INSTRUCTOR'S MANUAL TO ACCOMPANY

stein: calculus in the first three dimensions

McGraw-Hill Book Company

INSTRUCTOR'S MANUAL TO ACCOMPANY

calculus in the first three dimensions

Sherman K. Stein

Professor of Mathematics
University of California at Davis

McGraw-Hill Book Company

New York • St. Louis • San Francisco • Toronto • London • Sydney

Instructor's Manual to accompany

CALCULUS IN THE FIRST THREE DIMENSIONS

61004

1234567890WH72106987

CONTENTS

I. RELATION TO CUPM RECOMMENDATIONS

In a 76-page report, A General Curriculum in Mathematics for Colleges, published in 1965, CUPM prepared for consideration by the mathematical community a "minimal" undergraduate curriculum in mathematics, consisting of fourteen semester courses, that could be offered by even a small department. Five of these are lower-division courses: Math 1, Introductory Calculus; Math 2P, Probability; Math 2 and 4, Mathematical Analysis; Math 3, Linear Algebra.

The hypotheses from which CUPM drew its conclusions are important and simple. Let us recall a few of them:

> "Every college, large or small, needs a basic mathematics program simple enough to staff and operate, yet substantial and flexible enough to accommodate to today's diversity of students and their objectives." (Page 3.)
>
> ". . . there are now many more kinds of mathematical knowledge which [the incoming students] seek. . . . This is brought about by the computer, the increasing mathematization of the biological, management, and social sciences. . . . Then there is the explosive increase in the number of students who wish to major in mathematics." (Page 4.)
>
> ". . . one cannot assume that the student entering at eighteen knows what his professional interests will be. . ." (Page 5.)
>
> ". . . the five basic courses should not be taught in different styles for students with differing major fields and professional objectives." (Page 5.)

CUPM then makes its recommendations, which begin with Math 0, Elementary Functions and Coordinate Geometry, precalculus mathematics which should be, and frequently is, taught in high school (page 11). Math 1, 2, and 4 form a calculus sequence of some 90-120 class meetings. Two versions of Math 2, 4 are suggested, the preferred version introducing multivariable calculus in Math 2.

> "There are several reasons for believing that such an arrangement may be better than the conventional one in which single variable calculus is thoroughly exhausted before turning to multivariable calculus as a more advanced subject. . . the multivariable calculus need not be as difficult as these ramifications of single variable calculus, techniques of integration and geometric applications, and offers more new ideas to contribute to the first year calculus. . . .
>
> "Furthermore, the proposed arrangement is attractive for better service to engineering and physics for it sets up early many things that engineers and physicists want to do; for example, it permits an early introduction of the moments of solid figures. It is also more useful for students in other areas, such as social science and business." (Pages 11-12.)

Math 3, Linear Algebra (30 class meetings), is placed in the first two years, not so much as a tool to be used in the calculus, but to broaden the lower-division experience on the basis of which a student may decide to major in mathematics, and to serve physics, engineering, and the social sciences.

Math 2P, Probability (30 class meetings), uses the calculus and may follow Math 1.

Before we examine Math 1, 2, 4 in detail, let us recall a few remarks on rigor made in the report:

> "... it is the level of rigor in the student's understanding which counts and not only the rigor of the text or lecture presented to him. ... The problem, as we see it, is to devise a presentation which will convey the ideas of the calculus in forms which are intuitively valid, can later be made exact, and are made rigorous as the student advances." (Page 22.)
>
> "... limits, and all other concepts of the calculus, should be taught as concepts in some form at every stage. For example, the fundamental theorem of the integral calculus involves two concepts: the "limit" of a sum and the antiderivative. ... It is dishonest to conceal the connection between the two concepts by conditioning the student to accept the formalism without his being aware that the concepts are there." (Pages 22-23.)
>
> "... a student who stops with Mathematics 2 will at least know something useful, have enough technique to use it, and will have a grasp of the basic concepts of the calculus in a form precise as he can appreciate and precise enough so that it can be made rigorous." (Page 24.)

CUPM gives two sample outlines for Math 1 and two for Math 2, 4, preceded by this advice:

> "The intent is not prescriptive. For this reason we give more than one outline for some of the courses. ... Nothing in these outlines is intended to discourage original course design or further experimentation." (Page 29.)

Though I had the CUPM recommendations while writing the text, I did not try to follow one of their sample outlines but rather to implement their objectives—which were also mine—with an organization and approach based on my own experience in teaching the calculus (off and on) for twenty years. For instance, in addition to the spiral approach which is suggested in the report, I have used a "stereo-approach," which introduces major concepts by at least two different interpretations or applications.

On pages 31-34, the report presents two sample outlines of Math 1. The core of both outlines is the derivative, the definite integral, and the fundamental theorem of calculus. *Calculus in the First Three Dimensions* covers these in Chapters 1 to 6.

On pages 37-41, the report presents two sample outlines of Math 2, 4. Both give about twenty-four out of seventy-eight lectures to differential equations. Almost all the remaining material is included in the text. Though no chapter in the text is given completely to differential equations, it does solve several of them. In Chapter 5 we solve the equation $dy/dx = ky$; we apply it in Chapter 21 ("Growth in the Natural World") and Chapter 25 ("Rockets"). In Chapter 13 we solve the equation $d^2y/dx^2 = -ky$, using power series; we apply it in Chapter 26 (Gravity). In Chapter 19 the differential equation of a hanging cable is discussed, and in Chapter 24 (Traffic) the differential equations of a continuous Poisson distribution are solved.

The text, in its elementary and intuitive treatment of the algebra of vectors (including the dot product) in Chapter 16, prepares the student for CUPM's Math 3 (Linear Algebra). In the treatment of curve integrals (Chapter 18) and of Green's theorem in the plane (Chapter 19) it prepares the student for CUPM's Math 5, *Advanced Multivariable Calculus.* In the section on probability in Chapter 8 and in Chapter 24, "Traffic," the student is introduced to CUPM's Math 2P, Probability. In Chapter 20, The Interchange of Limits, the student is oriented toward CUPM's Math 11, 12, Real Variable Theory. In Chapters 21-26, which comprise Part III, the applications lead into CUPM's Math 10, Applied Mathematics.

II. STRUCTURE AND USE OF
CALCULUS IN THE FIRST THREE DIMENSIONS

The text consists of three parts, each of a distinctive character.

Part I. The Core of the Calculus (Chapters 1-9) focuses on the fundamental theorem of calculus and its applications. This part, which introduces the definite integral in dimensions 1,2, and 3 in Chapter 1 and the derivative in Chapter 2, reaches the fundamental theorem of calculus in Chapter 6, gives more applications of it in Chapter 8, and evaluates 2- and 3-dimensional integrals in Chapter 9.

Part II. Topics in the Calculus (Chapters 10-20) treats higher derivatives, maxima, series, Taylor's series, partial derivatives, estimating a definite integral, the algebra and calculus of vectors, curve integrals and Green's theorem, and the interchange of limits. As will be shown below in the discussion of the individual chapters, many of these topics may be presented before Part I is completed. For instance, the second derivative and maxima may follow Chapter 5.

Part III. Further Applications of the Calculus (Chapters 21-26) consists of six independent applications of the material developed earlier (mainly in Part I).

Now we will examine each chapter, keeping in mind such questions as: What is the relation of this chapter to the rest of the book? If there is a rush, what can be left out? Why is the calculus presented this way?

The number of lectures suggested depends on how much is covered, and how thoroughly. As a rule of thumb, one section corresponds to one lecture.

Part I: The Core of the Calculus

Chapter 1, The Definite Integral (3-5 lessons)

Sections 1, 2, and 3 gently lead up to the definite integral over an interval. Four major illustrations are used so that the student will not identify it with any particular one, such as area. Throughout the text we reserve the words *definite integral* for the limit of sums, and *antiderivative* for a function whose derviative is the integrand. The word *mesh* corresponds to the mesh of a screen that permits pebbles up to a certain size to pass through.

The limit concept is not made precise until Chapter 3, after the student has accumulated down-to-earth experience with the notion in Chapters 1 and 2. However, nothing has to be redone later. Chapters 1 and 2 are not a prologue; they are, so to speak, Act I.

It is possible to go directly from Section 3 of Chapter 1 to Chapter 2, "The Derivative." However, Section 4 ("Average of a Function over an Interval"), Section 5 ("The Definite Integral over a Set in the Plane"), and Section 6 ("The Definite Integral over a Set in Space"), do provide additional perspective. Perhaps the first time the instructor uses the text, he may wish to include say, only the first three sections, or, perhaps, in addition, one of Sections 4 and 5. (The first time I use a text, I usually find it hard to predict the difficulties and the time required for covering the various parts; the second time through, I feel more at home.)

Note that summation notation and functions appear in appendices.

Chapter 2, The Derivative (2-4 lessons)

Again we start with four applications. Slope, while certainly important, is not the application that most students will use. The "magnification" interpretation (what CUPM calls "scale-factor" of a mapping) is used in Chapter 4 to make the chain rule plausible and in Chapter 19 to lead up to the Jacobian in the plane. Section 4 may be omitted or may be suggested as reading.

Chapter 3, Limits and Continuous Functions (3-4 lessons)

This chapter provides mathematical definitions of concepts that were developed intuitively in Chapters 1 and 2 and sets the stage for Chapter 4, where we compute the derivatives of the elementary functions. If more rigor is desired, Appendix F may be included or referred to. My feeling is that there isn't enough time, nor is the audience sufficiently homogeneous, to do much with ϵ, δ in a lower-division course. I have preferred to use many pathological counterexamples instead, as a warning to the student.

In Section 1, be sure to include Example 6, $\lim_{n \to \infty} (1 + 1/n)^n$, since it will be referred to in Section 2 and needed in Chapter 4. There is no need to cover all six examples. Note that Exercise 28 outlines a proof that $\lim_{n \to \infty} (1 + 1/n)^n$, exists, and it may be included as a lecture. I prefer to forgo the proof and, instead, give greater attention to Chapter 6, "The Fundamental Theorem of Calculus."

In Section 2, Examples 1 and 3 should be emphasized, for they will be used in Chapter 4. Section 3 contains statements of the intermediate- and maximum-value theorems, which will be needed in Chapter 6. The theorem proved in Section 4 (namely, that if $\int_a^b f(x)\,dx$ exists, then f is bounded) is referred to later on two occasions: in Chapter 6, where we show that if $F(x) = \frac{1}{2} x^2 \sin(1/x^2)$ then $\int_0^1 F'(x)\,dx$ does not exist, and in Chapter 8 in the discussion of improper integrals. The theorem need not be mentioned until Chapter 6. If time is a factor, its proof may be omitted.

Chapter 4, The Computation of Derivatives (4-5 lessons)

We obtain the derivatives of all the elementary functions at once for several reasons: (1) the differentiation of polynomials conceals the limit concept because of the easy cancellations, (2) the transcendental functions are more important in applications, (3) to give maximum practice, (4) to have all the derivatives available.

The differentiation of $\log_a x$ is done in the classical manner not only to obtain the formula early, but also because the approach through definite integrals is too sophisticated for most students. The second approach is included in Chapter 6, and the instructor who intends to include it can simply state now that $D(\log_e x) = 1/x$.

The more skill the student has in computing derivatives, the easier will he find subsequent chapters, especially Chapter 7. Rather than dwell too long on Chapter 4, however, I would suggest that while Chapters 5 and 6 are being studied, a few problems be assigned from Chapter 4.

Chapter 5, The Law of the Mean (1-2 lessons)

The law of the mean runs as a subtheme through the rest of the book. In Chapter 6 it is used to prove that if $f = F'$ is an increasing function, then $\int_a^b f(x)\,dx$ exists and equals $F(b) - F(a)$. In Chapter 8 it is used, for instance, to obtain the formula for arc length. In Chapter 20 it is used to prove that $f_{xy} = f_{yx}$, to justify differentiation under the integral sign, and (in a generalized form) to obtain L'hôpital's rule. Once the differential equation $dy/dx = ky$ is solved, some of its applications from Chapter 21 can be included. After Chapter 5 one is ready for Chapter 10 (Higher Derivatives) and Chapter 11 (The Maximum and Minimum of a Function). Of course, such a detour would break the continuity of Part I. In Part I, all we want from the law of the mean is: two functions that have the same derivative differ by a constant and Corollary 5, which relates certain approximating sums to $F(b) - F(a)$. For this purpose the swiftest way to cover this chapter would be to state the law of the mean (which students would readily accept), then obtain Corollary 5. This would mean that only the special case of the fundamental theorem of calculus in Chapter 6 could be proved.

Chapter 6, The Fundamental Theorem of Calculus (2-4 lessons)

Considering only theory—not applications—Part I could end with this chapter, which relates the derivative and the definite integral. Rather than spend time on lower and upper sums, or step functions, or trying to prove that a continuous function has a definite integral, we prefer to give a complete proof of a special case ($f = F'$, for f increasing) and the traditional argument which assumes that a continuous function has a definite integral and deduces that $d[\int_a^x f(t)\,dt]/dx = f(x)$.

Moreover, it doesn't take much time to remark that there are functions that have a definite integral but no antiderivative (Exercise 15 in Section 1); functions whose derivative does not have a definite integral (Example 1 in Section 1); functions, such as e^{x^2}, whose antiderivatives are not elementary.

Theorem 1 (concerning continuous functions), Theorem 2 (the special case), and Theorem 3, which is implicit in Theorem 1 and asserts that the derivative of the definite integral $\int_a^x f(t)\,dt$ with respect to x is $f(x)$, we refer to as the fundamental theorem of calculus: all relate the derivative to the definite integral.

Section 4, "The Integral Approach to the Logarithm," may be omitted without affecting subsequent chapters, as long as the student knows that $D(\ln x) = 1/x$, a result obtained less rigorously in Chapter 4. Indeed, I would omit it if the time saved could be devoted to a more general topic in Part II or to an application in Part III.

Chapter 7, Computing Antiderivatives (2-5 lessons)

The techniques of substitution and integration by parts (Sections 1 and 2) and a table of integrals are adequate for practically all of the later development in the text. In this chapter the student should learn how to use integral tables and become familiar with their limitations.

In Section 3 ("Partial Fractions") the student is studying algebra, not calculus. The cookbook use of partial fractions is given in the section, but the student may be referred to Appendix G for a discussion of the algebra. A second lesson on this topic, including review exercises for Sections 1 and 2 may be appreciated.

Section 4 consists of special techniques.

Chapter 8, Computing and Applying Definite Integrals over Intervals (4-11 lessons)

Here the student develops skill in recognizing when quantities can be expressed as definite integrals and in applying the fundamental theorem. There is no need to cover all nine sections. The main innovations are the emphasis on the first moment (and higher moments) and the introduction of probability distribution and density (which is used in Chapter 24, "Traffic").

For later applications the most important sections are: volume, average value of a function, and improper integrals. In particular, they make Chapters 21-25 accessible. Section 1 ("Area") includes a discussion of substitution in a definite integral. The instructor may choose to cover some of the sections and finish the remaining ones after a break of a few chapters.

Chapter 9, Computing and Applying Definite Integrals over Plane and Solid Sets (2-6 lessons)

If time is short, this chapter may be cut or omitted. Section 2, integration via rectangular coordinates in the plane, is used in Chapter 19 ("Green's Theorem"). Integration via spherical coordinates is used in Chapter 26 ("Gravity") for computing the attraction of a sphere.

I suggest that, before beginning this chapter, the instructor review the definition of $\int_R f(P)\ dA$ from Chapter 1. This will also remind the student that we are now solving a problem posed in Chapter 1 and will help him to distinguish $\int_R f(P)\ dA$ from the repeated integral.

In Section 1 we prove that all balancing lines pass through a single point, which we then call the center of gravity. The argument is quite simple and shows the importance of plane definite integrals. Both physics and mathematics texts slide over this technicality (like two outfielders, each assuming the other will make the catch), but it merits attention and provides a meaningful introduction to Chapter 9—before the consideration of repeated integrals. In Section 1 we obtain the formulas for $(\overline{x}, \overline{y})$. If time is short, the instructor may simply state the formulas for use in the remaining sections. (They were stated in the case of constant density in Chapter 8.)

A minimal treatment of Chapter 9 consists of Section 2, "Computing Plane Integrals via Rectangular Coordinates," for this does convey the notion of a repeated integral.

Observe that the notation $\int_R f(P)\ dA$ is used for the definite integral, here as in Chapter 1, *not* the notations $\int\int_R f(P)\ dA$ or $\int\int f(x, y)\ dx\, dy$. Repeated integrals are written for emphasis with the insertion of parentheses, e.g., $\int_a^b (\int_c^d f(x,y)\ dy)\ dx$.

Part II: Topics in the Calculus

Chapter 10, The Higher Derivatives (2-4 lessons)

Here we meet the second derivative, a topic which may be introduced after Chapter 5. The first three sections, devoted to the second derivative, are independent. Section 4, which discusses the higher derivatives, sets the stage for Taylor's series by developing the formula $a_j = f_j(a)/j!$ for polynominals.

Section 1 discusses the relation between y" and concavity and maxima. However, it is not needed in Chapter 11, where maxima are dealt with in terms of f'. Section 2 uses y" to study constantly accelerated linear motion. Though it ties in nicely with Chapter 26 ("Gravity"), it is not needed in any later devlopments. Section 3 ("Curvature") can be omitted. The concept of curvature appears again in Chapter 17, where we obtain the normal component of acceleration. In Chapter 17 all that we need of curvature is that it is defined as $d\phi/ds$.

Chapter 11, The Maximum and Minimum of a Function (2-4 lessons)

It may seem surprising to find this subject, historically and traditionally placed at the center of the calculus, delayed until Chapter 11. I had several reasons for not putting it in Part I: (1) I wanted to treat f(x), f(x,y), and linear programming together; (2) its insertion in Part I would disrupt the continuity of the first nine chapters; (3) the student should have lots of practice with derivatives before applying them to maxima problems.

As a matter of fact, the instructor who wishes to consider Section 1 ("Maximum of f(x)") may do so as early as immediately after Chapter 5 ("Law of the Mean"), where it is shown that at an interior maximum a differentiable function has derivative equal to 0. Section 2, which introduces partial derivatives and applies them to maximizing f(x, y), can also be introduced then.

While Chapters 15-20 make use of partial derivatives, most of the applications in Part III do not.

Section 3, linear programming, is included as a reminder that in many practical cases extrema occur on the boundaries. It may be omitted.

Chapter 12, Series (3-4 lessons)

The student has met sequences in Section 1 of Chapter 3. It may be wise to assign homework back in that section just before starting this chapter—as a reminder—and to discuss some of the six examples in that section that may not have been treated before.

Example 3 in Section 2 shows that $\lim_{n\to\infty} x^n/n! = 0$, a result referred to in Chapter 13 ("Taylor's Series"). Section 3, which examines the error $E_n = S - S_n$, is not needed for later work. Section 4 ("Power Series"), shows that $\tan^{-1} x = x - x^3/3 + x^5/5 - \cdots$ and sets the stage for Taylor's series (but is not used in the development of Taylor's series).

Chapter 13, Taylor's Series (1-3 lessons)

We begin with the integral formula for R_n because of the (relative) simplicity of its proof and its connection with the fundamental theorem of calculus, which it generalizes. The derivative form of R_n generalizes the law of the mean.

Chapter 14, Estimating the Definite Integral (1-2 lessons)

This "practical" chapter ties together several ideas: the definite integral as a limit of sums, the limitations of the fundamental theorem as a computational tool, higher derivatives, and Taylor's series. It is not referred to later.

Chapters 15 through 19, going from partial derivatives and vectors to Green's theorem in the plane, form a unit which the mathematician or physical scientist should meet early in his career. The social scientist would benefit from Chapter 15 ("Partial Derivatives") and Chapter 16 ("The Algebra of Vectors"), the latter an introduction to linear algebra (CUPM's Math 3). Chapter 15-17 are needed in Chapter 26 ("Gravity"). The natural or social scientist may benefit from replacing Chapter 18 ("Curve Integrals") and Chapter 19 ("Green's Theorem in the Plane") with some chapters from Part III. On the other hand, a first-hand knowledge of Chapter 26 may be of value to anyone who will be using mathematics, for it is an example that may develop high standards and broad perspective in the applications of mathematics. Besides, neither we nor the students should stereotype them when they are, as CUPM succinctly puts it, eighteen years old.

Chapter 15, Further Applications of Partial Derivatives (3 lessons)

Note that the proof of the equality $f_{xy} = f_{yx}$ is delayed until Chapter 20. The proof is, however, self-contained and could be presented with Chapter 15, especially if Chapter 20 will not be covered.

Chapter 16, Algebraic Operations on Vectors (4 lessons)

For several reasons I feel that an introductory calculus course is not the place to introduce vectors as "equivalence classes of ordered pairs of points." (What is an ordered pair? What is an equivalence relation? Is addition of vectors well defined?) Rather I have chosen to include more physical motivation than is customary. In particular, fluid flow is referred to again in Chapters 18 and 19 to provide a plausibility argument for Green's theorem before its proof.

Chapter 17, The derivative of a vector function (3 lessons)

The development of A_N and A_T is included mainly to illustrate the power of vector techniques. (A_N and A_T are not used later in the book.) The components A_r and A_θ are obtained in Chapter 26, during the study of planetary orbits. It is possible to omit A_N and A_T and go directly to Chapter 26 for A_r A_θ .

Chapter 18, Curve Integrals (2 lessons)

It should be emphasized that a curve integral is a definite integral. This chapter and Chapter 19 generalize the fundamental theorem to the plane and provide a background for those who will go on to the higher-dimensional versions of Green's theorem.

Chapter 19, Green's Theorem in the Plane (3-4 lessons)

Here we use Green's theorem to develop the Jacobian in two dimensions. We do not use matrices or determinants. The material in this chapter (the

Jacobian and an example of a linear transformation) will provide some background for the subsequent study of linear algebra.

Chapter 20, The Interchange of Limits (1-3 lessons)

More than one analyst has felt that the interchange of limits is a major part of analysis. In this chapter, which is kept as simple as possible by "stacking the hypotheses" so that the law of the mean can be made the basis of the proofs, we provide a prelude to advanced calculus. Note that the proof of the generalized law of the mean is almost identical with that of the law of the mean. (No "skyhooks" are used to introduce a function from out of nowhere.)

Part III: Further Applications of the Calculus

These six independent chapters introduce no new mathematical ideas; hence, the instructor is free to cover them as he wishes. Perhaps some chapters may be assigned as term papers (the student solving a certain number of exercises in a chapter). Since the time to be devoted to a chapter is quite arbitrary, no suggested numbers of lessons will be offered.

Chapter 21, Growth in the Natural World

This is what most biologists want from the calculus. It could be covered after Chapter 5.

Chapter 22, Business Management and Economics

This chapter applies maximum-minimum arguments, improper integrals, and partial derivatives. Even the mathematics major may be interested to know that the Laplace transform of the revenue function is the present-value function.

Chapter 23, Psychology

This chapter provides illustrations of the use of the derivative and the definite integral.

Chapter 24, Traffic

This chapter calls attention to a relatively new field of engineering. With a million acres (1600 square miles) of road built every year in the U. S. A. and the population increasing at the rate of two million per year, this field will become increasingly prominent. The section on the Poisson distribution has applications to many fields, as is shown in the exercises. The section on cross-traffic provides a nice application of the infinite series $\sum_{n=1}^{\infty} np^n$. The student who hasn't the time to take a probability course (such as CUPM's Math 2P) would find here the basic concepts of continuous distributions.

Chapter 25, Rockets

This chapter should be of interest to many citizens of the twentieth century. It develops the basic equation of rocket propulsion in free space, mass-ratio, escape velocity, and drag. It uses no calculus beyond Chapter 8 and develops the little physics that is needed.

Chapter 26, Gravity

The historical introduction describes what Newton did and what he did not do, a subject over which a haze has settled; this approach is used to make the physical problem as vivid as possible, for the mathematics that follows is rather involved. The chapter depends on Chapters 8, 9, and 17 and is a fitting conclusion to a calculus course.

