i(kz - \omega t + \phi))$

a) Using $u = \frac{1}{2}(\varepsilon_o\vec{E}\cdot\vec{E} + \mu_o\vec{H}\cdot\vec{H})$, and using the real parts i.e.

$\vec{E} = E_o\hat{x}\cos(kz-\omega t) + \ldots$ etc. we get $u = \frac{1}{2}[\varepsilon_o(E_o^2\cos^2(kz - \omega t) + E_o'^2\cos^2(kz - \omega t + \phi) + \mu_o(H_o^2\cos^2(kz - \omega t) + H_o'^2\cos^2(kz - \omega t + \phi)]$. For the waves individually we have

$u_1 = \frac{1}{2}(\varepsilon_o E_o^2 + \mu_o H_o^2)\cos^2(kz - \omega t)$, and

$u_2 = \frac{1}{2}(\varepsilon_o E_o'^2 + \mu_o H_o'^2)\cos^2(kz - \omega t + \phi)$, therefore

$u = u_1 + u_2$. Now $\vec{S} = \vec{E}\times\vec{H} = u\,c\hat{z}$. From above we have

$u = u_1 + u_2$ so $\vec{S} = (u_1 + u_2)c\hat{z} = \vec{S}_1 + \vec{S}_2$, therefore

$$\langle\vec{S}\rangle = \frac{1}{2}\,\mathrm{Re}\,\langle\vec{E}\times\vec{H}^*\rangle = \frac{1}{2}(E_oH_o + E'_oH'_o)\hat{z} = \langle\vec{S}_1\rangle + \langle\vec{S}_2\rangle.$$

b) Elliptic polarization in general.

c) The electric and magnetic waves are perpendicular and $H = E/c\mu_o$.

d) $\vec{E}_{real} = E_o\hat{x}\cos\omega t + 2E_o\hat{y}\cos(\omega t - \frac{\pi}{4})$.

14.10 $\vec{E} = E_o\hat{x}\exp(i(kz - \omega t)) + E_o\hat{x}\exp(i(-kz - \omega t))$

a) We rewrite the electric field as

$$\vec{E} = E_o\hat{x}\exp(-i\omega t)[\exp(kz) + \exp(-kz)] = 2E_o\hat{x}\exp(-i\omega t)\cos kz$$

$$\vec{B} = \frac{E_o\hat{y}}{c}\exp(i\omega t)(\exp(kz)-\exp(-kz)) = \frac{2iE_o\hat{y}}{c}\exp(i\omega t)\sin kz$$

Thus the real fields are $\vec{E} = 2E_o\hat{x}\cos\omega t\,\cos kz$ and

$\vec{B} = \frac{2E_o}{c}\hat{y}\sin\omega t\,\sin kz$. The energy density is

$$u = \epsilon_o E^2/2 + B^2/2\mu_o = \epsilon_o^2 E_o^{\,2}\cos^2\omega t\,\cos^2 kz +$$

$$\frac{2E_o^2}{\mu_o c^2}\sin^2\omega t\,\sin^2 kz = 2\epsilon_o E_o^2(\cos^2\omega t\cos^2 kz + \sin^2\omega t\sin^2 kz)$$

$$\vec{S} = \frac{1}{\mu_o}\vec{E}\times\vec{B} = \frac{1}{\mu_o}(2E_o\hat{x}\cos\omega t\,\cos kz)\times(\frac{2E_o}{c}\hat{y}\sin\omega t\,\sin kz)$$

$$= \frac{E_o^{\,2}}{\mu_o c}\sin(2\omega t)\sin(2kz)\hat{z}.$$

b) We see that the $\vec{E}$ field goes as $\cos\omega t$ and the $\vec{B}$ field goes as $\sin\omega t$. Since $u = (\epsilon_o E^2 + B^2/\mu_o)/2$ we see that the energy oscillates between purely magnetic and purely electric limits.

c) and d) We see that for the planes $kz = {}^{n\pi}/2$ that $\vec{E} = 0$ and the energy is purely magnetic. When $\vec{E} = 0$ then $\vec{S} = \vec{E}\times\vec{B}/\mu_o = 0$. We see that there is no energy flow across this plane. Similarly for $kz = n\pi$ we have $\vec{B} = 0$, and also $\vec{S} = 0$.

14.11 $\vec{E} = \hat{x}\exp(i(ky - \omega t)) + \hat{y}\exp(i(kx - \omega t))$

a) $\vec{E} = \exp(-i\omega t)(\hat{x}\exp(iky) + \hat{y}\exp(ikx))$

$= \vec{E}_o \exp(-i\omega t)$ with $\vec{E}_o = (\hat{x}\exp(iky) + \hat{y}\exp(ikx))$.

14.12 $\langle S\rangle = 1.4 kW/m^2$. We use MKS units

$$E^2 = \frac{2\langle S\rangle}{\varepsilon_o c} = \frac{2 \times 1400}{8.85 \times 10^{-12} \quad 3 \times 10^8} = 1.05 \times 10^6\ V^2/m^2$$

This gives $E = 1.03$ kV/m, $B_o = E_o/c = 3.42 \times 10^{-6}$ T and

$H_o = B_o/\mu_o = 2.72$ Amps/m.

b) The area of sphere with radius $r = 1.5 \times 10^{11}$m is

$4\pi r^2 = 2.83 \times 10^{23} m^2$. Taking radiation to be isotropic gives

power $= 1400 \times 2.8 \times 10^{23} = 3.96 \times 10^{26}$ Watts.

14.13 **a)** $\vec{E} = 50\ \hat{x}\cos(\omega t - kz)$V/m. The wave is travelling in $\hat{z}$

direction. Thus using MKS

$$\langle S\rangle = \frac{c\varepsilon_o}{2} E_o^2 = 3 \times 10^8 \times 8.854 \times 10^{-12}(50)^2/2 = 3.307\ W/m^2$$

Now $A = \pi\left(\frac{5m}{2}\right)^2 = 19.6\ m^2$, thus $P = S \cdot A = 65.2$ Watts.

b) $\vec{E} = 2 \times 10^2\ \hat{\theta}\sin\theta\cos(\omega t - kr)/r$,

$\vec{H} = .53\ \hat{\phi}\sin\theta\cos(\omega t - kr)/r$, $\vec{S} = \vec{E} \times \vec{H} =$

$(.53 \times 200)\hat{r}/\sin^2\theta\cos^2(\omega t - kr)/r^2$,

$\langle S(\theta)\rangle = \hat{r}\ .53 \times 200 \times \sin^2\theta/2r^2$. The average power =

$$\int_0^{2\pi}\int_0^{\pi/2} \langle\vec{S}(\theta)\rangle r^2\sin\theta\, d\theta\, d\phi = 222 \text{ Watts, independent of } r.$$

14.14 Given $\sigma_c = 10^{-3}\ (\Omega\cdot m)^{-1}$, $K = 2.5 = \varepsilon/\varepsilon_o$, $E = E_o\sin\omega t$,

$E_o = 6 \times 10^{-6}$V/m, $\omega = .9 \times 10^9$rad/s.

$$|\vec{J}_d| = \frac{\varepsilon\,\partial|E|}{\partial t} = \varepsilon E_o \omega\cos\omega t = 1.2 \times 10^{-7}\cos\omega t$$

$$|\vec{J}_c| = \sigma_c E = (10^{-3})(6 \times 10^{-6})\sin(\omega t) = 6 \times 10^{-9}\sin\omega t.$$

14.15 $f = 1.6 \times 10^6$ Hz, $\sigma_c = 38.2 \times 10^6(\Omega\cdot m)^{-1}$, $K_m = \mu/\mu_o = 1.$

a) At $z = 0$ we have $\vec{E}_o = E_o\hat{x}$, thus from Eq. 14.92 $\vec{E} = E_o\hat{x}$ $\cos(\frac{n\omega z}{c} - \omega t)e^{-\omega kz/c}$, where $\omega = 2\pi f$. From Eq. 14.100 we have for a good conductor

$$k = n \cong n_o \sqrt{\sigma_c/\omega\varepsilon} \text{ where } n_o = \sqrt{K K_m} = \sqrt{K} \text{ and } \hat{K} = \frac{\omega}{c}(n + ik)$$

b) For the skin depth, we have from Eq. 14.100, for a good conductor, $\delta = \sqrt{2/\mu_o\omega\sigma_c} = 6.44 \times 10^{-5}$m, $v = c/n =$ $\frac{c\sqrt{2}}{n_o}\sqrt{\frac{\omega\varepsilon}{\sigma_c}} \simeq 650\ \frac{m}{s}$; $\lambda = v/f = 4.06 \times 10^{-4}$ m.

c) We have from Eq. 14.94: $i\hat{K}\hat{z} \times \vec{E} = i\omega\vec{B}$. Thus

$$\vec{B} = \hat{y}\,\frac{\hat{K}\,E_o}{\omega}\cos(\omega(t - \frac{n}{c}z))e^{-\omega kz/c}$$

d) Phase difference: since $\frac{\sigma_c}{\varepsilon\omega} >> 1$, then $\tan\phi = \frac{\sigma_c}{\omega\varepsilon} >> 1.$ Thus $\phi = \pi/2$ or $\phi/2$ the phase difference is $\pi/4$.

14.16 $f = 1 \times 10^9$Hz, $\varepsilon = 18\varepsilon_o$, $\mu = 800\ \mu_o$, $\sigma_c/\varepsilon\omega = 1.$

a) The damped wave equation,

$$\nabla^2\vec{E} - \mu\sigma_c \frac{d\vec{E}}{dt} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \text{ gives } \vec{E}(z,t) = \vec{E}_0 \exp(i\hat{K}z - \omega t)$$

with $\hat{K}$ the complex wave vector. Now

$$\hat{K} = \frac{n_o\omega}{c}\left(1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2\right)^{1/4} e^{i\phi/2} = \frac{\omega}{c}(n + ik),$$

with $\tan\phi = \left(\frac{\sigma_c}{\varepsilon\omega}\right)$.

b) $\delta = \frac{c}{\omega k}$, $v = \frac{c}{n}$, $\frac{\sigma_c}{\omega\varepsilon} = 1$,

$n_o = \sqrt{18 \times 800} = 120$, and $\phi = \tan^{-1}(1) = \frac{\pi}{4}$. Thus

$$n = n_o\left(1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2\right)^{1/4} \cos\frac{\phi}{2} = 131.8, \quad k = n_o 2^{1/4} \sin\frac{\phi}{2} = 54.6,$$

$$\delta = \frac{3 \times 10^8}{2\pi \times 10^9 \times 54.6} = .874 \text{ mm}; \quad v = \frac{3 \times 10^8}{n} = 2.28 \times 10^6 \text{ m/s}$$

c) At $z = 0$ we have $I = 1\ \text{W/m}^2$, $I \propto |E|^2$, and hence $I = I_o e^{-2z/\delta}$

Thus $I = 1\text{W/m}^2 \exp(-2/.874) = .318\ \text{W/m}^2$.

d) $E = \hat{\eta}\, H$ where

$$\hat{\eta} = \sqrt{(\mu/\varepsilon)}(1 + i\sigma_c/\omega\varepsilon)^{-1/2} \text{ so } |\hat{\eta}| = \sqrt{(\mu/\varepsilon)}(1 + \sigma_c^2/\varepsilon^2\omega^2)^{-1/4}$$

and $\hat{\eta} = |\hat{\eta}|\ e^{-i\frac{\phi}{2}} = 2.13\ \Omega\ e^{-i\pi/4}$.

e) $\langle S\rangle = |\vec{E}^* \times \vec{H}|/2 = E^*H\cos(\tfrac{1}{2}\phi)/2$ where $\phi/2 = \pi/4$ is the phase between E and H. Thus $\langle S\rangle = E^2\cos(\tfrac{1}{2}\phi)/2|\hat{\eta}|$ so

$E^2 = 4575\ V^2/m^2$ or $E = 67.6$ V/m.

f) The phase is $\frac{\phi}{2} = \pi/4$ (see part d).