III. OUTLINES OF VARIOUS COURSES BASED ON THE TEXT

Since many instructors are called on to teach the calculus more frequently than any other course, it seems to me that they should, for their own pleasure (and hence their students') be able to vary the course from year to year without disruptively changing texts every couple of years. For this reason each chapter, section, and example is written to offer the maximum choice of order, of omission or inclusion, and of degree of thoroughness or coverage. The appendixes increase this flexibility. The instructor who would like to include more analytic geometry, a discussion of the real numbers, or more material on the algebra of partial fractions may choose to include one or more of the appendixes.

Even if some chapters in Parts II or III or the appendixes are covered before Part I is complete, it is important that the student realize that Part I is a unit whose subtitle might be "The fundamental theorem of calculus and some of its applications."

Part I takes from twenty-four to forty-six lectures, depending on how much of each chapter is covered and how intensively. Teaching and learning this material is simplified by its unity: relatively few symbols, terms, and definitions are introduced. Moreover, the student knows *where* he is and *how* the chapters fit together. This may perhaps be called Math 1.

Part II takes from twenty-seven to thirty-nine lectures and may be called Math 2. Both Math 1 and Math 2 may incorporate some chapters from Part III. I think of the text as completing the basis of the calculus (CUPM's Math 1, 2, 4) in one year instead of CUPM's year and a half. This would include about four lectures on differential equations instead of CUPM's suggested twenty-four. CUPM seeks flexibility and smaller texts; I feel that any text that tries to cover more than two semesters is too big. Moreover, the attempt to shove such topics as analytic geometry, linear algebra, or differential equations into the calculus course frequently results in their inadequate treatment.

Various courses may easily be extracted from the text, as the following outlines suggest. The first course is included for the convenient description of the other courses following it; I am not advocating its inclusion in the curriculum.

A. <u>Minimum course</u>, 18 lectures: the derivative, the definite integral, and the fundamental theorem of calculus

The first six chapters contain the essence of the calculus, as expressed in the fundamental theorem. It is, therefore, a minimal self-contianed extract. In eighteen lectures one could cover Chapter 1 (omitting 2- and 3- dimensional definite integrals), Chapter 2 (omitting set derivatives), most of Chapters 3 and 4, the law of the mean and Corollary 5 from Chapter 5, and only the special case of the fundamental theorem from Chapter 6.

B. <u>Maximum course</u>, Parts I and II in 85 lectures plus Part III

C. <u>A course aiming at the Poisson distribution in Chapter 24, "Traffic,"</u> 25 lectures

The minimal course described in A, plus improper integrals, and probability from Chapter 8, plus Sections 1 and 2 of Chapter 24.

D. <u>A course omitting vectors</u>, 72 lectures plus Part III

This would differ from the maximum course in the omission of Chapters 16, 17, 18, 19, and 26.

E. <u>A course omitting plane and solid definite integrals</u>, 74 lectures plus Part III

This would eliminate part of Chapter 1, all of Chapters 9 and 19, and the attraction of a sphere in Chapter 26, and would suggest the omission of Chapter 18 ("Curve Integrals").

F. <u>A course emphasizing the applications in Part III excluding Chapter 26, "Gravity,"</u> 25 lectures plus the first five chapters of Part III

The minimal course, Sections 1 and 2 of Chapter 7, Sections 8 and 9 of Chapter 8, Section 1 from Chapter 11, the chain rule from Chapter 15, $\sum_{n=1}^{\infty} np^n$ from Chapter 12, plus Chapters 21-25 from Part III.

Many rearrangements are possible. For instance Chapter 10, "Higher Derivatives" (other than Section 3, "Curvature"), and Chapter 11, "Maximum and Minimum of a Function," can be placed right after Chapter 5. The chart below shows the interdependence of the chapters:

Chapter	may follow Chapter
10 Higher Derivatives	4 (the derivative)
11 Maxima	10 (or 5, if the second derivative is not used)
12 Series	7 and improper integrals from 8
13 Taylor's Series	10 (higher derivatives only), 12
14 Estimating the Definite Integral	7 (except for the part of Chapter 14 that uses Chapter 13, "Taylor's Series")
15 Partial Derivatives (Section 1);	13
remainder of chapter	5
16 Algebra of Vectors (Sections 1, 2);	any
remainder of chapter	15
17 Differentiation of Vectors	16
18 Curve Integrals	17
19 Green's Theorem	18
20 Interchange of Limits	15
<u>Part III</u>	
21 Natural Growth	5
22 Business and Economics	11, improper integrals from 8, chain rule from 15
23 Psychology	6
24 Traffic	8 (improper integrals and probability distribution and density only), 12 ($\sum_{n=1}^{\infty} np^n$ only)

Chapter	may follow Chapter
25 Rockets	5, 8 (improper integrals only)
26 Gravity	9 (integration by spherical coordinates only), 17 [D(f(t) **F**(t)) only]

V. SOLUTIONS TO EXERCISES

If time is a limiting factor, assign exercises from the second group sparingly--perhaps not all. However, they may serve as examples in lectures, as review problems, or as voluntary problems (solutions posted, so as not to use up class time). Solutions to many problems are given in detail in order that the instructor may decide with a minimum expenditure of time which exercises will best serve his purposes.

CHAPTER 1: The Definite Integral

SECTION 2: Precise Answers to the Four Problems

1. (a) $x_2 = 1$, $x_3 = 3/2$, $x_4 = 2$, $x_5 = 5/2$, $x_6 = 3$. (b) 6.
   (c) $X_1 = 1/4$, $X_2 = 3/4$, $X_3 = 5/4$, $X_4 = 7/4$, $X_5 = 9/4$, $X_6 = 11/4$.
3. (b) "Two triangles fill a square."
5. 232.4.
6. (a) $7^3/3$. (b) $7^3/3$.
7. (a) $b^3/3$. (b) $b^3/3$.
9. 9π.
10. 9 million dollars.

SECTION 3: The Definite Integral over an Interval

2. (b) $77/60 = 1.283$.
4. For instance, $m = 1.283$ and $n = 2.083$.
5. (a) 1.47. (b) 1.43. (c) 1.93.
16. (a) $\int_0^3 \pi x^2\,dx$. (b) 9π.
17. (a) $\int_0^3 t^2\,dt$. (b) 9 (millions of dollars).
18. $\int_0^9 (3 - \sqrt{y})\,dy$.
19. (a) $\int_{3/2}^3 5x^2\,dx$. (b) $\int_0^{3/2} 5x^2\,dx$.
20. $\int_0^1 x^3\,dx$. (For each n use the partition given by $x_i = i/n$ and $X_i = x_i$).
21. $\int_{-5}^5 2\sqrt{25 - x^2}\,dx$.
22. $\int_1^3 t^2\,dt$ (millions of dollars).

□ □ □

23. For convenience, consider the circle and ellipse to be cut from a right circular cylinder by two planes, one perpendicular to the axis of the cylinder, the other intersecting it at an angle A. The area of the circle may be expressed as $\int_{-a}^a c(x)\,dx$, where c is the cross section function; the area of the ellipse is $\int_{-a}^a c^*(x)\,dx$, where $c^*(x)$ is the appropriate cross section of the ellipse. Since $c^*(x) = (\sec A)\,c(x) = (b/a)\,c(x)$, we have $\int_{-a}^a c^*(x)\,dx = (b/a) \int_{-a}^a c(x)\,dx = (b/a)\,\pi a^2 = \pi ab$.

25. If you intend to cover Section 4 of Chapter 6, this exercise may be especially appropriate. Observe that if $X_1, \ldots, X_n$ are sampling points in $[x_0, x_1], \ldots, [x_{n-1}, x_n]$, then $3X_1, \ldots, 3X_n$ are sampling points in $[3x_0, 3x_1], \ldots, [3x_{n-1}, 3x_n]$. Thus

$$\sum_{i=1}^{n} (1/3X_i)(3x_i - 3x_{i-1})$$

is an approximating sum for $\int_3^6 1/x\, dx$. Cancellation of the 3's in $(1/3X_i)(3x_i - 3x_{i-1}) = (1/X)(x - x_{i-1})$, almost completes the exercise.

26. See the discussion of Exercise 25.

27. It does not matter at what speed he travels, since the volume of the set of raindrops that strike the windshield equals: "length of trip" × "area of windshield." An easy way to see this is to consider the set occupied at the beginning of the trip by all the raindrops that will eventually strike the windshield.

28. Arguing intuitively, we see that the area of the ring whose inner radius is x_{i-1} and outer radius is x_i is about $2\pi x_i(x_i - x_{i-1})$. (When straightened it resembles a rectangle of length $2\pi x_i$ and width $x_i - x_{i-1}$.) Thus,

$$\sum_{i=1}^{n} 2\pi x_i (x_i - x_{i-1})$$

is an estimate of the area of a disk, whence its area is $\int_0^r 2\pi x\, dx$.

29. If X_i is in $[x_{i-1}, x_i]$, then $2X_i$ is in $[x_{i-1}, x_i]$.

$$\sum_{i=1}^{n} (2X_i)^5 (2x_i - 2x_{i-1})$$

is therefore an approximation of $\int_0^2 x^5 dx$. From this it follows that $\int_0^2 x^5\, dx = (2^5 \cdot 2) \quad \int_0^1 x^5 dx = 64/6$.

SECTION 4: The Average of a Function over an Interval

6. Yes, and its value is 0.
7. (a) 3. (b) 1.
8. 24 miles per hour.
9. (a) 12 and 28. (b) $12 \leq \int_1^5 f(x)dx \leq 28$. (c) It is between 3 and 7.
10. (a) 21. (b) 4.2.
11. (a) 10. (b) 9. (c) 139/19 = 7.316.

□ □ □

13. Since the values of x at which f is known are not equally spaced, this exercise should lead to some open-ended discussion. "Should we give greater weight to the interval [0, 1/2] where we know more?" "Should we disregard f(1/4) and f(1/2)?" In any case, simply averaging 3, 7/2, 4, 6, 8 is not right, since doing so gives too much weight to small numbers. (1) Perhaps use $X_1 = 0$, $X_2 = 1/4$, $X_3 = 1/2$, $X_4 = 1$, $X_5 = 2$ and $x_0 = 0$, $x_1 = 1/8$, $x_2 = 3/8$, $x_3 = 3/4$, $x_4 = 3/2$, $x_5 = 2$.
(2) Perhaps graph the five known points (x, f(x)), join them by straight line segments and compute: (area under graph)/(width of interval).

14. (b) Partition [0, 2] into 3m sections all of equal length 2/3m. In precisely m of these sections choose X_i to be irrational.
15. $\pi ab/2a = \pi b/2$.
16. (This is an occasion to remind the student that the areas of similar plane figures are proportional to the *squares* of their linear dimensions.) The area of the cross section made by a plane parallel to the base and at a distance x from the vertex is therefore $(x^2/h^2)A$. Thus, the volume equals $\int_0^h (x^2/h^2) A\,dx = hA/3$. Note that this is the 3-dimensional analog of "the area of a triangle is half its base times its height."

SECTION 5: The Definite Integral of a Function over a Set in the Plane

5. 16, 20, 6 respectively.
6. (a) It equals 5A. (b) 5A.
7. (a) Any circumscribing rectangle has area $\leq d^2$, (b) A circle, (c) In Chapter 8, Section 1, Exercise 18 it is shown that its area is at most the area of a circle of diameter d.
9. By considering the minimum and maximum of f over the appropriate R, one shows that the definite integral is between (a) 6 and 15, (b) $9\pi(8 - 3\sqrt{2})$ and $9\pi(8 + 3\sqrt{2})$.
11. The first implies the second (for the area of each set becomes "small"), but the second does not imply the first (keep one set in the partition fixed, partition the rest).

□ □ □

15. (An exercise that may interest the non-physical scientist.) It is simply another version of the two problems discussed in the section.
16. Let the radius of R be a. An approximating sum for this f and R formed on a partition of mesh p, has a non-negative value that is less than $3(2a\cdot 2p)$, since a region R_i for which $f(P_i) \neq 0$ lies in the rectangle of length 2a and width 2p lying along the diameter. Since $12ap \to 0$ as $p \to 0$, the definite integral exists and equals 0.
17. No. Use the same argument as in Example 2 of Section 4.
18. The cross section for each x in [0, 4] is a triangle of base 2 and height 2x, hence of area 2x. Thus the volume equals $\int_0^4 2x\,dx = 16$.

SECTION 6: The Definite Integral of a Function over a Set in Three-dimensional Space

1. (Volume) $\times$ (maximum density) $= 4^3\ (4^2 + 4^2 + 4^2) = 64 \times 48$.
3. $(20/3)\pi r^3$, since each approximating sum has that value.
4. If R is the set and f(P) = distance from P to F, then the average distance is $\int_R f(R)\,dV$/volume of R.
6. It is between 384 and 1120 (ounces).
7. (a) It is at most 8(volume of R). (b) It is at most 8.
8. (b) As the density of a homogeneous potato occupying the same set and having the same mass.

□ □ □

9. Since the solid lies within a cube of side 10, the volume is less than 10^3. (This is not the best possible result.)
10. If R is the region in space occupied by the mountain, then the work equals $\int_R h(P)\,g(P)\,dV$. Energy considerations like this have been used in geological theory.

CHAPTER 2: The Derivative

SECTION 1: Estimates in Four Problems

3. (a) 5.999. (b) 80(5.999) feet per second. (c) 5.999, (d) 5.999 ounces per inch.
8. (c) 45° or $\pi/4$ radians.
9. (a) Using intervals to the right of the mentioned points, one obtains the estimates 2.001, 4.001, 6.001, 7.001 ounces per inch. (b) 2s ounces per inch.

□ □ □

12. (c) 6. (d) It is 6. In general, this method of estimating the slope of the tangent line at a point P is more accurate than the method that uses a chord one of whose ends is P.
15. (See Exercise 12.) (a) $12 + h^2$. (b) It approaches 12.

SECTION 2: The Derivative

1. (a) $3t^2$ miles per hour. (b) The derivative.
2. Using the idea that $\Delta V/\Delta x$ is an estimate of cross-sectional area, and thus that the derivative of V with respect to x is the area, one obtains (a) 3 square inches, (b) 12 square inches, (c) when $3x^2 = 3/4$, that is, $x = 1/2$ inch.
3. $\sum_{i=1}^{n} [1/f(X_i)](x_i - x_{i-1})$ is an estimate of the time; hence the time is a definite integral $\int_a^b [1/f(x)]dx$, assuming, of course, $f(x) \neq 0$.
4. (e) $-1/(4 + 2\Delta x)$.
5. (a) $-1/9$. (b) $-1/x^2$.
7. (c) 0.414. (d) 0.107. (e) c/x where c is fixed at about 0.4 (actually $c = 0.434$, to three decimals).
13. (a) 5. (b) 5. (c) The line itself.
14. (d) Using a common logarithm table, one sees that $\log_{10} 1.259$ is close to 0.1, thus $10^{0.1} = 1.259$ and $10^{1.1} = 12.59$. Thus the first line has slope 2.59 and the second, 25.9.
15. (b) When $x = 4$, (c) $t = 4$, (d) Only (4, 256).

□ □ □

19. $(10^{x+\Delta x} - 10^x)/\Delta x = 10^x[(10^{\Delta x} - 1)/\Delta x]$. Hence, the derivative at x is (10^x)(derivative at 0).
23. (a) $f(1) = f(1/1) = f(1) - f(1)$, hence $f(1) = 0$. (b) $f(1/x) = f(1) - f(x) = -f(x)$. (c) In the identity $f(x/y) = f(x) - f(y)$, replace y by $1/y$ and use (b). (d) $[f(x + \Delta x) - f(x)]/\Delta x = f((x + \Delta x)/x)/\Delta x = f(1 + \Delta x/x)/\Delta x$ $= (1/x)f(1 + \Delta x/x)/(\Delta x/x) = 1/x\,[f(1 + \Delta x/x) - f(1)]/(\Delta x/x)$. Let $\Delta x \to 0$ and recall that $f'(1) = 1$. (e) $\log_{10}x$.

SECTION 3: Standard Notations Related to the Derivative; the Differential

2.

dx	dy	Δy
3	9	63
2	6	26
1	3	7
0.1	0.3	0.331
0.01	0.03	0.030301
0	0	0
-1	-3	-1

4.

dx	dy	Δy
0.5	-0.5	-1/3 = 0.333
0.1	-0.1	-1/11 = 0.091

6.

dx	dy	Δy	$\Delta y/dy$
1	4	5	1.25
-1	-4	-3	0.75
0.1	0.4	0.41	1.025
-0.1	-0.4	-0.39	0.975
0	0	0	meaningless

9. (b) It approaches $3x_1^2$, (c) It is $3x^2$ at x.
10. It approaches $-1/x_1^2$.

11.

y=f(x)	D(y)	dy	Δy
x^2	$2x$	$2xdx$	$2xdx + (dx)^2$
x^3	$3x^2$	$3x^2 dx$	$3x^2 dx + 3x(dx)^2 + (dx)^3$
$1/x$	$-1/x^2$	$(-1/x^2)dx$	$[-1/(x^2 + x\,dx)]dx$
$\sqrt{x}$	$1/2\sqrt{x}$	$(1/2\sqrt{x})dx$	$\sqrt{x + dx} - \sqrt{x}$

(The lower right entry, as shown in Section 3, equals $[1/(\sqrt{x + dx} + \sqrt{x})]dx$.)

12. (a) The differential dy represents the mass of the string in $[x, x+\Delta x]$ if the density of that section is constant (and equals the density at x.) The increment Δy is the mass of the string in $[x, x + \Delta x]$.

14. $\dfrac{\Delta y - dy}{dx} = \dfrac{\Delta y}{dx} - \dfrac{f'(x)\,dx}{dx} = \dfrac{\Delta y}{dx} - f'(x)$, which approaches 0 as $dx(= \Delta x) \to 0$.

SECTION 4: A Generalization of the Derivative

5. We know that $[f(x + \Delta x) - f(x)]/\Delta x \to a$, a certain number. Now $f(x + \Delta x) - f(x - \Delta x)$, $\Delta x > 0$, may be written as $[f(x + \Delta x) - f(x)] + [f(x) - f(x - \Delta x)]$. From this it follows that

$$[f(x + \Delta x) - f(x - \Delta x)]/2\Delta x \to a/2 + a/2 = a$$

6. Similar to Exercise 5, but use for $\Delta x > 0$, $\Delta t > 0$ the identity

$$\frac{f(x + \Delta x) - f(x - \Delta t)}{\Delta x + \Delta t} = \frac{\Delta x}{\Delta x + \Delta t}\,\frac{[f(x+\Delta x) - f(x)]}{\Delta x} + \frac{\Delta t}{\Delta x + \Delta t}\,\frac{[f(x) - f(x - \Delta t)]}{\Delta t}$$

CHAPTER 3: Limits and Continuous Functions

SECTION 1: The Limit of a Sequence

1. 10.
6. The distance remaining at the n^{th} stage is $(1 - 1/4)(1 - 1/16) \ldots (1 - 1/(2n)^2)$, which does *not* approach 0 as $n \to \infty$. It approaches $2/\pi$.
8. (b) $a_n = 2$ when n is even and 0 when n is odd, (c) No.
9. (a) $a_2 = 2$, $a_3 = 7/4 = 1.75$, $a_4 = 97/56 = 1.73214$, $a_5 = 18,817/10,864 = 1.73205$. (b) Same as (a), (c) $a_n = \sqrt{3}$ for all n, (d) All (seem to) have the limit $\sqrt{3}$ ((c) obviously).
10. (b) $a_1 = 1/2$, $a_2 = 2/3$, $a_3 = 3/4$, $a_4 = 4/5$. (c) $a_n = n/(n+1)$, (d) First rewrite $a_n = 1/[1 + 1/n]$, perhaps.
11. (a) $a_1 = 1/4 = 0.250$, $a_2 = 13/36 = 0.361$, $a_3 = 61/144 = 0.424$, $a_4 = 1669/3600 = 0.464$. (c) The property used is that an increasing bounded sequence of real numbers has a limit. This is a consequence of the completeness of the real numbers, discussed in Appendix B and in Chapter 12.
13. (c) Yes, since $\lim_{n\to\infty} (n + 1)/2n = 1/2$.
14.

n	1	2	3	4	5	6
$(1 + 1/n)^n$	2	2.250	2.370	2.441	2.488	2.522

15. (a) $a_4 = 5/3 = 1.667$, $a_5 = 8/5 = 1.600$, $a_6 = 13/8 = 1.625$, $a_7 = 21/13 = 1.615$.
16. $a_3 = 2$, $a_4 = 3$, $a_5 = 2$, $a_6 = 1$, $a_7 = 1$. Thus, since $a_6 = a_1$ and $a_7 = a_2$, the sequence repeats in blocks of five. It is not convergent. If it were convergent and its limit were L, then $L^2 = (1 + L)/L$, hence $L = (1 + \sqrt{5})/2$, the positive sign in front of $\sqrt{5}$ since all the a_n's are clearly positive.
17. Since $a_2 = 1$ and $a_3 = 1$, it behaves like the sequence in Exercise 16.
18. (a) $a_1 = 1$, $a_2 = 1.5$, $a_3 = 1.833$, $a_4 = 2.083$, $a_5 = 2.283$. (c) No, a_n becomes arbitrarily large.
19. (a) $a_n = (1^4 + 2^4 + \ldots + n^4)/n^5$. (b) $a_1 = 1$, $a_2 = 17/32 = 0.531$, $a_3 = 0.403$.
20. (a) $a_n = 3/10 + 3/10^2 + \ldots + 3/10^n$.
22. $a_2 = -1/2$, $a_3 = 2/3$, $a_4 = 3$, and the sequence is periodic. It is not convergent.
23. (a) $a_2 = 2/3$, $a_3 = 3/4$, $a_4 = 4/5$, $a_5 = 5/6$, $a_6 = 6/7$. (b) Since we would have $L = 1/(2 - L)$, L would be 1. (c) The limit exists, since one may (inductively) show that $a_n = n/(n+1)$.

□ □ □

24. This is the Buffon needle problem. The probability turns out to be $2/\pi$, as is shown in Chapter 6, Section 1, Exercise 20.
27. (a) 1.500, 1.083, 0.950, 0.884, 0.846. (b) The number of terms increases but the terms get small. (c) The sequence approaches $0.693 = \int_1^2 1/x\,dx$, as will be shown in Exercise 31. (Note that $a_n > 1/2$.)
28. This exercise shows that e exists. If there is ample time, it may be covered as a lecture. If not, it may be suggested as a reference. Note that it makes use of the fact that an increasing bounded sequence has a limit, a result discussed in Appendix B.

30. (c) Assume $r > 1$. From (a) it follows that $s < r$. Clearing denominators, we have $s!(r + 1)^r = r^r N$, where N is an integer. Since r and r + 1 are relatively prime, r^r divides s! But $s! < s^s < r^r$ if $s > 1$.

31. Note that a_n in this exercise differs from a_n in Exercise 27 by 1/n.

(a) $a_n = \sum_{i=1}^{n} 1/(n + i) = \sum_{i=1}^{n} (1 + i/n)^{-1}(1/n)$, an approximation to $\int_1^2 1/x\, dx$ made with $x_i = 1 + i/n$, $i = 0, 1, \ldots, n$, $X_i = x_i$, $i = 1, 2, \ldots, n$.

(b) $a_n < 1/(n + 1) + 1/(n + 1) + \ldots + 1/(n + 1) = n/(n + 1) < 1$.

(c) $a_n - a_{n+1} = 1/n - 1/(2n + 1) - 1/(2n + 2) = (3n + 2)/n(2n + 1)(2n+2) > 0$.

32. In Chapter 12, Section 2, Example 2, it is shown, for $0 < p < 1$, that $p + 2p^2 + 3p^3 + \ldots = p/(1 - p)^2$. Thus, the sum in Exercise 32 is 3. The experiments sometimes suggest such guesses as 3, π, $\sqrt{10}$.