14.17 $\sigma_c = 5 \times 10^7 (\Omega/m)^{-1}$, $\mu = \mu_o$, and $\omega = 2\pi \times 10^8$ rad/s

$$\hat{\eta} = \sqrt{\frac{\mu}{\varepsilon}}\,(1 + \frac{i\sigma_c}{\varepsilon\omega})^{-1/2},\quad |\hat{\eta}| = \sqrt{\mu/\varepsilon}\,[1 + (\frac{\sigma_c}{\varepsilon\omega})^2)^{-1/4} =$$

$3.98 \times 10^{-3}\Omega$. Note $\frac{\sigma_c}{\varepsilon\omega} >> 1$ for a good conductor so

$n = \frac{\sqrt{KKm}}{2}\ \sqrt{\sigma_c/\varepsilon\omega} = 6.7\text{x}10^4$ and $v = c/n = 4.47 \times 10^3$ m/sec.

14.18 $f = 3 \times 10^8$ Hz, $K = 20$, $K_m = 1000$, and $\sigma_c = 2(\Omega/m)^{-1}$.

a) $\hat{n} = n_o(1+i\sigma_c/\omega\varepsilon)^{1/2} = = n + ik$ with $n = 265$ and $k = 225$.

Therefore $n_{eff} = 265$ and absorption constant $= \omega k/c = 1410\ m^{-1}$.

c) $|\hat{\eta}| = \sqrt{\frac{\mu}{\varepsilon}}\,(1 + (\frac{\sigma_c}{\omega\varepsilon})^2)^{-1/4} = 1.08 \times 10^3\Omega$.

14.19 **a)** Neglect displacement current : $\nabla \times \vec{B} - \mu_o\vec{J} = \mu_o\sigma_c\vec{E}$

Also we have $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$, therefore

$$\nabla \times \nabla \times \vec{B} = -\nabla^2\vec{B} = \frac{-\partial^2 B}{\partial y^2} = -\mu_o\sigma_c\frac{\partial B}{\partial t} \text{ or } \mu_o\sigma_c\frac{\partial B}{\partial t} = \frac{\partial^2 B}{\partial y^2}$$

b) $\xi = (\mu_o\sigma_c/4t)^{1/2}y$. Substitute into the D.E. to get

$\frac{d^2B}{d\xi^2} = -\,2\xi\,\frac{dB}{d\xi}$. This has a solution $\frac{dB}{d\xi} = C_i e^{-\xi^2}$ and hence

$B = C_2 + C_1 \int_0^{\xi} e^{-\xi^2} d\xi$. Applying the boundary condition which states that at t = 0 or $\xi = \infty$, B is zero, and at y = 0 or $\xi = 0$, B is equal to 2 we get $C_2 = 0$ and $0 = C_1 \int_0^{\xi} e^{-\xi^2} d\xi + 2$ which

gives $C_1 = \frac{-4}{\sqrt{\pi}}$, and thus $B = 2[1 - \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\xi^2} d\xi]$.

c) $P_m|_{y=0} = B^2/2\mu_o = 8 \times 10^5\ N/m^2$. Thus $F = P_m \times A = 8 \times 10^5 N$.

d) At t = 1 we have $\xi = .056$ which gives $B = 2\ (1-.063) = 1.874$ so $F = 2 \times 10^5 N$.

CHAPTER 15

15.1 $\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0.$ Since $\Phi = f(t - r/c)/r$ is ϕ and θ independent then $\nabla^2\Phi \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r})$. Now $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(f(t - r/c)/r) = 1/(rc^2)f''$ where f'' is second derivative with respect to its argument.

$$\frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f, \quad r^2\frac{\partial\Phi}{\partial r} = r\frac{\partial f}{\partial r} - f, \quad \frac{\partial f}{\partial r} = \frac{-1}{c}f'$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r}) = \frac{1}{r^2}(\frac{\partial f}{\partial r} + r\frac{\partial^2 f}{\partial r^2} - \frac{\partial f}{\partial r}) = \frac{1}{r}\frac{\partial^2 f}{\partial r^2} = \frac{1}{rc^2}f''$$

15.2 Using $\vec{A} = \frac{\mu_o}{4\pi}\int\frac{\vec{J}\,dv}{r} = \frac{\mu_o}{4\pi}\int\frac{\vec{I}\,d\ell}{r} = \frac{\mu_o}{4\pi}\int\frac{I(t-r/c)}{r}d\vec{\ell}$

Note: the <u>integral</u> must be done in cartesian coordinates because the unit vectors ($\hat{\phi}$, here) change the position. We consider curved segment r = a,b first. In this case

$$\frac{\mu_o}{4\pi}\int_o^\pi \frac{I\,(t-r/c)}{r}\,r\hat{\phi}(-\hat{d}\phi) \text{ where } \hat{\phi} = \hat{y}\cos\phi - \hat{x}\sin\phi$$

$$= \frac{\mu_o}{4\pi}I(t - \frac{r}{c})\int_o^\pi[-\hat{y}\cos\phi + \hat{x}\sin\phi]d\phi = \frac{\mu}{4\pi}I(t - \frac{r}{c})(+2\hat{x})$$

For the straight segments we have:

$\frac{\mu_o}{4\pi}2\int_a^b\frac{I(t-x/c)}{x}\hat{x}\,dx.$ Thus the total vector potential is

$$\vec{A}(t) = \frac{\mu_o \hat{x}}{2\pi}\left[I(t - \frac{a}{c}) - I(t - \frac{b}{c}) + \int_a^b \frac{I(t - \frac{x}{c})dx}{x}\right]$$

For near field $\lambda >> b$ or $\omega << \frac{c}{b}$, thus

$$\vec{A}(t) = \frac{\mu_o \hat{x}}{2\pi} I(t)\, \ell n(\frac{b}{a}) \qquad I(t) \sim I(t + \frac{b}{c})$$

15.3 a) In cylindrical coordinates

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\hat{\phi}\frac{\partial A_z}{\partial \rho} = -\rho\hat{\phi}\sin(\alpha t)$$

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}\frac{\nabla \partial\psi}{\partial t} = -\frac{1}{2}\hat{z}\rho^2\alpha\cos(\alpha t) - \frac{\partial}{\partial t}\nabla\psi = -\frac{1}{2}\hat{z}\rho^2\alpha\cos(\alpha t)$$

b) If $\Psi = 0$ show that $\nabla\cdot\vec{A} + \epsilon\mu\frac{\partial\Phi}{\partial t} = 0$ (Lorentz condition). In this case $\frac{\partial\Phi}{\partial t} = 0$, $\vec{A} = \frac{1}{2}\hat{z}\rho^2\sin(\alpha t)$ and therefore $\vec{\nabla}\cdot\vec{A} = 0$. To check the wave equation, $\nabla^2\vec{A} - \epsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}_f$, we note that $\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A}) = -\nabla\times\vec{B}$. Using for $\vec{B}$ from part a, we get:

$$-\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\rho\hat{\phi}\sin(\alpha t)) = 2\hat{z}\sin\alpha t$$

$$\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{1}{2}\hat{z}\rho^2\alpha^2\sin\alpha t$$

$$\nabla^2\vec{A} - \epsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} = \hat{z}(-2 + \frac{1}{2}\rho^2\alpha^2\epsilon\mu)\sin\alpha t = -\mu\vec{J}_f$$

c) $f(\vec{r},t) = \rho\vec{E} + \vec{J}\times\vec{B} = \frac{\text{force}}{\text{volume}}$

$$\vec{J}_f = \hat{z}\frac{1}{\mu}(2 - \frac{1}{2}\rho^2\alpha^2\epsilon\mu)\sin(\alpha t),\quad \vec{B} = -\rho\hat{\phi}\sin\alpha t$$

$$\vec{J}_f\times\vec{B} = \hat{\rho}\frac{1}{\mu}(2\rho - \frac{1}{2}\rho^3\alpha^3\epsilon\mu)\sin^2(\alpha t)$$

15.4 Maxwell's divergence equations in free space are satisfied by these expressions because $\nabla\cdot$(curl of a vector) $\equiv 0$.

a) We now check the Maxwell's curl equations.

$$\nabla \times \vec{E} = \nabla \times (\nabla \times \nabla \times \hat{K}\Phi) = \nabla[\nabla\cdot(\nabla \times \hat{K}\Phi)]$$

$$- \nabla^2[\nabla \times (\hat{K}\Phi)] = -\nabla^2[\nabla \times (\hat{K}\Phi)] = -\nabla \times \hat{K}\, \nabla^2\Phi \,.$$

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{c^2}\frac{\partial}{\partial t}[\nabla \times (\hat{K}\frac{\partial\Phi}{\partial t})] = \frac{1}{c^2}\nabla \times (\hat{K}\frac{\partial^2\Phi}{\partial t^2})$$

Since Φ satisfies the wave equation then

$$-\frac{\partial \vec{B}}{\partial t} = -\nabla \times (\hat{K}\,\nabla^2\Phi) = \nabla \times \vec{E}$$

Now we check the $\nabla \times \vec{E}$ equation.

$$\nabla \times \vec{B} = \frac{1}{c^2}\nabla \times [\nabla \times (\hat{K}\frac{\partial\Phi}{\partial t})] = \frac{1}{c^2}\frac{\partial}{\partial t}[\nabla \times \nabla \times (\hat{K}\Phi)]$$

$$= \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu E}\frac{\partial \vec{E}}{\partial t}$$

b) $\nabla \times \vec{B} = \nabla \times \nabla \times [\nabla \times (\hat{K}\Phi)]$

$$= \nabla[\nabla\cdot(\nabla \times \hat{K}\Phi)] - \nabla^2[\nabla \times (\hat{K}\Phi)]$$

$$= -\nabla \times (\hat{K}\nabla^2\Phi) = -\frac{1}{c^2}\nabla \times (\hat{K}\frac{\partial^2\Phi}{\partial t^2})$$

$$= -\frac{\partial}{\partial t}\nabla \times (\hat{K}\frac{\partial\Phi}{\partial t}) = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Similar procedure can confirm the other Maxwell's curl equations.