33. Though the gaps "tend to increase," U's do *not* always lead D's.

SECTION 2: The Limit of a Function of a Real Variable

3.

x	1	0.1	0.01	0
cos x	0.8415	0.998	0.9999599996	1
$1 - x^2/2$	0.500	0.995	0.99995	1

4. (a)

x	-1/2	-1/3	-1/4	-1/5	-1/6
$(1 + x)^{1/x}$	4	3.375	3.160	3.052	2.986

(b) Yes.

5.

x	-7/8	-1/2	-1/5	1/3	1	2	3	4	1000
$(1 + x)^{1/x}$	6.179	4	3.052	2.370	2	1.732	1.587	1.495	close to 1

6. (a) 80.
8. (a) 405. (c) $d(x^5)/dx = 5x^4$.

□□□

11. [In Chapter 6, Section 1, we will cite the function $y = (1/2)x^2 \sin(1/x^2)$ as a counterexample. This exercise is good preparation if you want to discuss that counterexample.] Point out that this graph oscillates between the lines $y = x$ and $y = -x$.
12. (a) No, since $f(x) \to 0$ as $x \to 0$ from right and $f(x) \to 1$ as $x \to 0$ from left. (b) Yes, 1/2. (c) Yes, 1/2.
14. Rewrite the quotient as $[(x^5 - 2^5)/(x - 2)]/[(x^2 - 2^2)/(x - 2)]$, which approaches $(5 \cdot 2^4)/(2 \cdot 2) = 20$ as $x \to 2$.

SECTION 3: Continuous Functions

1. Since $\lim_{x \to 0} (1 - \cos x)/x = 0$, we must set $f(0) = 0$.
3. (a) The population is not continuous. (b) Speed is continuous (even in a crash). (c) There may be some debate. Since a person is composed of a finite number of molecules, the weight is in fact not continuous. However, growth charts approximate it with a function that is intended to be continuous.
4. (b) At any x not an integer. (c) At all a, and $\lim_{x \to a} f(x) = 0$.

5. (b) Everywhere except at the integers.
6. (b) Since $\lim_{x \to 0} f(x)$ does not exist, there is no way of defining f(0) to provide a continuous f.
7. For m = 1, X = 1, -1; for $1 < m \le 4$, $X = \sqrt{m}$.
8. (b) X = 0, π, 2π. (c) For m = 1, two X's; for $-1 < m < 0$, two X's; for $0 < m < 1$, three X's.
9. There is no X. Note that f is not defined at 0.
10. No. The maximum value theorem refers to closed intervals.
11. The intermediate value theorem.
13. (a) f(0) = 0, f(1/2) = 0, f(-2) = 0, f(1) = 1/2. (b) For $x \neq -1$. (c) $x \neq -1$, 1.
14. (a) f(1/2) = 0, f(2) = 1, f(1) = 1/2, (b) All x; f(x) = 1 if $|x| > 1$, f(x) = 0 if $|x| < 1$, f(x) = 1/2 if $|x| = 1$. (c) Everywhere except at x = 1, -1

□ □ □

15. Encourage the students to graph f, which is continuous only at x = 1. They sometimes try to do it in their heads.
17. Introduce g, defined by g(x) = - f(x).
18. This is Brouwer's fixed point theorem in dimension 1. (A graph of f makes it plausible.) If $f(0) \neq 0$ then $f(0) > 0$. If $f(1) \neq 1$, then $f(1) < 1$. Then g defined by g(x) = f(x) - x is positive at x = 0 and negative at x = 1. The intermediate value theorem applied to g says that there is X in [0, 1] such that g(X) = 0, that is, X = f(X).
19. If f(0) = 0, let X = 0. If not, we may assume $f(0) = a > 0$. Then $f(a) = 0 < a$. Let g(x) = f(x) - x. We have $g(0) > 0$ and $g(a) < 0$. Apply the intermediate value theorem to g. (The result is a special case of P. A. Smith's fixed-point theorem: A hemeomorphism of Euclidean space that is periodic with prime-power period has a fixed point.)
20. (a) Let g(x) = f(x + 1/2) - f(x) for x in [0, 1/2]. Then g(0) = f(1/2) and g(1/2) = -f(1/2). The intermediate value theorem implies that there is X in [0, 1/2], such that g(X) = 0, which was to be proved. (b) Consider f(i/n), i = 0, 1, ..., n. Pick m such that $f((m-1)/n) \le f(m/n) \ge f((m+1)/n)$ and introduce g as in (a), g(x) defined for x in [(m - 1)/n, m/n]. (c) No. The function $\sin 2\pi x$ is a counterexample. (d) Yes. Consider g(x) = f(x + 2/3) - f(x) for x in [0, 1/3].
21. Pick a small $h > 0$ such that $f(h)/h < 0.1$ and $[f(1) - f(1-h)]/h > 0.9$, say. Let $g(x) = [f(x+h) - f(x)]/h$ for x in [0, 1-h]. We have $g(0) < 0.1$, $g(1-h) > 0.9$. Applying the intermediate value theorem to g, we see that there is X in (0, 1 - h) such that g(X) = 1/2. The same type of argument works for any preassigned slope m, $0 < m < 1$. There need not be a chord of slope 0 or 1, as $f(x) = (1/2)x^2$ shows.
22. We have $f(x^2) = (f(x))^2$. Thus $f(0))^2 = (f(0))^2$ and $f(1) = (f(1))^2$. Thus f(0) = 0 or 1 and f(1) = 0 or 1. The fixed points of g are only 0 and 1. We must prove that at least one of 0 and 1 is fixed under f. Say $f(1) \neq 1$, that is, $f(1) < 1$. Then, since f is continuous, there is a number $a < 1$, $f(a) = b < 1$. Since $f(x^2) = (f(x))^2$ we have $f(a^2) = b^2$, $f(a^4) = b^4$, ... Since $a^{2n} \to 0$ and $b^{2^n} \to 0$ as $n \to \infty$ and f is continous at 0, we have f(0) = 0 (= g(0)).

SECTION 4: Precise Definitions of Limits

3. Any $\epsilon > 0$ would work.
13. Encourage the student to write in newspaper English. There should be no mathematical symbols, such as f and g, in his answers.

CHAPTER 4: The Computation of Derivatives

SECTION 1: The Derivatives of Some Basic Functions

6. $\log_2(4) = 2$, $\log_4(2) = 1/2$, $\log_{10}\sqrt{10} = 1/2$, $\log_3(1/3) = -1$, $\log_e(1) = 0$, $2^{\log_2 8} = 8$, $e^{\ln x} = x$, $10^{\log_{10} 17} = 17$, James.
12. (a) 6, (b) 108 ounces per inch, (c) 1/16 miles per hour, (d) 100.
13. (a) 1/2, (b) 1/2 foot per second, (c) 1/2.

SECTION 2: The Derivative of the Sum, Difference, Product, and Quotient of Functions

1. (a) $2x + 3x^2$. (b) $2x + 3x^2$.
2. (a) $4x^3$. (b) $4x^3$.
4. $-\sin x + 2x$.
7. (b) $[(x^2 + \sin x)(3) - (1 + 3x)(2x + \cos x)]/(x^2 + \sin x)^2$
12. (a) $4/(1 - 2x)^2$, (b) $4x^4(-x \sin x + 5 \cos x)$, (c) $5x^4 - 1/x$.
13. (a) $4x^3 - \sec^2 x$, (b) $4x^3 + \sec^2 x$, (c) $(4x^3 \tan x - x^4 \sec x^2)/(\tan x)^2$. (d) $x^4 \sec^2 x + 4x^3 \tan x$.

□ □ □

17. The tangent line at (a, a^n) passes through (a, a^n) and $(0, -(n - 1)a^n)$.
18. (a) Write $(\tan \theta)/\theta$ as $[(\sin \theta)/\theta]/\cos \theta$.

SECTION 3: The Chain Rule

1. $y = u^2$, $u = x^6$; $y = u^3$, $u = x^4$; $y = u^4$, $u = x^3$; $y = u^6$, $u = x^2$. Of course, it is also true that $y = u^{12}$, $u = x$; $y = u$, $u = x^{12}$.
6. (b) Let $y = \ln |u|$, $u = \cos x$. Then $dy/du = 1/u$, $du/dx = -\sin x$, hence $dy/dx = -\sin x/\cos x = -\tan x$. In this, as in (e) and (f), *justifications* should be shown briefly beneath the equals sign.
7. (a) $[(6x^2 + 2)(10x + 1) - (5x^2 + x + 1)(12x)]/(6x^2 + 2)^2$.
   (b) $[(6x^2 + 2)(10x + 1) - (5x^2 + x + 1)(1200x)]/(6x^2 + 2)^{101}$.
   (c) $4(\log_{10} e)\, x^3/(x^4 + 1)$, (d) $2x \cos x^2/\sin x^2$, (e) $10 \tan^4 2x \sec^2 2x$,
   (f) $(5x^3 + 4)^{14}(3x^2 + 1)^9[225x^2(3x^2 + 1) + 60x(5x^3 + 4)]$.
8. (a) In (b), (c), (d), (e) and (f). (b) In (d), (e), and (f).
9. $dV/dt = 15s^2$.
10. (a) $n(1 + x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \ldots + (n-1)\binom{n}{n-1}x^{n-2} + nx^{n-1}$.
    (b) $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \ldots + (n-1)\binom{n}{n-1} + n$. (c) Both sides equal 32.
    (d) $0 = \binom{n}{1} - 2\binom{n}{2} + \ldots + (-1)^{n-2}(n-1)\binom{n}{n-1} + (-1)^{n-1}n$.

SECTION 4: The Derivative of the Inverse Function

4. $\cos^{-1}(0) = \pi/2$, $\cos^{-1}(1/2) = \pi/3$, $\cos^{-1}(1) = 0$, $\cos^{-1}(-1/2) = 2\pi/3$, $\cos^{-1}(-1) = \pi$.
6. (b) $\theta = \tan^{-1}(x/3)$. (c) $dx/dt = 2$, $d\theta/dx = 3/(x^2 + 9)$.
7. The limit is the derivative of 10^x at $x = 0$, hence $10^0 \ln 10 = \ln 10$.
11. Both (a) and (b) are the derivative of e^x at $x = 2$, hence e^2 or 7.3891.

12. (b) $(\ln 2)(2^{\sin^{-1}x})(1/\sqrt{1-x^2})$, (e) $(\ln 10)(10^{1/x})(-1/x^2)$.
13. (a) $(\log_{10}e)(12x^2)/(1+3x^4)$, (c) $e^x/(1+e^{2x})$.
20. (a) 4/5, (b) 4/5, (c) 4/5, (d) 4/5, (e) 4/5, (f) letting $f(x) = \sqrt{9+x^2}$ and using differentials at $x = 4$, we obtain $f(4.1) \sim 5 + (4/5)(0.1) = 5.08$ and $f(3.9) \sim 4.92$.

32. $\lim_{h \to 0} (1+2h)^{1/h} = \lim_{h \to 0} [(1+2h)^{1/2h}]^2 = e^2$.

CHAPTER 5: The Law of the Mean

SECTION 1: Rolle's Theorem

2 and 3. These should be expressed in nonmathematical language, e. g., for 2: If after starting and before finishing its journey, the particle reaches its farthest position to the right, then at that moment its speed is 0.

6. $X = 0, 1, -1$.
7. Since f is not differentiable on (2, -2) Rolle's theorem says nothing.

□ □ □

9. Apply Rolle's theorem to g defined by $g(x) = a_0 x + \frac{a_1 x^2}{2} + \ldots + \frac{a_n x^{n+1}}{n+1}$. Note that $g(0) = 0 = g(1)$.

SECTION 2: The Law of the Mean

1. If the change in position is d and the duration of the trip is t, then at some instant during the trip the particle had the velocity d/t.
2. At some point in the string the density equals total mass/length of string.
3. If the projector magnifies a certain interval by a factor m, then at some point in that interval the magnification is m.
5. (b) 4.375 (if differentials are computed at $x = 16$).
6. (a) Yes, by the law of the mean. (b) No, consider $y = x^3$, and its horizontal tangent.
7. There is X, $3 < X < 8$, such that $f'(X) = 2$.
8. (a) Cor. 2, (b) Cor. 1, (c) Cor. 1, (d) Cor. 2.
9. (a), (b), (c). That for any partition, there is a choice of sampling points such that the approximating sum yields the exact answer, that is, "has no error."
10. (a) The definite integral may not exist. (b) That the definite integral exists.
11. (b) No. Write $10^x = e^{(\ln 10)x}$.
19. $X_1 = 2$, $X_2 = 7/2$, $X_3 = 9/2$, $X_4 = 11/2$.

□ □ □

24. First show that there is a chord of slope 1/2 (using the technique described in the solution of Exercise 21 in Chapter 3, Section 2). Then apply the law of the mean.
25. Let f be a differentiable function and m a number between f' (a) and f' (b). Then there is X, $a < X < b$, such that $f'(X) = m$.
26. (c) One may prove that $f'(x) \geq 0$.

27. $[f(1+\Delta x) - f(1)]/\Delta x = f'(X)$, for some X between 1 and $1+\Delta x$. Since $\lim_{x\to 1} f'(x)$ exists, $f'(1)$ exists.

28. Let $h(x) = f(x) - g(x)$. We have $h(0) = 0$, $h'(x) \geq 0$. Hence $f(x) \geq g(x)$ for $x \geq 0$ and $f(x) \leq g(x)$ for $x \leq 0$.

29. $f(3) = f(0) + (3-0)f'(X) \geq 3$. The function $y = 3x$ shows that we cannot claim more.

30. (a) $D(f(x) \cdot f(k-x)) = f(x)(-f(k-x)) + f(k-x)f(x) = 0$. (b) Constant. (c) If A is the constant value of $f(x)f(k-x)$ we have $f(0)f(k-0) = A$, whence $f(k) = A$. Hence $f(x)f(k-x) = f(k)$. (d) Letting $k = x+y$, we obtain from (c) $f(x)f(y) = f(x+y)$.

31. (a) $D(f(x) + f(k/x)) = 1/x + (1/(k/x))(-k/x^2) = 0$. Thus $f(x) + f(k/x) = A$, constant. Setting $x = 1$, we see that $A = f(k)$. Then let $k = xy$. (b) $f(x)$ and $\ln x$ have the same derivative and agree at $x = 1$.

32. (a) $D(\tan x - x) = \sec^2 x - 1 > 0$ for x in $(0, \pi/2)$. (c) Follows from (b). (d) Follows from (c) by considering $D(\sin x/x)$.

33. (a) Yes. (b) No. Apply the law of the mean to h defined by $h(x) = f(x) - g(x)$.

34. $\prod_{i=1}^{k} (1-u_i) < e^{-u_1-u_2-\cdots-u_k}$.

35. For convenience, let the distance from the pessimist to the chair be 1. His first step has length u_1 and $1-u_1$ remains. His second step has length $u_2(1-u_1)$ and $(1-u_1)-u_2(1-u_1) = (1-u_2)(1-u_1)$ remains, etc. Then apply Exercise 34 to show that the distance remaining approaches 0.

36. For $0 < x < 1$, $|\ln(1-x)| = -\ln(1-x)$, whose derivative is $1/(1-x)$. Thus $f(x) = x/(1-x) - |\ln(1-x)|$ has the derivative

$$1/(1-x)^2 - 1/(1-x) > 0 \text{ for } 0 < x < 1.$$

Thus f is an increasing function for $0 \leq x < 1$, and since $f(0) = 0$, $f(x)$ is positive.

38. $f'(0) = \lim_{h\to 0} [f(h) - f(0)]/h$. Since $[f(h) - f(0)]/h \geq 0$ by the law of the mean, $f'(0) \geq 0$.

CHAPTER 6: The Fundamental Theorem of Calculus

SECTION 1: Various Forms of the Fundamental Theorem of Calculus

3. Select $0 < x < 1/1000$ such that $1/x^2 = 2n\pi$, for instance any $x = \sqrt{1/2n\pi}$ where $2n\pi > (1000)^2$, in particular $n = 1{,}000{,}000$ and $x = \sqrt{1/2{,}000{,}000\pi}$.

12. (a) $e^{x^5}/5$, (b) $-1/x$, (c) $-e^{-x}$.

13. (d) Both.

15. This is a good illustration of the limitation of the FTC and a reminder that a definite integral is a limit of sums. (a) For a partition of mesh p,

$$0 \leq \sum_{i=1}^{n} f(X_i)(x_i - x_{i-1}) \leq 6p$$

since at most two of the X_i's are equal to 1. Thus the sums $\to 0$ as $p \to 0$. (b) If $F' = f$, then $F(x) = c$ for $x < 1$ and $F(x) = k$ for $x > 1$, where c and k are constants. Since F is continuous, $c = k$, whence F is constant. Thus $F'(x) = 0$ for all x; in particular $F'(1) = 0 \neq f(1)$.

□ □ □

18. (a) Since f satisfies the conclusion of the intermediate value theorem it is continuous. (Expect only a plausibility argument.)

19. Clearly $\int_{-1}^{0} x^{15}e^{-x^4}\,dx = -\int_{0}^{1} x^{15}e^{-x^4}\,dx$, thus $\int_{-1}^{1} x^{15}e^{-x^4}\,dx = 0$.

SECTION 2: Proof of a Special Case of the Fundamental Theorem of Calculus

6. Since the mesh is $1/n$, the sum differs from the definite integral by at most $|f(1) - f(0)|/n$.
7. Yes.
9. $\int_2^3 e^t\,dt = e^3 - e^2$, by Theorem 2.
10. The volume, which is $\int_0^{\pi/2} \pi \sin x\,dx = \pi$. No elementary function has a derivative equal to $\sqrt{\sin x}$.
11. (b) $1 - e^{-b}$.
12. (b) $2\sqrt{b} - 2$, (c) It becomes arbitrarily large.
13. (a) The first; the second cannot be evaluated by the fundamental theorem of calculus (See Exercise 10).
14. (a) $a_1 = 1/4 = 0.250$, $a_2 = 25/72 = 0.347$, $a_3 = 469/1200 = 0.391$.

(b) $\frac{1}{(1+1/n)^2}\frac{1}{n} + \frac{1}{(1+2/n)^2}\frac{1}{n} + \dots + \frac{1}{(1+n/n)^2}\frac{1}{n}$, an approximation to $\int_1^2 (1/x^2)\,dx$, formed with $x_i = 1 + i/n$, $i = 0, 1, \ , \dots, n$, $X_i = x_i$.

□ □ □

15. This exercise may be generalized to: If $f = F'$ and the slope of the chords of f are bounded (in absolute value) by M, then $\int_a^b f(x)dx$ exists, and equals $F(b) - F(a)$. Proof. Use the X_i and X_i^* of the proof of Theorem 2. Consider

$$\left|\sum_{i=1}^{n} f(X_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(X_i^*)(x_i - x_{i-1})\right| \leq$$
$$\sum_{i=1}^{n} |f(X_i) - f(X_i^*)|(x_i - x_{i-1}) \leq p\sum_{i=1}^{n} |f(X_i) - f(X_i^*)| \leq$$
$$p\sum_{i=1}^{n} M|X_i - X_i^*| \leq p\sum_{i=1}^{n} M(x_i - x_{i-1}) = pM(b-a),$$

which approaches 0 as $p \to 0$.

17. $f(x) = x = g(x)$.

SECTION 3: A Different View of the Fundamental Theorem of Calculus

4. Let $G(x) = \int_0^x e^{t^2}\,dt$. Then the limit is $G'(x_1) = e^{x_1^2}$.
5. (a) Since $A + \int_0^T f(t)\,dt$ is the cost during a period T, $g(T)$ is the average cost per unit time. (b) $g'(T) = [Tf(T) - (A + \int_0^T f(t)\,dt)]/T^2$. (c) Set the numerator in (b) equal to 0. (d) Yes: if $f(T) < g(T)$ then we would lower the cost per unit time by keeping the machine longer. If $f(T) < g(T)$, we should have overhauled the machine earlier.

6. Hopefully, the student will use the fundamental theorem, and conclude: $f(1) - f(0) = 1/2 - 0$.
8. We have $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F' = f$. Differentiate both sides of the equation.
9. Necessarily, f must be the derivative of $(\sin t)/(1 + t^2)$.
10. (b) $f'(t) = kf(t)$, hence $f(t) = Ae^{kt}$.

□ □ □

14. (Note in this case that the continuity of v implies its differentiability.) Pursuing the hint, we obtain

$$v(b) = v((a+b)/2) + (b-a)(1/2)v'((a+b)/2)$$

and

$$-v(a) = -v((a+b)/2 + (b-a)(1/2)v'/((a+b)/2).$$

Subtracting, we find $v(a) + v(b) = 2v((a+b)/2)$. (The only continuous solutions of this functional equation for v are $v(t) = ct + d$.) Or, differentiate: $v'(a) = v'((a+b)/2)$. Since b is arbitrary, v' is constant, hence $v = ct + d$.
15. (See note concerning Exercise 14.) We have

$$v(b) = \frac{v(a) + v(b)}{2} + \frac{(b-a)v'(b)}{2}$$

$$-v(a) = -\frac{v(a) + v(b)}{2} + \frac{(b-a)v'(a)}{2},$$

which we rewrite (respectively) as

$$v(b) - v(a) = (b-a)v'(b)$$

and

$$v(b) - v(a) = (b-a)v'(a)$$

whence $v'(b) = v'(a)$. Thus v' is constant and v is of the form $ct + d$.
16. (b) Rewrite A_n as

$$[(2n)/(n-1)\pi] \sum_{k=0}^{n-1} \sin(k\pi/n)\,\pi/n,$$

noticing that $\sin(0\pi/n) = 0$. The sum approaches $\int_0^\pi \sin x\,dx$ as $n \to \infty$.
17. Break $[a, b]$ into intervals determined by the values of x where $f'(x) = 0$. On each interval apply Theorem 2, since f' does not change sign in any of those intervals.
18. Differentiating with respect to a, we obtain $bf(ab) = f(a)$. Now hold a fixed and let $x = ab$. We have $f(x) = f(a)/b = af(a)/x$. Thus, $f(x)$ is of the form k/x for some constant $k(= af(a))$.

SECTION 4: An Alternative Approach to the Logarithm and Exponential Function

1. It is differentiable and its derivative, $1/x$, is positive.
8. Rewrite the expression in brackets in the form of an approximating sum to $\int_1^2 1/x\,dx$, namely as

$$\frac{1}{1+1/n}\,\frac{1}{n} + \frac{1}{1+2/n}\,\frac{1}{n} + \dots + \frac{1}{1+n/n}\,\frac{1}{n}$$

10. (a) $\ln(1 + x)$ is the area of the region under the curve $y = 1/x$ from 1 to $1 + x$. This region is contained in a rectangle of base x, height 1, and contains a rectangle of base x, height $1/(1 + x)$.

12. Let $y = E(x)$. Then $x = L(y)$. Thus $1 = dx/dx = (dL/dy)(dy/dx) = (1/y)(dy/dx)$, whence $dy/dx = y = E(x)$. This is the same proof as that in Chapter 4 for $d(e^x)/dx = e^x$.

□ □ □

16. $$\frac{1}{n+1} + \dots + \frac{1}{2n} = (1 + \frac{1}{2} + \dots + \frac{1}{2n}) - (1 + \frac{1}{2} + \dots + \frac{1}{n}) =$$

$$1 + (\frac{1}{2} - 1) + \frac{1}{3} + (\frac{1}{4} - \frac{1}{2}) + \frac{1}{5} + (\frac{1}{6} - \frac{1}{3}) + \dots + (\frac{1}{2n} - \frac{1}{n}) =$$

$$1 - 1/2 + 1/3 - 1/4 + \dots - 1/2n.$$

19. $L(a^x) = L(E^{xL(a)}) = xL(a)$.

20. $a^{xy} = E(xyL(a))$ and $(a^x)^y = E(yL(a^x))$, which by Exercise 19 equals $E(y(xL(a))) = E(xyL(a))$.