15.5 To calculate the fields produced by the charge, we first calculate the first two multipoles:

$$q = \int \rho(\vec{r}', t - r/c)dv' = Q$$

$$\vec{p}(t - r/c) = \int \vec{r}' \rho(\vec{r}', t - \frac{r}{c})dv'$$

Because $\rho(\vec{r}', t - r/c) = \rho_o(r') + \rho_1(r') \cos(\omega t - kr)$ which depends on r' but not on its direction, then $\vec{p}(t - r/c) = 0$. Because of the spherical symmetry higher multipoles vanish too. Thus

$$\Phi = \frac{1}{4\pi\varepsilon_o} \frac{Q}{r}, \text{ and } \vec{E} = \frac{Q\hat{r}}{4\pi\varepsilon_o r^2}$$

The $\vec{B}$ field can now be shown to vanish, since $\vec{A}$ vanishes:

$$\vec{A} = \frac{\mu_o}{4\pi r} \int \vec{J}(\vec{r}', t - r/c)dv' = \frac{d}{dt} \int \vec{r}'\rho \, dv' = 0$$

15.6 $x = \dfrac{1 + t^4}{(1 + t^2)^2}$, $y = \dfrac{2t}{(1 + t^2)^{3/2}}$, $z = \dfrac{2t^3}{(1 + t^2)^2}$

$$x^2 + y^2 + z^2 = [(1 + t^4)^2 + (2t)^2(1 + t^2) + (2t^3)^2)]/(1 + t^2)^4$$

$$= [1 + 2t^4 + t^8 + 4t^2 + 4t^4 + 4t^6]/(1 + t^2)^4 = 1$$

This means that the wire is wrapped on a unit sphere.

$$\vec{A} = \frac{\mu_o}{4\pi} \int \frac{I \, d\vec{\ell}}{|\vec{r} - \vec{r}'|} = \frac{\mu_o}{4\pi} \int \frac{I(t - r/c)d\vec{\ell}}{|\vec{r} - \vec{r}'|}$$

Here, r' = 1 and in the $\lim(|r| \to \infty)\vec{A} \to 0$. Now for $\vec{r} = 0$ and $\vec{r}'(t = -\infty) = \vec{r}'(t = \infty)$, the integral around a closed loop is zero: $\oint d\vec{\ell} = 0$.

15.7 For a single wire, turned on at $t = 0$ over the whole length. (Note: since the fact that the current is on cannoe travel faster than c, a current cannot instaneously start flowing.) Only parts of the wire closer than ct can contribute. (See Ex. 15.2.) We have $\vec{A} = 0$ for $t < \rho/c$, and for $t > \rho/c$

$$\vec{A} = \frac{\mu_o I_o \hat{z}}{2\pi} \ln \left(\frac{\sqrt{c^2t^2 - \rho^2} + ct}{\rho}\right)$$

For two wires

$$\vec{A} = \frac{\mu_o I_o}{2\pi} [\ln\left(\frac{\sqrt{c^2t^2 - \rho_1^2} + ct}{\rho_1}\right) +$$

$$\ln \left(\frac{\sqrt{c^2(t - t_o)^2 - \rho_2^2} + c(t - t_o)}{\rho_2}\right)$$

The two vector potentials are equal when

$$\frac{\sqrt{c^2t^2 - \rho_1^2} + ct}{\rho_1} = \frac{\sqrt{c^2(t - t_o)^2 - \rho_2^2} + c\,(t - t_o)}{\rho_2}$$

where $t > \frac{\rho_1}{c}$, and $t > t_o + \frac{\rho_2}{c}$.

15.8 From Ex. 15.3, the electric field of an electric dipole $\hat{z}p(t)$ is given by Eq. 15.55

$$\vec{E} = \frac{\hat{z}\cdot\hat{r}}{4\pi\varepsilon_o}\left(\frac{3p}{r^3} + \frac{3\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right)\hat{r} - \frac{\hat{z}}{4\pi\varepsilon_o}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right)$$

$$= A_1\hat{r} + A_2\hat{z}.$$

In the plane of interest, we utilize an orthogonal coordinate composed of the r axis and an axis in the θ direction, thus $\hat{z} = \cos\theta\ \hat{r} + \sin\theta\hat{\theta}$, and $\vec{E} = A_1\hat{r} + A_2\hat{z} = (A_1 + A_2\cos\theta)\hat{r} + (A_2\ \sin\theta)\hat{\theta}$. Now from Eq. 2.6 defining the line of force

$$\frac{dr}{A_1 + A_2\cos\theta} = \frac{-rd\theta}{A_2\sin\theta}\ ,\ \text{thus}$$

$$\frac{dr}{(2p/r^3 + 2\dot{p}/cr^2)\cos\theta} = \frac{rd\theta}{(p/r^3 + \dot{p}/cr^2 + \ddot{p}/cr^2)\sin\theta}$$

$$2\cot\theta d\theta - \frac{(p/r^4 + \dot{p}/cr^3 + \ddot{p}/cr^2)}{p/r^3 + \dot{p}/cr^2}\,dr = 0$$

$$2\cot\theta d\theta + \frac{d(\dot{p}/c + p/r)}{\dot{p}/c + p/r} = 0$$

which integrates to $(\dot{p}/c + p/r)\sin^2\theta = \text{constant}.$

To get the static electric dipole results, we take the limit of $r \to 0$. In this case one can drop the $\dot{p}/c$ term in comparison with the p/r term and we get $(p/r)\sin^2\theta = \text{constant}.$

15.9 a) From Eqs. 15.112, we have

$$\vec{E} = \frac{p_o k^3}{4\pi\varepsilon_o} \frac{\cos(\omega t - kr)}{kr} \sin\theta\hat{\theta}, \quad \vec{B} = \frac{p_o k^3}{4\pi\varepsilon_o c} \frac{\cos(\omega t - kr)}{kr} \sin\phi\hat{\phi}$$

Along the dipoles, $\theta = 0$, $\vec{E} = 0$ and $\vec{B} = 0$.

b) For $\theta = \pi/2$ we have

$$\vec{E} = -\frac{p_o k^3}{4\pi\varepsilon_o} \frac{\cos(\omega t - kr)}{kr} \hat{z}, \quad \vec{B} = -\frac{p_o k^3}{4\pi\varepsilon_o c} \frac{\cos(\omega t - kr)}{kr} \hat{\phi},$$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_o} = \frac{p_o^2 k^6}{(4\pi\varepsilon_o)^2 \mu_o c} \frac{\cos(\omega t - k\rho)}{k\rho} \hat{\rho}$$

c) $$\vec{S} = \frac{p_o^2 k^4}{(4\pi\varepsilon_o)^2 \mu_o c r^2} \cos^2(\omega t - kr) \sin^2\theta \, \hat{r}.$$

d) Total power radiated is $P = \oint \vec{S}\cdot\hat{n}da = \dfrac{p_o^2 k^4}{6\pi\varepsilon_o^2 \mu_o c} \cos(\omega t - kr)$

15.10 Radiating <u>magnetic</u> dipole and image (see Ex. 15.5). Take the plane to be in the x-y plane. Then for an electric dipole $p'_x = -p_x$, $p'_y = -p_y$, $p'_z = p_z$. For a magnetic dipole $m'_x = m_x$, $m'_y = m_y$, $m'_z = -m_z$.

a) $\vec{m}_T = \vec{m} + \vec{m}' = 2\vec{m}$ when it is parallel to the surface.

b) $\vec{m}_T = \vec{m} - \vec{m}' = 0$ when it is perpendicular to the surface.

Generally, we have $\vec{p}_T = 2\hat{n}(\vec{p}\cdot\hat{n})$ and $\vec{m}_T = 2\hat{n} \times (\vec{m} \times \hat{n})$ where $\hat{n}$ is the unit vector normal to the surface.

c) For electric dipoles (see Eq. 15.114)

$$\vec{S}(r,\theta,\phi) = \frac{p_o^2 \, k^4 \cos^2(\omega t - kr)\sin^2\theta \, \hat{r}}{(4\pi\varepsilon_o)^2 \mu_o c r^2}$$

The energy flux or power density is proportional to p_o^2. If we time average we get $\langle P\rangle = p_o^2 k^4 c/12\pi\varepsilon_o$.

d) For magnetic dipoles we make the transformation

$$\frac{p_o^2}{4\pi\varepsilon_o} \rightarrow \frac{\mu_o m_o^2}{4\pi} \text{ so } \langle P\rangle = \frac{\mu_o m_o^2 k^4 c}{12\pi} \text{ (magnetic).}$$

For our problem, $\vec{p}_T = 2\vec{p}$, but we are only radiating into $\frac{1}{2}(4\pi)$ solid angle, so our power density is 4 times as high, but total power is only twice: $\langle P\rangle = \mu_o m_o^2 k^4 c/6\pi$

15.11 Using complex notations we get: $\vec{p}_1 = p_o \hat{x} \, e^{i\omega t}$, $\vec{p}_2 = \vec{p}_o(\hat{x}\cos\phi_o + \hat{y}\sin\phi_o)(-ie^{i\omega t})$,

$$\vec{E} \propto \hat{\phi}[\sin\phi - i\sin(\phi - \phi_o)] - \hat{\theta}[\cos\phi - i\cos(\phi - \phi_o)]\cos\theta$$

where θ, ϕ are observer position and ϕ_o is dipole orientation.

a) For $\theta = \frac{\pi}{2}$, $\cos\theta = 0$, $\vec{E} \propto \hat{\phi}[\sin\phi - i\sin(\phi - \phi_o)]$ which is linear polarizion.

b) For $\theta = 0$ and π, $\cos\theta = \pm 1$, and for $\phi_o = \frac{\pi}{2}$ we have, $\vec{E} \propto \hat{x} - i\hat{y}$ which is circular polarizion.

c) For $\theta = 0, \pi$ we have $\vec{E}\ \alpha(\hat{x} - i\hat{x}\cos\phi_o + \hat{y}\sin\phi_o)$, which is elliptical polarizion.

15.14 For a small sphere where $r \ll \lambda$ we use the dipole approximation. (See Ex. 15.7.)

$$\vec{E}_e = \frac{\beta_e E_o k^2}{4\pi\varepsilon_o r}[\hat{\phi}\sin\phi - \hat{\theta}\cos\phi\cos\theta]e^{-i(\omega t - kr)}$$

$$\vec{B}_e = \frac{\beta_e E_o k^2}{4\pi\varepsilon_o rc}[\hat{\phi}\cos\phi\cos\theta + \hat{\theta}\sin\phi]e^{-i(\omega t - kr)}$$

$$\vec{B}_m = \frac{\beta_m H_o k^2}{4\pi\varepsilon_o c^2 r}[\hat{\phi}\cos\phi + \hat{\theta}\sin\phi\cos\theta]e^{-i(\omega t - kr)}$$

$$\vec{E}_m = \frac{\beta_m H_o k^2}{4\pi\varepsilon_o cr}[-\hat{\phi}\sin\phi\cos\theta + \hat{\theta}\cos\phi]e^{-i(\omega t - kr)}$$

Note that on the x-axis, $\theta = \frac{\pi}{2}$ and $\phi = 0$, thus $\vec{E}_e \to 0$, $\vec{B}_e \to 0$,

$$B_m \to \frac{\beta_m H_o k^2}{4\pi\varepsilon_o c^2 r}\hat{\phi}\cos(\omega t - kr),\quad \vec{E}_m = \frac{\beta_m H_o k^2}{4\pi\varepsilon_o cr}\hat{\theta}\cos(\omega t - kr),$$

$$\vec{S} = \vec{E}\times\vec{H} \doteq \frac{1}{\mu_o}\vec{E}\times\vec{B} = \frac{\beta_m^2 H_o^2 k^4}{(4\pi\varepsilon_o)^2 c^3 r^2 \mu_o}\hat{\rho}\cos^2(\omega t - kr).$$ This is independent of β_e.