SECTION 5: The Antiderivative

1. As proved in Chapter 3, a function unbounded on $[a,b]$ lacks a definite integral over $[a,b]$. Secondly, $1/x^2$ is *not* the derivative of $-1/x$ for $x = 0$.
4. Since $F' = f = G'$, F and G have the same derivative, hence by Cor. 2 to the law of the mean, F differs from G by a constant.
5. (a) $7/3$, (b) $\ln 2$, (c) $\pi/4$.
8. (a) Yes, $\int f(x)\,dx - \int g(x)dx$ is an antiderivative of $f - g$. (b) None that I know of. The complication is due to the fact that $D(uv)$ is not necessarily $D(u)D(v)$.

□ □ □

10. Let M be the maximum value of $f(x)$ for x in $[0, 1]$ and let $f(a) = M$, $1 \geq a > 0$. Since $\int_0^0 f(x)\,dx \geq f(0)$, we have $f(0) = 0$. Thus

$$\int_0^a f(x)\,dx < aM \leq M = f(a)\text{, a contradiction.}$$

11. (a) implies (b) by the law of the mean. (b) implies (a), as follows:

$$\lim_{y\to x} [f(x) - f(y)]/(x - y) = \lim_{y\to x} g(T) = g(x).$$

Hence $f'(x)$ exists and equals $g(x)$. (a) implies (c) by FTC. (c) implies (a) by Theorem 3.

CHAPTER 7: Computing Antiderivatives

SECTION 1: Some Basic Facts

1. (a) $x^4/4$, (b) $5x^4/4$, (c) x^8, (d) $x^8/8$.
3. (a) $\sin x - 2\cos x$, (b) $5\tan^{-1} x$.
4. (a) $6e^x$, (b) $-6e^{-x}$, (c) $-(5/2)\cos 2x$.
5. (a) $x + x^3/3 - 3x^2$, (b) $x^6/6$, (e) $x + 2x^3/3 + x^5/5$.
6. (a) $x + 2x^2$, (b) $5x^3/3$, (c) $-\sqrt{1 - x^2}$, (d) $-1/x$.

SECTION 2: The Substitution Technique

3. $(2x + 5)^{3/2}/3$, $(1/4)\ln|4x + 7|$, $2\sqrt{1 + x^2}$, $-\sqrt{1 - x^2}$.
4. $(-1/3)/(1 + x^2)^3$, $-(1/10)/(5x^2 + 1)$, $-(\cos^4 x)/4$.
5. (a) $-e^{-x}$, (b) $(1/3)e^{3x}$, (c) $10^x/\ln 10$, (d) $2^x/\ln 2$.
6. $\int (e^{1/x}/x^2)\,dx = -e^{1/x}$.
8. (b) $\int [\ln(x^2)/x]dx = 2\int[\ln x/x]dx = (\ln x)^2$, (c) $(1/2)(\ln x)^2$.
9. (a) $a = 2$, (b) $a = -1$, (c) $a = 0$ or $-1/2$, for instance.
10. (a) $(\tan^2 5x)/10$, (b) $e^{-7x}/(-7)$, (c) $(5/2)(\ln x)^2$.
11. (a) $\tan^{-1} x$, (b) $(1/3)\tan^{-1} 3x$, (c) $1/\sqrt{5}\tan^{-1}(\sqrt{5}\,x)$,

    (d) $(1/\sqrt{5})\tan^{-1}(x/\sqrt{5})$.
12. Generally, no. $\int x \cdot x^2\,dx = x^4/4 + C$ while $x\int x^2\,dx = x^4/3 + K$ which are not equal for any choice of C and K.

□ □ □

13. $2\tan^{-1}\sqrt{x}$.
14. Use the substitution $u = x^2$.
15. Use the substitution $u = e^x$.
16. We would have $(e^x p/q)' = e^x/x$, hence the identity $xqp' - pq'x + pqx = q^2$. Thus x divides q^2, hence q. (Simply set $x = 0$.) Write $q = x^i r$, $i > 0$, x not dividing r. Then the identity becomes $x^{i+1}rp' - ipx^i - pr'x^{i+1} + prx^{i+1} = x^{2i}r^2$. Thus, x^{i+1} divides the left side, hence divides prx^i, hence r, a contradiction.
17. From the first to the second use the substitution $u = x^2$; from the second to the third use the substitution $x = \tan\theta$.

SECTION 3: Integration by Parts

1. (a) $xe^x - e^x$, (b) $e^x(x^2 - 2x + 2)$.
3. $x\sin^{-1} x + \sqrt{1 - x^2}$.
4. $x\ln^2 x - 2x\ln x + 2x$.
5. (a) $e^{-kx}(-kx - 1)/k^2$, (b) $e^{-kx}(k^2x^2 + 2kx + 2)/(-k^3)$.
6. (a) $\sin x - x\cos x$, (b) $2x\cos x - 2\sin x + x^2\sin x$.
7. $x\ln(4 + x^2) - \int 2x^2/(4 + x^2)dx$, and

$$\int \frac{2x^2}{4 + x^2}\,dx = \int \frac{2x^2 + 8}{4 + x^2}\,dx - \int \frac{8}{4 + x^2}\,dx = 2x - 8\int \frac{1}{4 + x^2}\,dx$$

$$= 2x - 4\tan^{-1}(x/2)$$

   Hence, the answer is $x\ln(4 + x^2) - 2x + 4\tan^{-1}(x/2)$.
8. $\int \sin^2 x\,dx = x/2 - (\sin 2x)/4$

   $\int \sin^3 x\,dx = -\cos x + (\cos^3 x)/3$

   $\int \sin^4 x\,dx = 3x/8 - 3\sin 2x/16 - (\sin^3 x\cos x)/4$

   $\int \sin^5 x\,dx = -(\sin^4 x\cos x)/5 + (4/5)[-\cos x + (\cos^3 x)/3]$.
9. The symbol $\int x^{-1}dx$ is multiple-valued. It is true that $\int x^{-1}dx + 1$ is an antiderivative of x^{-1}, which we write (loosely) as $\int x^{-1}dx + 1 = \int x^{-1}dx$.

□ □ □

12. Use the substitution $u = \sqrt[3]{x}$. (The same method works for $\int e^{\sqrt[n]{x}}\,dx$, n a positive integer.) The answer is $3e^{\sqrt[3]{x}}[(\sqrt[3]{x})^2 - 2\sqrt[3]{x} + 2]$.

SECTION 4: The Antidifferentiation of Rational Functions by Partial Fractions

1. (a) $-1/(4x - 2)$, (b) $x/2 + (1/4)\ln|2x-1| - 1/(4x - 2)$,
   (c) $(1/8)[(2x - 1) + 2\ln(2x - 1) - 1/(2x - 1)]$.
2. (a) $(2/\sqrt{8})\tan^{-1}[(2x + 2)/\sqrt{8}]$, (b) $(2/\sqrt{8})\tan^{-1}[(4x + 4)/\sqrt{8}]$.
4. (a) $(1/7)\ln\left|\dfrac{x - 1}{x + 6}\right|$, (b) $\ln\left|\dfrac{x + 2}{x + 3}\right|$.
6. $\tan^{-1}(2x - 3)$.
7. (a) $-1/x + (1/2)/(x + 1) + (1/2)/(x - 1)$, (b) $-\ln|x| + (1/2)\ln|x + 1|$
   $+ (1/2)\ln|x - 1|$.
8. $2\ln|x - 1| - \ln(x^2 + 1) - 6\tan^{-1} x$.
9. $x^2/2 - 2x + (1/6)\ln|x - 1| - (1/2)\ln|x + 1| + (16/3)\ln|x + 2|$.
10. $x + 4\ln|x + 2| + 4/(x + 2)$.
11. $(2x + 1)/[3(x^2 + x + 1)] + [4/(3\sqrt{3})]\tan^{-1}((2x + 1)/\sqrt{3})$.
12. $-(2 + x)/[3(x^2 + x + 1)] - (2/(3\sqrt{3}))\tan^{-1}((2x + 1)/\sqrt{3})$.
13. (a) $(1/2)\ln(x^4 + x^2 + 1)$, (b) The partial fraction decomposition is

$$\frac{1}{x^4 + x^2 + 1} = \frac{(\frac{1}{2})x + \frac{1}{2}}{x^2 + x + 1} + \frac{(-\frac{1}{2})x + \frac{1}{2}}{x^2 - x + 1},$$

and $\int [1/(x^4 + x^2 + 1)]dx = (1/4)\ln(x^2 + x + 1) - (1/4)\ln(x^2 - x + 1)$

$$+ 1/(2\sqrt{3})\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + 1/(2\sqrt{3})\tan\left(\frac{2x - 1}{\sqrt{3}}\right).$$

14. The integrand equals $x - 1 + (-2x + \frac{1}{6})/(x^2 + x + 1)$; its antiderivative is $x^2/2 - x - \ln(x^2 + x + 1) + (14/\sqrt{3})\tan^{-1}((2x + 1)/\sqrt{3})$.

SECTION 5: Some Special Techniques

2. (a) $1/2[-\cos 2x + (2/3)\cos^3 2x - (1/5)\cos^5 2x]$,
   (b) $1/2[\sin 2x - (2/3)\sin^3 2x + (1/5)\sin^5 2x]$, (c) $(-1/3)\cos^3 x + (1/5)\cos^5 x$.
3. (a) $(1/2)[(1/5)\sec^5 2x - (1/3)\sec^3 2x]$, (b) $(1/5)\sec^5\theta - (2/3)\sec^3\theta + \sec\theta$.
4. $-\ln|\cos\theta| = \ln|\sec\theta|$
5. (a) $(1/2)[x\sqrt{4 - x^2} + 4\sin^{-1}(x/2)]$, (b) $\ln|x - \sqrt{x^2 - 4}|$,
   (c) $\ln|x + \sqrt{x^2 + 4}|$.
6. $(3/7)(x + 1)^{7/3} - (3/4)(x + 1)^{4/3}$.
7. The integrand in terms of u has the partial fraction decomposition $3 + 12/(u - 2) + (-2u - 8)/(u^2 + 2u + 4)$. The answer (in terms of u) is $3u + 12\ln|u - 2| - \ln(u^2 + 2u + 4) - \sqrt{12}\tan^{-1}((2u + 2)/\sqrt{12})$.

SECTION 6: Summary and Practice

1. (a) $(1/3)\int[\sec^3\theta/\tan^3\theta]d\theta = 1/3\int(1/\sin^3\theta)\,d\theta$, which leads to a rational function under the substitution $u = \cos\theta$. (b) $\int u^2/(u^2 - 9)^2\,du$.
   (c) Since

$$\frac{u^2}{(u^2 - 9)^2} = \frac{1/12}{u - 3} + \frac{1/4}{(u - 3)^2} + \frac{-1/12}{u + 3} + \frac{1/4}{(u + 3)^2}$$

the antiderivative in (b) is

$$1/2 \ln|u-3| - (1/4)/(u-3) + (-1/12)\ln|u+3| - (1/4)/(u+3).$$

2. (a) $\int 2(u^2-1)^2 du$, (b) $\int (u^{3/2} - 2u^{1/2} + u^{-1/2})du$, (c) $\int 2\tan^5\theta \sec\theta \, d\theta$. All are easily evaluated, perhaps (a) the most easily. The answer, in terms of x, is $2(1+x)^{5/2}/5 - 4(1+x)^{3/2}/3 + 2(1+x)^{1/2}$.
3. $(2/3)(x-2)\sqrt{x+1}$.
4. (a) $(2/7)\tan^{-1}\left(\frac{2x+5}{\sqrt{7}}\right)$, (b) $(1/4)\ln(2x^2+5x+6) - (5/14)\tan^{-1}\left(\frac{2x+5}{\sqrt{7}}\right)$.
5. (a) $x - \tan^{-1}x$, (b) $(1/2)\ln(x^2+1)$, (c) $x + (1/2)\ln\left|\frac{x+1}{1-x}\right|$.
6. (a) $\ln\left|\frac{x+2}{x+3}\right|$, (b) $2\ln|x^2+5x+6|$.
7. (a) The partial-fraction decomposition is

$$\frac{x^2}{x^4-1} = \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{1/2}{x^2+1}$$

Hence, the antiderivative is $(1/4)\ln|x-1| - (1/4)\ln|x+1| + (1/2)\tan^{-1}x$, (b) $(1/4)\ln|x^4-1|$.
8. Use the factorization $x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$ described in Appendix G. A laborious problem.
9. Let $u = x^2$. Answer is $(1/2)\tan^{-1}x^2$.
10. $(1/4)\ln(x^4+1)$.
11. $(1/4)\ln|x| - (1/8)\tan^{-1}(x/2)$.
19. (a) $x^2\sqrt{1+x^2} - 2\int\sqrt{1+x^2}\,x\,dx$, (b) $\int\tan^3\theta\sec\theta\,d\theta$, (c) $\int(u^2-1)du$.
20. $(1+x^2)^{3/2}/3 - (1+x^2)^{1/2}$.
21. $e^{12x}/12$.
22. $\sqrt{4-x^2} + 2\ln\left|\frac{\sqrt{4-x^2}-2}{x}\right|$
23. (a) The substitution $u = \sqrt[3]{x}$ sends this into $\int[3u^4/(u^3+1)]du =$

$$3u^2/2 - \ln|u+1| + (1/2)\ln(u^2+u+1) + (1/\sqrt{3})\tan^{-1}((2u+1)/\sqrt{3}).$$

(b) Usually nothing.
24. (a) $(1/2)[-\cos 2x + \cos^3 2x - (3/5)\cos^5 2x + (1/7)\cos^7(2x)]$, (b) $(1/16)\sin^8 2x$.
25. (a) Use the recursion formula for $\int\sec^n x\,dx$, (b) $(1/5)\sec^5 x$, (c) $(1/5)\cos^{-5}x$.
26. Method (b), which is easier. Answer:

$$(x-1)^2/2 + 3(x-1) + 3\ln|x-1| - 1/(x-1).$$

27. The second, which can be evaluated by an integration by parts. Answer: $\sin x - x\cos x$.
28. (a) For $m = 0$, clear. Then argue inductively using integration by parts. (b) Follow by the substitution $v = (m+1)u$ and use (a).
29. Let $u = e^x$.
31. (a) $\ln|\ln|x||$, (b) $(1/2)(\ln|x|)^2$.
33. $(1/2)x^2\sin^{-1}x - (1/4)\sin^{-1}x + (1/4)x\sqrt{1-x^2}$.
34. $x\sin^{-1}x + \sqrt{1-x^2}$.
35. $x\ln(x^2+5) - 2x + (10/\sqrt{5})\tan^{-1}(x/\sqrt{5})$ (by an integration by parts).

36. Let $u = x - 1$. Answer: $(x - 1) + 3 \ln|x - 1| - 3/(x - 1) - (1/2)/(x - 1)^2$.
37. Let $u = \sqrt{3x + 2}$. Answer: (in terms of u) $(4/45)u^5 - (2/27)u^3$.
38. The partial fraction decomposition is

$$2 + \frac{-1/16}{x} + \frac{-1/4}{x^2} + \frac{129/16}{x - 4} \quad .$$

39. $(1/2)e^x(\sin x - \cos x)$.
40. (a) $\ln(x + \sqrt{9 + x^2})$, (b) $1/2[x\sqrt{x^2 - 9} - 9 \ln(x + \sqrt{x^2 - 9})]$.
41. Let $u = e^x$ and use partial fractions. Answer: $2 \ln|e^x - 1| - x$.
42. (a) $(1/8) \ln(4x^2 + 1)$, (b) $(1/2) \tan^{-1} 2x$.
43. $(1/54)(1 + 3x^2)^9$.
44. Expand the integrand. Answer: $x + 2x^3 + (9/5)x^5$.
45. (a) $(1/3) \ln|x^3 + 1|$, (b) $(-1/3) \ln|x + 1| + (1/6) \ln(x^2 - x + 1)$ $+ (1/\sqrt{3})\tan^{-1}((2x - 1)/\sqrt{3})$,
(c) $x - (1/3)\ln|x + 1| + (1/6) \ln(x^2 - x + 1) - (4/(9\sqrt{3}))\tan^{-1}((2x - 1)/\sqrt{3})$.
46. (a) Let $u = \sqrt{2x + 1}$. Answer: (In terms of u)$(1/4)[u^5/5 - 2u^3/3 + u]$,
(b) $\sqrt{2x + 1}$, (c) $(1/2) \ln|2x + 1|$.
47. Express as $\int (1/\sqrt{-x^2 + 5x - 6})\,dx$ and use integral table or complete the square.
48. Express as $\int \sqrt{1 - 4x^2}\,dx$ and use integral table or the substitution $4x^2 = \sin^2\theta$.
49. (a) $(-\cos 3x)/3$, (b) $(-1/3)x \cos 3x + (1/9) \sin 3x$.
50. (a) $2\sqrt{x^2 + 1}$, (b) Use integral table or the substitution $x = \tan\theta$;

the latter transforms it to $\int 2 \sec\theta\, d\theta = 2 \ln|\sec\theta + \tan\theta|$ $= 2 \ln|x + \sqrt{x^2 + 1}|$.
51. (a) $(1/25) \ln|5x + 2| + (2/25)/(5x + 2)$, (b) $(-1/5)/(5x + 2)$.

□ □ □

52. Use the substitution $u = \sqrt[4]{x}$.
54. (a) If $v^t = 1 - x$, hence $tv^{t-1}\,dv = -dx$ and $\int x^p(1 - x)^{s/t}\,dx =$

$\int (1 - v^t)^p v^s(-tv^{t-1}\,dt)$, which has a rational integrand. (b) is similar.
(c) The substitution $x = 1/t$ reduces this case to case (a).
This theorem can be used to show that for rational p and q,
$\int \sin^p x \cos^q x\,dx$ is elementary if and only if p or q is odd or p + q is even. To demonstrate this, make the substitution $u = \sin^2 x$.
56. $1/1^2 + 1/2^2 + 1/3^2 + \ldots = (1/1^2 + 1/3^2 + 1/5^2 + \ldots) + (1/2^2 + 1/4^2 + \ldots) =$ $(1/1^2 + 1/3^2 + 1/5^2 + \ldots) + (1/4)(1 + 1/2^2 + 1/3^2 + \ldots)$. Denoting the sum $1/1^2 + 1/3^2 + 1/5^2 + \ldots$ by "x", we have $\pi^2/6 = x + (1/4)(\pi^2/6)$. Solve for x.

CHAPTER 8: Computing and Applying Definite Integrals Over Intervals

SECTION 1: Area
6. $\sqrt{2} - 1$.

9. (a) $\int_1^4 (1 + u)^{10}du$, (b) $\int_2^5 u^{10}du$.

10. (a) $(1/2)\int_0^{1/4}[u/(1-u)]du$, (b) $(-1/2)\int_{3/4}^{1}[(1-u)/u]du$,

(c) $\int_0^{\pi/6}[\sin^3\theta/\cos\theta]d\theta$, (d) $\int_{\pi/3}^{\pi/2}[\cos^3\theta/\sin\theta]d\theta$. The value is

$(1/2)[\ln(4/3) - 1/4]$.

15. $\dfrac{214\sqrt{7} - 512}{405}$.

16. (a) 1/5, (b) 4/5.

□ □ □

18. Area $= \int_0^{\pi}(1/2)(f(\theta))^2 d\theta = \int_0^{\pi/2}(1/2)[(f(\theta))^2 + (f(\theta+\pi/2))^2]\,d\theta$, which by the Pythagorean theorem is not larger than $\int_0^{\pi/2}(1/2)d^2\,d\theta = \pi d^2/4$.

This solution is due to W. Feller.

23. For x in $[2\pi, 4\pi]$, $\sin x/2 = -\sqrt{(1-\cos x)/2}$.

25. Yes. For any α we have $(1/2)\int_0^{\alpha}[(f(\theta))^2 - (f(\theta+\pi))^2]d\theta = 0$. Differentiation with respect to α yields $(f(\alpha))^2 - (f(\alpha+\pi))^2 = 0$, thus $f(\alpha) = f(\alpha+\pi)$.

26. Since area $= \int_0^{2\pi}(1/2)[f(\theta)]^2\,d\theta = \int_0^{\pi}(1/2)\{[f(\theta)]^2 + [f(\theta)+\pi)]^2\}\,d\theta$ and $f(\theta) + f(\theta+\pi) \geq a$, the area is $\geq \pi a^2/4$. We used the fact that if $x + y = a$, $x \geq 0$, $y \geq 0$, then $x^2 + y^2 \geq a^2/2$. To establish this, note that $x^2 + y^2 = x^2 + (a-x)^2 = 2x^2 - 2ax + a^2 = 2(x^2 - ax) + a^2 = 2(x-a/2)^2 - a^2/2 + a^2 = 2(x-a/2)^2 + a^2/2 \geq a^2/2$.

SECTION 2: Volume

3. (a) $\dfrac{32\pi}{5}$, (b) 8π, (c) $\dfrac{176\pi}{15}$, (d) 8π.
6. Half the volume of the cylinder.
7. $(2/3\pi)$(volume of the cylinder).
10. (c) $4\pi^2$.
11. $14\pi^2$.

□ □ □

15. (a) Consider the solid formed by revolving about the y axis the region bounded by $y = f(x)$ and the coordinate axes.
16. Examples are $f(x) = \sin x$ or $\tan x$. The assumption $f(0) = 0$ is only a convenience; e^x would be another example. One has $tf(t) = \int_0^{f(t)} f^{-1}(x)dx + F(t) - F(0)$, where $F' = f$. Replace t by $f^{-1}(u)$ and obtain $f^{-1}(u)u = \int_0^u f^{-1}(x)\,dx + F(f^{-1}(u)) - F(0)$ and solve for $\int_0^u f^{-1}(x)\,dx$.

SECTION 3: The First Moment

6. (a) Area $= \int_1^2 e^{x^2}\,dx$, (b) moment $= \int_1^2 xe^{x^2}dx$,

(c) moment $= \int_1^2 (x+1)e^{x^2}\,dx$, (d) part (b), answer: $(1/2)(e^4 - e)$.

11. $22\pi/3$.
12. 4π.
13. $c(y) = 8 + 3y$.

16. (d) $\frac{(9)(62.4)(\pi)}{64} \int_0^8 (x + 20)x^3\, dx.$

□ □ □

19. Volume = $(\text{area})(2\pi\bar{y}) = (2\pi)(\text{moment})$. But volume = $\int_a^b \pi(f(x))^2\, dx$; solve for moment.
20. Consider the region bounded by $x = \sqrt{\ln y}$, $x = 1$, $x = 2$, $y = 0$. Or, for a simpler region but more involved function, consider the region bounded by the positive coordinate axes and $x = \sqrt{\ln(2 - y)}$.
22. (a) No. (b) Furnish the end R with a density 50 D and determine the depth of the center of gravity of the end. (This will involve the new idea of center of gravity of a non-homogeneous material.) Theorem 2 is valid relative to this center of gravity.

SECTION 4: Arc Length

2. $|v(0)| = 32$, $|v(1)| = 32\sqrt{2}$, $|v(2)| = 32\sqrt{5}$.
3. $(8/27)\,[(10)^{3/2} - (13/4)^{3/2}]$.
5. (b) $|t(4 + 9t^2)^{1/2}|$, (d) $y = x^{3/2}$.
13. $|v(t)| = \{[g(t)\, h'(t)]^2 + [g'(t)]^2\}^{1/2}$.