15.15 $\omega = 5 \times 10^{10}$ rad/s, $I_o = 1$A. Now we have

$$\ell \ll c/\omega = \frac{3 \times 10^{10}\ \text{cm/s}}{5 \times 10^{10}\ /\text{s}} = 0.6\ \text{cm}$$

a) Use $\ell = 0.01$ cm $= 10^{-4}$ m.

b) From Eq. 15.150 $p_o = \ell I_o/\omega = 1.2 \times 10^{-13}$ coul/m

c) From 15.154 we have $R = 789(\frac{\ell}{\lambda})^2$ohms $= 5.6 \times 10^{-3}\ \Omega$.

15.16 **a)** The current distribution in the antenna is represented by

$$I = I_o \sin[\frac{m\pi}{\ell}(z + \ell/2)]e^{-i\omega t}$$

where m is an integer. Thus at $z = \pm\ \ell/2$ the current is zero.

b) Consider a small element at distance z' and length dz. From $\ell/2$ Eq. 15.150 $dp/dz' = I/\omega$.

c) From Eq. 15.157 we have

$$dB_\phi = \frac{\mu_o I_o \omega^2}{4\pi c^2 kr} \int \cos(\omega t - kr + kz' \cos\theta) \sin \frac{m\pi}{\ell} [(z' + \ell/2)]dz'$$

d) A corresponding expression for the electric field E_θ can be easily arrived at in a similar way. Integrating, and using $S = E_\theta B_\phi/\mu_o$, we get

$$\vec{S} = \frac{I_o^2}{4\pi\varepsilon_o cr^2} \cos^2(\omega t - kr) \begin{cases} \dfrac{\cos^2(\frac{m\pi}{2} \cos\theta)}{\sin^2\theta} & \text{odd m} \\[2ex] \dfrac{\sin^2(\frac{m\pi}{2} \cos\theta)}{\sin^2\theta} & \text{even m} \end{cases}$$

15.18 From Eq. 15.168 we have $|\vec{E}| = |\vec{E}_o| \dfrac{\sin(\frac{\pi}{2} N \sin\theta)}{\sin(\frac{\pi}{2} \sin\theta)}$. Thus for N = 4 we get $|\vec{E}| = |\vec{E}_o| \dfrac{\sin(2\pi \sin\theta)}{\sin(\frac{\pi}{2} \sin\theta)}$

15.19 The antenna can be decomposed into two components. One is a quarter wave which is normal to the ground, and a $\sqrt{3}/2$ wave parallel to the ground. The dipole emission from the horizontal component vanishes because of its image components whereas the vertical component behaves as a half wave when its image antenna is added to it. Thus we expect the pattern of radiation to be that of a half wave placed along the vertical direction. See Examples 15.5 and 15.10.

15.20 Given $q = q_o \sin\omega t$

a) $\vec{p} = \ell q_o \hat{z} \sin \omega t$

b) $I = dq/dt = \ell q_o \omega \cos \omega t$

c) $\ddot{p} = - \ell q_o \omega^2 \sin \omega t$, thus

$$\vec{E}_R = -\frac{\ell q_o \omega^2}{4\pi \partial_o c^2 r} [\hat{r}(\hat{r}\cdot\hat{z}) - \hat{z}] \sin(\omega t - kr)$$

$$\vec{B}_R = -\frac{\ell \mu_o q_o \omega^2}{4\pi c r} \hat{r} \times \hat{z} \quad \sin(\omega t - kr)$$

Along the line joining the charge $\theta = 0$, $\hat{r} \times \hat{z} = 0$, and $\hat{r}(\hat{r}\cdot\hat{z}) - \hat{z} = 0$, thus the fields vanish.

15.21 The dipole moment of the charge is $\vec{p}(t - r/c) = \int \vec{r}'\ \rho(\vec{r}', t - \vec{r}/c)d\vec{r}'$. Since $\vec{r}' = (vt - \frac{1}{2}at^2)\hat{z}$, and $\rho = q\delta(\vec{r}' - (vt + \frac{1}{2}at^2)\hat{z})$ then $\vec{p}(t - \vec{r}/c) = \int \vec{r}'\ q\ \delta(\vec{r}' - (vt + \frac{1}{2}at^2)\hat{z})$ or $\vec{p} = q(\vec{v}t + \frac{1}{2}at^2)\hat{z}$; hence $\ddot{p} = aq\hat{z}$. The radiation electric field of the charge is

$$\vec{E}_R = \frac{1}{4\pi\varepsilon_o c^2 r} [\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}] = \frac{qa}{4\pi\varepsilon_o c^2 r} [\hat{r}\cos\theta - \hat{r}\cos\theta + \hat{\theta}\sin\theta].$$

Since $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$. Thus $\vec{E}_R = \frac{qa}{4\pi\varepsilon_o c^2 r}\sin\theta\hat{\theta}$. The magnetic field and hence $\vec{S}$ are $\vec{B}_R = \frac{1}{c}\hat{r} \times \vec{E}_R = \frac{qa \sin\theta}{4\pi\varepsilon_o c^3 r}\hat{\phi}$,

$\vec{S} = \vec{E} \times \vec{H} = \frac{q^2 a^2 \sin^2\theta}{(4\pi)^2 \varepsilon_o c^3 r^2}\hat{r}$. The radiated power is found:

$\frac{d^2W}{dt'dA} = \frac{dP}{dA}$. For low velocities we have

$\frac{dW}{dt} = \vec{S}\cdot\hat{n}\ dA = \vec{S}\cdot\hat{r}\ r^2 d\Omega = dP$. Therefore

$\frac{dP}{d\Omega} = \vec{S}\cdot\hat{r}\ r^2 = \frac{q^2 a^2}{(4\pi)^2 \varepsilon_o c^3}\sin^2\theta$

The angular distribution is proportional to $\sin^2\theta$, and the radiation is independent of the initial velocity v_o.

b) Total power

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \int_0^{2\pi} d\phi \int_{-1}^{1} \frac{q^2 a^2 \sin^2\theta}{(4\pi)^2 \varepsilon_o c^3}\, d(\cos\theta).$$ Let $x = \cos\theta$, then

$1 - x^2 = \sin^2\theta$. Thus $P = \frac{q^2 a^2\, 2\pi}{(4\pi)^2 \varepsilon_o c^3} \int_{-1}^{1} (1 - x^2)dx = \frac{q^2 a^2}{6\pi\varepsilon_o c^3}$.

15.22 Consider a particle moving with $\vec{v}_o$, where the acceleration is parallel to $\vec{v}_o$. v_o does not have to be << c, therefore we use Lienard-Weichert results. From Eq. 15.228

a) $$\frac{dP}{d\Omega} = \frac{q^2 a^2 \sin^2\theta}{(4\pi\varepsilon_o)^2 \mu_o c^5 (1 - \beta\cos)^5}$$

b) Integrate to determine the total power:

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{q^2 a^2}{(4\pi\varepsilon_o)^2 \mu_o} \int_0^{2\pi} d\phi \int_{-1}^{1} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}\, d(\cos\theta)$$

$$P = \frac{2\pi q^2 a^2}{(4\pi\varepsilon_o)^2 \mu_o} \int_{-1}^{1} \frac{(1 - x^2)dx}{(1 - \beta x)^5}$$

For evaluating the $\int dx$ we let $u = 1 - \beta x$, and hence $du = -\beta dx$. Therefore

$$I = \int_{-1}^{1} \frac{1 - x^2}{(1 - \beta x)^5}\, dx$$

$$I = \frac{1}{\beta} \int_{1-\beta}^{1+\beta} \frac{1}{u^5}\, du - \frac{1}{\beta^3} \int_{1-\beta}^{1+\beta} \frac{1 - 2u + u^2}{u^5}\, du.$$

$$I = -\frac{1}{\beta^3} \Big|_{1-\beta}^{1+\beta} \frac{\beta^2 - 1}{4u^4} + \frac{2}{3u^3} - \frac{1}{2u^2} = -\frac{4\beta^3}{3\beta^3(\beta^2 - 1)^3} = \frac{4}{3}\gamma^6$$

$$P = \frac{q^2 a^2 c^2}{6\pi\varepsilon_o} \gamma^6 \text{ with } \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

15.23 We will find energy radiated in stopping a particle using $E = \int_o^t P(t')dt'$. $\beta = \frac{v_o - at'}{c}$ therefore $d\beta = -\frac{a}{c} dt'$. Using the results of problem 15.22 we write

$$E = -\int_{v_o/c}^{0} \frac{c}{a} P d\beta = \frac{c}{a} \frac{q^2 a^2 c^2}{6\pi\varepsilon_o} \int_o^{\beta_o} \frac{1}{(1 - \beta^2)^3} d\beta$$ where $\beta_o = v_o/c$. Using integral tables:

$$E = \frac{c}{a} \frac{q^2 a^2 c^2}{6\pi\varepsilon_o} \Big|_o^{\beta_o} \left(\frac{\beta}{4(1 - \beta^2)^2} + \frac{3\beta}{8(1 - \beta^2)} + \frac{3}{16a^2} \log\left(\frac{1 + \beta}{1 - \beta}\right)\right)$$

$$E = \frac{q^2 a c^3}{6\pi\varepsilon_o} \left(\frac{\beta_o^2}{4(1 - \beta_o^2)^2} + \frac{3\beta_o}{8(1 - \beta_o^2)} + \frac{3}{16a^5} \log\left(\frac{1 + \beta_o}{1 - \beta_o}\right)\right)$$

15.24 Using 15.228

$$\frac{dP}{d\Omega} = \frac{q^2 a^2 \sin^2\theta}{(4\pi\varepsilon_o)^2 \mu_o c R^2 (1 - \beta\cos\theta)^6} \propto \frac{\sin^2\theta}{(1 - \beta\cos\theta)^6}$$

$$\frac{d}{d\theta}\frac{dP}{d\Omega} \propto \frac{2\sin\theta\cos\theta(1 - \beta\cos\theta)^6 - \sin^2\theta\, 6(1 - \beta\cos\theta)^5 \beta\sin\theta}{(1 - \beta\cos\theta)^{12}} = 0$$

$\cos\theta(1 - \beta\cos\theta) - 3\beta\sin^2\theta = 0$ or $\cos\theta(1 - \beta\cos\theta) - 3\beta(1 - \cos^2\theta) = 0$, or $2\beta\cos^2\theta + \cos\theta - 3\beta = 0$ which solves to

$$\cos\theta = \frac{-1 \pm \sqrt{1 + 24\beta^2}}{4\beta}.$$

We use the plus sign when $\lim \beta \to 1$: $\frac{\sqrt{25} - 1}{4} = 1$ or $\cos\theta_o = 1$ or $\theta_o = 0$. On the other hand when $\lim \beta \to 0$:

$$\sqrt{1 + 24\beta^2} \sim 1 + 12\beta^2 \qquad \cos\theta_o \sim 3\beta \to 0 \text{ or } \theta_o = \frac{\pi}{2}.$$

15.25 **a)** $\vec{p} = \hat{x}\, p_o \cos\omega t + \hat{y}\, p_o \sin\omega t$.

b) The phase difference is $\pi/2$.

c) Example 15.6 dealt with two dipoles located at the origin, one of them is along x and the other makes an angle ϕ. Thus taking $\phi_o = \pi/2$ gives our present configuration. The angular distribution is calculated from 15.129 taking $\phi_o = \pi/2$; thus

$$\frac{\langle dP \rangle}{d\Omega} = \frac{p_o^2 k^4 c}{32\pi^2 \varepsilon_o} (1 + \cos^2\theta)$$

d) $\ddot{\vec{p}} = p_o \omega^2 (\hat{x} + i\hat{y}) e^{i\omega t}$ (in complex notation). Now $\langle |\hat{r} \times \ddot{\vec{p}}|^2 \rangle = \frac{1}{2} |(\hat{x} \sin\theta\cos\phi + \hat{y} \sin\theta\sin\phi + \hat{z} \cos\theta) \times \ddot{\vec{p}}| = \frac{1}{2} p_o^2 \omega^4 (1 + \cos^2\theta)$. Thus $\langle dP \rangle / d\Omega$ is the same as we got in (a).