□ □ □

16. The area of the figure is $\pi ab + (a - b)^2$. By the isoperimetric theorem, if $\pi ab + (a - b)^2 = \pi R^2$, then the perimeter of the figure is greater than $2\pi R$. To show that $R > (a + b)/2$ show that $\pi[(a + b)/2]^2 < \pi ab + (a - b)^2$, or equivalently, that $\pi[(a - b)/2]^2 < (a - b)^2$, which is valid since $\pi < 4$.
17. The length of the inscribed polygon is precisely

$$\sum_{i=1}^{n} \sqrt{r_i^2 + r_{i-1}^2 - 2r_i r_{i-1} \cos(\theta_i - \theta_{i-1})},$$

which may be rewritten as

$$\sum_{i=1}^{n} \sqrt{\left(\frac{r_i - r_{i-1}}{\theta_i - \theta_{i-1}}\right)^2 + 4\, r_i r_{i-1} \left\{\frac{\sin[(\theta_i - \theta_{i-1})/2]}{\theta_i - \theta_{i-1}}\right\}^2}\ (\theta_i - \theta_{i-1}).$$

Arguing for plausibility, one observes that $[(r_i - r_{i-1})/(\theta_i - \theta_{i-1})] = r'(\theta_i^*)$, while the expression in { } is "close" to 1/2. The difficulty of this approach should enhance the appreciation of the indirect approach used in the text.
18. Assume dx/dt and dy/dt are continuous and at P are not both equal to 0. Arc length equals $\sqrt{[\dot{x}(T)]^2 + [\dot{y}(T)]^2}\,(t_2 - t_1)$ for some T in $[t_1, t_2]$ while $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \{[\dot{x}(T_1)]^2 + [\dot{y}(T_2)]^2\}\,(t_2 - t_1)$ for some T_1 and T_2 in $[t_1, t_2]$.
19. Most students give up when they see that they cannot compute the arc length with the aid of the FTC. Perhaps, assign it a second time with the hint "draw the graph." Clearly the arc length is larger than the sum of the ordinates at $x_n = 1/[\pi(2n + 1/2)]$, $n = 1, 2, 3, \ldots$, that is, larger than $\sum_{n=1}^{\infty} 1/[\pi(2n + 1/2)]$, a series whose divergence follows from Chapter 3, Section 1, Exercise 18.

SECTION 5: Area of a Surface of Revolution

4. (a) V(sphere) = (2/3) V (tin can), (b) equal.

5. (a) $2\pi \int_c^d g(t) \, [(g'(t))^2 + (h'(t))^2]^{1/2} dt$,

   (b) $\pi[2\sqrt{145} + (1/6)\ln(12 + \sqrt{145}) - (1/2)\sqrt{10} - (1/6)\ln(3 + \sqrt{10})]$.

6. (b) $2\pi \int_1^6 (\sqrt{x} + 1)\sqrt{1 + \frac{1}{4x}}\, dx$, (c) $\frac{\pi}{6}[125 + 60\sqrt{6} - 17\sqrt{5}$

   $+ (1/6)\ln(\frac{2\sqrt{6} + 5}{\sqrt{5} + 2})]$.

7. (b) $2\pi \int_1^{\sqrt{6}} y^2 \, [1 + (2y)^2]^{1/2} dy$, (c) $\frac{\pi}{4}[\frac{245}{4}\sqrt{6} - \frac{3}{8}\sqrt{2} + (1/8)\ln(\frac{\sqrt{2} + 1}{2\sqrt{6} + 5})]$.

8. They are equal.

9. It is the portion of a ring bounded by two concentric circles and two rays from their center. Area = $2\pi rh$, where r and h are defined in the text.

13. If $a > b$, and $c = \sqrt{a^2 - b^2}$, the surface area is $2\pi a^2 + [\pi ab^2/c]\ln((a+c)/(a-c))$.

15. $\pi^2(r_2^2 - r_1^2)$, where r_1 and r_2 are the inner and outer radii.

17. On axis of symmetry and at a distance $2a/\pi$ from the center of the circle, whose radius is a.

19. $S + 6\pi A$.

□ □ □

20. This is a special case of Tarski's plank problem. Consider the surface of the sphere whose equatorial disk is the given disk. With each strip associate the band on the surface bounded by the two planes perpendicular to the equator and bounding the strip. Note that if the strips cover the disk, then the bands cover the surface of the sphere. Hence, if their widths are $d_1, d_2, \ldots, d_n$, we have

$$\sum_{i=1}^{n} 2\pi a d_i \geq 4\pi a^2, \text{ and thus } \sum_{i=1}^{n} d_i \geq 2.$$

21. We have, on differentiation with respect to b, $2\pi y\sqrt{1 + (dy/dx)^2} = k$, whence (assuming non-negative dy/dx and positive y) $dy/dx = \sqrt{k^2 - 4\pi^2 y^2}/2\pi y$. One solution is $y = k/2\pi$ and we have a cylinder. Otherwise, we have

$$\frac{2\pi y \, dy}{\sqrt{k^2 - 4\pi^2 y^2}} = dx,$$

which yields $(-1/2\pi)\sqrt{k^2 - 4\pi^2 y^2} = x + c$. Squaring shows that this is a circle.

SECTION 6: The Higher Moments of a Function

2. $M_0 = 2$, $M_1 = 0$, $M_2 = \frac{\pi^2}{2} - 4$.

3. No. (It should be emphasized that the "centroid" is related to first moment. There is no corresponding point for higher moments.)

7. (a) $\frac{3m\omega^2}{4}$, (b) $\frac{m\omega^2}{3}$.

9. $30x^2 - 24x + 3$.

□ □ □

12. (a) If f has no roots, then by the intermediate value theorem it is always positive in [a, b] or always negative and $\int_a^b f(x)\,dx$ would not be 0. Note that there is a root c, $a < c < b$. (b) If c is the only root of $f(x) = 0$ in [a, b], consider $\int_a^b f(x)(x - c)dx$. The sign of the integrand $f(x)(x - c)$ remains fixed since f(x) changes sign at c. Thus $\int_a^b f(x)(x - c)\,dx \neq 0$, contradicting the assumption that $\int_a^b xf(x)dx = 0 = \int_a^b f(x)\,dx$. (c) If $\int_a^b x^i f(x)dx = 0$, $i = 0, 1, \ldots, n-1$, then f has at least n roots in [a, b], in fact, at least n roots at which f changes sign. To prove this, assume that f changes sign only at $c_1, c_2, \ldots, c_{n-1}$ and consider $\int_a^b f(x)(x - c_1)(x - c_2)\ldots(x - c_{n-1})dx$.
13. Let h = f - g. According to Exercise 12, there is an infinite set of roots of the equation $h(x) = 0$, hence $h \equiv 0$, since h is a polynomial.

SECTION 7: Average Value of a Function

8. $\int_0^L r\,ds/L = \int_0^{2\pi} r\sqrt{r^2 + (r')^2}\,d\theta/L \geq \int_0^{2\pi} r^2\,d\theta/L = 2A/L$. Equality holds only when $r'(\theta) = 0$ for all θ, that is, in the case of a circle.
12. f(a).

□ □ □

13. This is an instance of substitution in a definite integral.
16. (a) $$\sum_{i=1}^n f(X_i)\, g(X_i)(x_i - x_{i-1}) = \sum_{i=1}^n f(X_i)\sqrt{(x_i - x_{i-1})}\, g(X_i)\sqrt{(x_i - x_{i-1})}$$ $$\leq \left[\sum_{i=1}^n [f(X_i)]^2 (x_i - x_{i-1})\right]^{1/2} \left[\sum_{i=1}^n [g(X_i)]^2 (x_i - x_{i-1})\right]^{1/2}$$ and let the mesh $\to 0$. (c) When equality holds and f(x) is not 0 for all x in [a, b], the polynomial h(t) has degree two and a root, c. We have $cf(x) = g(x)$ for all x in [a, b]. Equality holds also if $f(x) = 0$ for all x in [a,b].
17. We let $x = f(t)$ and for convenience assume that dx/dt is non-negative and wish to prove that

$$\frac{\int_a^b \frac{dx}{dt}\,dt}{b - a} \leq \frac{\int_{f(a)}^{f(b)} \frac{dx}{dt}\,dx}{f(b) - f(a)}$$

Rewriting $\int_{f(a)}^{f(b)} \frac{dx}{dt}\,dx$ as $\int_a^b \left(\frac{dx}{dt}\right)^2 dt$ and doing a little algebra, we wish to prove that

$$\int_a^b \frac{dx}{dt}\,dt \leq \left(\int_a^b 1\,dt\right)^{1/2} \left[\int_a^b \left(\frac{dx}{dt}\right)^2 dt\right]^{1/2}.$$

This inequality follows from Schwarz's inequality of Exercise 16 by letting $f(x) = 1$ and $g(x) = dx/dt$.
18. We wish to show that $\int_0^a xh(x)\,dx/\int_0^a h(x)\,dx \leq \int_0^a x^2 h(x)\,dx/\int_0^a xh(x)\,dx$, where h is the density function. This inequality is a consequence of Schwarz's inequality of Exercise 16 in case $f(x) = x\sqrt{h(x)}$ and $g(x) = \sqrt{h(x)}$.
19. Let $g(x) = 1/f(x)$ in Schwarz's inequality of Exercise 16.

SECTION 8: Improper Integrals

1. π.
2. 2.
6. (a) $G(0) = 0$; (b) $-\pi/2$, (c) $\pi/2$.
10. $a < -1$.
11. $a > -1$.
13. (a) $M_2 = 2/c^2$, (b) $M_0 = 1$, $M_1 = 1/c$, $M_2 = 2/c^2$.
17. 1.

□ □ □

18. Let $u = x^2$.
19. Let $u = x^2$.
20. Let $x = \sin\theta$.
21. From a table of integrals or by an integration by parts, $\int x^4 \ln x\, dx = x^5 [(\ln x)/5 - 1/25]$, which we call $F(x)$. Now $F(1) = -1/25$ and $\lim_{x\to 0} F(x) = (1/5) \lim_{x\to 0} x^5 \ln x = (1/5) \lim_{y\to -\infty} e^{5y} y = -(1/5) \lim_{t\to\infty} t/e^{5t} = 0$, by Example 5, in Chapter 5, Section 2. Hence, the improper integral has the value $-1/25$.
22. (c) Observe that $G(a) + G(a) = \int_0^\infty (1 + x^a)/[(1 + x^a)(1 + x^2)]dx = \pi/2$.
24. Factor $x^4 + 1 = (x^2 + \sqrt{2}\, x + 1)(x^2 - \sqrt{2}\, x + 1)$ and use partial fractions.
25. Use the substitution $u = \sqrt{x}$.
26. First use integration by parts with $u = \sin^2 x$ and $dv = (1/x^2)\, dx$. Then note that $du = 2 \sin x \cos x = \sin 2x$, and make the change of variable $w = 2x$.
27. Integration by parts.

SECTION 9: Probability Distribution and Density

1. (b) $1/(1 + t)^2$.
2. (b) $(2 \tan^{-1} x)/\pi$.
4. (a) $e^{-k(n-1)} - e^{-kn}$, (c) 1.
5. (a) $F(0) = 0$; $F(1) = \frac{26}{1000}$; $F(5) = \frac{30}{1000}$; $F(10) = \frac{32}{1000}$; $F(20) = \frac{39}{1000}$

   $F(30) = \frac{51}{1000}$; $F(40) = \frac{69}{1000}$; $F(50) = \frac{112}{1000}$; $F(60) = \frac{209}{1000}$

   $F(70) = \frac{391}{1000}$; $F(80) = \frac{664}{1000}$; $F(90) = \frac{929}{1000}$; $F(100) = 1$.

   (c) $f(0) = \frac{26}{1000}$; $f(5) = \frac{2}{(5)1000}$; $f(10) = \frac{7}{10000}$; $f(20) = \frac{12}{10000}$

   $f(30) = \frac{18}{10000}$; $f(40) = \frac{43}{10000}$; $f(50) = \frac{97}{10000}$; $f(60) = \frac{182}{10000}$

   $f(70) = \frac{273}{10000}$; $f(80) = \frac{265}{10000}$; $f(90) = \frac{71}{10000}$; $f(100) = \frac{71}{10000}$.

   (e) About 65, the result depending on the choice of approximating points.

   (f) $m(0) = \frac{26}{1000}$; $m(5) = \frac{2}{970}$; $m(20) = \frac{12}{9610}$; $m(60) = \frac{182}{7910}$.

6. (a) $f(t) = (a/b)t^{a-1} e^{-t^a/b}$, $z(t) = (a/b)t^{a-1}$.
7. (a) 0.91, (b) 0.005, (c) 0.17 = F(40) - F(20).
8. (a) About 31.
9. (a) About 30.
10. (a) F(0) = 0; F(20) = 0.1, F(80) = 0.5.
11. (b) About 41; (c) About 65.

12. (a) $F(0) = 0$; $F(1) = \frac{133}{1000}$; $F(5) = \frac{191}{1000}$; $F(10) = \frac{209}{1000}$; $F(20) = \frac{237}{1000}$;

$F(30) = \frac{287}{1000}$; $F(40) = \frac{351}{1000}$; $F(50) = \frac{428}{1000}$; $F(60) = \frac{536}{1000}$; $F(70) = \frac{694}{1000}$

$F(80) = \frac{860}{1000}$; $F(90) = \frac{962}{1000}$; $F(100) = 1$.

(c) $f(0) = \frac{133}{1000}$; $f(5) = \frac{18}{5000}$; $f(60) = \frac{158}{10000}$ and see answer to Exercise 5.

15. (b) Smokers $m(40) = \frac{44}{10000}$; nonsmokers $m(40) = \frac{11}{10000}$.

□ □ □

16. (a) From $f(t)/[1 - F(t)] = z(t)$ it follows that $\int_0^x f(t)/[1 - F(t)]dt = \int_0^x z(t)dt$, hence $-\ln[1 - F(x)] + \ln[1 - F(0)] = \int_0^x z(t)\,dt$. Solve for F(x). (b) Differentiate F(x) given in (a).
18. (a) The fraction of the original population that reaches age a is 1 - F(a). Of this fraction approximately $f(t)\,\Delta t$ have between t - a and $t + \Delta t - a$ years remaining. (b) Integrate by parts with $u = t - a$, $dv = f(t)\,dt$ and $v = F(t)$ or $F(t) - 1$.
19. Note that this provides a formula for expectation directly in terms of F.
21. (a) Surely the expected life of the device is not longer than that of a device which turns on the second component only when the first fails. With this alternative arrangement the device has an expected life A + B. (b) Let the distribution of the first component be F(t) and of the second component, G(t). Then F(t) G(t) is the distribution function of the device, hence $F(t)\,dG/dt + G(t)\,dF/dt$ is the probability density of the device. The expected life of the device is

$$\int_0^\infty t\,[F(t)\,dG/dt + G(t)\,dF/dt]\,dt;$$

since $F(t) \leq 1$ and $G(t) \leq 1$, the expected life is at most

$$\int_0^\infty t\,dG/dt\,dt + \int_0^\infty t\,dF/dt\,dt = B + A.$$

CHAPTER 9: Computing and Applying Definite Integrals Over Plane and Solid Sets

SECTION 1: The Center of Gravity of a Flat Object (lamina)

2. 6 feet to the right of m_1.

3. $\bar{x} = \left(\sum_{i=1}^{n} m_i x_i \right) / \sum_{i=1}^{n} m_i$.

□ □ □

18. At the center of gravity. It minimizes the polar moment of inertia. "It is easiest to spin something around its center of gravity."
19. The center of gravity of the forest.
20. Many students leave their common sense and intuition behind them when they study calculus. One answer to this exercise is, "Draw the forest on tagboard, cut it out, and find two balancing lines. The center of gravity is at their intersection." Put more layers of tagboard at the denser or more valuable parts.

SECTION 2: Computing $\int_R f(P)\, dA$ by Introducing Rectangular Coordinates

8. (a) $(1/x^2)[1 - \cos(x^2)]$.
9. (a) 32/9, (b) 64/15, (c) 128/21, (d) 352/315, (e) 512/525.
12. (a) 4/27, (b) 4/65, (c) 4/119, (d) 12,296/208,845,
(e) 4,048/502,775.
13. (a) $0 \le y \le 1,\ 0 \le x \le y^2$. (b) $0 \le y \le 1,\ 0 \le x \le 1$.

□ □ □

14. (a) Clearly f is continuous, hence has a definite integral over an interval. (b) $|G(x) - G(x^*)| = |\int_0^2 [f(x, y) - f(x^*, y)]dy| \le \int_0^2 |f(x, y) - f(x^*, y)|dy \le \int_0^2 2|x - x^*|dy = 4|x - x^*|$. Thus G is continuous. This shows that the repeated integral $\int_0^3 [\int_0^2 f(x, y)\, dy]\, dx$ exists.
15. (a) The sum is of the form $\sum_{k=1}^{r} f(P_k) A_k$. (b) Write the sum as $\sum_{i=1}^{n} \left[\sum_{j=1}^{n} f(x_i, y_j)(y_j - y_{j-1}) \right] (x_i - x_{i-1})$. The inner sum is "close" to $\int_0^2 f(x_i, y)\, dy = G(x_i)$. The repeated sum is then "close" to $\int_0^3 G(x)\, dx$. A measure of the closeness could be worked out.

SECTION 3: Computing $\int_R f(P)\, dA$ by Introducing Polar Coordinates

2. For $0 \le r \le 1$, we have $0 \le \theta \le \pi/4$.
For $1 \le r \le \sqrt{2}$, we have $\tan^{-1}[(r^2 - 1)^{1/2}] \le \theta \le \pi/4$.
3. (a) For each y between -a and a, x varies between $-\sqrt{a^2 - y^2}$ and $\sqrt{a^2 - y^2}$.
(b) For each θ between 0 and 2π, r varies between 0 and a.
(c) For each r between 0 and a, θ varies between 0 and 2π.
4. (a) rectangular: $f(P) = x^2$ where $P = (x, y)$.
polar: $f(P) = r^2 \cos^2\theta$ where $P = (r, \theta)$.
(b) rectangular: $f(P) = x^2 + y^2$ where $P = (x, y)$.
polar: $f(P) = r^2$ where $P = (r, \theta)$.
5. (a) $f(r, \theta) = r\cos\theta$, (b) $f(r, \theta) = r$.
15. (a) 1/12, (b) 1/20.
18. (a) For each x between 0 and a, y varies from 0 to $\sqrt{2a^2 - x^2}$.
(b) For each y between 0 and a, x varies between 0 and a; for each y between a and $a\sqrt{2}$, x varies between 0 and $\sqrt{2a^2 - y^2}$.

(c) For each θ between 0 and $\pi/4$, r varies between 0 and a sec θ. For each θ between $\pi/4$ and $\pi/2$, r varies between 0 and a $\sqrt{2}$.

(d) For each r between 0 and a, θ assumes all values between 0 and $\pi/2$; for each r between a and $a\sqrt{2}$, θ assumes all values between $\cos^{-1}(a/r)$ and $\pi/2$.

19. (b) $4\pi/3$; $2\pi - (32/9)$, (c) edge of town.

□ □ □

21. Since $\int_0^\pi f(\theta)\,d\theta = 0$, there is a, $0 < a < \pi$, $f(a) = 0$. Assume only one such root. At a, $f(\theta)$ changes sign; hence $\int_0^\pi f(x)\sin(x-a)\,dx$ is *not* 0 since $\sin(x-a)$ also changes sign only at a (for x in $[0, \pi]$). But $\sin(x-a) = \sin x \cos a - \cos x \sin a$. Since $\int_0^\pi f(x)\sin x\,dx = 0 = \int_0^\pi f(x)\cos x\,dx$, the definite integral $\int_0^\pi f(x)\sin(x-a)\,dx$ would be 0. Contradiction.

22. From Exercise 11 we know that $\int_0^\pi [g^3(\theta+\pi) - g^3(\theta)]\cos\theta\,d\theta = 0 = \int_0^\pi [g^3(\theta+\pi) - g^3(\theta)]\sin\theta\,d\theta$. To apply Exercise 21 to $f(\theta) = g^3(\theta+\pi) - g^3(\theta)$ we need to make $\int_0^\pi [g^3(\theta+\pi) - g^3(\theta)]\,d\theta = 0$. This we may achieve in advance by choosing the polar axis properly. (The existence of such a choice may be established by considering the function

$$h(\alpha) = \int_{\alpha+\pi}^{\alpha+2\pi} g^3(\theta)\,d\theta - \int_\alpha^{\alpha+\pi} g^3(\theta)\,d\theta.$$

(Note that $h(\alpha+\pi) = -h(\alpha)$ and apply the intermediate value theorem.)

23. Assume $f(0) > 0$. Hence $f(2\pi) > 0$. Since $\int_0^{2\pi} f(\theta)\,d\theta = 0$, there is b, $0 < b < 2\pi$, $f(b) < 0$. By the intermediate value theorem there are numbers a_1 and a_2, $0 < a_1 < b < a_2 < 2\pi$, $f(a_1) = 0 = f(a_2)$.

24. By Exercise 23 there are at least two roots a_1 and a_2, $0 < a_1 < a_2 < 2\pi$. If f has precisely two roots in $(0, 2\pi)$ or precisely three roots in $(0, 2\pi)$ then f changes sign at exactly two of them (as a diagram will show). Let us assume that a_1 and a_2 are the roots where $f(\theta)$ changes sign. Then $f(\theta)$ and $\sin((\theta - a_1)/2)\sin((\theta-a_2)/2)$ change sign only at a_1 and a_2 for θ in $[0, 2\pi]$. Argue as in Exercise 21 concerning

$$\int_0^{2\pi} f(\theta)\sin\left(\frac{\theta - a_1}{2}\right)\sin\left(\frac{\theta - a_2}{2}\right)d\theta.$$

25. $\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{2\pi} g^3(\theta)g'(\theta)\,d\theta = [g^4(2\pi) - g^4(0)]/4 = 0$. By Exercise 11,

$\int_0^{2\pi} g^3(\theta)\cos\theta\,d\theta = 0 = \int_0^{2\pi} g^3(\theta)\sin\theta\,d\theta$; integration by parts applied to the first integral, $u = g^3(\theta)$ and $dv = \cos\theta\,d\theta$, yields $0 = g^3(\theta)(\sin\theta)\Big|_0^{2\pi} - \int_0^{2\pi} 3g^2(\theta)g'(\theta)\sin\theta\,d\theta$, hence $\int_0^{2\pi} g^2(\theta)g'(\theta)\sin\theta\,d\theta = 0$. Similarly $\int_0^{2\pi} g^2(\theta)g'(\theta)\cos\theta\,d\theta = 0$. By Exercise 24, $g^2(\theta)g'(\theta)$ (hence $g'(\theta)$) vanishes at at least four numbers that are distinct (mod 2π). As the little "dr, $rd\theta$"-triangle suggests, the radius is perpendicular to the curve when $g'(\theta) = 0$.

26. Let $f(\theta) = g(\theta) - g(\theta + (\pi/2))$ and apply Exercise 24.

28. For (a) use the substitution $x = 2u$; for (b) the substitution $u = x^2$; for (c) integration by parts, with $u = x$ and $dv = xe^{-x^2}dx$; for (d) use the result of (c) and the substitution $u = x^2$.
29. Simply replace the variables r and θ by x and y--that is, interpret r and θ as rectangular coordinates.

SECTION 4: Coordinate Systems in Three Dimensions and Their Volume Elements

1. $0 \le z \le 1; 0 \le y \le 1 - z; 0 \le x \le 1-z-y$.
2. $0 \le z \le 4; 0 \le y \le 3 -(3z/4),\ 0 \le x \le 2- z/2 - 2y/3$.
3. $0 \le x \le 5; 0 \le y \le \sqrt{25 -x^2}; 0 \le z \le \sqrt{25- x^2 -y^2}$.
4. (a) $r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}(y/x),\ z = z$, (b) $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.
5. $0 \le r \le 5,\ 0 \le \theta \le 2\pi; 0 \le z \le \sqrt{25 - r^2}$.
6. (a) $\rho = \sqrt{x^2 + y^2 + z^2},\ \phi = \tan^{-1}(\sqrt{x^2 + y^2}/\sqrt{x^2 + y^2 + z^2})$, $\theta = \tan^{-1}(y/\sqrt{x^2 + y^2})$ (b) $r = \rho\sin\phi,\ \theta = \theta,\ z = \rho\cos\phi$.
7. $-4 \le z \le 4,\ -3 \le x \le 3,\ -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}$ and $4 \le |z| \le 5,\ -\sqrt{25 - z^2} \le x \le \sqrt{25 - z^2},\ -\sqrt{25 - z^2 - x^2} \le y \le \sqrt{25 - z^2 - x^2}$.
8. $0 \le z \le 4 - \frac{\sqrt{3}z}{2} \le x \le \frac{\sqrt{3}}{2}z,\ -\sqrt{\frac{3z^2}{4} - x^2} \le y \le \sqrt{\frac{3z^2}{4} - x^2}$.
9. $0 \le z \le 4,\ 0 \le \theta \le 2\pi,\ 0 \le r \le z/\sqrt{3}$.
11. (a) $\phi = \pi/2$, (b) $\phi = 0$, (c) $\phi = \pi/2,\ \theta = 0$.
12. (a) $z = 0$, (b) $\theta = \pi/4$ or $5\pi/4$, (c) $z = 0,\ \theta = 0$.