CHAPTER 16

16.1 Near the origin we have $\vec{H}_1 = \vec{B}_1/1.5\ \mu_o$ and $\vec{H}_2 = \vec{B}_2/5\mu_o$ or $\vec{H}_1 = (1.6\ \hat{x} + 6.67\ \hat{z})/\mu_o$, $\vec{H}_2 = (5\ \hat{x} - 3.5\ \hat{y} + 2\ \hat{z})/\mu_o$. The current distribution is therefore: $\vec{K} = (\vec{H}_1 - \vec{H}_2) \times \hat{z} = (-3.4\ \hat{x} + 3.5\ \hat{y} + 4.67\ \hat{z}) \times \hat{z}/\mu_o = (3.5\ \hat{x} + 3.4\ \hat{y})\mu_o$ A/m.

16.2 a) From Eqs. 16.50 and 16.51, we find that the angle of refraction in this case is complex, with $\sin\theta_2 = \sin\theta_1/\sin\theta_c = \alpha > 1$, and $\cos\theta_2 = (1 - \beta^2)^{1/2} = i\beta$ where θ_1 and θ_c are the incidence and critical angles. Inside the n_2 material we can write (the z axis is normal to the interface plane)

$$\vec{E}_2 = \vec{E}_{2o}e^{i(\vec{k}_2\cdot\vec{r}-\omega t)} = \vec{E}_{2o}e^{i(k_2\alpha x-\omega t)}\ e^{-k_2\beta z}.$$

The direction of propagation is along x, that is parallel to the interface plane. The phase velocity is calculated from $k_2\alpha x-\omega t$ = constant: $v_p = \omega/k_2\alpha = n_2\omega/(k_2\sin\theta_1)$ which is a function of the angle of incidence.

b) The attenuation takes place along the normal to the interface (i.e. z axis), and the coefficient of attenuation is $k_2\beta$.

c) Calculate $\langle\vec{S}_2\rangle\cdot\hat{z} = \text{Re}\ (\vec{E}_2 \times \vec{H}_2^*)\cdot\hat{z}$. But $\nabla \times \vec{E}_2 = i\vec{k}_2 x\vec{E}_2 = -\dfrac{\partial\vec{B}_2}{\partial t} = i\omega\vec{B}_2 = i\omega\mu_o\vec{H}_2$; thus

$$\langle\vec{S}_2\rangle\cdot\hat{z} = \frac{1}{\omega\mu o}\ \text{Re}\ [\vec{E}_2 x\ (\vec{k}_2 x\ \vec{E}_2)]\cdot\hat{z} = \text{Re}\ \frac{1}{\omega\mu_o}\ |E_2|^2\ (\vec{k}_2\cdot\hat{z})$$

$$= \text{Re}\,\frac{1}{\omega\mu_o} k_2|E_2|^2 \cos\theta_2 = \text{Re}\,\frac{i}{\omega\mu_o} k_2|E_2|^2\beta = 0$$

16.3 **a)** We have from Ex. 16.4:

$$\tan\frac{\phi_s}{2} = \frac{(\sin^2\theta_1 - (n_2/n_1)^2)^{1/2}}{\cos\theta_1}, \quad \tan\frac{\phi_p}{2} = \frac{(\sin^2\theta_1 - (n_2/n_1)^2)^{1/2}}{(n_2/n_1)^2\cos\theta_1}$$

Thus $\phi_s = \phi_p$ when $\sin\theta_1 = n_2/n_1$ or when $\theta_1 = \pi/2$.

b) We maximize $(\phi_p - \phi_s)/2$. Now $\delta/2 = (\phi_p - \phi_s)/2 = \tan^{-1}[\gamma(n_1/n_2)^2] - \tan^{-1}\gamma$ where $\tan(\phi_s/2) = \gamma$. Taking the derivitive of $\delta/2$ and equating it to zero gives $\gamma = n_2/n_1$. Thus

$\delta_{max}/2 =$

$\tan^{-1}\frac{n_1}{n_2} - \tan^{-1}\frac{n_2}{n_1}$ or $\tan(\delta_{max}/2) = (1 - (n_2/n_1)^2/2(n_2/n_1)$

c) Yes if also $E_{op} = E_{os}$. Since $\delta = 0$ when $\theta_1 = \sin^{-1}n_2/n_1$ and when $\theta_1 = \pi/2$, and since the $\tan(\delta_{max}/2)$ as given above occur in between, then it is possible to have $\tan(\delta/2) = 1$ provided that

$1 - (\frac{n_2}{n_1})^2 \geq 2\frac{n_2}{n_1}$ and $\frac{n_2}{n_1} < .414$

16.4 From Ex. 16.6 we find that the boundary conditions at the two boundaries give

$E_1 - E_1' = E_2 - E_2'$ $n_1(E_1 + E_1') = n_2(E_2 + E_2')$

$E_2e^{i\theta_2} - E_2'e^{-i\theta_2} = E_3e^{i\theta_3}$, $n_2(E_2e^{i\theta_2} + E_2'e^{-i\theta_2}) = n_3e^{i\theta_3} E_3$

where $\theta_2 = \frac{n_2\omega d}{c}$, $\theta_3 = \frac{n_3\omega d}{c}$, $n_1 = n_3 = 1$, and $n_2 = 2$

Solving these four equations for E_1'/E_1 gives

$$\frac{E_1'}{E_1} = \frac{3(e^{-2i\theta_2} - 1)}{(9e^{-2i\theta_2} - 1)} = -\frac{3}{5 + 4\ i\cot\theta_2} \quad \text{thus}$$

$$\left(\frac{E_1'}{E_1}\right)^2 = \frac{1}{1 + 16/9\ \csc^2\theta_2}$$

16.5 From ex. 16.6

$$\frac{E_1^2}{E_3^2} = \frac{1}{4}\left(1 + \frac{n_3}{n_1}\right)^2 \cos^2\left(\frac{n_2\omega d}{c}\right) + \frac{1}{4}\left(\frac{n_2}{n_1} + \frac{n_3}{n_2}\right)^2 \sin^2\left(\frac{n_2\omega d}{c}\right)$$

$$T = \frac{n_3}{n_1}\ \frac{\langle E_3^2\rangle}{\langle E_1^2\rangle}$$

1) For $n_1 = 1$, $n_2 = 2$, $n_3 = 3$ we have $T = \dfrac{3}{4 - \frac{15}{16}\sin^2\left(\frac{2\omega d}{c}\right)}$

2) For $n_1 = 2$, $n_2 = 4$, $n_3 = 1$ we have $T = \dfrac{8}{9 + 72\sin^2\left(\frac{4\omega d}{c}\right)}$

3) For $n_1 = 3$, $n_2 = 2$, $n_3 = 1$ we have $T = \dfrac{1/3}{\frac{4}{9} - \frac{135}{1296}\sin^2\left(\frac{2\omega d}{c}\right)}$

16.6 Notice first that the incident flux undergoes a phase shift at boundary #1 (because $n_2 > n_1$) but does not shift phase at interface #2 ($n_3 < n_2$). Thus we have: minimum A requires $2n_2 d = m$ ($m = 0,1,2$), maximum A requires $2n_2 d = (m + 1/2)$ ($m = 0,1,2$). But $\lambda = \frac{2\pi c}{\omega_o}$, then for a minimum we have $n_2 d = \frac{\pi c m}{\omega_o}$ and for a maximum we have $n_2 d = \frac{\pi c(m + 1/2)}{\omega_o}$.

16.7 The critical angle $\theta_c = \sin^{-1} n_2/n_1$. The largest angle is $\theta = \pi/2 - \theta_c$. Thus for $n_2 = 1$, $\theta = 49.8$, for $n_2 = 1.52$,

$\theta = 11.3$ and for $n_2 = 1.50$, $\theta = 14.6$.

16.8 The analysis is very similar to that given in Sec. 16.2.1 leading to Eq. 16.22, except n_2 becomes complex, and we take $n_1 = 1$. Thus $\hat{r}_{12} = (\hat{n}_2 - 1)/(\hat{n}_2 + 1)$.

16.12 **a)** For p polarization incident at the analog of the Brewster angle we have: $\tan\delta_p = -2\phi_m n_1 \eta_2'/(\phi_m^2 - n_1^2|\eta_2|)$. Since ϕ_m approaches $n_1/|\hat{n}_2| = n_1|\hat{n}_2|$, then $\tan\delta_p \to \infty$ or $\delta_p = \pi/2$.

b) For s polarization we have $\tan\delta_s = \dfrac{2\eta_2' \cos\theta_1}{-1 + 2\eta_2 \cos\theta_1}$. For highly conducting surfaces η_2' and η_2 become very small, and $\tan\delta_s$ is negative and approaches zero. Thus $\tan\delta_s = \pi$.

c) If the amplitudes are equal and the initial phases are the same, then upon reflection they will be $\pi/2$ out of phase, producting circular polarization. When one of the incident amplitudes, either E_{op} or E_{os} is zero, then we have linear polarization.

16.13 Using the same procedure followed in Ex. 16.12 except we are given here the amplitude of the incident H field instead of the E field. Thus we first calculate $E_o = B_o/c = \mu_o H_o/c = 3\mu_o/c$. Thus from Ex. 16.12 we have

$$E_1/E_o = \frac{2}{1 + \sqrt{\sigma_c/\varepsilon_o \omega}\; e^{i\pi/2}}, \quad E_2/E_1 = e^{-d/\delta}$$

$$E_3/E_2 = \frac{2\hat{n}_2}{\hat{n}_2 + n_1} = 2/[1 + (\varepsilon_o \omega/\sigma_c)e^{-i\pi/4}] \text{ where } \delta = \sqrt{2/\mu_o \sigma_c \omega}.$$

16.14 From Eq. 16.111 we have $\lambda_c = 2a/n = 16/2 = 8\text{cm}$ for $n = 2$. Now from the same Eq. we have $\lambda_g = (1/\lambda_o^2 - n^2/4a^2)^{-1/2}$ which is equal to 12.8 cm for $n = 1$.

16.15 Considering TE waves, then we can write $\vec{E}(x,y,z) = E'(x,y)\, \hat{x}\, e^{i(k_g z-\omega t)}$. Thus $\nabla\cdot\vec{E} = 0$ gives $\partial E'/\partial x = 0$ indicating E' being independent of x.

b) The wave equation for $\vec{E}$ gives $\nabla^2\vec{E} + (\omega^2/c^2)\vec{E} = 0$

$d^2E'/dy^2 + (\omega^2/c^2 - k_g^2)E' = 0$, which has the solution:

$E'(y) = E_o\sin(k_c y) + E_o'\cos(k_c y)$, $k_c = \frac{\omega}{c}(1 - k_g^2/k^2)^{1/2}$.