□ □ □

SECTION 4: Coordinate Systems in Three Dimensions and Their Volume Elements

13. (a) The right triangle in the yz-plane whose vertices are (1, 1), (1, 2), (2, 2) is revolved a quarter of a complete turn around the z-axis. (b) $1 \le r \le 2,\ r \le z \le 2,\ 0 \le \theta \le \pi/2$, (c) $\tan^{-1}1/2 \le \phi \le \pi/4$, $0 \le \theta \le \pi/2,\ 1/\sin\phi \le \rho \le 2/\cos\phi$. (d) $0 \le x \le 2$, $\sqrt{1 - x^2} \le y \le \sqrt{4 - x^2}$ if $0 \le x \le 1$ and $0 \le y \le \sqrt{4 - x^2}$ if $1 \le x \le 2$, $\sqrt{x^2 + y^2} \le z \le 2$.
14. A cone, a half plane, a sphere; a plane, a half plane, a (surface of a) cylinder.

SECTION 5: Computing $\int_R f(P)\,dV$ by Introducing Coordinates in R

9. $122\,\pi/3$.
11. $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^3 + xz^2)\,dz\,dy\,dx$.
12. 1/24.
22. πa^4 (hence $\int_R x^2\,dV = \pi a^4/3$).

□ □ □

26. This result is important in the theory of gravity. (The potential due to a homogeneous sphere is the same as if all its mass were at its center.)

Locate the point Q on the positive z axis. Use spherical coordinates and integrate first with respect to ϕ.

28. The average distance from Q at a distance q from the origin to points in $[-a, a]$ is $\ln[(q + a)/(q - a)]/2a$, and involves a. (Nor is the analog for Q in the plane of a disk true.)

29. (a) The set of "points" (x, y, q, t) such that $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, $0 \leq t \leq 1$. (b) $\int_R 1\, dV$ where R is the "3-cell." (c) 1.

30. (a) The set of "points" (x, y, z, t) such that $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, $0 \leq t \leq 1$, $0 \leq x + y + z + t \leq 1$. (b) $\int_R (1 - x - y - z)dV$ where R is the "3-simplex" described in the exercise.

(c) $\int_0^1 (\int_0^{1-x} (\int_0^{1-x-y} (1 - x - y - z)\, dz)\, dy)\, dx$. (d) 1/24.

31. (a) The set of "points" (x, y, z, t) such that $x^2 + y^2 + z^2 + t^2 \leq 1$.

(b) $\int_R 2\sqrt{1 - x^2 - y^2 - z^2}\, dV$ where R is the "3-ball". (c) For each of the three coordinate systems there will be a repeated integral

(d) $\int_R \pi(1 - r^2)\, dA$ where R is the "2-ball" and r is polar radius.

For if we denote $x^2 + y^2$ by r^2 and $z^2 + t^2$ by s^2, then $r^2 + s^2 \leq 1$; that is, introduce polar coordinates into the xy-plane and the zt-plane.

CHAPTER 10: The Higher Derivatives

SECTION 1: The Geometric Significance of the Sign of the Second Derivative

2. (a) $x > 1$, (b) $x > 2/3$ and $x < 0$, (c) $x = 0$ and $x = 2$. (d) $x = 1$, (e) $x = 0$ and $x = 2/3$.

5. $f(x) > 0$ when $x = 2n\pi + y$, $0 < y < \pi$; $f'(x) > 0$ when $x = 2n\pi + y$, $\pi/2 < y < 3\pi/2$; $f''(x) > 0$ when $x = 2n\pi + y$, $\pi < y < 2\pi$. f and f'' change sign at $n\pi$; f' changes sign at $n\pi + \pi/2$.

6. No, y'' never changes sign.

7. y' changes sign at 0; y'' changes sign at $x = \pm 1/\sqrt{3}$.

8. Since $f'(x) = 1 + x^2$, $f''(x) = 2x$.

11. Since there is some "margin of interpretation" in specifying the coordinates, the answer is not included.

12. (a) Yes, (b) No; y'' does not change sign (at 0).

13. (c) No. Pick a tangent line on the graph. Since $f'(x) < 0$, this line cuts the x-axis. Since $f''(x) < 0$, the graph lies below the chosen tangent line; it, too, must cut the x-axis.

14. (d) $f(1) = -20$, $f(5) = -100$, $f(2) = -46$.

15. (a) Inflection points at $x = \pm 1/\sqrt{3}$, (b) Inflection points at $x = \pm 1/\sqrt{2}$.

□ □ □

16. (a) Slopes upward, for $g'(f(x)) = 1/f'(x)$. (b) is concave downward, for $g''(f(x))f'(x) = -f''(x)/[f'(x)]^2$. One may also solve the problem geometrically, since the graph of g is the reflection in the line y = x of the graph of f.

17. (b) If $f(a) < 0$, $0 < a < 1$ then by the law of the mean there are x_1, $0 < x_1 < a$, $f'(x_1) < 0$, and x_2, $a < x_2 < 1$, $f'(x_2) > 0$. The law of the mean applied to the function f' implies the existence of x_3, $x_1 < x_3 < x_2$, $f''(x_3) > 0$.

18. This generalization of Exercise 17 may be obtained from it by applying it to the function g, where $g(a) = f(ax_1 + (1-a)x_2) - af(x_1) - (1-a)f(x_2)$. The second derivative of g with respect to a is $(x_1 - x_2)^2 f''(ax_1 + (1-a)x_2) \geq 0$. Thus $g(a) \geq 0$, which completes the exercise.
19. Consider $x_1 < x_2$. Since $(x_2, f(x_2))$ lies above the tangent at $(x_1, f(x_1))$ and $(x_1, f(x_1))$ lies above the tangent line at $(x_2, f(x_2))$ we have $f(x_2) > f(x_1) + (x_2 - x_1)f'(x_1)$ and $f(x_1) > f(x_2) + (x_1 - x_2)f'(x_2)$, inequalities which imply that $f'(x_2) > f'(x_1)$. Since f' is increasing, its derivative f'' is never negative.

SECTION 2: The Significance in Motion of the Second Derivative

1. $y = -16t^2 + 64t$.
3. $y = 96 - 16t^2$; for $\sqrt{6}$ seconds.
5. A ball thrown up at that time and with the same speed as that with which the ball in the example strikes the ground would incorporate the path of the example.
10. The velocities have the same magnitude but opposite signs; the speeds are equal.
11. The change in coordinate; the distance travelled (as registered by an odometer, say).
12. Distance travelled; acceleration.
13. (b) 6, (d) $y = 0$, (e) $y = \pm 6$.

□ □ □

14. The constant acceleration is irrelevant; assume only that the acceleration and deceleration is the same at both stops and that the maximum speed is reached both times. Thus it travels 30 - 24 = 6 blocks in 120 - 96 = 24 seconds at its maximum speed.
15. (c) Use integration by parts.
16. Consider the graph of $y = v(t)$. If $|v'(t)| \leq 4$ for all t in $[0, 1]$, then the graph is properly within the triangle whose vertices are (0, 0), (1,0), (1/2, 2); hence $\int_0^1 v(t)dt$ is less than the area of the triangle, and hence less than 1. (b) Sketch a graph looking like the triangle described in (a) and meeting the conditions.

SECTION 3: The Second Derivative and Curvature

7. The absolute value of the curvature is $ba^4/(a^4 - a^2x^2 + b^2x^2)^{3/2}$.
8. The absolute value of the curvature is $ba^4/(a^4 \sin^2\theta + b^2a^2\cos^2\theta)^{3/2}$.
9. (b) Probably not. If the wheel rotates "instantaneously at rest," then the tack rotates on a circle of diameter 2. Since the hub of the wheel moves, the radius of curvature is presumably larger than 2.

□ □ □

15. Let $x = f(s)$ and $y = g(s)$ and use the method of Example 2.
16. (b) Differentiate (a) with respect to s. (c) Solve for x'' in (b) and substitute in $x'y'' - y'x''$.

SECTION 4: Information Supplied by the Higher Derivatives

4. (a) 0, (b) e^x, (c) $-e^{-x}$, (d) $-\pi^{31}\cos\pi x$.
5. (a) 30!, (b) 31!x, (c) 0, (d) e^{-x}, (e) $-\pi^{30}\sin\pi x$.
6. (a) $60(x-4)^2$ and 0, (b) $120(x-4)$ and 0, (c) 120 = 5! and 120, (d) (32)(5!), (e) 0.

8. $(x - 2)^3 - 5(x - 2) - 3$.
9. (a) $2(x - 1)^2 - 1$, (b) $2(x + 2)^2 - 12(x + 2) + 17$.
11. $f^{(1)}$ vanishes at least three times; $f^{(2)}$ at least twice; $f^{(3)}$ at least once.
13. (a) $4(x - 2)^3 + 30(x - 2)^2 + 67(x - 2) + 48$,
    (b) $f(2.1) \doteq 48 + 6.7 = 54.7$ $\qquad f(1.9) \doteq 48 - 6.7 = 41.3$.
17. $-e^{-x}(f + f^{(1)} + f^{(2)} + \ldots)$
20. $D^{20}[(\cos 3x + 3 \cos x)/4] = (1/4)(3^{20} \cos 3x + 3 \cos x)$.

□□□

22. (a) $f'(x) = a$, constant. Thus $f(x) - ax$ has a derivative equal to 0 for all x. (b) Similar.
23. If $D^n(f(x)) = 0$ for all x then f is a polynomial of degree at most $n - 1$.
26. (a) If r is the root of the polynomial equation $f(x) = 0$ and $D^n f = 0$ for some r. (b) $\sin \pi = 0$, all the derivatives of sin x at 0 are 0 or ± 1, and $(1 + D^2) \sin x = 0$. It is not known whether e is quasialgebraic.
28. $f(x) = x^n/n! - nx^{n+1}/n! + \ldots + (-1)^n x^{2n}/n!$ (a) clearly $f^{(j)}(0) = 0$ $j < n$ while $f^{(j)}(0) = \pm C^n_j D^j(x^j)/n!$ evaluated at 0 for $2n \geq j \geq n$, which is an integer since $D^j(x^j) = j!$ is divisible by $n!$ Furthermore $f^{(j)}(x) = 0$ for all x and all $j > 2n$.
31. The statement "π^2 is irrational," for if a number is rational its square is. However, $\sqrt{2}$ has a rational square.
32. We have $a_n = f^n(0)/n!$; hence $f(x) = 3 + 2x + (5/2)x^2 + (1/12)x^3$.

CHAPTER 11: The Maximum and Minimum of a Function

SECTION 1: Maximum and Minimum of f(x)

11. The remaining fence should form the three other walls.
18. At $x = 0$. Neither.
21. (a) The derivative is 0 when $\cos x = \sin x$. Local maxima occur when $x = 2n\pi + \pi/4$. The global maximum occurs at $x = \pi/4$.

□□□

22. Volume as a function of θ (half the angle subtended by the base of the typical inscribed cone at the center) is $\pi(a \sin\theta)^2(a + a \cos\theta)/3$. The maximum occurs when $\cos\theta = 1/3$. The volume is then $32\pi a^3/81$. An algebraic solution can be obtained also by working from the equations $V = \pi r^2 h/3$ and $(h - a)^2 = a^2 - r^2$, implicitly or explicitly.
24. Let θ be the angle subtended by arc BP at the center. We wish to minimize $f(\theta) = 200 \cos(\theta/2)/100 + 100\,\theta/200$, subject to $0 \leq \theta \leq \pi$. The maximum of f occurs when $\theta = \pi/3$ and the minimum when $\theta = \pi$. He should not swim at all.
25. Cut along PQ and lay it flat to become a rectangle of width L and height h. Then glue n such rectangles together to become a rectangle of width nL and height h. Interpret the path on this rectangle: it will be a straight line.
26. Let $f(x) = \sin 5x/\sin x$. We have $f(\pi/2) = 1$ and $f(\pi - x) = f(x)$, $\lim_{x\to 0} f(x) = \lim_{x\to 0} [(\sin 5x)/5x]/[(\sin x)/5x] = 5$, hence $\lim_{x\to\pi} f(x) = 5$. Consider only x in $(0, \pi/2)$. We have $\sin 2x/\sin x = 2 \sin x \cos x/\sin x = 2 \cos x < 2$. Then $\sin 3x/\sin x = (\sin x \cos 2x + \cos x \sin 2x)/\sin x = \cos 2x + \cos x(\sin 2x/\sin x) < 1 + 2 = 3$. Continue inductively. (Calculus is not always the best way.)

28. Since the car can be anywhere on the circle of radius 1 and center at his camp, he should walk 1 mile in a straight line, then walk around the circle. If he walks at most $1 + 2\pi$ miles he will surely reach his car. No shorter route will do since he may have to make this complete trip around the circle. If he only wants to reach the road he has a shorter route, for instance: travel $\sqrt{2}$ miles in a straight line, then 1 mile along a tangent to the circle, then continue three quarters of the way around the circle for a total distance of $\sqrt{2} + 1 + (3/2)\pi = 7.13$ miles, which is less than $1 + 2\pi = 7.28$. If he travels in a direction inclined at an angle θ to some radius until he meets the tangent line at that radius, then walks on this tangent line until he meets the circle, then walks $2\pi - 2\theta$ miles on the circle, he will meet the road. The total distance is $\sec\theta + \tan\theta + 2\pi - 2\theta$, which has a minimum value $\sqrt{3} + (5/3)\pi = 6.97$ miles when $\theta = \pi/6$. This may be the shortest route.

SECTION 2: Maximum and Minimum of f(x, y)

1. (a) $z_x = 1/x$, $z_y = -1/y$; (b) $z_x = 2x/y$, $z_y = -(x^2 + 1)/y^2$;
   (c) $z_x = \tan^{-1} y$, $z_y = x/(1 + y^2)$.
2. (a) $z_x = 2x\cos(x^2 y)$, $z_y = x^2\cos(x^2 y)$; (c) $z_x = yx^{y-1}$, $z_y = x^y \ln x$;
   (d) $z_x = 1/y$, $z_y = -x/y^2$.
6. This is a study of a saddle point.
7. Since f_x and f_y simultaneously vanish only at (0, 0), the maximum cannot occur within the triangle. It must occur on the border. On the two legs, f(x, y) = 0. Examination of f(x, y) for (x, y) on the hypotenuse shows that the maximum occurs at (1/2, 1/2) where f(x, y) = 1/4.
12. Use (4). This means that

$$\sum_{i=1}^{n} d_i = 0,$$

where d_i is the vertical deviation from (x_i, y_i) to the line of regression.

□ □ □

16. Let $x = cy + d$ be the line, assuming that it is not vertical. Then we wish to minimize

$$f(c, d) = \sum_{i=1}^{n} (cy_i + d - x_i)^2.$$

The formulas for c and d are obtainable from those for m and b by interchanging the roles of x and y.

SECTION 3: Maximum and Minimum of ax + by + c; Linear Programming

3. (b) (1, 2), (2, 4), (3, 1).
5. Evaluating $-x + 3y + 6$ at each of the vertices, we see that the maximum is 10, occurring at (5, 3).
6. (b) (1, 1), (5, 2), (4, 6), (2, 3); (c) 4 (at (5, 2)).

CHAPTER 12: Series

SECTION 1: The n^{th} Term, Integral, Alternating Series Tests

2. (b) Use S_4 and S_5.

6. (b), (c), (f) converge; the others diverge.
10. Note that $1/7! < 0.001$. We have
$0.8415 = 1 - 1/3! + 1/5! . 1/7! < \sin 1 < 1 - 1/3! + 1/5! = 0.8417$.
12. In (a) and (b) a_n does not approach 0. (c) converges.

□□□

14. Recall that $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges. This suggests the example

$p_1 = 2/\sqrt{1}$, $p_2 = 1/\sqrt{1}$, $p_3 = 2/\sqrt{2}$, $p_4 = 1/\sqrt{2}$, ... $p_{2n-1} = 2/\sqrt{n}$, $p_{2n} = 1/\sqrt{n}$, ...
15. (a) Rewrite $1/(1 - 1/p_i^2)$ as $1 + 1/p_i^2 + 1/p_i^4 = \ldots$ and multiply the m series. (A simpler proof can be obtained similarly by considering $\prod_{i=1}^{m} 1/[1 - 1/p_i]$ and the divergence of the harmonic series.)
17. The series of positive terms diverges; so does the series of negative terms. Take enough positive terms so that their sum exceeds 2, then the first negative term. Then take from the remaining positive terms enough so that their sum exceeds 2, then the second negative term. Etc...

SECTION 2: Comparison and Ratio Tests, Absolute Convergence
6. Converges only for x in $[-1, 1]$.
11. (a) 4, 8, 64/6 = 10.7, 256/24 = 10.7, 1024/120 = 8.5, 4,096/720 = 5.7, 16,384/5,040 = 3.2.

14. (a) Since $\lim_{n\to\infty} (|u_{n+1}|/|u_n|) = 1/2$, $\sum_{n=1}^{\infty} u_n$ converges absolutely.

15. (a) 3, 9/2, = 4.5, 27/6 = 4.5, 81/24 = 3.4, 243/120 = 2.0, 729/720 = 1.0.

□□□

18. Select N such that $c_n^2 \leq 1$ for $n > N$. Then

$$\sum_{n=N+1}^{\infty} a_n c_n^2 \leq \sum_{n=N+1}^{\infty} a_n ,$$

which is convergent.
19. Differentiate the second identity in Exercise 9(a).
20. In one-sixth of the experiments an ace shows on the first throw. In (5/6)(1/6) of the experiments an ace appears the first time on the second throw, etc. Then use Example 2.
21. Both involve geometric series. The answer to (a) is (13/5)h and to (b) is $5\sqrt{h}/4$ seconds. For the latter we assume that an object falls $16t^2$ feet in t seconds. Note that the ball covers 5/13 of the length of its journey during the first descent, which requires only 1/5 of the total time.

SECTION 3: Truncation Error E_n
3. We seek n such that $|r^{n+1}/(1 - r)| < 0.001$. Since $|1 - r| < 2.5$, it suffices to have $|r^{n+1}| < 0.0025$ or $(0.6)^{n+1} < 0.0025$ (n = 11 is adequate).
5. 1000.

□□□

7. Let m be an integer greater than n. Then

$$\sum_{n+1}^{m} u_i \le \int_n^m f(x)\,dx \le \int_n^\infty f(x)\,dx.$$

Thus

$$E_n = \sum_{n+1}^{\infty} u_i \le \int_n^\infty f(x)\,dx.$$

SECTION 4: Power Series

13. This method is due to Newton.

CHAPTER 13: Taylor's Series

SECTION 1: Taylor's Series in x and in x - a

1. (a) Since $1/11! < 0.00001$, the first five terms suffice, giving sin (1) = 0.84147.

16. $\sin x = 1 - (x - \pi/2)^2/2! + (x - \pi/2)^4/4! - (x - \pi/2)^6/6!$ (showing first seven terms).

17. $\cos x = 1/2 - (\sqrt{3}/2)(x - \pi/3) - (1/2)(x - \pi/3)^2/2! + (\sqrt{3}/2)(x - \pi/3)^3/3!$
$+ 1/2(x - \pi/3)^4/4! - (\sqrt{3}/2)(x - \pi/3)^5/5!$

□ □ □

28. The assumption that the rule D(f + g) = Df + Dg extends to

$$D\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} D(f_n).$$

(The interchange of limits, of which this is an instance, is discussed in Chapter 20.)

29. $f'(0) = \lim_{x\to 0} e^{-1/x^2}/x$. Let $u = 1/x$ and consider $\lim_{n\to\infty} u/e^{u^2}$, making use of the fact that $e^v > v^2/2$, proved in Chapter 5. Treat $f''(0)$ and $f'''(0)$ similarly; more generally, $f^{(n)}(x) = e^{-1/x^2} g_n(x)$, $x \neq 0$, where $g_n(x)$ is a rational function of x, hence $f^{(n+1)}(0) = 0$.

30. This exercise was contributed by an electrical engineer. (a) The best second degree polynomial is $(60 - 3\pi^2)/\pi^3 - [(6 - 72/\pi^2)/\pi^3]x^2 = 0.980 - 0.042x^2$. Note that it is not the front end of the Taylor's series for cos x. (b) Introduction of x^3 does not provide a better fit. After differentiation under the integral sign is discussed in Chapter 20 this exercise becomes much easier.

31. (a) $(12/\pi^2)x = 1.216x$. (b) Yes.

32. This is a restatement of $\sin \pi = 0$.

SECTION 2: $R_n(x)$ in Terms of a Derivative; Newton's Method

4. 1.375, 1.04875, 1.0049875.

6. (a) 0, 1, 0, 1, (b) $0 + x + 0x^2$ and $R_2(x) = [(5X^2 - 1)/(1 - X^2)^{5/2}]x^3/3!$.

10. (Rounded off to one decimal) $x_2 = 25.0$, $x_3 = 12.6$, $x_4 = 6.4$.

11. (b) (Rounded off) $x_1 = 1$, $x_2 = 3$, $x_3 = 2.26$; (c) (rounded off) $x_1 = 2$, $x_2 = 1.92$, $x_3 = 1.91$.
16. 1.146.
17. (Rounded off to one decimal) $x_1 = 1$, $x_2 = 95.5$, $x_3 = 48.7$, $x_4 = 26.3$, $x_5 = 16.8$.

□□□

18. We have $f(a+1) = f(a) + f'(a) + (1/2)f''(a) + (1/6)f'''(b)$ and $f(a-1) = f(a) - f'(a) + (1/2\ f''(a) - (1/6)f'''(c)$ where $a < b < a+1$ and $a-1 < c < a$. From consideration of $f(a+1) - f(a-1)$ and $f(a+1) + f(a-1)$ the solution follows.
19. From $f(2) = f(x) + f'(x)(2-x) + (1/2)f''(c)(2-x)^2$, $x < c < 2$ and $f(0) = f(x) + f'(x)(0-x) + (1/2)f''(d)(0-x)^2$, $0 < d < x$ it follows that $f(2) - f(0) = 2f'(x) - (1/2)f''(d)x^2 + (1/2)f''(c)(x-2)^2$. Thus $|2f'(x)| \le |f(2)| + |f(0)| + (1/2)x^2 + (1/2)(x-2)^2$, which is at most 4 since $x^2 + (x-2)^2 \le 4$ for x in $[0, 2]$.
20. Clear for $k = 0, 1, 2$. Assume $F_{2n}(x) = 0$ has no roots. Now $F_{2n+1}(x)$ has at least one root. If it had two roots, then by Rolle's theorem, $F_{2n}(x) = 0$ would have a root, since $F_{2n} = F'_{2n+1}$. Consider next $F_{2n+2}(x) = 0$. At the minimum value of $F_{2n+2}(x)$, say at $x = a$ we have $F_{2n+1}(a) = F'_{2n+2}(a) = 0$. But $F_{2n+2}(a) = F_{2n+1}(a) + a^{2n}/(2n)!$ $= a^{2n}/(2n)! > 0$. Thus, the minimum value of $F_{2n+2}(x)$ is greater than 0.
21. (a) Express $f(a + \Delta x)$ as $f(a) + f'(a)\ \Delta x + (1/2)f''(X_1)(\Delta x)^2$.
(b) Express $f(a + \Delta x)$ and $f(a - \Delta x)$ in terms of derivatives at a and a third derivative at X_3 and at X_4. Subtract and note that $[f^{(3)}(X_3) + f^{(3)}(X_4)]/2 = f^{(3)}(X_2)$ for some X_2 between X_3 and X_4.
22. (a) Since f' is increasing (hence one-to-one) for small h. (b) We have

$$f(a+h) = f(a) + hf'(a) + (1/2)f^{(2)}(X_1)h^2, \quad a < X_1 < a+h.$$

Also, replacing $f'(a + \theta h)$ by $f'(a) + \theta hf''(X_2)$, $a < X_2 < a + \theta h$ in the equation $f(a+h) = f(a) + hf'(a + \theta h)$, we obtain

$f(a+h) = f(a) + h[f'(a) + \theta hf''(X_2)] = f(a) + hf'(a) + (1/2)h^2 f''(X_1)$.