Applying the boundary condition that E' should vanish at $y = 0$ and $y = a$ requires that $k_c = n\pi/a$ and $E_o' = 0$. Thus $\vec{E}(x,y,z,t) = E_o\sin(k_c y)\, e^{i(k_g z-\omega t)}$.

c) k_g has to be real quantity for propagation to take place. Thus from $k_c = (\omega/c)(1 - k_g^2/k^2)^{1/2}$ we get $k_g = (1 - k_c^2/k^2)^{1/2}$, or $k_g = (1 - n\pi c/a\omega)^{1/2}$. Thus for $n = 1$, we have $n\pi c/a\omega < 1$ or $\omega = \pi c/a$ is the smallest frequency.

16.16 a) Using $\nabla \times \vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$, we get

$$\frac{\partial H_x}{\partial z} = -\varepsilon_o \frac{\partial E_y}{\partial t},\ \frac{\partial H_x}{\partial y} = \varepsilon_o \frac{\partial E_z}{\partial t},\ \frac{\partial E_x}{\partial t} = 0.$$ For example the first

equation gives $\varepsilon_o \frac{\partial E_y}{\partial t} = \mu_o k\sin\beta\, H_o\cos(ky\cos\beta)\, \sin(kz\, \sin\beta-\omega t)$, and $E_y = -(\mu_o k \sin\beta/\varepsilon_o\omega)\, H_o \cos(ky\cos\beta)\, \cos(kz\sin\beta-\omega t)$.

Similarily we find $E_x = 0$, and

$E_z = (\mu_o k\cos\beta/\varepsilon_o\omega) H_o \sin(ky\cos\beta)\, \sin(kz\sin\beta-\omega t)$.

b) At $y = 0$, and $y = a$, we should have $E_z = 0$. Thus

$\sin(ka\cos\beta) = 0$ or $ka\cos\beta = n\pi$ where n is an integer.

c) $\langle\vec{S}\rangle = \langle\vec{E} \times \vec{H}\rangle$, thus

$= \hat{z}\, (\mu_o k \sin\beta\, H_o^2/\varepsilon_o\omega)\, \langle\cos^2(ky\cos\beta)\, \cos^2(kz\sin\beta-\omega t)\rangle$

$+ \hat{y}(\mu_o k\cos\beta\, H_o^2/4\varepsilon_o\omega)\langle\sin(2ky\cos\beta)\, \sin(2kz\sin\beta-2\omega t)\rangle$. It is easy to show that the second term vanishes; hence the time average of the $\vec{S}$ is entirely in the z direction.

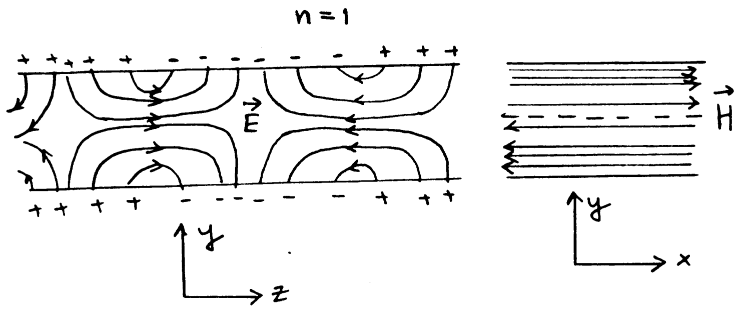
$$\textbf{d)}\quad \int_0^b dx \int dy\, \langle S\rangle = \frac{\mu_o k\sin\beta\, H_o^2 ab}{8\varepsilon_o\omega} = \frac{\mu_o \sin\beta\, H_o^2 abc}{8\varepsilon_o}$$

16.17 a) In the case of TE, we had $\vec{E}$ tangent to the plate, that is $\vec{E} = E\hat{x}$, and hence $\vec{H} = H\hat{y}$ (normal to the plate). In the TM case, $\vec{H}$ is taken tangent to the plate while $\vec{E}$ is taken normal to it. Thus we write $\vec{H} = \hat{x}[H_o\cos(k_c y) + H_o' \sin(k_c y)]e^{i(k_g z-\omega t)}$ where $k_c = k\cos\theta$, and $k_g = k\sin\theta$. But from $\nabla \times \vec{H} = \hat{y}\, \partial H_x/\partial z - \hat{z}\, \partial H_x/\partial y = -i\varepsilon_o\omega\vec{E}$ we conclude that $\partial H_x/\partial y$ vanishes on the plates since the tangential electric field vanishes there. Thus H_o' is set to zero, and $k_c a = ka\cos\theta = n\pi$ giving $k_c = n\pi/a$ where n is an integer. Thus $\vec{H} = \hat{x}\, H_o \cos\frac{n\pi}{a} y\, e^{i(k_g z-\omega t)}$ and

$\vec{E} = (H_o/\varepsilon_o\omega)\, [-\hat{y}k_g \cos\frac{n\pi y}{a} + \frac{izn\pi}{a} \sin\frac{n\pi y}{a}]e^{i(k_g z-\omega t)}$

b) Even if $n = 0$, $\vec{H}$ and $\vec{E}$ do not vanish in the TM case whereas all fields vanish if $n = 0$ in the TE case.

c) We sketch the pattern of the n = 0, and n = 1 modes



16.18 **a)** From the results given between Eqs. 16.114 and 16.115 we have $\langle u\rangle/a = \varepsilon_o|E_o|^2/4$ and $\langle\vec{S}\rangle/a = \varepsilon_o c|E_o|^2 k_g\,\hat{z}/4k$ where $k = \omega/c$ (that of the free wave). The lowest two modes are n = 1 and n = 2. Thus using $k_g^2 = k^2-(n\pi/a)^2$, we get

$\langle u\rangle/a = \varepsilon_o|E_o|^2/4$, and $\langle\vec{S}\rangle/a = \varepsilon_o c|E_o|^2\,\hat{z}(k^2-(n\pi/a)^2)^{1/2}/4k$

where n = 1, and n = 2.

b) $v_p = c/\sin\theta = ck/k_g = ck(k^2 - n^2\pi^2/a^2)^{-1/2}$ and

$v_g = c\sin\theta = c(k^2- n^2\pi^2/a^2)^{1/2}/k$.

16.19 The solution must satisfy the boundary conditions $E_y = 0 = H_x = \partial H_y/\partial x$ at the surface of the plates at x = 0, and x = a. Thus $\vec{E} = \hat{x}\,E_o\,e^{i(kz-\omega t)}$, and $\vec{H} = \hat{y}(E_o/v_o\mu_o)\,e^{i(kz-\omega t)}$. This solution satisfies all of Maxwell's equations and the boundary conditions. Moreoever it has no restriction on ω and can even work for dc current flow ($\omega = 0$).

16.20 From Eq. 16.22 we have $k_g^2 = k^2 - k_x{}^2 = k_y{}^2$ or $k_g = [\omega^2/c^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2]^{1/2}$. Thus

$$v_p = \frac{\omega}{k_g} = \frac{c}{\sqrt{1 - (\omega_c/\omega)^2}}, \quad v_g = \frac{d\omega}{dk_g} = \omega(1 - (\omega_c/\omega)^2)^{1/2}$$ where

$$\omega_c = \frac{1}{c}\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$$ is the cutoff frequency.

16.21 **a)** From Eq. 16.126, the cutoff wavelength is $1/\lambda_c^2 = (\frac{m}{2a})^2 + (\frac{n}{2b})^2$. For TE_{10}, and TE_{01} we have $\lambda_c = 2a$ and $2b$ respectively. Thus if $a > b$, then TE_{10} has the largest λ_c.

b) From Eq. 16.126 we have for $a = 2b$,

$\Omega_c = 2b\omega_c/\pi c = (m^2 + 4n^2)^{1/2}$ or $m^2 + 4n^2 = 1/(\lambda_c/2a)^2$

		Ω_c			
	n = 0	1	2	3	4
m = 0		2	4	6	8
1	1	2.24	4.13	6.08	8.06
2	2	2.84	4.48	6.32	8.25
3	3	3.61	5	6.71	8.54
4	4	4.48	5.66	7.21	8.94

c) $(\lambda_c/2a) = (m^2 + 4n^2)^{-1/2}$. The cutoff wavelength is given in the following table.

	λ_c				
	n = 0	1	2	3	4
m = 0		1	.5	.33	.25
1	2	.89	.48	.33	.25
2	1	.70	.45	.32	.24
3	.67	.55	.4	.30	.23
4	.5	.45	.35	.28	.22

Thus we see that in the first case only the mode TE_{10} will propagate, while in the second case we have a number of modes: all modes listed in the first two columns, and the top three of the third column.

16.22 a) The wave equation of the x component of $\vec{H}$, $\nabla^2 H_x - \varepsilon_o\mu_o \dfrac{\partial^2 H_x}{\partial t^2} = 0$ will be first shown to reduce to

$$\frac{\partial^2 H_x}{\partial z^2} = \varepsilon_o\mu_o \frac{\partial^2 H_x}{\partial t^2}$$ by showing that $$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} = 0.$$ This can

be shown by noting that $(\nabla \times \vec{H})_z = \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = \varepsilon_o \dfrac{\partial E_z}{\partial t} = 0$ and

$$\nabla\cdot\vec{H} = 0 = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}.$$ Therefore

$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$. Differentiating the first equation with respect to y and the second with respect to x and adding gives $\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} = 0$. Now the solution of the above wave equation for H_x is just $H_x = H_x(x,y)\ e^{i(kz-\omega t)}$ with $\omega/k = v_o = 1/\sqrt{\varepsilon_o \mu_o}$ which is not restricted to a discrete set of values.

b) Solutions of the form $H_x = H_{ox} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}\ e^{i(kz-\omega t)}$ and $H_y = H_{oy} \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\ e^{i(kz-\omega t)}$ where m, n = 1,2,...

c) Using $\nabla \cdot \vec{H} = 0$, and $(\nabla \times \vec{H})_z = 0$ gives $(H_{ox} = -am/bn)\ H_{oy}$, $H_{ox} = (bn/am)\ H_{oy}$ Dividing these two equations gives $1 = -(am/bn)^2$. which cannot be satisfied for real values of m, n, a, and b.

16.23 Consider the following solutions of the wave equation (m = n = 0): $H_z = H_{oz}\ e^{i(k_g z-\omega t)}$, $H_x = H_{ox}\ e^{i(k_g z-\omega t)}$, and $H_y = H_{oy}\ e^{i(k_g z-\omega t)}$. Now one can see that $\nabla \cdot \vec{B} \neq 0$ for this solution. $\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 + 0 + ik_g H_{oz} e^{i(k_g z-\omega t)} \neq 0$. A similar argument applies to cavity modes TE_{001}, etc, therefore those modes are prohibited.