Thus $\theta h^2 f''(X_2) = (1/2)h^2 f''(X_1)$ or $2\theta = f''(X_1)/f''(X_2)$. Let $h \to 0$.
24. The correct one is $b^2/2$. In the approximation of $1 - e^{-b}(1+b)$ use $1 - b + b^2/2$ for e^{-b}.
25. (b) Since $e^{-x} > 0$ and $e^{-x} = 1 - x + x^2/2! - x^3/3! + \ldots$ the series of even power terms is larger (but differs little from the otner series when x is large).
26. Using $1 - x + x^2/2$ as an approximation of e^{-x} we have approximately $1 - y = 1 - (1+k)y + (1+k)^2 y^2/2$, which implies $2ky = (1+k)^2 y^2$, hence $2k = (1+k)^2 y$ or y is approximately 2k.
27. Rewrite $\ln(x+p) = \ln[p(1 + x/p)] = \ln p + \ln(1 + x/p)$, etc.

CHAPTER 14: Estimating the Definite Integral

2. The trapezoidal rule.
7. By rectangles, 0.535; by trapezoids, 1.035; by Simpson's rule, 1.026.
10. (a) 0.842, (b) 0.836.
13. (a) Rectangular, 0.996; trapezoidal 1.107; Simpson's rule, 1.096. (b) 1/4, 1/12, 1/60. (c) Since ln 3 = 1.099, the errors are 0.103, 0.008, 0.003.

□ □ □

20. (a) $e^x = e^a + e^a(x-a)/2! + e^a(x-a)^2/2! + \dots$. (b) Factor e^a out of the series in (a) and notice that this yields $e^x = e^a e^{x-a}$. Replace a by -y.

CHAPTER 15: Further Applications of Partial Derivatives

SECTION 1: The Charge Δz and the Differential dz

2. (a) $u_x = 2xyz$, $u_y = x^2z$, $u_z = x^2y$. (b) $u_x = -y \sin x$, $u_y = \cos x$, $u_z = 1$.

9.

Δr	Δh	dV	ΔV	ΔV/dV (rounded off)
0.1	0.2	4.2π	4.482π	1.07
0.01	0.03	0.51π	0.5122π	1.00
0.001	-0.001	0.015π	0.014994999π	1.00

18. $z = x^2$ + any function of y.

□ □ □

23. (a) Since f_{xy} would have to equal f_{yx} the answer is 'no'. (b) $f(x, y) = x^2y + y^3$ is an example.

25. We have $2 = \frac{dx}{dx} + \frac{dy}{dy} = \frac{\partial x}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial y}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial x}{\partial Z}\frac{\partial Z}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial y}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial y}{\partial Z}\frac{\partial Z}{\partial y}$;

the same six summands make up dX/dX + dY/dY + dZ/dZ = 3, as the chain rule shows in a similar manner. (This is an instance of the invariance of dimension under homeomorphisms, a topological theorem whose proof does not require the assumption of differentiability.)

26. To obtain (b) differentiate the relation in (a) with respect to x_j.

SECTION 2: Higher Partial Derivatives and Taylor's Series

3. $f_x = 9x^2 + 2y^2 + 12x - 5y + 6$ $\quad f_y = 4xy - 5x$

   $f_{xx} = 18x + 12$ $\quad f_{yx} = 4y - 5$ $\quad f_{xy} = 4y - 5$ $\quad f_{yy} = 4x$

   $f_{xxx} = 18$ $\quad f_{xyx} = 0$ $\quad f_{yxx} = 0$ $\quad f_{yyx} = 4$

   $f_{xxy} = 0$ $\quad f_{xyy} = 4$ $\quad f_{yxy} = 4$ $\quad f_{yyy} = 0$

   All higher partials are 0.
5. (a) $1 + x^2/2 + y^2/2$.
7. $1 + x + x^2/2 + y^2/2$.
9. 4(0.1) + 0.1 + 2(0.01) + 2(0.01) + (0.01)(0.1) = 0.4 + 0.1 + 0.02 + 0.02 + 0.001 = 0.541.

12. $1 + 2(x - 1) + 2(y - 1) + (x - 1)^2 + 4(x - 1)(y - 1) + (y - 1)^2$
$+ 2(x - 1)^2(y - 1) + 2(x - 1)(y - 1)^2 + (x - 1)^2(y - 1)^2$

□ □ □

15. Completing the square we have $A(Ax^2 + 2Bxy + Cy^2) = A^2x^2 + 2ABxy + ACy^2 = (Ax + By)^2 + (AC - B^2)y^2$ and consider (x, y) with $x = 0$ and (x, y) with $y = 0$.
16. A test for maximum or saddle point. Most advanced calculus books cover this result.

CHAPTER 16: Algebraic Operations on Vectors

SECTION 1: The Algebra of Vectors

1. Scalars.
2. (b) 10, direction $\tan^{-1}(-8/6)$, approximately south-east.
3. (a) $\overrightarrow{(-5\sqrt{2}, 5\sqrt{2})}$, (c) $\overrightarrow{(9\sqrt{2}/2, -9\sqrt{2}/2)}$, (d) $\overrightarrow{(5, 0)}$.
4. (a) $\overrightarrow{(2, 3)}$, (c) $\overrightarrow{(0, 3)}$, (d) $\overrightarrow{(-6, -9)}$.
5. (c) $\overrightarrow{(1, 3, 2)}$.
11. (a) Their total production, (b) The first firm tripling its production.
12. (a) $\overrightarrow{(30, 21, 50, 75, 18)}$, (b) $\overrightarrow{(3, 2, 4, 7, 1)}$.
13. (a) The first vector has magnitude $\sqrt{208}$, the second, $\sqrt{200}$. (b) The second, (c) the first.
16. c, $|c|$, $|\mathbf{A}|$.
18. $\overrightarrow{(4, 5)}/3$, $3\overrightarrow{(4, 5)}$, $\overrightarrow{(4, 0)} - \overrightarrow{(3, 5)}$.

□ □ □

23. (a) It is usually defined as $\sqrt{\sum_{i=1}^{\infty} x_i^2}$ (if the series converges).
(b) $\sqrt{\sum_{n=1}^{\infty} 1/4^n} = \sqrt{1/3}$. (c) $\sum_{n=1}^{\infty} 1/n^2 < 1 + \sum_{n=2}^{\infty} 1/[n(n-1)] = 2$.
Hence, the vector has length less than $\sqrt{2}$; in fact its length is $\sqrt{\pi^2/6}$.

SECTION 2: The Dot Product of Two Vectors

6. (c) The dot product is $3 \neq 0$.
16. (a) 1, (b) 1.
18. $\overrightarrow{(3, 4)} + \overrightarrow{(5, 7)} \cdot \overrightarrow{(6, 8)}$, $3[\overrightarrow{(4, 5)}\overrightarrow{(6, 7)}]$.

□ □ □

21. (a) Revenue from chairs. (b) Revenue from chairs and desks. (c) Cost of producing chairs and desks. (d) Revenue is more than cost, a desirable situation in a business.
27. It breaks even.

SECTION 3: Directional Derivatives and the Gradient

6. (a) Two, (b) perpendicular to the direction of ∇f.
9. $\nabla f = \overrightarrow{(4x, -6y)} = \overrightarrow{(8, -6)}$ gives the direction, $\tan^{-1}(-6/8)$.

12. All are 0 (recall that we assume that they are defined).

13. $-\overrightarrow{(\alpha \cos \alpha x \cos \beta y, \ -\beta \sin \alpha x \sin \beta y)}$

CHAPTER 17: The Derivative of a Vector Function

SECTION 1: The Position and Velocity Vectors

1. (a) The hyperbola $xy = 1$ to the right of $x = 1$. (d) $dx/dt = 1$, $dy/dt = -1/t^2 \to 0$ as $t \to \infty$, $|\underset{\sim}{V}| \to 1$, and $\underset{\sim}{V} \to \overrightarrow{(1, 0)}$.

5. (b) $\underset{\sim}{R}(t) = \overrightarrow{(\cos t^3, \ \sin t^3)}$, $\underset{\sim}{V}(t) = 3t^2\overrightarrow{(-\sin t^3, \ \cos t^3)}$.

   (c) $|\underset{\sim}{V}(t)| \to \infty$ as $t \to \infty$; the particle moves arbitrarily fast.

6. (c) $|\underset{\sim}{V}(t)|$ behaves like $32t$ and the direction of $\underset{\sim}{V}(t)$ approaches the vertical.

7. (a) $\underset{\sim}{R}(0) = \overrightarrow{(0, \ 0)}$, $\underset{\sim}{V}(0) = \overrightarrow{(100, 100\sqrt{3})}$

   (b) $\underset{\sim}{R} = \overrightarrow{(10000\sqrt{3}/32, \ 15000/32)}$ $\quad \underset{\sim}{V} = \overrightarrow{(100, 0)}$

   (c) $\underset{\sim}{R} = \overrightarrow{(10000\sqrt{3}/16, 0)}$ $\quad \underset{\sim}{V} = \overrightarrow{(100, \ -100\sqrt{3})}$.

□ □ □

9. (b) Consider the line L passing through the bow and the initial position of the ball. The vertical deviation of the arrow from this line at time t is the same as the distance the ball has fallen at time t. (If there were no acceleration, then the arrow would remain on L and the ball at rest.) (c) All that we need is that the arrow and ball are subject at all times to equal accelerations. See discussion of (b).

SECTION 2: The Derivative of a Vector Function

1. (a) Circle of radius 10.

   (b) $\underset{\sim}{V} = 20\pi\overrightarrow{(-\sin 2\pi t, \ \cos 2\pi t)}$ $\quad \underset{\sim}{A} = 40\pi^2\overrightarrow{(-\cos 2\pi t, \ -\sin 2\pi t)}$

10. If $\underset{\sim}{F}$ is perpendicular to $\underset{\sim}{V}$, then $\underset{\sim}{A} \cdot \underset{\sim}{V} = 0$ and $d(\underset{\sim}{V} \cdot \underset{\sim}{V})/dt = 0$. Hence $\underset{\sim}{V} \cdot \underset{\sim}{V}$ is constant.

11. (a) Differentiate the relation $\underset{\sim}{R} \cdot \underset{\sim}{V} = 0$.

12. (a) $\underset{\sim}{V} = 2\overrightarrow{(-t \sin t^2, \ t \cos t^2)}$

    $\underset{\sim}{A} = 2\overrightarrow{(-2t^2 \cos t^2 - \sin t^2, \ -2t^2 \sin t^2 + \cos t^2)}$

13. (a) $d\underset{\sim}{R}/dt$ is perpendicular to $\underset{\sim}{R}$, (b) No, e.g. $\underset{\sim}{R} = \overrightarrow{(\cos(t^3), \ \sin(t^3))}$.

□ □ □

19. (a) Yes. If $\underset{\sim}{V}(t) = \overrightarrow{(x(t), \ y(t))}$ and $y(t) = mx(t)$, m constant, then $\dot{y}(t) = m\dot{x}(t)$. (b) No. Consider circular motion with constant speed.

SECTION 3: Tangential and Normal Components of **A**

3. $A_N = \dfrac{ds}{dt}\dfrac{ds}{dt}\dfrac{d\phi}{ds} = \dfrac{ds}{dt}\dfrac{d\phi}{dt} = v\dfrac{d\phi}{dt}$.

4. Use Exercise 3.

5. Quadrupled; halved.
9. (b) Speeding up.

□ □ □

17. (a) Since $A_T = dv/dt = k$ we have $v = kt + v_0$. (b) Since $A_N = k$ $= v^2/(\text{radius of curvature}) = (\frac{ds}{dt})^2(\frac{d\phi}{ds}) = \frac{ds}{dt} \cdot \frac{d\phi}{dt} = v\frac{d\phi}{dt}$, we have $d\phi/dt = k/v$. (c) From $d\phi/dt = k/(kt + v_0)$ it follows that $\phi = \ln(kt+v_0) + \phi_0$. (d) Since $v = kt + v_0$ his speed is getting arbitrarily great (but, let us agree, not to exceed the speed of light). By (c) ϕ increases without bound, hence the spiral. Since radius of curvature equals v^2/k, the path continually gets straighter.
18. See discussion of Exercise 17.
19. Differentiate the equation $(ds/dt)^2 = (dx/dt)^2 + (dy/dt)^2$ with respect to t and solve for d^2s/dt^2.
22. Differentiate the equation $\underset{\sim}{T} \cdot \underset{\sim}{T} = 1$ with respect to t.

CHAPTER 18: Curve Integrals

SECTION 1: The Curve Integral of a Vector Field $(\overrightarrow{P,Q})$

4. (c) The work done by $\underset{\sim}{F} = \overrightarrow{(x^2, y+1)}$ in pushing a particle once around the circle. (d) The net loss of fluid over the circle if the velocity vector is $\underset{\sim}{V} = \overrightarrow{(y+1, -x^2)}$.
12. $7(2) + 10(3) = 44$.
15. Using arc length s as a parameter, we have $\int_C (\frac{dx}{ds}\, dx + \frac{dy}{ds}\, dy)$ $= \int_C (\frac{dx}{ds}\frac{dx}{ds} + \frac{dy}{ds}\frac{dy}{ds})\, ds = \int_C 1\, ds =$ arc length of C.
17. Write $\cos\phi = dx/ds$ and $\sin\phi = dy/ds$. (This result is needed in Exercise 10 of the next section.)

SECTION 2: The Curve Integral of a Gradient Field ∇F

2. One such F is $x^2/2 + 3xy + 2y^2$.
3. One such F is $-e^{-x}\cos y$.
5. (a) $q/(8\pi\epsilon)$, (b) equal.
6. $\underset{\sim}{V}$ = gradient F will do; pick F.
7. (a) Yes, $F(x,y) = x^3y$, (b) No, $P_y \neq Q_x$, (c) Yes, $F(x, y) = x^2 + 3xy - 2y^2$.

□ □ □

9. Let w be the absolute value of the component of $\underset{\sim}{W}$ along the route. Then the total time is $1/(V+w) + 1/(V-w) = 2/(V - w/V)$ which is less than $2/V$ if $w \neq 0$.
10. Let ϕ be the angle from $\underset{\sim}{W}$ to a tangent line on the curve and v be the forward speed of the plane along the curve. Inspection of the triangle whose three sides are V, W, v and which has angle ϕ opposite V shows that $v = \sqrt{V^2 - W^2\sin^2\phi} + W\cos\phi$. Thus, the time for the trip is

$$\oint_c \frac{ds}{v} = \oint_c \frac{ds}{\sqrt{V^2 - W^2 \sin^2\phi} + W\cos\phi} = \oint_c \frac{\sqrt{V^2 - W^2 \sin^2\phi} - W\cos\phi}{V^2 - W^2\sin^2\phi - W^2\cos^2\phi}\, ds$$

$$= \oint_c \frac{\sqrt{V^2 - W^2 \sin^2\phi} - W\cos\phi}{V^2 - W^2}\, ds = \oint_c \frac{\sqrt{V^2 - W^2\sin^2\phi}}{V^2 - W^2}\, ds$$

$$> \oint_c \frac{\sqrt{V^2 - W^2}}{V^2 - W^2}\, ds = \oint_c \frac{1}{\sqrt{V^2 - W^2}}\, ds > \oint \frac{ds}{V} = \text{time for the trip}$$

when there is no wind. We assumed that $W > 0$ and also made use of Exercise 17 of the preceding section. The formula for v may also be obtained from the law of cosines: $V^2 = v^2 + W^2 - 2vW\cos\phi$ (solve for v using the quadratic formula;

$$v = W\cos\phi \pm \sqrt{W^2\cos^2\phi - (W^2 - V^2)}$$

and since v is positive, the + is used.)

SECTION 3: Other Notations for Curve Integrals

2. $P = 3x^2y + f(x)$, f arbitrary.
3. $3x^2y^2/2 + f(y)$, f arbitrary.
4. (a) $[2x/(x^2 + y^2)]dx + [2y/(x^2 + y^2)]dy$, (b) $\int_c dF = 0$ (since $F(3, 0) = F(0, 3)$).
7. $F(4, 8) - F(1, 1) = e^4 \sin 8 + \ln(81) - e \sin 1 - \ln 3$.
8. It is sensitive only to the direction of flow along (parallel to) the border.
9. Since F is constant, $F(B) - F(A) = 0$, where A and B are the ends of C. So the integral is 0. Or one could observe that since the level curve is perpendicular to the gradient, the integrand $F_x\, dx/dt + F_y\, dy/dt$ is identically 0.

CHAPTER 19: Green's Theorem

SECTION 1: Green's Theorem

7. $3\pi/2$. Note that (a) is easier.
12. Letting the velocity vector of heat flow be $(\overrightarrow{T_x, T_y})$ we have $\oint(-T_y\, dx + T_x\, dy) = 0$ for each closed curve, hence, by Green's theorem $\int_R (T_{yy} + T_{xx})\, dA = 0$ for each region bounded by such curves. Considering small R shows that $T_{xx} + T_{yy} = 0$.
14. $1/\pi$. Use of a table of integrals materially shortens the computations in (c).

□□□

18. (a) For instance, $(\frac{1}{2})\int_A^B (-y\,dx + x\,dy) = (1/2)(a_1b_2 - a_2b_1)$, as is computed by setting $y = a_2 + [(x - a_1)/(b_1 - a_1)](b_2 - a_2)$ and therefore $dy = [(b_2 - a_2)/(b_1 - a_1)]dx$. The area is

$$(1/2)[(a_1b_2 - a_2b_1) - (a_1c_2 - a_2c_1) + (b_1c_2 - b_2c_1)].$$

(d) No. The area of an equilateral triangle of side $\underline{s}$ is $\sqrt{3}\, s^2/4$, which is irrational when s^2 is rational. If the triangle has integral coordinates, then the Pythagorean theorem shows that s^2 is an integer. This violates (c).

23. If the formula for area found in Exercise 18(a) has a negative value, then ABCA is clockwise; if positive, counterclockwise. (If 0, then the three points are colinear, as may be shown by analytic geometry.)

SECTION 2: Magnification in the Plane: the Jacobian

2. (a) The rectangle parallel to axes with opposite vertices (2, 0) and (4, $3\pi/2$). (b) The parallelogram with vertices (2, 5), (4, 10), ($2-3\pi/2$), $5-7\pi/2$), ($4-3\pi/2$, $10-7\pi/2$).
4. The area is 4/3.
7. (c) 3, and the direction of C_S is counterclockwise.

□ □ □

13. From (1)(1) = 1 we obtain $1 = x_u u_x y_u u_y + x_u u_x y_v v_y + x_v v_x y_u u_y + x_v v_x y_v v_y$. Replace $u_x y_u$ in the first monomial by $- v_x y_v$ and $v_x y_v$ in the fourth monomial by $- u_x y_u$. The result coincides with the product of the two Jacobians.
(c) If you look through the wrong end of a telescope that magnifies by the factor a you will see everything "magnified" by the factor 1/a, that is, shrunk by the factor a. Part (b) shows that this is also so for magnification that varies from point to point, as in an imperfect lens.

16. This is analogous to substitution in the definite integral $\int_a^b f(x)\,dx$, $x = h(u)$, where we have $\int_a^b f(x)\,dx = \int_A^B (f \circ h)(dx/du)du$. The two-dimensional formula is plausible, for if

$$\sum_{i=1}^{n} f(F(u_i,\ v_i)) \left(\partial(x,\ y)/\partial(u,\ v)\right) \Big|_{(u_i, v_i)} A_i$$

is an approximation of $\int_R (f \cdot F)(\partial(x,\ y)/\partial(u,\ v))\,dA$ based on the partition $R_1, R_2, \ldots, R_n$ and points $P_i = (u_i, v_i)$, then $F(R_1), F(R_2), \ldots$, $F(R_n)$ is a partition of S and the area of $F(R_i)$ is approximately

$\partial(x,\ y)/\partial(u,\ v) \Big|_{(u_i, v_i)} A_i$. Hence, the approximating sum is approximately

$$\sum_{i=1}^{n} f(x_i,\ y_i)\,B_i$$

where B_i = area of $F(R_i)$ and $(x_i, y_i) = F(u_i, v_i)$, a point in $F(R_i)$; the latter sum is an approximation of $\int_S f\,dA$.

17. (a) Because "magnification" ought to behave like that. Also, in one-dimensional magnification, we have the chain rule $dy/dx = (dy/du)(du/dx)$.
(c) Chain rule.

SECTION 3: Hyperbolic Functions

6. (b) Recall that $\lim_{u \to \infty} \tanh u = 1$.

□ □ □

10. Use the formulas for cosh and sinh in terms of e^x and e^{-x}.

CHAPTER 20: The Interchange of Limits

7. (b) If $|x| > 1$, then $f(x) = 1$; $f(1) = f(-1) = 1/2$; if $|x| < 1$, $f(x) = 0$. Thus at all a other than 1 and -1, f is continuous.
10. (a) $f(0) = 1$. (b) Since $(1 + nx^2)/(1 + nx) = (1/n + x^2)/(1/n + x)$, $f(x) = x^2/x = x$ when $x > 0$.

13. (a)
$$\lim_{y\to 0}\left[\lim_{x\to\infty}\frac{(1+\frac{2y}{x})^x - (1+\frac{y}{x})^x}{y}\right] = \lim_{y\to 0}\frac{e^{2y}-e^y}{y}$$
$$= \lim_{y\to 0}\frac{e^y(e^y-1)}{y} = 1,$$

since $\lim_{y\to 0}(e^y-1)/y$ = derivative of e^y at $y = 1$. (b) If a and b are constants, $\lim_{y\to 0}[(1 + ay)^b - 1]/y = d(1 + ay)^b/dy$ at $y = 0$, thus ba. Thus
$$\lim_{y\to 0}\frac{(1+\frac{2y}{x})^x - (1+\frac{y}{x})^x}{y} = \lim_{y\to 0}\frac{[(1+\frac{2y}{x})^x - 1] - [(1+\frac{y}{x})^x - 1]}{y}$$
$= 2 - 1 = 1$, and $\lim_{x\to\infty} 1 = 1$.

14. Let $g(x) = x$.
15. (a) 0 and 1, (b) 0 and 0.
16. (a) Somewhere a tangent line is parallel to the chord from (g(a),f(a)) to (g(b), f(b)). (b) No. Consider two points on a helix such that the chord joining them is parallel to the axis of the helix.

□ □ □

20. Let θ = angle BOP. Then $Q = (1, \theta)$ and $P = (\cos\theta, \sin\theta)$, whence R has the x coordinate $(\theta\cos\theta - \sin\theta)/(\theta - \sin\theta)$, which approaches -2 as $\theta \to 0$. (Use L'hôpital's rule three times, or, perhaps, the approximations $\cos\theta = 1 - \theta^2/2$ and $\sin\theta = \theta - \theta^3/6$.)

CHAPTER 21: Growth in the Natural World

1. $t = 38.3$.
13. 200.
16. From $dP/P^{1.01} = kdt$, we deduce that $-100P^{-0.01} = kt + C$ where $C < 0$. Thus $P = [-100/(kt + C)]^{100}$, whose denominator becomes 0 when $kt + C = 0$; $t = -C/k$ is doomsday.
18. (a) About 41. (b) About 65.