16.24 The mode structure was given in Eqs. 16.130 - 16.140. The $\nabla \cdot \vec{E} = 0$ equation leads to Eq. 16.138, that is

$$\frac{\ell\pi}{a} E_{ox} + \frac{m\pi}{b} E_{oy} + \frac{n\pi}{c} E_{oz} = 0$$

a) Consider the case ℓ, m, and n $\neq 0$. Two of the E_{ox}, E_{oy}, and E_{oz} constants are independent, and hence the modes are in general doubly degenerate, i.e. to each value of the allowed mode frequency there corresponds two oscillations with different configurations of the electric and magnetic fields.

b) If one of the integers ℓ, m, n is zero, then there will be no degeneracy.

c) Now if the dimensions a,b,c are in ratio of integers then the order of degeneracy increases.

16.25 a) The wave equation for H_z, $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right] H_z = 0$, will be solved under the following boundary conditions on the six surface: $H_z = 0$ at $z = 0$ and $z = c$ (normal component vanishes), $\partial H_z/\partial y = 0$ at $y = 0$ and $y = b$, $\partial H_z/\partial x = 0$ at $x = 0$ and $x = a$. Since the solution will be sine and cosine functions, then we can immediately write $H_z(\vec{r},t) = H_{oz} \cos\frac{\ell\pi x}{a} \cos\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \; e^{-i\omega t}$ with $\omega^2/c^2 = (\ell\pi/a)^2 + (m\pi/b)^2 + (n\pi/c)^2$. The other fields can be determined using Eqs. 16.118 - 16.119.

b) The fields are given in Eqs. 16.130 - 16.140 with E_{oz} is taken zero. It is, however, customary to express the amplitudes of all components in terms of that of H_z. From Eq. 16.138 we have $(\ell\pi/a) E_{ox} = -(m\pi/b) E_{oy}$, and $-i[(\ell\pi/a) E_{oy} - (m\pi/b) E_{ox}]/\mu_o\omega = H_{oz}$; thus

$$E_{ox} = \frac{\mu_o\omega}{\pi} \frac{a^2 bm}{a^2m^2 + b^2\ell^2} H_{oz}, \quad E_{oy} = -\frac{\mu_o\omega}{\pi} \frac{ab^2\ell}{a^2m^2 + b^2\ell^2} H_{oz}$$

$$H_{ox} = -\left(\frac{b}{c}\right)\frac{ab\ell n}{a^2m^2 + b^2\ell^2}H_{oz},\quad H_{oy} = -\left(\frac{a}{c}\right)\frac{abmn}{a^2m^2 + b^2\ell^2}$$

c) Now n cannot be zero because H_z vanishes and hence all fields vanish. On the other hand if m and ℓ vanish, then all fields vanish too. Thus the lowest mode is when m or ℓ vanishes only. Now if $a < b < c$, then from

$\frac{\omega^2}{\pi^2c^2} = \frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}$, the lowest mode is for $\ell = 0$, $m = 1 = n$.

d) If $a = b = c$, $\Omega^2 = (a^2\omega^2/\pi^2c^2) = \ell^2 + m^2 + n^2$. Thus the spectrum is $\Omega^2 = 2, 3, 5$ and so on.

16.26 $\langle\vec{S}\rangle = \frac{1}{2}\,\mathrm{Re}(\vec{E}^* \times \vec{H})$. Inspecting Eqs. 16.130 - 16.140 which give the fields in the cavity, shows that $\vec{H}$ and $\vec{E}$ are $\pi/2$ out of phase. Thus $\langle\vec{S}\rangle = 0$.

CHAPTER 17

17.1 An observer moving with the muon finds a life time of $t' = 1.52 \times 10^{-6}$ sec. But he sees the distance to the earth shorter (due to space-contraction) by the $(1 - v^2/c^2)^{1/2}$ factor. Although the two observers, one on earth and one on the muon, disagree on the measurements of length and time intervals, they both agree if they measure the fraction of muons reaching the earth before they decay.

17.2 a) Because of space contraction, the observer on ruler 1, sees the length of the second ruler $\ell = \ell_o(1 - v^2/c^2)^{1/2}$. He needs to find the time it takes the ruler to move $\ell - \ell_o(1 - v^2/c^2)^{1/2}$. Thus we have

$$\Delta t = \frac{\ell_o - \ell_o(1 - v^2/c^2)^{1/2}}{v} = \frac{\ell_o}{v}\left(1 - (1 - v^2/c^2)^{1/2}\right) \text{ or}$$

$v = 2\ell_o\Delta t/(\Delta t^2 + \ell_o^2/c^2)$. The same is true for the other observer.

b) According to 1 the right-hand ends will meet first. According to 2 the left-hand ends will meet first. According to the third observer, the left and right ends will coincide simultaneously.

17.4 For two sucessive transformations, we write using Eqs. 17.2: $z' = (z - v_1 t)\gamma_1$, $t' = (t - v_1 z/c^2)\gamma_1$

$z'' = (z' - v_2 t)\gamma_2$, $t'' = (t' - v_2 z'/c^2)\gamma_2$ where $\gamma_1 = (1 - v_1^2/c^2)^{1/2}$, and $\gamma_2 = (1 - v_2^2/c^2)^{1/2}$. Eliminating the prime variables and getting a direct relationship between (z'', t'') and (z,t) we find $z'' = (z - vt)\gamma$, and $t'' = (t - vz/c^2)\gamma$ where $v = (v_1 + v_2)/(1 + v_1 v_2/c^2)$, and $\gamma = (1 - v^2/c^2)^{1/2}$.

17.5 $ds'^2 = dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2$. But $dx' = dx$, $dy' = dy$, $dz' = \gamma(dz - vdt)$, and $dt' = \gamma(dt - vdz/c^2)$. Thus $dz'^2 - c^2 dt'^2 = \gamma^2(dz^2 + v^2 dt^2 - 2v\, dz\, dt) - c^2\gamma^2(dt^2 + v^2 dz^2/c^4 - 2vdtdz/c^2) = dz^2(\gamma^2 - v^2/c^2) - c^2 dt^2(\gamma^2 - v^2/c^2) = dz^2 - c^2 dt^2$. Therefore $ds'^2 = ds^2$ or $ds' = ds$.

17.6 The number of wave crests in a wave is $N = (\vec{k}\cdot\vec{r} - \omega t)/2\pi$. Let us evaluate $N' = \vec{k}'\cdot\vec{r}' - \omega' t'$ using the transformation given in Eqs. 17.7 and 17.25 - 17.26, we can verify that $N = N'$. Of course these transformations for ω' and k' were derived from this condition in the first place.

17.7 Taking the inverse of Eq. 17.28, that is interchanging the prime ($\theta \to \theta'$, $\omega' \to \omega$, $\omega \to \omega'$), and taking $v \to -v$, we get: $\omega = \omega'(1 + \frac{v}{c}\cos\theta')/(1 - v^2/c^2)^{1/2}$ which gives the required result.

17.8 From Eq. 17.28, and using $\theta = \pi/2$, we get $\tan\theta' = -(1 - v_o^2/c^2)^{1/2}/(v_o/c)$ or $\cos\theta' = -v_o/c$.

17.9 We start with uniformly distributed stars, that is $N_o/4\pi$ is their density per solid angle, and N_o is their total number. $dN/d\Omega' = (N_o/4\pi)\cdot(d\Omega/d\Omega')$. But we substitute for $d\Omega/d\Omega'$ from Eq. 17.33, thus $\frac{dN}{d\Omega'} = \frac{N_o}{4\pi}\frac{(1 - v^2/c^2)}{[1 - (v/c)\cos\theta]^2}$.

17.10 From Ex. 17.4, if $\theta_1 = 0$, $\cos\theta_1 = 1$, then $\cos\theta_2 = -(1 - 2\beta + \beta^2)/(1 - 2\beta + \beta^2) = -1$. Thus $\theta_2 = 0$. Also the frequencies are related by Eq. 17.36:
$\omega_2/\omega_1 = (1 - \beta\cos\theta_1)/(1 - \beta\cos\theta_2) = (1 - \beta)/(1 + \beta)$. If it is approaching, then β is negative, and if it is receding, then β is positive, therefore $\omega_2/\omega_1 = (1 + |\beta|)/(1 + |\beta|)$ for approaching, and $\omega_2/\omega_1 = (1 - |\beta|)/(1 + |\beta|)$ for receding. If $\beta \to 1$, then we have $\omega_2 \to \infty$, and $\omega_2 \to 0$ respectively.

17.11 $\omega_1 = \omega_2$, and the angle of incidence is equal to the angle of reflection. This can be seen from the results of Ex. 17.4 when taking $v_z = 0$ or $\beta = 0$.

17.12 For simplicity let us consider a two-dimensional motion. Reference S_2 is moving along the z axis with velocity v, while S_1 is stationary with respect to the laboratory. Both origins coincide at time zero, and stay parallel. Thus we write

$z_2 = A_1 z_1 + A_2 t_1 + A_3 x_1$, $x_2 = D_1 x_1 + D_2 t_1 + D_3 z_1$, $t_2 = B_1 t_1 + B_2 z_2 + B_3 x_1$, where A_i and B_i are constants independent of the space and time coordinates. 1) Now let us consider the motion of the origin of S_1. As seen by S_1 the origin is always at rest where as S_2 sees it moving along the $-z$ axis at speed v. Thus $dz_2/dt_2 = -v = A_2/B_1$ or $A_2 - -vB_1$ and $D_2/B_1 = 0$ or $D_2 = 0$. 2) In a similar way, the motion of the origin of S_2, gives $\frac{dz_2}{dt_1} = \frac{dz_2}{dt_2} \times \frac{dt_2}{dt_1} = 0 = A_1 v + A_2$ or $A_2 = -vA_1$ and $D_3 v + D_2 = 0$ or $D_3 = 0$. Thus we have after these two conditions: $z_2 = A_1(z_1 - vt_1)$, $t_2 = B_2 z_1 + A_1 t_1$ and $x_2 = D_1 x_1$. 3) We now determine A_1 and B_2 from the constancy of the speed of light. If observer S_1 shines a beam of light along the z axis, then both he and observer S_2 will measure the same speed for the light that is c. Thus $dz_1/dt_1 = c = dz_2/dt_2$, and

$$\frac{dz_2}{dt_2} = A_1\left(\frac{dz_1}{dt_2} - v\frac{dt_1}{dt_2}\right) = A_1\left(\frac{dz_1}{dt_1} - v\right)\frac{dt_1}{dt_2} \quad \text{or}$$

$c = A_1(c - v)(dt_1/dt_2) = A_2(c - v)/(B_2 c + A_1)$. This solves to $B_2 = -(v/c^2)A_1$. Thus $z_2 = A_1(z - vt_1)$, and $t_2 = A_1(t_1 - (v/c^2)z_1)$. 4) It can be shown that $D_1 = 1$ by considering the measurement of the length of an identical rod when measured in both systems. A cross measurement however gives $L_2 = D_1 L_o$, and $L_o = D_1 L_1$. But $L_2/L_o = L_1/L_o$ because we cannot distinguish between the inertial system, then $D_1 = 1$.

5) Suppose S_1 projects a light beam along x_1 then S_1 will measure the speed as c while S_2 will measure a speed

$(\dot{z}_2^2 + \dot{x}_2^2)^{1/2} = c$, or $(v^2 + c^2/A_1^2)^{1/2} = c$ or

$A_1 = (1 - v^2/c^2)^{-1/2}$.