□ □ □

23. The inflection point suggests (34).
31. (a) Assuming that the death rate remains constant, we have $dP/dt = kP^2 - cP$, where k and c are positive constants. (b) There are three cases: $P(0) = c/k$, $P(0) > c/k$, $P(0) < c/k$. If $P(0) = c/k$, then P is constant. [This depends on the uniqueness of solutions of differential equations of the form $dy/dx = f(x, y)$]. If $P(0) > c/k$, then, since $dP/dt = kP(P - c/k)$, as long as $P(t) > c/k$, $P(t)$ is increasing. Thus $P(t) - c/k > 0$ and, applying the same method we used in solving Eq. (27), we obtain $P(t) = (c/k)/$

$(1 - e^{ct+q})$, where q is a negative constant. Doomsday occurs at $t = -q/c$. If $P(0) < c/k$, similar reasoning shows that $P(t) \to 0$ as $t \to \infty$.

CHAPTER 22: Business Management and Economics

1. (a) Double the order size and halve the frequency. (b) Halve the order size and double the frequency.
2. The cost due to "P" is proportional to A, hence cannot be influenced.
7. (a) A debatable and crude assumption. As area A is expanded about a warehouse, the distances increase and are proportional to $\sqrt{A}$, hence to the average distance, too.
14. (a) One dollar kept in the bank would provide a rate of profit r. Hence the present value of the (constant)profit function r is 1. Thus

    $1 = \int_0^\infty e^{-rt}\, r\, dt.$
15. (a) g(T) represents the present value of the three factors in the decision. (b) When we scrap we would expect the revenue rate to exceed the rate at which the scrap value declines by exactly the rate of return on the scrap.

□ □ □

16. (a) The present value of G dollars plus an expenditure at the rate of B dollars per year is $G + \int_0^\infty e^{-rt} B dt = G + B/r$; similarly, the present value of the electric stove and electricity to be used is $E + A/r$. The electric stove is more economical when $E + A/r < G + B/r$, that is $r(G - E) > A - B$. (b) Yes, for $r(G - E)$ is the rate of return of investment of $G - E$, and $A - B$ is the initial rate of extra cost for electricity. The return on the investment of $G - E$ should more than cover the extra cost of electricity.
17. (b) If p is the lower price and p + 1 the higher price, then the number sold at the higher price is $9 - (p + 1)^2$ and the number sold at the lower price is $(9 - p^2) - (9 - (p + 1)^2)$. Hence, the revenue function is $(p + 1)[9 - (p + 1)^2] + p[(p + 1)^2 - p^2]$, which has a maximum at $p = (\sqrt{22} - 1)/3$. (His revenue is then 11.7).
20. To simplify the arithmetic maximize $u^6 = x^3 y^2 [(180 - 2x - y)/5]$.
21. (b) In this case (merger) note that by cutting production in half the total profit is increased.

CHAPTER 23: Psychology

2. For the octave from 500 to 1000 cycles we would evaluate $\int_{500}^{1000} (1/2.22)\, x^{-0.308}\, dx$, as in (13). As in (25), this is $(0.651)(1000^{0.692} - 500^{0.692}) = 0.651(119.1 - 73.8) = 29.5$; hence around 30 or 31 keys. Similar arithmetic shows that the octave from 1000 to 2000 cycles requires about $2^{0.692}$ as much, and hence, since $2^{0.692} = 1.61$, about 49 keys.
3. $1 + \int_{500}^{1000} (1/2.22)\, x^{-0.308}\, dx$ (about 31), and

    $1 + \int_{1000}^{2000} (1/2.22)\, x^{-0.308}\, dx$ (about 49).

7. (b) Since $\log_{10} y = \log_{10} a + n \log_{10} x$, n is the slope of the line when the data is plotted on log-log paper. (c) Semilog, with the x axis logarithmic.
10. (b) $f(x) = c + (t/k) \ln x$.

□ □ □

14. (c) $\sqrt{x^2 + y^2}/2$
16. (a) 0 (none knew at the beginning). (b) 1 (all know). (c) Yes, as long as $p \neq 1$. (Since $0.80p + 0.20 > 0.80p + 0.20p = p$, and $0.92p + 0.08 > p$). In case of avoidance we have $(0.80p + 0.20) - p = 0.20 (1 - p)$; in the case of shock we have $(0.92p + 0.08) - p = 0.08 (1 - p)$. The dog "learns more" by avoiding the shock than by being shocked. (d) After avoidance, $p = 0.20$; after shock, $p = 0.8$. The smallest increase in p occurs after shock. After two shocks, p is $(0.92)(0.08) + 0.08 = 0.1536$ ... etc. (e) Since p increases at each stage and is bounded by 1, it has a limiting value. More precisely if p_n denotes p at n^{th} stage, we have $p_1 = 0 < p_1 < p_2 < \dots < 1$. Let $P = \lim_{n \to \infty} p_n$. Since $p_{n+1} \geq 1.08 - 0.08p_n$ we have $P \geq 1.08 - 0.08P$, hence $P \geq 1$. Thus $P = 1$.
17. (a) $-k_2 x$ and $-c_2 y$ represent "fatigue"--the effect of previous buildups; (b) $k_1 y$ and $c_1 x$; (c) k_3 and c_3, which are independent of x and y.

CHAPTER 24: Traffic

SECTION 1: Preliminaries

2. (c) 1/2.
3. (b) The improper integral $\int_0^\infty (2x/[\pi(1 + x^2)])\,dx$ is divergent; we may say that the expected gap is infinite. (The experimental average gap would tend to get arbitrarily large as more experiments are performed.)
9. (c) 7, since the first die can show any one of its six faces. Rather than speak of the "first" and "second" dice, it may be clearer to speak of one die "thrown with the left hand" and the other with the "right hand," so that the feeling of simultaneity is preserved. The experiments may be performed that way.

10. (a) $1/2^n$, (b) $\sum_{n=1}^{\infty} n/2^n = 2$.
12. (d) $1/e = e^{-1}$. Actually (b) does hold for all n.

SECTION 2: The Exponential (Poisson) Model of Random Traffic.

8. From $\sum_{n=0}^{\infty} P_n(x) = 1$ and (20) (for k = 1) it follows that $\sum_{n=0}^{\infty} (x^n/n!)e^{-x} = 1$.

□ □ □

21. Since $f(0 + 0) = f(0)f(0)$ we have $f(0) = 0$ or 1. If $f(0) = 0$, then $f(0 + y) = f(0)f(y)$, that is, $f(y) = 0$; thus $f(y) = 0^y$ for all y (if we agree to interpret $0^0 = 0$). If $f(0) = 1$ let $f(1) = a$. Thus $f(2) = f(1 + 1) = f(a)f(1) = a^2$,

and similarly, $f(n) = a^n$ for any positive integer n. Since $1 = f(-n + n) = f(-n)f(n)$, $f(-n) = a^{-n}$ for any negative integer -n. Also $f(1/n + 1/n + \ldots + 1/n) = a = [f(1/n)]^n$, thus $f(1/n) = \sqrt[n]{a}$. (Observe that $f(x) \geq 0$ since $f(x) = [f(x/2)]^2$.) Thus $f(m/n) = (\sqrt[n]{a})^m = a^{m/n}$. By continuity, $f(x) = a^x$ for all x.

SECTION 3: Cross Traffic and the Gap between Cars

4. When c is large, then w is close to "expected initial distance" 1/k.
9. The probability equals (probability that the car from the west is at least a distance c from him) × (probability that the car from the east is at least a distance c from him). (Note: same as if eastbound traffic doubled in density.)
13. (b) $p(2) = p(3) = \cdots = 0$. It is interesting to compare W for this traffic to W for the traffic randomly distributed, same k and c. For this uniform traffic $W = (1/2)(0) + (1/2)(25) = 12.5$. For random traffic, $W = 100(e^{1/2} - 1 - 1/2) = 14.9$, which is larger. However, random traffic has the advantage that any northbound driver eventually can cross.
15. (b) $10 \times 4 = 40$ minutes.

□ □ □

16. (c) By Schwarz's inequality we have

$$\int_0^\infty xf(x)\,dx = \int_0^\infty [x\sqrt{f(x)}][\sqrt{f(x)}]\,dx$$

$$\leq \left\{\int_0^\infty [x\sqrt{f(x)}]^2\,dx\right\}^{1/2} \left\{\int_0 [\sqrt{f(x)}]^2\,dx\right\} = [\int_0^\infty x^2 f(x)\,dx]^{1/2} [\int_0^\infty f(x)\,dx]^{1/2},$$

which establishes the desired inequality. (d) Equality occurs only if $x\sqrt{f(x)}$ is proportional to $\sqrt{f(x)}$, that is: x is constant, which is not the case.

CHAPTER 25: Rockets

2. Doubling c doubles $v_1 - v_0$; doubling dm/dt has no effect; doubling $m_1 - m_0$ has an intermediate effect since $\ln(1 + 2x) < 2\ln(1 + x)$ for $x > 0$ $(x = m_1 - m_0)$. To obtain the inequality take logs of $1 + 2x < (1 + x)^2$.
3. Both produce the same effect, as (10) shows, increasing $v_1 - v_0$ by 50%.
6. (b) The work would be finite.
8. The precise answer is (82)(4000/4082) mile-pounds.
9. 4 miles per second.
12. 0 miles per second.
17. It is smaller than the escape velocity. (Consider the reversed trip.)

□ □ □

23. To show that for $y > 0$, $y^2 < 2(e^y - 1 - y)$, use Taylor's series for e^y or else Example 5, Section 2, Chapter 5, where it is shown that $e^y > 1 + y + y^2/2$ for $y > 0$.
24. (b) The time T_1 required to reach the maximum height mentioned in Exercise 22 is $(1/k)\ln(C/32)$. The time T_2 required to reach the same

height when there is no air resistance is (using formula $y = 16t^2$) $(1/4)\sqrt{(C - 32)/k^2 - (32/k^2)\ln(C/32)}$. We show that T_2 is greater than T_1. The inequality $T_2^2 > T_1^2$ is equivalent to

$$\frac{1}{16}\left[\frac{C-32}{k^2} - \frac{32}{k^2}\ln\left(\frac{C}{32}\right)\right] > \frac{1}{k^2}\left[\ln\left(\frac{C}{32}\right)\right]^2.$$

Note that $C > 32$, as was shown in Exercise 21, and let $C = 32x$, $x > 1$. The above inequality becomes

$$\frac{1}{16}\left[\frac{32(x-1)}{k^2} - \frac{32}{k^2}\ln x\right] > \frac{1}{k^2}(\ln x)^2,$$

a simple consequence of Exercise 23.

25. No. Consider the extreme case, $v_0^2 = (4000)(0.012)$. The (16) gives

$$v^2 = (4000)(0.012) + (4000)^2(0.012)\left(\frac{1}{r} - \frac{1}{4000}\right) = (4000)^2(0.012)\left(\frac{1}{r}\right).$$

If r approaches a limiting position r^* then v approaches a limiting non-zero velocity v^*. Indeed $v > v^*$ and $v^* = \sqrt{(4000)^2(0.012)(1/r^*)}$. But an object travelling with a velocity that is always greater than some fixed positive number travels an infinite distance, in particular goes beyond r^*, its alleged limiting position.

26. Using (16) we have $v^2 = 47.2 + (4000)^2(0.012)\left(\frac{1}{r} - \frac{1}{4000}\right)$ and total time is $\int_{4,000}^{240,000}(1/v)\,dr$. Now $v^2 = -0.8 + 192,000/r$. We must evaluate $\int_{4,000}^{240,000}\sqrt{r/(-0.8r + 192,000)}\,dr = 1/\sqrt{0.8}\int_{4,000}^{240,000}\sqrt{r/(240,000-r)}$. Evaluate this either by a substitution $r = 240,000\sin^2\theta$ or use of the formula for $\int\sqrt{(p+x)/(q-x)}\,dx$ from an integral table. The time is $(1/\sqrt{0.8})[\sqrt{4000}\sqrt{236,000} + 240,000\sin^{-1}(236/240)]$ seconds, or $(1.12)[(63.2)(486) + (240,000)(1.39)] = (1.12)(30,700 + 334,000) = 408,000$ seconds or about 113 hours.

CHAPTER 26: Gravity

1. (a) $r\dot{\theta}$ would be velocity of particle moving on circle of radius r and angular velocity $\dot{\theta}$; $\dot{r}$ would be velocity if particle were restrained to the ray. (b) If the particle moved on a circle of radius r and angular velocity $\dot{\theta}$ it would have velocity $v = r\dot{\theta}$, hence $r(\dot{\theta})^2 = r(v/r)^2 = v^2/r$, in agreement with results in Chapter 17. The expression $r\ddot{\theta}$ is equal to the acceleration if r were constant.
5. Success is not expected.
7. The latter, by Kepler's third law. First mean radius is r + [(1/2)(200 + 600)]; the second is r + [(1/2)(150 + 700)], where r is the radius of Earth.
8. $\frac{96^2}{(4317.5)^3} = \frac{T^2}{(4400.5)^3}$, whence T = 98.8 minutes.

13. This reduces to the equation $\int_{-a}^{a} f(x)dx = 0$ if $f(-x) = f(x)$, a result obtainable by the substitution $u = -x$ or by showing that $\int_0^a f(x)dx = -\int_{-a}^{0} f(x)dx$.

15. (a) The 'area is swept out at constant rate' (II). (b) I and II. (c) I, II, III.

□ □ □

17. (a) We have $d^2x/dt^2 = -kx$ and $d^2y/dt^2 = -ky$. According to Exercise 16, $x = a_1 \cos\sqrt{k}t + a_2 \sin\sqrt{k}t$ and $y = b_1 \cos\sqrt{k}t + b_2 \sin\sqrt{k}t$. At $t = 0$, $y = 0$ and $dx/dt = 0$; thus $b_1 = 0$ and $a_2 = 0$. We have $x/a_1 = \cos\sqrt{k}t$ and $y/b_2 = \sin\sqrt{k}t$ and thus $x^2/a_1^2 + y^2/b_2^2 = 1$. (b) The time to complete an orbit is determined by k; the period is $2\pi/\sqrt{k}$.

19. From formula (19) $A_r = -4\pi^2 a^3/T^2 r^2$. In this case $a = r = 240,000$ and $A_r = (4000/240,000)^2(-32)/5280$ miles per second per second or $A_r = -(1/60)^2(0.006)$. Thus

$$\frac{0.006}{(60)^2} = \frac{4\pi^2(240,000)}{T^2},$$

whence T may be found. (About 27.8 days, with this data.)

20. $\dot{r} = ake^{k\theta}\dot{\theta} = ake^{k\theta}h/(e^{k\theta})^2 = akh\,e^{-k\theta}$
$\ddot{r} = -ak^2h\,e^{-k\theta}\dot{\theta} = -ak^2h\,e^{-k\theta}h/r^2 = -h^2k^2/r^3$.
Thus $\ddot{r} - r\dot{\theta}^2 = -h^2k^2/r^3 - rh^2/r^4 = -(h^2k^2 + h^2)/r^3$.

Appendix A: Analytic Geometry

2. (a) $(x-1)^2 + (y+1)^2 = 25$. (b) No, for $(4.5-1)^2 + (2.5+1)^2 = 24.5 \neq 25$.

5. They may look colinear but (3) shows that they are not.

7. (a) $y - 5 = (-3/2)(x-2)$; (c) yes.

15. (d) (6,0,0), (0,3,0), (0,0,2) respectively.

17. (a) The plane parallel to the xz plane, passing through (0,3,0), (b) The plane parallel to the xy plane, passing through (0,0,2), (c) The plane perpendicular to the xy plane and passing through (1,1,0) and (1,0,0), (d) Empty, (e) The spherical surface of radius $\sqrt{5}$ and center at the origin.

18. (a) (straight) line, (b) (straight) line, (c) horizontal curve congruent to the curve $1 = xy$, (d) the axes, (e) a curve whose shadow on the xz plane by light parallel to the y axis is the curve $z = x^2$.

24. (a) $r^2 = 1/(\sin 2\theta)$, (b) $r = 1/(\cos\theta + \sin\theta)$, (c) $r = 2\cos\theta + 4\sin\theta$.

25. (a) $x^2 + y^2 = 4y$, (b) $\sqrt{x^2 + y^2} = e^{\tan^{-1}(y/x)}$, (c) $(x^2 + y^2)^2 = 2xy$.

28. Length of string in both cases is 10. In the first case use foci (4,0) and (-4,0); in the second case, (0,4) and (0,-4).

29. The tacks would be 6 feet apart and the string would be 10 feet long.

33. Simply use the definition of the hyperbola.

35. (b) $F = (1/2, 0)$, directrix $x = -1/2$.

37. The length of string 2a is now the "slant" distance along a generator of the cone between the equators of the two spheres perpendicular to the axis of the cone.

40. (a) Rewrite each in the form $r = pe/[1 + \cos(\theta - B)]$. Thus $r = 5/(3 + 4\cos\theta) = (5/3)/[1 + (4/3)\cos(\theta - 0)]$, whence $e = 4/3$ (and $p = 5/4$). The eccentricities of the others are 3/4, 1, 4/3. (b) hyperbola, ellipse, parabola, hyperbola.

43. (a) $r^2 = 3\,r\cos\theta + 4\,r\sin\theta$, hence $x^2 + y^2 = 3x + 4y$. Complete the square. (b) $r(3\cos\theta + 4\sin\theta) = 1$ or $3x + 4y = 1$.

Appendix B: The Real Numbers

1. (b) For instance, $a > 0$, $b > 0$, $a - b < 0$. Then $|a - b| = b - a$, $|a| = a$, $|b| = b$. From $b - a < c$ we wish to deduce $a < b + c$. But this is clear for $a - b < 0$ implies $a - b < c$, hence $a < b + c$.

2. (a) $3.\overline{769230}$, (b) $0.\overline{428571}$, (c) $0.625\overline{0}$, (d) $0.\overline{1176470588235294}$.
3. (a) (d) (e) (f) can easily be put in the form m/n. In the text it is proved that π is irrational; we do not prove that $\sqrt{2}$ is irrational.
5. (a) 62,395/9990, (b) 20,162/990.
7. Yes; for instance, the set of negative irrational numbers.
9. (a) 10. (b) 10.

Appendix C: Functions

2. -1, 4, -21 respectively.
6. $f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$.
8. (a), (d) and (e) are one-to-one; (a), (b) and (e) have their range Y; (a) and (e) are one-to-one correspondences between X and Y.
9. (15,1), (1,2), (4,3), (27,4).
11. (a) 120, (b) 24.
12. (a) Vertical, (b) at most once, (c) exactly once.
13. $f^{-1}(7) = 3$ $f^{-1}(6) = 2$.
14. They are equal.
15. (b) 1/2 and 1/2.
16. 4.
21. f is also a one-to-one correspondence. Proof: If $f(x_1) = f(x_2)$ then $f(f(x_1)) = f(f(x_2))$ or $(f \circ f)(x_1) = (f \circ f)(x_2)$, hence $x_1 = x_2$. Also if y is in X, then $y = (f \circ f)(x)$ for some x in X, hence $y = f(f(x))$; thus y is the image of an element on X, namely the image of f(x).
23. (a) Maternal grandmother, (b) paternal grandmother, (c) maternal grandfather, (d) siblings, (e) cousins.
24. This is the Newton recursion for $\sqrt{4}$. (a) $5/2 = 2.5$, $41/20 = 2.05$, $3281/1640 = 2.0006$. (b) The values approach 2. (No proof expected.)
25. (a) All. $h = h \circ f$ where $f(1) = 1$ and $f(2) = 2$. (b) All, in fact. Let $g(1) = 2$ and $g(2) = 1$. Then

$$h = h \circ f = (h \circ g) \circ g.$$

26. They are one-to-one correspondences and $f = g^{-1}$, $g = f^{-1}$.

Appendix D:

4. (a) $1 - 1/5 = 4/5$.

Appendix E: Length, Area, and Volume

11. The expected number of intersections of a flat surface randomly placed in space with the lines is (intuitively) proportional to its area. Moreover, the expected number of intersections of a convex surface with the

lines is proportional to its average shadow on the xy plane; for if the shadow meets a line, the surface cuts it, generally, twice. For a sphere of radius a, the average shadow is πa^2 and the surface area is $4\pi a^2$. Thus, the surface area of any convex body is four times its average shadow.

Appendix F:

8. (a) $|\sqrt{x} - \sqrt{9}| = \dfrac{|x - 9|}{\sqrt{x} + \sqrt{9}} \leq \dfrac{|x - 9|}{\sqrt{9}} = \dfrac{|x - 9|}{3}$

Thus, if we make $|x - 9| < (3)(0.04)$, we have $|x - 9|/3 < 0.04$. Thus $\delta = 3(0.04) = 0.12$ suffices.

11. Since $|r^n - 0| = \big| |r|^n - 0 \big|$, the result for $-1 < r < 0$ follows from Theorem 8. The case $r = 0$ is trivial.

17. (b) No.

18. (a) $f(x) = x$ in this case, hence differentiable, (b) trouble in defining $f(x)$, (c) yes, the resulting f would be continuous and nowhere differentiable.

Appendix G:

2. (a) $1/2 - 1/3$ or $-1/2 + 2/3$, (b) $2/3 - 2/5$ or $-1/3 + 3/5$. (c) $1/2 + 1/4 - 1/3 - 1/9$ or $-1/4 + 1/3 + 2/9$, (d) $2/3 + 1/27$.

4. (a) $(x + 2)(2x - 1)(x + 2)$, (b) it is prime, (c) since $B^2 - 4AC = 57 > 0$ it is not prime. Write it as $2(x^2 + (1/2)x - 7/2)$ $= 2(x - (-1 + \sqrt{57})/4)(x - (-1 - \sqrt{57})/4) = (2x - (-1 + \sqrt{57})/2)(x - (-1 - \sqrt{57})/4)$.

5. (a) $(x - 2)(x^2 + 2x + 4)$, (b) $(x - \sqrt[3]{4})(x^2 + \sqrt[3]{4}\,x + \sqrt[3]{16})$.

6. (b) $\dfrac{6x + 1}{x^4 + x^2 + 1} = \dfrac{\frac{1}{2}x - \frac{5}{2}}{x^2 + x + 1} + \dfrac{-\frac{1}{2}x + \frac{7}{2}}{x^2 - x + 1}$

9. (a) $\dfrac{2x^2 + 1}{(x - 2)^3} = \dfrac{2}{x - 2} + \dfrac{8}{(x - 2)^2} + \dfrac{9}{(x - 2)^3}$

(b) $\dfrac{x}{(x + 1)^2} = \dfrac{1}{x + 1} - \dfrac{1}{(x + 1)^2}$

(c) $\dfrac{x^3 - 5x^2 + 9x + 1}{(x^2 + 1)(x - 3)^2} = \dfrac{1}{(x - 3)^2} + \dfrac{x}{x^2 + 1}$

10. 2 is a root. The representation is

$$\frac{1/23}{x - 2} + \frac{(-4/23)\,x - 10/23}{4x^2 + 2x + 3}$$

11. Twenty in either case.

12. (a) $$\frac{x^2}{(x^2 + 3x + 1)^2} = \frac{-2\sqrt{5}}{25}\,\frac{1}{\left(x - \left(\frac{-3 + \sqrt{5}}{2}\right)\right)} + \frac{7 - 3\sqrt{5}}{10}\,\frac{1}{\left(x - \left(\frac{-3 + \sqrt{5}}{2}\right)\right)^2}$$
$$+ \frac{2\sqrt{5}}{25}\,\frac{1}{\left(x - \left(\frac{-3 - \sqrt{5}}{2}\right)\right)} + \frac{7 + 3\sqrt{5}}{10}\,\frac{1}{\left(x - \left(\frac{-3 - \sqrt{5}}{2}\right)\right)^2}$$

(b) $$\frac{x}{(x-1)^2(x^3+1)} = \frac{1}{12}\left[\frac{-3}{(x-1)} + \frac{6}{(x-1)^2} - \frac{1}{(x+1)} + \frac{4x-8}{(x^2-x+1)}\right]$$

13. (a) $$\frac{x^5+1}{x^3+1} = x^2 - \frac{x-1}{x^2-x+1}$$

(b) $$\frac{x^4}{(x^2+1)^2} = 1 - \frac{2}{x^2+1} + \frac{1}{(x^2+1)^2} .$$

61004