17.13 From Eq. 17.27 we have $\omega' = \omega_o\gamma\,(1 - \frac{v}{c}\cos\theta)$. For $\theta = \pi$, and $\theta = 0$ and since $\lambda'/\lambda_o = \omega_o/\omega'$ then $\lambda' = \sqrt{(1 - v/c)/(1 + v/c)}$ and $\lambda' = \lambda_o\sqrt{(1 + v/c)/(1 - v/c)}$, for approaching and receding observers respectively.

17.14 **a)** Since from Ex. 17.7 we have $d4x = -idx_1dx_2dx_3dx_4$ as a Lorentz invariant, and since $\rho dx_1dx_2dx_3$ is invariant, then ρ will transform like x_4, the time coordinate. Hence $J_\mu = (\vec{J}, ic\rho)$ is a legitimate four vector. The transformation can be easily seen from Eq. 17.2: $J_x' = J_x$, $J_y' = J_y$, $\rho' = (\rho - (v/c^2)J_z)\gamma$, and $J_z' = (J_z - v\rho)\gamma$.

b) Using the transformation for ρ, and taking $J_z = 0$, we get $\lambda = \lambda_o\gamma = \lambda_o(1 - v^2/c^2)^{1/2}$.

17.15 From Eq. 17.98 we have

$$E'^2 - c^2B'^2 = \gamma^2(E_1 - c\beta B_2)^2 + \gamma^2(E_2 + c\beta B_1)^2 + E_3^2 -$$
$$c^2\gamma^2(B_1 + \beta E_2/c)^2 - c^2\gamma^2(B_2 - \beta E_1/c)^2 - c^2B_3^2 =$$
$$\gamma^2E_1^2(1 - \beta^2) + \gamma^2E_2^2(1 - \beta^2) + E_3^2 - \gamma^2c^2B_1^2(1 - \beta^2) =$$
$$\gamma^2c^2B_2^2(1 - \beta^2) - c^2B_3^2 + 2\gamma c\beta E_1B_2 + 2\gamma^2c\beta B_1E_2 - 2c\gamma^2\beta B_1E_2 =$$
$$E^2 - c^2B^2.$$

17.16 We use the transformations of $\vec{E}$ and $\vec{B}$ and we use the condition $\vec{E}' \times \vec{B}' = 0$. Moreoever let us first find $\vec{v}$ which is normal to $\vec{E}$ and $\vec{B}$, thus $[\vec{E} + \vec{v} \times \vec{B}] \times (\vec{B} - \vec{v} \times \vec{E}/c^2] = 0$. Expanding we get: $\vec{E} \times \vec{B} - \vec{E} \times (\vec{v} \times \vec{E})/c^2 + (\vec{v} \times \vec{B}) \times \vec{B} - (\vec{v} \times \vec{B}) \times (\vec{v} \times \vec{E})/c^2 = 0$, or $\hat{v}|\vec{E} \times \vec{B}| - \vec{v}E^2/c^2 - \vec{v}B^2 - \hat{v}\, v^2|\vec{E} \times \vec{B}|/c^2 = 0$. Thus we can now solve the quadratic equation for $\vec{v}$. Noting that $|\vec{E} \times \vec{B}|^2 = E^2B^2 \sin^2\theta = E^2B^2(1 - \cos^2\theta) = E^2B^2 - |\vec{E}\cdot\vec{B}|^2$ where θ is the angle between $\vec{E}$ and $\vec{B}$, then after multiplying by c^2 we get Eq. 17.105. This is not the only velocity that does the job. In fact any other motion parallel to the common direction of $\vec{E}'$ and $\vec{B}'$ will keep then aligned.

b) Substituting for v in Eqs. 17.99 or 17.100 we get

$$2E'^2 = E^2 - c^2B^2 + \sqrt{(E^2 - c^2B^2) + 4c^2(\vec{E}\cdot\vec{B})^2}$$

$$2c^2B'^2 = c^2B^2 - E^2 + \sqrt{(E^2 - c^2B^2)^2 + 4c^2(\vec{E}\cdot\vec{B})^2} \,.$$

17.17 From the results of problem 17.17, we have two cases. If $E > cB$, then B' can be zero, thus

$$E^2 - c^2B^2 = \sqrt{(E^2 - c^2B^2)^2 + 4c^2(\vec{E}\cdot\vec{B})^2}, \text{ or } \vec{E}\cdot\vec{B} = 0,$$

$$2E'^2 = 2(E^2 - c^2B^2) \text{ or } E' = (E^2 - c^2B^2)^{1/2}.$$

If $E < cB$, then $E' = 0$ and $cB' = (c^2B^2 - E^2)^{1/2}$ and $\vec{E}\cdot\vec{B} = 0$.

The velocities of the frames in these two cases respectively are: $\vec{v}/c = c\hat{n}B^2/|\vec{E} \times \vec{B}| = c(\vec{E} \times \vec{B})/E^2$ and $\vec{v}/c = (\vec{E} \times \vec{B})/cB^2$.

17.18 Consider the results of Ex. 17.14. **a)** Along the line of motion ($\theta = 0$ and π), the $\vec{E}$ field is $\vec{E} = q\,\vec{R}/4\pi\varepsilon_o R^3\gamma^2$. Thus it is smaller than Coulomb field $\vec{E} = q\,\vec{R}/4\pi\varepsilon_o R^3$ by the factor $(1 - v^2/c^2)$. On the other hand in the perpendicular direction to the motion ($\theta = \pi/2$), the $\vec{E}$ field is $\vec{E} = q\,\vec{R}\gamma/4\pi\varepsilon_o R^3$ which is larger than the Coulomb field by the factor $(1 - v^2/c^2)^{-1/2}$. For $v \sim c$, the field is only larger within an angular range $\Delta\theta \simeq (1 - v^2/c^2)^{1/2}$.

b) The condition $\vec{E}'_{\parallel} = \vec{E}_{\parallel}$ points to the same conclusions in the four space. In the rest frame of the charge, a point appears further from the charge by a factor $(1 - v^2/c^2)^{-1/2}$ than in the laboratory frame which accounts for the effect.

17.19 **a)** $\Phi = q/4\pi\varepsilon_o R^* = q\gamma/4\pi\varepsilon_o r'$ where $r' = [\gamma^2(z - vt)^2 + \rho^2]^{1/2}$. Thus $-\nabla\Phi = (q/4\pi\varepsilon_o\gamma^2 R^{*3})[\gamma^2(z - vt)\,\hat{z} + \rho\hat{\rho}]$.

$$-\frac{\partial\vec{A}}{\partial t} = -(\mu_o vq/4\pi R^{*3})\, v(z - vt)\hat{z} = -(v^2q/4\pi\varepsilon_o c^2R^{*3})(z - vt)\hat{z}$$

$= [-(q/4\pi\varepsilon_o\gamma^2R^{*3}) + (q/4\pi\varepsilon_o R^{*3})](z - vt)\hat{z}$. Thus one can see that the lines of $-\nabla\Phi$, and the lines of $-\partial\vec{A}/\partial t$ are not separately straight lines emanating from the instantaneous location of the charge. However when both are added to get the total field, then the second piece of $-\partial\vec{A}/\partial t$ will cancel the z component of $-\nabla\Phi$ resulting in an overall field which has straight lines of force with center at the instantaneous position of the charge.

17.20 The potentials of a moving dipole were calculated in Ex. 17.15. The fields can be calculated from them. Here we calculate the fields directly. In the rest frame of the dipole,

we have $\vec{E}' = \dfrac{1}{4\pi\varepsilon_o r^3}[(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}]$, and $\vec{B}' = 0$ or

$$\vec{E}' = \frac{p_o}{4\pi\varepsilon_o r^3}[\cos\theta\sin\theta\hat{\rho} - \sin^2\theta\ \hat{z}] = \vec{E}_\perp + \vec{E}_\parallel \text{ where}$$

$\vec{E}_\perp = p_o\hat{\rho}\cos\theta\sin\theta/4\pi\varepsilon_o r^3$ and $\vec{E}_\parallel = -\ p_o\hat{z}\sin^2\theta/4\pi\varepsilon_o r^3$.

From Eq. 17.99, we then conclude that $\vec{E}_\parallel' = -\ p_o\hat{z}\sin^2\theta/4\pi\varepsilon_o r^3$,

$\vec{E}_\perp' = \gamma p_o\hat{\rho}\sin 2\theta/8\pi\varepsilon_o r^3$, $\vec{B}_\parallel' = 0$, and $\vec{B}_\perp' = -\ \gamma\vec{v}\times\vec{E}_\perp/c^2$.

17.21 From the definition $\vec{p} = d\vec{P}/dv$, and $\vec{m} = d\vec{M}/dv$ and from the Lorentz transformation of $\vec{P}$, $\vec{M}$, and the volume element which are given in Eqs. 17.110 and from Eq. 17.75 we get:

$$\vec{p}' = \vec{p} - \frac{v}{c^2}\times\vec{m} - (\gamma - 1)\ \vec{v}\ \frac{\vec{v}\cdot\vec{p}'}{\gamma v^2}$$

$$\vec{m}' = \vec{m} + v\times\vec{p}' - (\gamma - 1)\ \vec{v}\ \frac{\vec{v}\cdot\vec{m}'}{\gamma v^2} \quad \text{(See Eqs. 17.100).}$$

17.22 **a)** From Eq. 17.99, we have $E'_{\perp} = \gamma[\vec{v} \times B_{\perp}]$. Since $\mathcal{E} = \int\vec{E}\cdot d\vec{\ell}$, then $\mathcal{E} = \gamma(\vec{v} \times \vec{B})\cdot\vec{\ell} = \gamma vB\ell$.

b) Take $\vec{B}$ to be along x, and v along z, then in the laboratory frame the vector potential associated with the uniform B field is $\vec{A} = -\vec{r} \times \vec{B}/2 = (Bz\hat{y} - By\hat{z})/2$, and the scalar potential is zero. In a frame moving with the conductor, we get from the Lorentz transformation $f(\vec{A}, i\Phi/c)$. $\Phi' = \gamma vA_z$. Thus $\vec{E} = -\nabla\Phi' = -\gamma v\nabla A_z = \gamma vB\hat{y}$ and $\mathcal{E} = \gamma vB\ell$.

17.23 **a)** To calculate the charge densities we use the Lorentz transformation of the four vector $(\vec{J}, ic\rho)$. We use the filmanetary approximation where Jdv is replaced by $Id\ell$, and ρdv is replaced by $\lambda d\ell$. In S', $\lambda' = 0$, thus in S, and in sides 1 and 3 we have $\lambda = \pm (v/c^2)I'\gamma$. The total charge is therefore $\lambda\, a(1 - v^2/c^2)^{1/2}$ which takes into consideration the Lorentz contraction. Thus $q_1 = -q_3 = -v\, a\, I'/c^2$. The charge on the other two sides is zero.

b) The electric dipole moment is $q_1 b = I'vab/c^2$. The same result may be alternatively determined from the transformation of the polarization and magnetization given in Eq. 17.110.

c) From Eq. 17.110, we have $M = \gamma M'$. But $m' = abM' = I'ab$, and $m = \gamma I'ab/\gamma$ (Lorentz contraction is accounted for), then $m = m'$. The same results can also be arrived at using the transformation of $\vec{p}$ and $\vec{m}$ as given in problem 17.21.